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On the Schatten–von Neumann properties of some pseudo-differential operators

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ABSTRACT

We obtain a number of explicit estimates for quasi-norms of pseudo-differential operators in the Schatten–von Neumann classes $\mathfrak{S}_q$ with $0 < q \leqslant 1$. The estimates are applied to derive semi-classical bounds for operators with smooth or non-smooth symbols.

1. Introduction

When working with compact pseudo-differential operators it is often important to know how fast their singular values (or eigenvalues) decay. These properties are conveniently stated in terms of the classical Schatten–von Neumann classes $\mathfrak{S}_p$, $p > 0$, or even more general ideals $\mathfrak{S}_{p,q}$, $p, q > 0$. We refer to [3,4,9,24] for information on compact operator ideals.

The Schatten–von Neumann properties of pseudo-differential operators are often determined by smoothness of their symbols. The first bound in the trace class $\mathfrak{S}_1$ was obtained in [23], and later reproduced in [22, Proposition 27.3], and [18, Theorem II-49], see also [13]. Some useful $\mathfrak{S}_1$-bounds were obtained in the much more recent paper [21].

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More general ideals including $\mathfrak{S}_p$ were studied e.g. in [1,6,7,10,11,19], and there one can find further references. The fundamental paper [3] contains $\mathfrak{S}_{p,q}$-estimates for integral operators in terms of smoothness of their kernels.

In spite of a relatively large number of available results, they are not always practically useful since in applications one often needs more detailed information. In this paper we obtain some explicit bounds for Schatten–von Neumann norms of various pseudo-differential operators aiming at applications in semi-classical analysis. Let $p = p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})$, $\mathbf{x}, \mathbf{y}, \boldsymbol{\xi} \in \mathbb{R}^d$, $d \geqslant 1$, be a smooth *amplitude*. For any $\alpha > 0$ introduce the standard notation for the pseudo-differential operator with amplitude p:

$$\big(\mathrm{Op}^{\mathrm{a}}_\alpha(p)\big)u(\mathbf{x}) = \left(\frac{\alpha}{2\pi}\right)^d \iint e^{i\alpha(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) u(\mathbf{y})\, d\mathbf{y}\, d\boldsymbol{\xi}, \tag{1.1}$$

for any Schwartz class function u. In the literature one uses more often the reciprocal value α^{-1} which is interpreted as the Planck constant. It is natural for us to study a somewhat more general variant of the operator (1.1). Let $\mathbf{T} = \{t_{jk}\}$ be a non-degenerate (2×2)-matrix with real-valued entries. We concentrate on the operators

$$\begin{cases} \mathrm{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}}), \quad p_{\mathbf{T}}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}), \\ \quad \text{with } \mathbf{w} = t_{11}\mathbf{x} + t_{12}\mathbf{y}, \mathbf{z} = t_{21}\mathbf{x} + t_{22}\mathbf{y}. \end{cases} \tag{1.2}$$

This choice of the amplitude allows us to derive bounds for various standard quantizations of pseudo-differential operators. For a smooth *symbol* $a = a(\mathbf{x}, \boldsymbol{\xi})$ and a number $t \in [0, 1]$ we define the t-quantization as the pseudo-differential operator

$$\big(\mathrm{Op}_{\alpha,t}(a)u\big)(\mathbf{x}) = \left(\frac{\alpha}{2\pi}\right)^d \iint e^{i\alpha(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} a\big((1-t)\mathbf{x} + t\mathbf{y}, \boldsymbol{\xi}\big) u(\mathbf{y})\, d\mathbf{y}\, d\boldsymbol{\xi}, \tag{1.3}$$

for any Schwartz class function u, see e.g. [18, Chapter 2, §4], [6] or [27]. It is clear that this operator can be written as

$$\mathrm{Op}_{\alpha,t}(a) = \mathrm{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}}), \quad \text{with } p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) = a(\mathbf{w}, \boldsymbol{\xi}),\ \mathbf{T} = \begin{pmatrix} 1-t & t \\ -1 & 1 \end{pmatrix}. \tag{1.4}$$

In this formula the choice of the second row in the matrix $\mathbf{T}$ is unimportant as long as $\mathbf{T}$ remains non-degenerate. Note also that formally $(\mathrm{Op}_{\alpha,t}(a))^* = \mathrm{Op}_{\alpha,1-t}(\bar{a})$. The values $t = 0$ and $t = 1$ give the standard "left", or Kohn–Nirenberg, and "right" quantizations. In these cases the operator (1.3) has the symbol $a(\mathbf{x}, \boldsymbol{\xi})$ (for $t = 0$) or $a(\mathbf{y}, \boldsymbol{\xi})$ (for $t = 1$). In the literature one sometimes uses for them the notation $\mathrm{Op}^{\mathrm{l}}_\alpha(a)$ and $\mathrm{Op}^{\mathrm{r}}_\alpha(a)$ respectively. Another important example is the Weyl quantization:

$$\mathrm{Op}^{\mathrm{W}}_\alpha(a) = \mathrm{Op}_{\alpha,\frac{1}{2}}(a),$$

which has the advantage that $\mathbf{x}$ and $\mathbf{y}$ enter the definition (1.3) symmetrically. If the symbol a depends only on $\boldsymbol{\xi}$ then the operators (1.3) for different values of t coincide with each other and we write simply $\mathrm{Op}_\alpha(a)$.

If the functions p and a above are sufficiently smooth and decay sufficiently fast at infinity then the operators (1.2) and (1.3) belong to $\mathfrak{S}_q$ with a suitable $q > 0$. The aim of the paper is to study this property for $q \in (0, 1]$. Our results are divided in three groups. First in Section 2 we obtain general estimates in $\mathfrak{S}_q$ for $\alpha = 1$, see Theorems 2.5 and 2.6. The $\mathfrak{S}_q$-bounds for the operators (1.2) seem to be quite useful from the practical point of view. In particular they allow us to study the operators of the form $h_1 \mathrm{Op}_{1,t}(a)h_2$, $t \in [0, 1]$ with the weights h_1, h_2 whose supports are disjoint, and to control explicitly the dependence on the distance between the supports, see Theorem 2.6(2). Our approach stems from a simple idea suggested in the paper [21] where trace class properties of pseudo-differential operators were studied. In fact, our results can be viewed as quantitative variants of Proposition 3.2 and Theorem 3.5 from [21], extended to the ideals $\mathfrak{S}_q$, $q \leqslant 1$. As the classes $\mathfrak{S}_q$ with $q < 1$ are not normed, the obtained $\mathfrak{S}_q$-estimates for the operators (1.2) and (1.3) involve the so-called *lattice quasi-norms* (see (2.3)) for the amplitudes/symbols and their derivatives (for $q = 1$ these quasi-norms are simply L^1-integral norms). The estimates in $\mathfrak{S}_q$ with $q > 1$ are also of great interest, but they are likely to be stated in different terms, cf. [1,6,27], and thus they are not discussed here.

Sections 3 and 4 are devoted to applications. In Section 3 we use Theorems 2.5 and 2.6 to derive estimates for large values of the parameter α, which can be interpreted as the semi-classical regime. These results are stated in terms of the scaling properties of the symbols which makes them flexible and convenient for applications. Section 4 is concerned with semi-classical bounds for operators with discontinuous symbols. We emphasize that the term “discontinuous symbol” is not understood literally: we are interested in operators with smooth symbols a sandwiched between characteristic functions $\chi_\Lambda(\mathbf{x})$ and $\chi_\Omega(\boldsymbol{\xi})$ of some Lipschitz domains Λ and Ω. Precisely, we derive $\mathfrak{S}_q$-semi-classical estimates for the Hankel-type operators $\chi_\Lambda \mathrm{Op}_{\alpha,t}(a)(I - \chi_\Lambda)$ and $\chi_\Lambda P_{\Omega,\alpha}(I - \chi_\Lambda)$, $P_{\Omega,\alpha} = \mathrm{Op}_\alpha(\chi_\Omega)$, with a smooth symbol a. This study is motivated by the trace asymptotics for Wiener–Hopf and Hankel operators, both classical, see e.g. [16,28], and multi-dimensional, see [26,25]. The Schatten–von Neumann properties of classical Wiener–Hopf operators were also studied e.g. in [17]. Certain types of Toeplitz and Hankel operators were considered in [2,15]. In the literature one also finds results on the Schatten–von Neumann properties of the operators $\mathrm{Op}_{\alpha,t}(a)$ with genuinely discontinuous symbols a. An interesting special case of such a symbol is the characteristic function of a domain in $\mathbb{R}^d \times \mathbb{R}^d$. These issues are beyond the scope of the present paper, and we refer to [10] and also to more recent paper [27] where one can find further references.

A number of estimates similar to the ones in Sections 3 and 4 have been established in [26] for the trace class $\mathfrak{S}_1$. However some applications in Mathematical Physics, and in particular in Quantum Information Theory, call for estimates in the classes of compact operators with a faster decay of the singular values, see [8,12]. This was the main incentive for the current paper.

To conclude the Introduction we make some notational conventions. Throughout the paper we denote by C or c with or without indices various positive constants whose value is unimportant. The notation $B(\mathbf{u}, r)$ is used for the open ball in $\mathbb{R}^d$, $d \geqslant 1$, of radius $r > 0$ centred at the point $\mathbf{u} \in \mathbb{R}^d$. The characteristic function of the ball $B(\mathbf{u}, r)$ is denoted by $\chi_{\mathbf{u},r}$.

2. General estimates in $\mathfrak{S}_q$-ideals with $q \in (0, 1]$: smooth symbols

2.1. Ideals $\mathfrak{S}_q$

The notation $\mathfrak{S}_q$, $q > 0$, is standard for the set of all compact operators A on a Hilbert space with singular values $s_k(A)$, $k = 1, 2, \dots$, for which the functional

$$\|A\|_{\mathfrak{S}_q} = \left(\sum_{k=1}^{\infty} s_k(A)^q \right)^{\frac{1}{q}}$$

is finite. For $q \geqslant 1$ this functional defines a natural norm on $\mathfrak{S}_q$, whereas for $q < 1$ it defines a quasi-norm. Nevertheless one has the triangle inequality of the form

$$\|A_1 + A_2\|_{\mathfrak{S}_q}^q \leqslant \|A_1\|_{\mathfrak{S}_q}^q + \|A_2\|_{\mathfrak{S}_q}^q, \quad 0 < q \leqslant 1, \tag{2.1}$$

see [20] and [4, p. 262], and the following Hölder-type inequality:

$$\|A_1 A_2\|_{\mathfrak{S}_q} \leqslant \|A_1\|_{\mathfrak{S}_{q_1}} \|A_2\|_{\mathfrak{S}_{q_2}}, \quad q^{-1} = q_1^{-1} + q_2^{-1},\ 0 < q_1, q_2 \leqslant \infty, \tag{2.2}$$

see [4, p. 262].

A crucial technical point in the study of the operators (1.2) is to estimate suitable $\mathfrak{S}_q$-(quasi)-norms for the operators $h \operatorname{Op}_1(a)$, $h = h(\mathbf{x})$, $a = a(\boldsymbol{\xi})$, which have been studied quite extensively. We need the following estimate which is a slight generalization of the bound found in [3, Theorem 11.1] (see also [5, Section 5.8]), and quoted in [24, Theorem 4.5] for $s \in [1, 2]$.

Let $\mathcal{C}_{\mathbf{u}} \subset \mathbb{R}^m$ be a cube centred at $\mathbf{u} \in \mathbb{R}^m$ with the edge of unit length. For a function $h \in \mathsf{L}^r_{\mathrm{loc}}(\mathbb{R}^m)$, $r \in (0, \infty)$, denote

$$\begin{cases} \left|\!\left|\!\left| h \right|\!\right|\!\right|_{r,\delta} = \left[\sum_{\mathbf{n} \in \mathbb{Z}^d} \left(\int_{\mathcal{C}_{\mathbf{n}}} |h(\mathbf{x})|^r \, d\mathbf{x} \right)^{\frac{\delta}{r}} \right]^{\frac{1}{\delta}}, & 0 < \delta < \infty, \\ \left|\!\left|\!\left| h \right|\!\right|\!\right|_{r,\infty} = \sup_{\mathbf{u} \in \mathbb{R}^d} \left(\int_{\mathcal{C}_{\mathbf{u}}} |h(\mathbf{x})|^r \, d\mathbf{x} \right)^{\frac{1}{r}}, & \delta = \infty. \end{cases} \tag{2.3}$$

These functionals are sometimes called *lattice quasi-norms* (norms for $r, \delta \geqslant 1$). If $\|h\|_{r,\delta} < \infty$ we say that $h \in \mathsf{l}^\delta(\mathsf{L}^r)(\mathbb{R}^m)$.

Proposition 2.1. *Suppose that $f \in \mathsf{l}^q(\mathsf{L}^2)(\mathbb{R}^n)$ and $g \in \mathsf{l}^q(\mathsf{L}^2)(\mathbb{R}^m)$, with some $q \in (0,2]$. Let $K : \mathsf{L}^2(\mathbb{R}^m) \to \mathsf{L}^2(\mathbb{R}^n)$ be the operator with the kernel*

$$f(\mathbf{x}) e^{i\mathbf{x}\cdot\mathbf{S}\mathbf{y}} g(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n, \ \mathbf{y} \in \mathbb{R}^m,$$

where $\mathbf{S} : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map. Then

$$\|K\|_{\mathfrak{S}_q} \leqslant C_q \|f\|_{2,q} \|g\|_{2,q},$$

with a constant $C_q = C_q(\mathbf{S})$ depending only on the number s_0 in the bound $\max_{jk} |s_{jk}| \leqslant s_0$ for the entries s_{jk}, $j = 1, 2, \dots, n$; $k = 1, 2, \dots, m$, of the matrix $\mathbf{S}$.

We do not give the proof as it repeats that of [3, Theorem 11.1] almost word for word.

2.2. Estimates for the operators (1.2)

Now we need to specify the conditions on the matrix $\mathbf{T} = \{t_{jk}\}$, $j, k = 1, 2$. The end results require $\mathbf{T}$ to be non-degenerate, i.e. $\mathbf{T} \in GL(2, \mathbb{R})$. For convenience we sometimes assume that

$$t_{11} + t_{12} = 1, \tag{2.4}$$

and denote

$$\tau = t_{21} + t_{22}. \tag{2.5}$$

Using the inverse of $\mathbf{T}$, we can recover $\mathbf{x}$ and $\mathbf{y}$ from the vectors $\mathbf{w}$ and $\mathbf{z}$ defined in (1.2):

$$\begin{cases} (\det \mathbf{T})\mathbf{x} = t_{22}\mathbf{w} - t_{12}\mathbf{z}, \qquad (\det \mathbf{T})\mathbf{y} = -t_{21}\mathbf{w} + t_{11}\mathbf{z}, \\ \quad \text{so } (\det \mathbf{T})(\mathbf{x} - \mathbf{y}) = \tau\mathbf{w} - \mathbf{z}. \end{cases} \tag{2.6}$$

We assume that

$$\max_{jk} |t_{jk}| \leqslant t_0, \qquad |\det \mathbf{T}| \geqslant \delta_0, \tag{2.7}$$

with some fixed positive numbers t_0, δ_0. In the estimates below the constants may be dependent on t_0 and δ_0. We provide appropriate comments in every instance.

Assuming that $p(\cdot,\cdot,\boldsymbol{\xi}) \in \mathsf{L}^1(\mathbb{R}^{2d})$, introduce the "double" Fourier transform:

$$\hat{p}(\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^d} \iint e^{-i\mathbf{w}\cdot\boldsymbol{\eta} - i\mathbf{z}\cdot\boldsymbol{\mu}} p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) \, d\mathbf{w} \, d\mathbf{z}.$$

Lemma 2.2. *Let* $\mathbf{T}$ *be an arbitrary* (2×2)*-matrix with real-valued entries. Suppose that* $p(\cdot,\cdot,\boldsymbol{\xi}) \in \mathsf{L}^1(\mathbb{R}^{2d})$ *for a.e.* $\boldsymbol{\xi} \in \mathbb{R}^d$*. Let* $h_1, h_2 \in \mathsf{l}^{2q}(\mathsf{L}^2)(\mathbb{R}^d)$*, and let* $\hat{p} \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^{3d})$ *with some* $q \in (0,1]$*. Then the operator* $h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2$ *belongs to* $\mathfrak{S}_q$ *and*

$$\left\| h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2 \right\|_{\mathfrak{S}_q} \leqslant C_q \| h_1 \|_{2,2q} \| h_2 \|_{2,2q} \| \hat{p} \|_{1,q}, \tag{2.8}$$

with a constant $C_q = C_q(t_0)$.

Proof. Represent the amplitude a via its Fourier transform

$$p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^d} \iint e^{i\mathbf{z}\cdot\boldsymbol{\eta} + i\mathbf{w}\cdot\boldsymbol{\mu}} \hat{p}(\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi}) \, d\boldsymbol{\eta} \, d\boldsymbol{\mu},$$

and rewrite $A = h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2$ as follows:

$$A = B_1 B_2^*,$$

where $B_j : \mathsf{L}^2(\mathbb{R}^{3d}) \to \mathsf{L}^2(\mathbb{R}^d)$, $j = 1, 2$, are the operators with the kernels

$$b_1(\mathbf{x}; \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^d} h_1(\mathbf{x}) e^{i\mathbf{x}\cdot(\boldsymbol{\xi} + t_{11}\boldsymbol{\eta} + t_{21}\boldsymbol{\mu})} \hat{p}(\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi})^{\frac{1}{2}},$$
$$b_2(\mathbf{x}; \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^d} \overline{h_2(\mathbf{x})} e^{i\mathbf{x}\cdot(\boldsymbol{\xi} - t_{12}\boldsymbol{\eta} - t_{22}\boldsymbol{\mu})} \left| \hat{p}(\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi}) \right|^{\frac{1}{2}},$$

where $z^{1/2} = z|z|^{-1/2}$ for any $z \neq 0$. By Proposition 2.1,

$$\|B_j\|_{\mathfrak{S}_{2q}} \leqslant C_q(t_0) \| |\hat{p}|^{1/2} \|_{2,2q} \| h_j \|_{2,2q}, \quad j = 1, 2.$$

Now (2.8) follows from (2.2). $\square$

It is usually more convenient to write $\mathfrak{S}_q$-estimates in terms of the amplitudes themselves, and not their Fourier transforms. For $m, n = 0, 1, \dots,$ let

$$P_{n,m}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p) = \frac{1}{1 + |\mathbf{z} - \tau\mathbf{w}|^m} \sum_{n_1, n_2 = 0}^{n} \sum_{l=0}^{m} \left| \nabla_{\mathbf{w}}^{n_1} \nabla_{\mathbf{z}}^{n_2} \nabla_{\boldsymbol{\xi}}^{l} p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) \right|, \tag{2.9}$$

$$Q_{n,m}(\boldsymbol{\xi}; p) = \iint P_{n,m}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) \, d\mathbf{w} \, d\mathbf{z}. \tag{2.10}$$

The parameter τ is defined in (2.5).

Corollary 2.3. *Let the matrix* $\mathbf{T}$ *and the functions* h_1, h_2 *be as in Lemma 2.2, and let* p *be such that* $Q_{n,m}(p) \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^d)$ *with some* $q \in (0,1]$*, and*

$$n = \left[d q^{-1} \right] + 1. \tag{2.11}$$

Then

$$\big\| h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2 \big\|_{\mathfrak{S}_q} \leqslant C_q {\vert\!\vert\!\vert} h_1 {\vert\!\vert\!\vert}_{2,2q} {\vert\!\vert\!\vert} h_2 {\vert\!\vert\!\vert}_{2,2q} {\vert\!\vert\!\vert} Q_{n,0}(p) {\vert\!\vert\!\vert}_{1,q}, \tag{2.12}$$

with a constant $C_q = C_q(t_0)$.

Proof. Integrating by parts, we get

$$\big|\hat{p}(\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\xi})\big| \leqslant C(n)\big(1 + |\boldsymbol{\eta}|\big)^{-n}\big(1 + |\boldsymbol{\mu}|\big)^{-n} \sum_{n_1, n_2 = 0}^{n} \iint \big|\nabla_{\mathbf{w}}^{n_1} \nabla_{\mathbf{z}}^{n_2} p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi})\big| \, d\mathbf{w} \, d\mathbf{z}.$$

For $n = [dq^{-1}] + 1$ the function on the right-hand side belongs to $\mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^{3d})$, and its quasi-norm (2.3) does not exceed $C {\vert\!\vert\!\vert} Q_{n,0}(p) {\vert\!\vert\!\vert}_{1,q}$. Now (2.12) follows from Lemma 2.2. □

Lemma 2.4. *Suppose that* $\mathbf{T} \in GL(2, \mathbb{R})$ *satisfies (2.4). Let* h_1, h_2 *be as in Lemma 2.2, and let* p *be such that* $Q_{n,m}(p) \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^d)$ *with some* $q \in (0, 1]$, *with* n *satisfying (2.11), and some* $m = 0, 1, \ldots$. *Then*

$$\big\| h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2 \big\|_{\mathfrak{S}_q} \leqslant C_q {\vert\!\vert\!\vert} h_1 {\vert\!\vert\!\vert}_{2,2q} {\vert\!\vert\!\vert} h_2 {\vert\!\vert\!\vert}_{2,2q} {\vert\!\vert\!\vert} Q_{n,m}(p) {\vert\!\vert\!\vert}_{1,q}, \tag{2.13}$$

with a constant $C_q = C_q(t_0)$.

Proof. Let

$$\mathcal{P}_{\mathbf{x}}^{(\pm)} = \big(1 \pm i(\det \mathbf{T})^2 \mathbf{x} \cdot \nabla_{\boldsymbol{\xi}}\big)\big(1 + (\det \mathbf{T})^2 |\mathbf{x}|^2\big)^{-1}.$$

Clearly, $\mathcal{P}_{\mathbf{x}}^{(-)} e^{i\boldsymbol{\xi}\cdot\mathbf{x}} = e^{i\boldsymbol{\xi}\cdot\mathbf{x}}$, so integrating by parts m times, we get the following formula for the kernel of the operator $\operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}})$:

$$\frac{1}{(2\pi)^d} \int e^{i\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{y})} p_{\mathbf{T}}^{(m)}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \, d\boldsymbol{\xi},$$

with

$$p_{\mathbf{T}}^{(m)}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = \big(\mathcal{P}_{\mathbf{x}-\mathbf{y}}^{(+)}\big)^m p_{\mathbf{T}}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}),$$

so by (2.6)

$$p^{(m)}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) = \big(1 + |\tau \mathbf{w} - \mathbf{z}|^2\big)^{-m}\big(1 + i(\det \mathbf{T})(\tau \mathbf{w} - \mathbf{z}) \cdot \nabla_{\boldsymbol{\xi}}\big)^m p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}).$$

Now it is straightforward to see that

$$P_{n,0}\big(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p^{(m)}\big) \leqslant C(t_0) P_{n,m}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p).$$

By Corollary 2.3 this implies the proclaimed result. □

In the next theorem we replace the $(2, 2q)$-quasi-norms of functions h_1, h_2 by much weaker ones.

Theorem 2.5. *Suppose that* $\mathbf{T} \in GL(2, \mathbb{R})$ *satisfies (2.4). Let* $h_1, h_2 \in \mathsf{l}^\infty(\mathsf{L}^2)(\mathbb{R}^d)$*, and let* p *be such that* $P_{n,m} \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^{3d})$ *with some* $q \in (0, 1]$*, with* n *satisfying (2.11) and some* $m = 0, 1, \ldots$ *. Then*

$$\left\| h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2 \right\|_{\mathfrak{S}_q} \leqslant C_{q,m} \left\bracevert h_1 \right\bracevert_{2,\infty} \left\bracevert h_2 \right\bracevert_{2,\infty} \left\bracevert P_{n,m}(p) \right\bracevert_{1,q}, \tag{2.14}$$

with a constant $C_{q,m} = C_{q,m}(t_0, \delta_0)$ *depending on* t_0 *and* δ_0*.*

Proof. Let us define a convenient partition of unity. The open balls $B(\mathbf{j}, 2\sqrt{d}\,)$, $\mathbf{j} \in \mathbb{Z}^d$, form a covering of $\mathbb{R}^d$. Let $\{\psi_{\mathbf{j}}\}$ be an associated partition of unity such that

$$\left| \nabla_{\mathbf{x}}^k \psi_{\mathbf{j}}(\mathbf{x}) \right| \leqslant C_k, \quad k = 0, 1, \ldots, \tag{2.15}$$

uniformly in $\mathbf{j} \in \mathbb{Z}^d$. Let us consider the operator $\operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}^{(\mathbf{j},\mathbf{s})})$ with the amplitude

$$p^{(\mathbf{j},\mathbf{s})}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) = \psi_{\mathbf{j}}(\mathbf{w}) \psi_{\mathbf{s}}(\mathbf{z}) p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}).$$

Since $\mathbf{w} \in B(\mathbf{j}, 2\sqrt{d}\,)$, $\mathbf{z} \in B(\mathbf{s}, 2\sqrt{d}\,)$, we have

$$\mathbf{x} \in B(\mathbf{l}, R), \qquad \mathbf{y} \in B(\mathbf{n}, R), \quad \text{with } R = \frac{4t_0\sqrt{d}}{\delta_0},$$
$$\mathbf{l} = \frac{t_{22}\mathbf{j} - t_{21}\mathbf{s}}{\det \mathbf{T}}, \qquad \mathbf{n} = \frac{-t_{21}\mathbf{j} + t_{11}\mathbf{s}}{\det \mathbf{T}},$$

see (2.6). Consequently

$$h_1 \operatorname{Op}_1^{\mathrm{a}}\big(p_{\mathbf{T}}^{(\mathbf{j},\mathbf{s})}\big) h_2 = h_1 \chi_{\mathbf{l},R} \operatorname{Op}_1^{\mathrm{a}}\big(p_{\mathbf{T}}^{(\mathbf{j},\mathbf{s})}\big) h_2 \chi_{\mathbf{n},R},$$

and hence by Lemma 2.4,

$$\left\| h_1 \operatorname{Op}_1^{\mathrm{a}}\big(p_{\mathbf{T}}^{(\mathbf{j},\mathbf{s})}\big) h_2 \right\|_{\mathfrak{S}_q} \leqslant C_q \left\bracevert h_1 \chi_{\mathbf{l},R} \right\bracevert_{2,2q} \left\bracevert h_2 \chi_{\mathbf{n},R} \right\bracevert_{2,2q} \left\bracevert Q_{n,m}\big(p^{(\mathbf{j},\mathbf{s})}\big) \right\bracevert_{1,q}.$$

The first two factors are estimated by $C \bracevert h_1 \bracevert_{2,\infty}$ and $C \bracevert h_2 \bracevert_{2,\infty}$ respectively, with some constant $C = C(t_0, \delta_0)$. Thus by the triangle inequality (2.1)

$$\begin{aligned} \left\| h_1 \operatorname{Op}_1^{\mathrm{a}}(p_{\mathbf{T}}) h_2 \right\|_{\mathfrak{S}_q}^q &\leqslant \sum_{\mathbf{j},\mathbf{s}} \left\| h_1 \operatorname{Op}_1^{\mathrm{a}}\big(p_{\mathbf{T}}^{(\mathbf{j},\mathbf{s})}\big) h_2 \right\|_{\mathfrak{S}_q}^q \\ &\leqslant C_q \bracevert h_1 \bracevert_{2,\infty}^q \bracevert h_2 \bracevert_{2,\infty}^q \sum_{\mathbf{j},\mathbf{s}} \left\bracevert Q_{n,m}\big(p^{(\mathbf{j},\mathbf{s})}\big) \right\bracevert_{1,q}^q. \end{aligned}$$

Remembering that the number of intersecting balls $B(\mathbf{j}, 2\sqrt{d}\,)$ is uniformly bounded, we can estimate the sum on the right-hand side by $\tilde{C}{\mathbf{|}} P_{n,m}(p){\mathbf{|}}_{1,q}^q$. This completes the proof. □

2.3. Estimates for the operators (1.3)

Theorem 2.5 allows amplitudes independent of $\mathbf{z}$, e.g. it allows one to consider t-pseudo-differential operators (1.3). We isolate this observation in a separate theorem. For a symbol $a = a(\mathbf{x}, \boldsymbol{\xi})$ denote

$$F^{\circ}_{n,m}(\mathbf{w}, \boldsymbol{\xi}; a) = \sum_{k=0}^{n} \big|\nabla_{\mathbf{w}}^k \nabla_{\boldsymbol{\xi}}^m a(\mathbf{w}, \boldsymbol{\xi})\big|,$$

$$F_{n,m}(\mathbf{w}, \boldsymbol{\xi}; a) = \sum_{l=0}^{m} F^{\circ}_{n,l}(\mathbf{w}, \boldsymbol{\xi}; a), \quad n, m = 0, 1, \ldots. \tag{2.16}$$

The constants in the next theorem are independent of $t \in [0, 1]$.

Theorem 2.6. *Let $h_1, h_2 \in \mathsf{l}^\infty(\mathsf{L}^2)(\mathbb{R}^d)$, let n be as in (2.11), and $q \in (0, 1]$.*

(1) *Suppose that $F_{n,n}(a) \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^{2d})$. Then for any $t \in [0, 1]$ we have*

$$\big\|h_1 \operatorname{Op}_{1,t}(a) h_2\big\|_{\mathfrak{S}_q} \leqslant C_q {\mathbf{|}} h_1 {\mathbf{|}}_{2,\infty} {\mathbf{|}} h_2 {\mathbf{|}}_{2,\infty} {\mathbf{|}} F_{n,n}(a) {\mathbf{|}}_{1,q}. \tag{2.17}$$

(2) *Suppose that the distance between the supports of the functions h_1, h_2 is at least $r \geqslant 1$. If $F^{\circ}_{n,m}(a) \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^{2d})$, $m \geqslant n$, then for any $t \in [0, 1]$ we have*

$$\big\|h_1 \operatorname{Op}_{1,t}(a) h_2\big\|_{\mathfrak{S}_q} \leqslant C_{q,m} r^{\frac{d}{q} - m} {\mathbf{|}} h_1 {\mathbf{|}}_{2,\infty} {\mathbf{|}} h_2 {\mathbf{|}}_{2,\infty} {\mathbf{|}} F^{\circ}_{n,m}(a) {\mathbf{|}}_{1,q}. \tag{2.18}$$

Proof. Use Theorem 2.5 with $p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) = a(\mathbf{w}, \boldsymbol{\xi})$ and the matrix

$$\mathbf{T} = \begin{pmatrix} 1 - t & t \\ -1 & 1 \end{pmatrix}, \tag{2.19}$$

so that $\tau = 0$, see (2.5). By definitions (2.9) and (2.16),

$$P_{n,m}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p) \leqslant \frac{F_{n,m}(\mathbf{w}, \boldsymbol{\xi}; a)}{1 + |\mathbf{z}|^m}.$$

To estimate ${\mathbf{|}} P_{n,m}(p) {\mathbf{|}}_{1,q}$ write for any $\mathbf{k}, \mathbf{s}, \mathbf{j} \in \mathbb{Z}^d$:

$$\int_{\mathcal{C}_\mathbf{k}} \int_{\mathcal{C}_\mathbf{s}} \int_{\mathcal{C}_\mathbf{j}} P_{n,m}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p)\, d\mathbf{w}\, d\mathbf{z}\, d\boldsymbol{\xi} \leqslant C \frac{1}{1 + |\mathbf{s}|^m} \int_{\mathcal{C}_\mathbf{k}} \int_{\mathcal{C}_\mathbf{j}} F_{n,m}(\mathbf{w}, \boldsymbol{\xi}; a)\, d\mathbf{w}\, d\boldsymbol{\xi}.$$

Consequently,

$$\left\| P_{n,m}(p) \right\|_{1,q}^{q} \leqslant C \left\| F_{n,m}(a) \right\|_{1,q}^{q} \sum_{\mathbf{s}\in\mathbb{Z}^d} \frac{1}{1+|\mathbf{s}|^{mq}} \leqslant C' \left\| F_{n,m}(a) \right\|_{1,q}^{q}.$$

Here we have used the fact that $mq \geqslant nq > d$. Now Theorem 2.5 with $m = n$ implies (2.17).

Proof of (2.18). Let $\zeta \in \mathsf{C}^\infty(\mathbb{R})$ be a function such that $0 \leqslant \zeta \leqslant 1$ and

$$\zeta(u) = \begin{cases} 1, & |u| \geqslant 1; \\ 0, & |u| \leqslant \frac{1}{2}. \end{cases} \tag{2.20}$$

Note that

$$h_1 \operatorname{Op}_{1,t}(a) h_2 = h_1 \operatorname{Op}_1^{\mathrm{a}}(g_{\mathbf{T}}) h_2, \qquad g(\mathbf{w},\mathbf{z},\boldsymbol{\xi}) = \zeta\bigl(|\mathbf{z}|r^{-1}\bigr) a(\mathbf{w},\boldsymbol{\xi}),$$

where the matrix $\mathbf{T}$ is defined as in (2.19). We use Theorem 2.5 again but in a slightly different way than above – first we implement integration by parts similar to the one done in the proof of Lemma 2.4. Let $\mathcal{P}_{\mathbf{z}}^{(\pm)} = (\pm i\mathbf{z}\cdot\nabla_{\boldsymbol{\xi}})|\mathbf{z}|^{-2}$. Clearly, $\mathcal{P}_{\mathbf{z}}^{(+)} e^{-i\boldsymbol{\xi}\cdot\mathbf{z}} = e^{-i\boldsymbol{\xi}\cdot\mathbf{z}}$, so, integrating by parts m times, we get the following formula for the kernel of the operator $\operatorname{Op}_1^{\mathrm{a}}(g_{\mathbf{T}})$:

$$\frac{1}{(2\pi)^d} \int e^{i\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{y})} g_{\mathbf{T}}^{(m)}(\mathbf{x},\mathbf{y},\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

with

$$g^{(m)}(\mathbf{w},\mathbf{z},\boldsymbol{\xi}) = \bigl(\mathcal{P}_{\mathbf{z}}^{(-)}\bigr)^m g(\mathbf{w},\mathbf{z},\boldsymbol{\xi}).$$

It is straightforward to see that

$$P_{n,0}\bigl(\mathbf{w},\mathbf{z},\boldsymbol{\xi}; g^{(m)}\bigr) \leqslant C \frac{F_{n,m}^{\circ}(\mathbf{w},\boldsymbol{\xi}; a)}{r^m + |\mathbf{z}|^m},$$

with a constant independent of r. Arguing as in the first part of the proof we get the bound

$$\left\| P_{n,0}\bigl(g^{(m)}\bigr) \right\|_{1,q}^{q} \leqslant C \left\| F_{n,m}^{\circ}(a) \right\|_{1,q}^{q} \sum_{\mathbf{s}\in\mathbb{Z}^d} \frac{1}{r^{mq} + |\mathbf{s}|^{mq}} \leqslant C' \left\| F_{n,m}^{\circ}(a) \right\|_{1,q}^{q} r^{d-mq}.$$

Theorem 2.5 with $m \geqslant n$ leads to (2.18). □

As the next theorem shows, in the case $d = 1$, when h_1 and h_2 have disjoint supports, one can sometimes allow symbols a depending only on ξ. Here and below we use x and ξ for one-dimensional variables.

Theorem 2.7. *Let $h_1, h_2 \in \mathsf{l}^\infty(\mathsf{L}^2)(\mathbb{R})$ be two functions such that*

$$h_1(x) = 0, \quad a.e.\ x > -\frac{r}{2}, \qquad h_2(x) = 0, \quad a.e.\ x < \frac{r}{2},$$

with some $r \geqslant 1$. Let $q \in (0, 1]$ be some number, and let $n = [q^{-1}] + 1$. Suppose that $a = a(\xi)$ satisfies the condition $\partial^m a \in \mathsf{l}^q(\mathsf{L}^1)(\mathbb{R})$, for some $m \geqslant n$. Then we have

$$\left\| h_1 \operatorname{Op}_1(a) h_2 \right\|_{\mathfrak{S}_q} \leqslant C_{q,m} r^{\frac{1}{q}-m} \left| h_1 \right|_{2,\infty} \left| h_2 \right|_{2,\infty} \left| \partial^m a \right|_{1,q}. \tag{2.21}$$

Proof. As in the proof of the previous theorem,

$$h_1 \operatorname{Op}_1(a) h_2 = h_1 \operatorname{Op}_1^{\mathrm{a}}(g) h_2, \qquad g(x, y, \xi) = \zeta\big(|x - y| r^{-1}\big) a(\xi),$$

where ζ is as defined in (2.20). Furthermore, integrating by parts m times we get the following formula for the kernel:

$$\frac{1}{2\pi} \int e^{i\xi(x-y)} g^{(m)}(x, y, \xi)\, d\xi, \quad g^{(m)}(x, y, \xi) = i^m \frac{\partial^m a(\xi)}{(x - y)^m}.$$

By definition of h_1, h_2 we obtain

$$P_{n,0}\big(x, y, \xi; g^{(m)}\big) \leqslant C \frac{|\partial^m a(\xi)|}{|x|^m + |y|^m + r^m}.$$

Since $m \geqslant n = [q^{-1}] + 1$, the right-hand side belongs to $\mathsf{l}^q(\mathsf{L}^1)(\mathbb{R}^3)$, and the quasi-norm is bounded from above by $\left| \partial^m a \right|_{1,q}$. Now the estimate (2.21) follows from Theorem 2.5. □

2.4. Trace-class estimates

For $q = 1$ the lattice quasi-norms in Theorems 2.5 and 2.6 coincide with the standard L^1-norms. Due to the relative simplicity of these bounds it seems appropriate to write them out separately. Moreover making the change $\alpha\boldsymbol{\xi} = \boldsymbol{\xi}'$ we can immediately extend them to all values $\alpha \geqslant 1$:

Theorem 2.8. *Suppose that $\mathbf{T} \in GL(2, \mathbb{R})$ satisfies (2.4). Let $h_1, h_2 \in \mathsf{l}^\infty(\mathsf{L}^2)(\mathbb{R}^d)$, and $P_{d+1,m} \in \mathsf{L}^1(\mathbb{R}^{3d})$, with some $m = 0, 1, \ldots$. Then for any $\alpha \geqslant 1$ we have*

$$\begin{aligned} \left\| h_1 \operatorname{Op}_\alpha^{\mathrm{a}}(p_{\mathbf{T}}) h_2 \right\|_{\mathfrak{S}_1} &\leqslant C_m \alpha^d \left| h_1 \right|_{2,\infty} \left| h_2 \right|_{2,\infty} \\ &\quad \times \sum_{n_1, n_2 = 0}^{d+1} \sum_{l=0}^{m} \iiint \frac{|\nabla_{\mathbf{w}}^{n_1} \nabla_{\mathbf{z}}^{n_2} \nabla_{\boldsymbol{\xi}}^{l} p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi})|}{1 + |\tau \mathbf{w} - \mathbf{z}|^m}\, d\mathbf{w}\, d\mathbf{z}\, d\boldsymbol{\xi}, \end{aligned} \tag{2.22}$$

with a constant $C_m = C_m(t_0, \delta_0)$.

Theorem 2.9. *Let $h_1, h_2 \in \mathsf{l}^\infty(\mathsf{L}^2)(\mathbb{R}^d)$, and $\alpha \geqslant 1$.*

(1) *Suppose that $F_{d+1,d+1}(a) \in \mathsf{L}^1(\mathbb{R}^{2d})$. Then for any $t \in [0,1]$ we have*

$$\big\|h_1 \operatorname{Op}_{\alpha,t}(a) h_2\big\|_{\mathfrak{S}_1} \leqslant C\alpha^d {|\!|\!|} h_1 {|\!|\!|}_{2,\infty} {|\!|\!|} h_2 {|\!|\!|}_{2,\infty} \times \sum_{k,l=0}^{d+1} \iint \big|\nabla_{\mathbf{w}}^k \nabla_{\boldsymbol{\xi}}^l a(\mathbf{w}, \boldsymbol{\xi})\big| \, d\mathbf{w} \, d\boldsymbol{\xi}. \tag{2.23}$$

(2) *Suppose that the distance between the supports of the functions h_1, h_2 is at least $r \geqslant 1$. If $F^\circ_{d+1,m}(a) \in \mathsf{L}^1(\mathbb{R}^{2d})$, $m \geqslant d+1$, then for any $t \in [0,1]$ we have*

$$\big\|h_1 \operatorname{Op}_{\alpha,t}(a) h_2\big\|_{\mathfrak{S}_1} \leqslant C_m (\alpha r)^{d-m} {|\!|\!|} h_1 {|\!|\!|}_{2,\infty} {|\!|\!|} h_2 {|\!|\!|}_{2,\infty} \sum_{k=0}^{d+1} \iint \big|\nabla_{\mathbf{w}}^k \nabla_{\boldsymbol{\xi}}^m a(\mathbf{w}, \boldsymbol{\xi})\big| \, d\mathbf{w} \, d\boldsymbol{\xi}.$$

The constants C and C_m do not depend on $t \in [0,1]$.

For $\mathbf{T} = \mathbf{I}$ and $t = 0, 1$ the above estimates were obtained in [26].

An estimate similar to (2.22) can be found in [21, Theorem 3.5]. The estimate (2.23) for $h_1 = h_2 = 1$, $t = 0$ (with larger number of derivatives) has been known since [23].

3. Semi-classical estimates

3.1. Compactly supported amplitudes/symbols

Now we proceed to estimates for arbitrary $q \in (0,1]$ for the operators containing the parameter $\alpha > 0$. Due to the nature of the bounds derived in the previous section we do not expect the semi-classical bounds to look as simple as in Theorems 2.8 and 2.9. Thus we do not try to find integral bounds but instead we concentrate on the scaling properties of the $\mathfrak{S}_q$-estimates. For arbitrary numbers $\ell > 0$ and $\rho > 0$ introduce the following norms:

$$\mathsf{N}^{(n_1,n_2,m)}(p; \ell, \rho) = \max \sup_{\mathbf{w},\mathbf{z},\boldsymbol{\xi}} \ell^{n+k} \rho^r \big|\nabla_{\mathbf{w}}^n \nabla_{\mathbf{z}}^k \nabla_{\boldsymbol{\xi}}^r p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi})\big|, \tag{3.1}$$

where the maximum is taken over all $0 \leqslant n \leqslant n_1$, $0 \leqslant k \leqslant n_2$ and $0 \leqslant r \leqslant m$. We say that p belongs to the class $\mathbf{S}^{(n_1,n_2,m)}$ if the norm (3.1) is finite for some (and hence for all) positive ℓ, ρ. For a symbol $a = a(\mathbf{w}, \boldsymbol{\xi})$ (resp. function $a = a(\boldsymbol{\xi})$) we use the notation $\mathsf{N}^{(n,m)}(a; \ell, \rho)$ (resp. $\mathsf{N}^{(m)}(a; \rho)$). Accordingly, we define classes $\mathbf{S}^{(n,m)}$ and $\mathbf{S}^{(m)}$. The presence of the parameters ℓ, ρ allows one to consider amplitudes and symbols with different scaling properties.

Let U_ℓ be the unitary operator on $\mathsf{L}^2(\mathbb{R}^d)$ defined by

$$(U_\ell u)(\mathbf{x}) = \ell^{\frac{d}{2}} u(\ell \mathbf{x}).$$

Then a straightforward calculation gives for any $\ell, \rho > 0$ the following unitary equivalence:

$$U_\ell \operatorname{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}}) U_\ell^{-1} = \operatorname{Op}^{\mathrm{a}}_\beta\big(p^{(\ell,\rho)}_{\mathbf{T}}\big), \qquad p^{(\ell,\rho)}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) = p(\ell\mathbf{w}, \ell\mathbf{z}, \rho\boldsymbol{\xi}), \qquad \beta = \alpha\ell\rho. \tag{3.2}$$

The norms (3.1) are also invariant:

$$\mathsf{N}^{(n_1,n_2,m)}(p; \ell, \rho) = \mathsf{N}^{(n_1,n_2,m)}\big(p^{(\ell_1,\rho_1)}; \ell\ell_1^{-1}, \rho\rho_1^{-1}\big), \tag{3.3}$$

for arbitrary positive $\ell, \ell_1, \rho, \rho_1$.

The operators $\operatorname{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}})$ transform in a standard way under Euclidean isometries (i.e. orthogonal transformations and shifts), their norms (3.1) remain invariant. We use these facts regularly without introducing formal notation for these transformations.

All the $\mathfrak{S}_q$-bounds below will be derived under the following conditions on the amplitudes or symbols. For the operator $\operatorname{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}})$ we assume that

$$\text{the support of } p = p(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}) \text{ is contained in } B(\mathbf{u}, \ell) \times \mathbb{R}^d \times B(\boldsymbol{\mu}, \rho), \tag{3.4}$$

with some $\mathbf{u}, \boldsymbol{\mu} \in \mathbb{R}^d$ and some $\ell > 0$, $\rho > 0$. For the t-operators $\operatorname{Op}_{\alpha,t}(a)$ we assume that

$$\text{the support of } a = a(\mathbf{w}, \boldsymbol{\xi}) \text{ is contained in } B(\mathbf{u}, \ell) \times B(\boldsymbol{\mu}, \rho). \tag{3.5}$$

In what follows most of the bounds are obtained under the assumption that $\alpha\ell\rho \geqslant \ell_0$ with some fixed positive number ℓ_0. The constants featuring in all the estimates below are independent of the symbols involved as well as of the parameters $\mathbf{u}, \boldsymbol{\mu}, \alpha, \ell, \rho$ but may depend on the constant ℓ_0.

Theorem 3.1. *Let $\mathbf{T} \in GL(2, \mathbb{R})$ be a matrix satisfying (2.4), and let $s, t \in [0, 1]$. Let $q \in (0, 1]$ and $\alpha\ell\rho \geqslant \ell_0$. Let $p \in \mathbf{S}^{(n,n,n)}$, with n defined in (2.11), be an amplitude satisfying the condition (3.4), and let $a \in \mathbf{S}^{(n,n)}$ be a symbol satisfying the condition (3.5). Then $\operatorname{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}}) \in \mathfrak{S}_q$, $\operatorname{Op}_{\alpha,t}(a) \in \mathfrak{S}_q$, and*

$$\big\|\operatorname{Op}^{\mathrm{a}}_\alpha(p_{\mathbf{T}})\big\|_{\mathfrak{S}_q} \leqslant C_q (\alpha\ell\rho)^{\frac{d}{q}} \mathsf{N}^{(n,n,n)}(p; \ell, \rho), \tag{3.6}$$

with a constant $C_q = C_q(t_0, \delta_0)$ (see (2.7)), and

$$\big\|\operatorname{Op}_{\alpha,t}(a)\big\|_{\mathfrak{S}_q} \leqslant C_q (\alpha\ell\rho)^{\frac{d}{q}} \mathsf{N}^{(n,n)}(a; \ell, \rho), \tag{3.7}$$

with a constant C_q independent of t. If, in addition $a \in \mathbf{S}^{(n,n+1)}$ then

$$\big\|\operatorname{Op}_{\alpha,t}(a) - \operatorname{Op}_{\alpha,s}(a)\big\|_{\mathfrak{S}_q} \leqslant C_q (\alpha\ell\rho)^{\frac{d}{q}-1} \mathsf{N}^{(n,n+1)}(a; \ell, \rho), \tag{3.8}$$

with a constant C_q independent of s, t.

Proof. The estimate (3.7) is a special case of (3.6) with the matrix $\mathbf{T}$ defined in (1.4).

Without loss of generality we may assume that $\mathbf{u} = \boldsymbol{\mu} = \mathbf{0}$. Furthermore, using (3.3) and (3.2) with $\ell_1 = (\alpha\rho)^{-1}$, $\rho_1 = \rho$, we see that it suffices to prove the sought inequalities for $\alpha = 1$, $\rho = 1$ and arbitrary $\ell \geqslant \ell_0$ with a fixed $\ell_0 > 0$.

Proof of (3.6). We use Theorem 2.5 with $h_1 = h_2 = 1$ and $m = n$. Assume without loss of generality that $\mathsf{N}^{(n,n,n)}(p; \ell, 1) = 1$. As $\ell \geqslant \ell_0$, we have

$$P_{n,n}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p) \leqslant C \frac{\chi_{\mathbf{0},\ell}(\mathbf{w})\chi_{\mathbf{0},1}(\boldsymbol{\xi})}{1 + |\tau\mathbf{w} - \mathbf{z}|^n},$$

and hence, for any $\mathbf{k}, \mathbf{s}, \mathbf{j} \in \mathbb{Z}^d$ we have

$$\int_{\mathcal{C}_{\mathbf{k}}} \int_{\mathcal{C}_{\mathbf{s}}} \int_{\mathcal{C}_{\mathbf{j}}} P_{n,n}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; p)\, d\mathbf{w}\, d\mathbf{z}\, d\boldsymbol{\xi} \leqslant C \frac{\chi_{\mathbf{0},R\ell}(\mathbf{j})\chi_{\mathbf{0},2\sqrt{d}}(\mathbf{k})}{1 + |\tau\mathbf{j} - \mathbf{s}|^n},$$

where $R = R(\ell_0) = \ell_0^{-1}\sqrt{d} + 1$. As a consequence,

$$\big| P_{n,m}(p) \big|_{1,q} \leqslant C \biggl(\sum_{|\mathbf{j}| \leqslant R\ell} \sum_{\mathbf{s}} \frac{1}{1 + |\tau\mathbf{j} - \mathbf{s}|^{nq}} \biggr)^{\frac{1}{q}} \leqslant C\ell^{\frac{d}{q}}, \quad C = C(\ell_0),$$

as $n = [dq^{-1}] + 1 > dq^{-1}$. This leads to (3.6).

Proof of (3.8). We use Theorem 2.5 with $h_1 = h_2 = 1$ and $m = n + 1$. Without loss of generality assume temporarily that $\mathsf{N}^{(n,n+1)}(a) = 1$. Rewrite the difference on the left-hand side of (3.8) in the form

$$\mathrm{Op}_{\alpha,t}(a) - \mathrm{Op}_{\alpha,s}(a) = \mathrm{Op}_\alpha^{\mathrm{a}}(g_{\mathbf{S}}),$$

with $g(\mathbf{w}, \mathbf{z}) = a(\mathbf{w}, \boldsymbol{\xi}) - a(\mathbf{z}, \boldsymbol{\xi})$ and the matrix

$$\mathbf{S} = \begin{pmatrix} 1-t & t \\ 1-s & s \end{pmatrix}.$$

Note that $\det \mathbf{S} = s - t$, and assume that $|s - t| \geqslant 1/4$. For all $n_1, n_2 \leqslant n$, $l \leqslant n + 1$ we have

$$\big|\nabla_{\mathbf{w}}^{n_1} \nabla_{\mathbf{z}}^{n_2} \nabla_{\boldsymbol{\xi}}^{l} g(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi})\big| \leqslant \ell^{-n_1-n_2}\big(\chi_{\mathbf{0},\ell}(\mathbf{w}) + \chi_{\mathbf{0},\ell}(\mathbf{z})\big)\chi_{\mathbf{0},1}(\boldsymbol{\xi}),$$
$$\big|\nabla_{\boldsymbol{\xi}}^{l} g(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi})\big| \leqslant \ell^{-1}|\mathbf{w} - \mathbf{z}|\big(\chi_{\mathbf{0},\ell}(\mathbf{w}) + \chi_{\mathbf{0},\ell}(\mathbf{z})\big)\chi_{\mathbf{0},1}(\boldsymbol{\xi}).$$

Therefore

$$P_{n,n+1}(\mathbf{w}, \mathbf{z}, \boldsymbol{\xi}; g) \leqslant C\ell^{-1} \frac{(\chi_{\mathbf{0},\ell}(\mathbf{w}) + \chi_{\mathbf{0},\ell}(\mathbf{z}))\chi_{\mathbf{0},1}(\boldsymbol{\xi})}{1 + |\mathbf{w} - \mathbf{z}|^n}.$$

Arguing as in the first part of the proof we arrive at the estimate

$$\left|P_{n,n+1}(g)\right|_{1,q} \leqslant C\ell^{\frac{d}{q}-1}, \quad C = C(\ell_0),$$

which implies (3.8) by virtue of Theorem 2.5. As we have assumed that $|\det \mathbf{S}| \geqslant 1/4$, the constant in (3.8) does not depend on s, t.

If $|s-t| < 1/4$, then we choose a number $u \in [0,1]$ such that $|s-u| \geqslant 1/4$, $|t-u| \geqslant 1/4$, apply the estimate obtained in the first part of the proof to $\mathrm{Op}_{\alpha,s}(a) - \mathrm{Op}_{\alpha,u}(a)$ and $\mathrm{Op}_{\alpha,t}(a) - \mathrm{Op}_{\alpha,u}(a)$, and use the triangle inequality (2.1). □

Theorem 3.2. *Let $q \in (0,1]$, $\alpha\ell\rho \geqslant \ell_0$ and $R \geqslant \ell$. Let $h_1, h_2 \in \mathsf{L}^\infty(\mathbb{R}^d)$ be two functions such that the distance between their supports is at least R. Let $a \in \mathbf{S}^{(n,m)}$, $m \geqslant n$, be a symbol satisfying the condition (3.5). Then for any $t \in [0,1]$ we have*

$$\left\|h_1 \mathrm{Op}_{\alpha,t}(a) h_2\right\|_{\mathfrak{S}_q} \leqslant C_{q,m} \|h_1\|_{\mathsf{L}^\infty} \|h_2\|_{\mathsf{L}^\infty} (\alpha R\rho)^{\frac{d}{q}-m} \mathsf{N}^{(n,m)}(a; \ell, \rho),$$

with a constant $C_{q,m}$ independent of t.

Proof. Using (3.3) and (3.2) with $\ell_1 = \ell$, $\rho_1 = (\alpha\ell)^{-1}$, we see that it suffices to prove the sought inequality for $\alpha = 1$, $\ell = 1$, and arbitrary $\rho \geqslant \ell_0$ and $R \geqslant 1$. Again, without loss of generality assume that $\mathbf{u} = \boldsymbol{\mu} = \mathbf{0}$, $\|h_1\|_{\mathsf{L}^\infty} = \|h_2\|_{\mathsf{L}^\infty} = 1$, and $\mathsf{N}^{(n,m)}(a; 1, \rho) = 1$. Use Theorem 2.6(2) with $r = R$. It is straightforward to see that

$$F^\circ_{n,m}(\mathbf{w}, \boldsymbol{\xi}; a) \leqslant C\chi_{\mathbf{0},1}(\mathbf{w})\chi_{\mathbf{0},\rho}(\boldsymbol{\xi})\rho^{-m},$$

see (2.16) for definition, so that

$$\left|F^\circ_{n,m}(a)\right|^q_{1,q} \leqslant C_q \rho^{d-mq}.$$

By (2.12),

$$\left\|h_1 \mathrm{Op}_{1,t}(a) h_2\right\|_{\mathfrak{S}_q} \leqslant C_q (R\rho)^{\frac{d}{q}-m},$$

which leads to the sought estimate. □

3.2. Symbols with non-compact support

Here we illustrate the use of the obtained estimates and derive a semi-classical bound for the t-pseudo-differential operators whose symbols are not necessarily compactly supported. Suppose that for some constant $A > 0$, and some number $q \in (0,1]$ the symbol a satisfies the bound

$$\max_{\substack{0\leqslant k\leqslant n\\ 0\leqslant l\leqslant n}} \left|\nabla^k_{\mathbf{w}} \nabla^l_{\boldsymbol{\xi}} a(\mathbf{w}, \boldsymbol{\xi})\right| \leqslant A\big(1 + |\mathbf{w}|\big)^{-\gamma_1}\big(1 + |\boldsymbol{\xi}|\big)^{-\gamma_2}, \quad \gamma_1, \gamma_2 > dq^{-1}, \tag{3.9}$$

where n is as in (2.11).

Theorem 3.3. *Let the symbol a satisfy (3.9), and let $q \in (0,1]$. Then*

$$\|\mathrm{Op}_{\alpha,t}(a)\|_{\mathfrak{S}_q} \leqslant C_q A \alpha^{\frac{d}{q}},$$

with a constant C_q independent of $t \in [0,1]$.

Proof. As in the proof of Theorem 2.5 cover $\mathbb{R}^d$ with open balls $B(\mathbf{j}, 2\sqrt{d})$, $\mathbf{j} \in \mathbb{Z}^d$. Let $\psi_{\mathbf{j}} \in \mathsf{C}_0^\infty(\mathbb{R}^d)$, $\mathbf{j} \in \mathbb{Z}^d$, be an associated partition of unity satisfying (2.15). Consider the symbols

$$a^{(\mathbf{j},\mathbf{s})}(\mathbf{w}, \boldsymbol{\xi}) = \psi_{\mathbf{j}}(\mathbf{w})\psi_{\mathbf{s}}(\boldsymbol{\xi})a(\mathbf{w}, \boldsymbol{\xi}).$$

These symbols are compactly supported and

$$\mathsf{N}^{(n,n)}\big(a^{(\mathbf{j},\mathbf{s})}; 1, 1\big) \leqslant CA\big(1 + |\mathbf{j}|\big)^{-\gamma_1}\big(1 + |\mathbf{s}|\big)^{-\gamma_2}.$$

By (3.7),

$$\big\|\mathrm{Op}_{\alpha,t}\big(a^{(\mathbf{j},\mathbf{s})}\big)\big\|_{\mathfrak{S}_q}^q \leqslant CA^q\alpha^d\big(1 + |\mathbf{j}|\big)^{-\gamma_1 q}\big(1 + |\mathbf{s}|\big)^{-\gamma_2 q}.$$

By the triangle inequality (2.1) we have

$$\|\mathrm{Op}_{\alpha,t}(a)\|_{\mathfrak{S}_q}^q \leqslant CA^q\alpha^d \sum_{\mathbf{j},\mathbf{s}\in\mathbb{Z}^d} \big(1 + |\mathbf{j}|\big)^{-\gamma_1 q}\big(1 + |\mathbf{s}|\big)^{-\gamma_2 q} \leqslant C'A^q\alpha^d,$$

as claimed. □

4. Estimates for operators with non-smooth symbols

4.1. Admissible domains

Here we obtain $\mathfrak{S}_q$-estimates for operators with symbols having jump discontinuities. The discontinuities are introduced via the projections χ_Λ and/or $P_{\Omega,\alpha} = \mathrm{Op}_\alpha(\chi_\Omega)$ where Λ and Ω are some suitable domains whose properties are specified in the next definition.

Definition 4.1. Let $d \geqslant 2$. We say that $\Lambda \subset \mathbb{R}^d$ is a *basic domain* if there exists a Lipschitz function $\Phi = \Phi(\hat{\mathbf{x}})$, $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$, such that with a suitable choice of the Cartesian coordinates $\mathbf{x} = (\hat{\mathbf{x}}, x_d)$, $\hat{\mathbf{x}} = (x_1, x_2, \dots, x_{d-1})$ the domain Λ is represented as

$$\Lambda = \big\{\mathbf{x} \in \mathbb{R}^d\colon\ x_d > \Phi(\hat{\mathbf{x}})\big\}. \tag{4.1}$$

It is assumed that the function Φ is *uniformly* Lipschitz, i.e. the constant

$$M = M_\Phi = \sup_{\substack{\hat{\mathbf{x}},\hat{\mathbf{y}},\\ \hat{\mathbf{x}}\neq\hat{\mathbf{y}}}} \frac{|\Phi(\hat{\mathbf{x}}) - \Phi(\hat{\mathbf{y}})|}{|\hat{\mathbf{x}} - \hat{\mathbf{y}}|} \tag{4.2}$$

is finite. In this case we use the notation $\Lambda = \Gamma(\Phi)$. A domain Λ is said to be *admissible* if locally it can be represented by basic domains, i.e. for any $\mathbf{z} \in \mathbb{R}^d$ there is a radius $r > 0$ such that $B(\mathbf{z}, r) \cap \Lambda = B(\mathbf{z}, r) \cap \Lambda_0$ with some basic domain $\Lambda_0 = \Lambda_0(\mathbf{z})$.

Let $d = 1$. Then Λ is said to be a basic domain if Λ is either $(0, \infty)$ or $(-\infty, 0)$. A domain Λ is said to be *admissible* if $\Lambda = (0, L)$ with some $L \in (0, \infty)$.

This definition allows us to state the results for the cases $d \geqslant 2$ and $d = 1$ simultaneously.

Our objective is to obtain semi-classical $\mathfrak{S}_q$-estimates for the Hankel-type operators $\chi_\Lambda \operatorname{Op}_{\alpha,t}(a)(I - \chi_\Lambda)$, $P_{\Omega,\alpha} \operatorname{Op}_{\alpha,t}(a)(I - P_{\alpha,\Omega})$ and $\chi_\Lambda P_{\alpha,\Omega}(I - \chi_\Lambda)$, with suitable admissible domains Λ, Ω and suitable symbols a. We work either with $t = 0$ or $t = 1$. First we establish the sought estimates for basic domains Λ and Ω, and then extend the result to the general bounded admissible ones using appropriate partitions of unity.

For $d \geqslant 2$ all the $\mathfrak{S}_q$-estimates obtained for the basic domains are uniform in the Lipschitz constants M_Φ and M_Ψ satisfying the condition

$$\max(M_\Phi, M_\Psi) \leqslant M, \tag{4.3}$$

with some constant M. Needless to say, the choice of the coordinates for which Λ or Ω have the form (4.1) does not have to be the same for the domains Λ and Ω.

As in the previous section we assume as a rule that the symbols are compactly supported and satisfy the condition (3.5). The constants in the obtained estimates will be independent of the symbols, and of $\mathbf{u}$, $\boldsymbol{\mu}$ and ℓ, ρ but may depend on the constant ℓ_0 in the bound $\alpha\ell\rho \geqslant \ell_0$, and, for $d \geqslant 2$, on M. As mentioned in the Introduction some estimates were obtained in [26] for the class $\mathfrak{S}_1$. Note also that for $d \geqslant 2$ the results of [26] require C^1-smoothness of the domains Λ, Ω whereas in the current paper the Lipschitz property suffices.

We obtain consecutively estimates of two types. First we study the operators

$$\chi_\Lambda \operatorname{Op}_{\alpha,t}(a)(I - \chi_\Lambda) \quad \text{and} \quad P_{\Omega,\alpha} \operatorname{Op}_{\alpha,t}(a)(I - P_{\Omega,\alpha}).$$

Since these operators contain only one characteristic function we refer to this case as the case of discontinuity in one variable. Next we look at the operators of the form $\chi_\Lambda \operatorname{Op}_{\alpha,t}(a) P_{\Omega,\alpha}(I - \chi_\Lambda)$ which is naturally referred to as the case of discontinuity in two variables.

It is useful to remark on the scaling properties of basic domains in $d \geqslant 2$. Applying (3.2) to the characteristic function χ_Λ, $\Lambda = \Gamma(\Phi)$, we observe that under scaling U_ℓ the domain Λ transforms into $\Gamma(\tilde{\Phi})$, where $\tilde{\Phi}(\hat{\mathbf{x}}) = \ell\Phi(\ell^{-1}\hat{\mathbf{x}})$. It is obvious that $M_{\tilde{\Phi}} = M_\Phi$.

Let $\Lambda = \Gamma(\Phi) \subset \mathbb{R}^d$, $d \geqslant 2$, be a basic domain. By definition (4.2),

$$\left| x_d - \Phi(\hat{\mathbf{x}}) - \big(y_d - \Phi(\hat{\mathbf{y}})\big) \right| \leqslant \langle M \rangle |\mathbf{x} - \mathbf{y}|, \quad \langle M \rangle := \sqrt{1 + M^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, so that

$$|\mathbf{x} - \mathbf{y}| \geqslant \frac{1}{\langle M \rangle} \big(x_d - \Phi(\hat{\mathbf{x}})\big), \quad \text{for all } \mathbf{x} \in \Lambda,\ \mathbf{y} \notin \Lambda. \tag{4.4}$$

In the case $d = 1$, for a basic domain Λ the same type of bound is obvious:

$$|x - y| \geqslant |x|, \quad x \in \Lambda,\ y \notin \Lambda. \tag{4.5}$$

4.2. Discontinuity in one variable

Here we study the combinations involving an operator with a smooth symbol and one of the operators χ_Λ or $P_{\Omega,\alpha}$.

Theorem 4.2. *Let Λ and Ω be basic domains. Let $q \in (0, 1]$, $\alpha\ell\rho \geqslant \ell_0$, n be as in (2.11), and let*

$$m = \big[(d + 1)q^{-1}\big] + 1. \tag{4.6}$$

Suppose that the symbol $a \in \mathbf{S}^{(n,m)}$ satisfies (3.5). Then for $t = 0$ or 1 we have

$$\big\|\chi_\Lambda \operatorname{Op}_{\alpha,t}(a)(1 - \chi_\Lambda)\big\|_{\mathfrak{S}_q} \leqslant C_q(\alpha\ell\rho)^{\frac{d-1}{q}} \mathsf{N}^{(n,m)}(a; \ell, \rho), \tag{4.7}$$

$$\big\|P_{\Omega,\alpha} \operatorname{Op}_{\alpha,t}(a)(1 - P_{\Omega,\alpha})\big\|_{\mathfrak{S}_q} \leqslant C_q(\alpha\ell\rho)^{\frac{d-1}{q}} \mathsf{N}^{(m,n)}(a; \ell, \rho). \tag{4.8}$$

Proof. The bound (4.8) follows from (4.7) upon exchanging the roles of the variables $\mathbf{x}$ and $\boldsymbol{\xi}$. Thus it suffices to prove (4.7).

Proof of (4.7). Assume without loss of generality that $\mathsf{N}^{(n,m)}(a; \ell, \rho) = 1$. We prove (4.7) for the operator $\operatorname{Op}_{\alpha,0}(a)$ only, the case $t = 1$ is done in the same way.

Let $d \geqslant 2$. We use the same scaling argument as in the proof of Theorem 3.1, and the fact that the Lipschitz constant of the domain Λ does not change under scaling, see the remark at the end of Section 4.1. Thus it suffices to prove (4.7) for $\alpha = \rho = 1$ and arbitrary $\ell \geqslant \ell_0$ with an $\ell_0 > 0$. Moreover without loss of generality assume that $\mathbf{u} = \boldsymbol{\mu} = \mathbf{0}$.

Choose the coordinates in such a way that Λ is represented as in (4.1). Denote

$$\Lambda_s = \big\{\mathbf{x} \in \mathbb{R}^d\colon\ x_d > \Phi(\hat{\mathbf{x}}) + s\big\}, \quad s \in \mathbb{R}.$$

By virtue of (4.4),

$$|\mathbf{x} - \mathbf{y}| \geqslant \frac{s + |x_d - \Phi(\hat{\mathbf{x}}) - s|}{\langle M \rangle}, \quad \forall \mathbf{x} \in \Lambda_s,\ \mathbf{y} \notin \Lambda,\ s > 0.$$

Cover the closure $\overline{\Lambda}$ with open balls of radius $2\sqrt{d}$ centred at the lattice points $\mathbf{j} \in \mathbb{Z}^d$. Let $R = 4\langle M\rangle\sqrt{d}$ and denote

$$\Sigma = \{\mathbf{j} \in \mathbb{Z}^d\colon\ B(\mathbf{j}, 2\sqrt{d}\,) \cap \Lambda \neq \varnothing\}, \qquad \Sigma_0 = \{\mathbf{j} \in \mathbb{Z}^d\colon\ \mathbf{j} \in \Lambda_R\}, \qquad \Sigma_1 = \Sigma \setminus \Sigma_0.$$

These definitions ensure that

$$\operatorname{dist}\{B(\mathbf{j}, 2\sqrt{d}\,), \complement\Lambda\} \geqslant 2\sqrt{d} + \frac{|j_d - \Phi(\hat{\mathbf{j}}) - R|}{\langle M\rangle}, \quad \text{for all } \mathbf{j} \in \Sigma_0,$$

where $\complement\Lambda = \mathbb{R}^d \setminus \Lambda$. Let $\psi_{\mathbf{j}}$, $\mathbf{j} \in \Sigma$, be a smooth partition of unity subordinate to the introduced covering, such that

$$\left|\nabla_{\mathbf{x}}^k \psi_{\mathbf{j}}(\mathbf{x})\right| \leqslant C_k, \quad k = 0, 1, \dots,$$

uniformly in $\mathbf{j} \in \Sigma$. Denote $\Lambda_{\mathbf{j}} = \Lambda \cap B(\mathbf{j}, 2\sqrt{d}\,)$, and

$$T_{\mathbf{j}} = \chi_{\Lambda_{\mathbf{j}}} \operatorname{Op}_{1,0}(\psi_{\mathbf{j}} a)(I - \chi_\Lambda).$$

Since $\mathsf{N}^{(n,m)}(a; 1, 1) \leqslant C\mathsf{N}^{(n,m)}(a, \ell, 1) \leqslant C$, by Theorem 3.2 we obtain

$$\|T_{\mathbf{j}}\|_{\mathfrak{S}_q}^q \leqslant C\left(2\sqrt{d} + \frac{|j_d - \Phi(\hat{\mathbf{j}}) - R|}{\langle M\rangle}\right)^{d-mq}, \quad \mathbf{j} \in \Sigma_0.$$

By the triangle inequality (2.1),

$$\left\|\sum_{\mathbf{j}\in\Sigma_0} T_{\mathbf{j}}\right\|_{\mathfrak{S}_q}^q \leqslant C \sum_{|\hat{\mathbf{j}}|\leqslant C\ell} \sum_{j_d\in\mathbb{Z}} \left(2\sqrt{d} + \frac{|j_d - \Phi(\hat{\mathbf{j}}) - R|}{\langle M\rangle}\right)^{d-mq} \leqslant C'\ell^{d-1}, \tag{4.9}$$

where we have used the fact that $qm > d + 1$, see (4.6). For $\mathbf{j} \in \Sigma_1$ we use the bound

$$\|T_{\mathbf{j}}\|_{\mathfrak{S}_q} \leqslant \left\|\operatorname{Op}_{1,0}(\psi_{\mathbf{j}} a)\right\| \leqslant C,$$

which follows from (3.7). As $\#\Sigma_1 \leqslant C\ell^{d-1}$, $C = C(\ell_0)$, with the help of the triangle inequality we obtain

$$\left\|\sum_{\mathbf{j}\in\Sigma_1} T_{\mathbf{j}}\right\|_{\mathfrak{S}_q}^q \leqslant C \sum_{\mathbf{j}\in\Sigma_1} 1 \leqslant C'\ell^{d-1}.$$

Together with (4.9) this leads to

$$\left\|\chi_\Lambda \operatorname{Op}_{1,0}(a)(I - \chi_\Lambda)\right\|_{\mathfrak{S}_q}^q \leqslant C\ell^{d-1}.$$

As explained earlier this bound implies (4.7).

The proof in the case $d = 1$ is a simplified version of that for $d \geqslant 2$. In particular, instead of (4.4) one uses (4.5). We omit the details. □

Remark 4.3. It is immediate to obtain from Theorem 4.2 estimates of the form (4.7) and (4.8) for the commutators $[\mathrm{Op}_{\alpha,t}, \chi_\Lambda]$ and $[\mathrm{Op}_{\alpha,t}(a), P_{\Omega,\alpha}]$. Indeed, recall that $[A, \Pi] = (I - \Pi)A\Pi - \Pi A(I - \Pi)$ for any bounded operator A and any projection Π, and that $(\mathrm{Op}_{\alpha,t}(a))^* = \mathrm{Op}_{\alpha,1-t}(\bar{a})$. Thus for $t = 0$ or 1 it follows from (4.7) that

$$\big\| [\mathrm{Op}_{\alpha,t}(a), \chi_\Lambda] \big\|_{\mathfrak{S}_q} \leqslant C_q (\alpha\ell\rho)^{\frac{d-1}{q}} \mathsf{N}^{(n,m)}(a; \ell, \rho),$$

and the same estimate holds for the commutator with $P_{\Omega,\alpha}$.

The corollary below extends Theorem 4.2 to arbitrary bounded admissible domains.

Corollary 4.4. *Let Λ and Ω be bounded admissible domains. Let $q \in (0, 1]$, $\alpha\ell\rho \geqslant \ell_0$, n, m be as in (2.11) and (4.6) respectively. Suppose that the symbol $a \in \mathbf{S}^{(n,m)}$ satisfies (3.5). Then for $t = 0$ or 1 we have*

$$\big\| \chi_\Lambda \mathrm{Op}_{\alpha,t}(a)(1 - \chi_\Lambda) \big\|_{\mathfrak{S}_q} \leqslant C_q (\alpha\ell\rho)^{\frac{d-1}{q}} \mathsf{N}^{(n,m)}(a; \ell, \rho), \tag{4.10}$$

$$\big\| P_{\Omega,\alpha} \mathrm{Op}_{\alpha,t}(a)(1 - P_{\Omega,\alpha}) \big\|_{\mathfrak{S}_q} \leqslant C_q (\alpha\ell\rho)^{\frac{d-1}{q}} \mathsf{N}^{(m,n)}(a; \ell, \rho). \tag{4.11}$$

The constant C_q in the above estimates may depend on the domains Λ, Ω.

Proof. In the proof there is no difference between the cases $d = 1$ and $d \geqslant 2$. As in Theorem 4.2 the bound (4.11) follows from (4.10). Cover $\overline{\Lambda}$ with finitely open balls $B(\mathbf{z}_j, r)$, $j = 1, 2, \dots, J$ where r is chosen in such a way that for each j we have $B(\mathbf{z}_j, 4r) \cap \Lambda = B(\mathbf{z}_j, 4r) \cap \Lambda_0$ with some basic domain $\Lambda_0 = \Lambda_0(j)$. Let $\{\phi_j\}$, $j = 1, 2, \dots, J$, be a finite partition of unity subordinate to the above covering. Due to the triangle inequality (2.1) it suffices to obtain the bound (4.20) for the operators of the form

$$T_\alpha = \chi_\Lambda \mathrm{Op}_{\alpha,t}(b)(1 - \chi_\Lambda),$$

where $b(\mathbf{w}, \boldsymbol{\xi}) = \phi(\mathbf{w})a(\mathbf{w}, \boldsymbol{\xi})$, and ϕ is an element of the partition above supported in the ball $B(\mathbf{z}, r)$. Here we have omitted the index j for brevity. If Λ had been a basic domain then the required bound would have followed from (4.16). Let Λ_0 be a basic domain such that

$$B(\mathbf{z}, 4r) \cap \Lambda = B(\mathbf{z}, 4r) \cap \Lambda_0. \tag{4.12}$$

By construction,

$$T_\alpha = \chi_{\Lambda_0} \mathrm{Op}_{\alpha,t}(b)(I - \chi_\Lambda).$$

Now we need to show that the estimate (4.10) is preserved if one replaces Λ with Λ_0 in the last bracket on the right-hand side. Let $\zeta \in \mathsf{C}^\infty(\mathbb{R}^d)$ be as defined in (2.20), and let $h(\mathbf{x}) = \zeta((|\mathbf{x}-\mathbf{z}|(4r)^{-1}))$, $\tilde{h} = 1 - h$. Observe that the distance between the supports of ϕ and h is at least r. Thus by Theorem 3.2 we have

$$\begin{aligned}\big\|\chi_{\Lambda_0} \operatorname{Op}_{\alpha,t}(b)(I-\chi_\Lambda)\big\|_{\mathfrak{S}_q}^q &\leqslant \big\|\operatorname{Op}_{\alpha,t}(b)h\big\|_{\mathfrak{S}_q}^q + \big\|\chi_{\Lambda_0}\operatorname{Op}_{\alpha,t}(b)\tilde{h}(I-\chi_\Lambda)\big\|_{\mathfrak{S}_q}^q \\ &\leqslant C_m(\alpha r)^{d-mq} + \big\|\chi_{\Lambda_0}\operatorname{Op}_{\alpha,t}(b)\tilde{h}(I-\chi_{\Lambda_0})\big\|_{\mathfrak{S}_q}^q.\end{aligned}$$

Here we have used (4.12). The last term on the right-hand side is bounded by

$$\big\|\chi_{\Lambda_0}\operatorname{Op}_{\alpha,t}(b)(I-\chi_{\Lambda_0})\big\|_{\mathfrak{S}_q}^q.$$

Since Λ_0 is a basic domain we can use (4.7) to obtain (4.10) for the symbol b. As explained earlier, this leads to (4.10) for the symbol a. $\square$

4.3. Discontinuity in two variables

In this subsection we prove analogues of Theorem 4.2 and Corollary 4.4 with the smooth symbol a replaced by the symbol $a(\mathbf{x},\boldsymbol{\xi})\chi_\Omega(\boldsymbol{\xi})$. Now we need a partition of unity of a special type which is described in [14, Chapter 1].

Proposition 4.5. *Let $\tau = \tau(\boldsymbol{\xi}) > 0$ be a Lipschitz function on $\mathbb{R}^d$ such that*

$$\big|\tau(\boldsymbol{\xi}) - \tau(\boldsymbol{\eta})\big| \leqslant \varkappa|\boldsymbol{\xi}-\boldsymbol{\eta}|, \tag{4.13}$$

for all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^d$ with some $\varkappa \in [0,1)$. Then there exists a set $\boldsymbol{\xi}_j \in \mathbb{R}^d$, $j \in \mathbb{N}$ such that the balls $B(\boldsymbol{\xi}_j, \tau(\boldsymbol{\xi}_j))$ form a covering of $\mathbb{R}^d$ with the finite intersection property, i.e. each ball intersects no more than $N = N(\varkappa) < \infty$ other balls. Furthermore, there exist non-negative functions $\psi_j \in \mathsf{C}_0^\infty(\mathbb{R}^d)$, $j \in \mathbb{N}$, supported in $B(\boldsymbol{\xi}_j, \tau(\boldsymbol{\xi}_j))$ such that

$$\sum_j \psi_j(\boldsymbol{\xi}) = 1,$$

and

$$\big|\nabla^m \psi_j(\boldsymbol{\xi})\big| \leqslant C_m \tau(\boldsymbol{\xi})^{-m},$$

for all m uniformly in j.

Assume that $\Lambda, \Omega \subset \mathbb{R}^d$ are basic domains. For $d \geqslant 2$ we choose the coordinates in such way that

$$\Omega = \{\boldsymbol\xi = (\hat{\boldsymbol\xi}, \xi_d) \in \mathbb{R}^d\colon\ \xi_d > \Psi(\hat{\boldsymbol\xi})\},$$

with a Lipschitz function Ψ. For our purposes the convenient choice of $\tau(\boldsymbol\xi)$ for all $\boldsymbol\xi \in \mathbb{R}^d$ is

$$\tau(\boldsymbol\xi) = \frac{1}{32\langle M\rangle}\big(\big(\xi_d - \Psi(\hat{\boldsymbol\xi})\big)_+^2 + \alpha^{-2}\big)^{\frac{1}{2}}, \tag{4.14}$$

with the number M as in (4.3). Since $|\nabla\tau| \leqslant 1/16$, the condition (4.13) is satisfied with $\varkappa = 1/16$.

In the case $d = 1$ we let

$$\tau(\xi) = \frac{1}{32}\big(|\xi|^2 + \alpha^{-2}\big)^{\frac{1}{2}}. \tag{4.15}$$

Theorem 4.6. *Let Λ and Ω be basic domains. Let $q \in (0, 1]$, n be as in (2.11), and let m be as in (4.6). Suppose that the symbol $a \in \mathbf{S}^{(n,m)}$ satisfies (3.5). Assume that $\alpha\ell\rho \geqslant 2$. Then for $t = 0$ or 1 we have*

$$\big\|\chi_\Lambda \operatorname{Op}_{\alpha,t}(a)P_{\Omega,\alpha}(1-\chi_\Lambda)\big\|_{\mathfrak{S}_q} \leqslant C_q\big((\alpha\ell\rho)^{d-1}\log(\alpha\ell\rho)\big)^{\frac{1}{q}}\mathsf{N}^{(n,m)}(a;\ell,\rho). \tag{4.16}$$

Proof. Suppose that $d \geqslant 2$. Without loss of generality suppose that $\mathsf{N}^{(n,m)}(a;\ell,\rho) = 1$ and $\boldsymbol\mu = \mathbf{0}$. It suffices to prove the formula (4.16) for $\ell = \rho = 1$ and arbitrary $\alpha \geqslant 2$. Denote

$$T_\alpha = \chi_\Lambda \operatorname{Op}_{\alpha,t}(a)P_{\Omega,\alpha}(1-\chi_\Lambda).$$

Let ψ_j, $j = 1, 2, \ldots,$ be a partition of unity associated with the function (4.14). Let $\tau_j = \tau(\boldsymbol\xi_j)$ be the radii defined in Proposition 4.5. Then

$$T_\alpha = \sum_j T_\alpha^{(j)}, \qquad T_\alpha^{(j)} = \chi_\Lambda \operatorname{Op}_{\alpha,t}(a\psi_j)P_{\Omega,\alpha}(1-\chi_\Lambda). \tag{4.17}$$

Note that $\mathsf{N}^{(n,m)}(a\psi_j; 1, \tau_j) \leqslant C$ and $\alpha\tau_j \geqslant (32\langle M\rangle)^{-1}$ uniformly in j. We split the set of indices j in the sum (4.17) into two disjoint parts:

$$\Sigma_0 = \{j \in \mathbb{N}\colon\ \operatorname{supp}\psi_j \cap \partial\Omega \cap B(\mathbf{0},1) \neq \varnothing\},$$
$$\Sigma_1 = \{j \in \mathbb{N}\colon\ \chi_\Omega\psi_j = \psi_j,\ \operatorname{supp}\psi_j \cap B(\mathbf{0},1) \neq \varnothing\}.$$

First assume that $j \in \Sigma_0$. By (4.14) we have $c\alpha^{-1} \leqslant \tau_j \leqslant C\alpha^{-1}$ with some constants c, C. Thus by (3.7),

$$\big\|T_\alpha^{(j)}\big\|_{\mathfrak{S}_q} \leqslant \big\|\operatorname{Op}_{\alpha,t}(a\psi_j)\big\|_{\mathfrak{S}_q} \leqslant C(\alpha\tau_j)^{\frac{d}{q}} \leqslant \tilde{C},$$

uniformly in j. Since the boundary $\partial\Omega$ is Lipschitz, it is clear that $\#\Sigma_0 \leqslant C\alpha^{d-1}$, and hence by triangle inequality (2.1),

$$\Big\|\sum_{j\in\Sigma_0} T_\alpha^{(j)}\Big\|_{\mathfrak{S}_q}^q \leqslant \sum_{j\in\Sigma_0} \big\|T_\alpha^{(j)}\big\|_{\mathfrak{S}_q}^q \leqslant C\alpha^{d-1}. \tag{4.18}$$

Let us turn to the remaining indices, i.e. to $j \in \Sigma_1$. By definition of Σ_1 we have $T_\alpha^{(j)} = \chi_\Lambda \operatorname{Op}_{\alpha,t}(a\psi_j)(I - \chi_\Lambda)$, $j \in \Sigma_1$, and hence by Theorem 4.2,

$$\big\|T_\alpha^{(j)}\big\|_{\mathfrak{S}_q} \leqslant C(\alpha\tau_j)^{\frac{d-1}{q}}, \quad j \in \Sigma_1.$$

Let us sum up all the contributions using the triangle inequality (2.1):

$$\begin{aligned}\Big\|\sum_{j\in\Sigma_1} T_\alpha^{(j)}\Big\|_{\mathfrak{S}_q}^q \leqslant \sum_{j\in\Sigma_1} \big\|T_\alpha^{(j)}\big\|_{\mathfrak{S}_q}^q &\leqslant C_q\alpha^{d-1} \sum_{j:|\boldsymbol\xi_j|<2} \tau_j^{d-1} \\ &\leqslant \tilde C_q \alpha^{d-1} \int\limits_{\boldsymbol\xi\in\Omega,\, |\boldsymbol\xi|\leqslant 2} \tau(\boldsymbol\xi)^{-1}\, d\boldsymbol\xi. \end{aligned} \tag{4.19}$$

Here we have used the finite intersection property stated in Proposition 4.5 and the bounds

$$(1+\varkappa)^{-1}\tau(\boldsymbol\xi) \leqslant \tau(\boldsymbol\xi_j) \leqslant (1-\varkappa)^{-1}\tau(\boldsymbol\xi), \quad \boldsymbol\xi \in B\big(\boldsymbol\xi_j, \tau(\boldsymbol\xi_j)\big).$$

The integral on the right-hand side of (4.19) does not exceed

$$C \int\limits_{|\hat{\boldsymbol\xi}|\leqslant 2} \int\limits_{\substack{\xi_d>\Psi(\hat{\boldsymbol\xi}),\\ |\xi_d|\leqslant 2}} \frac{1}{\sqrt{\alpha^{-2} + (\xi_d - \Psi(\hat{\boldsymbol\xi}))^2}}\, d\xi_d\, d\hat{\boldsymbol\xi} \leqslant C' \int_0^4 \frac{1}{\sqrt{t^2+\alpha^{-2}}}\, dt \leqslant C'' \log(\alpha+1).$$

Together with (4.18) this leads to

$$\|T_\alpha\|_{\mathfrak{S}_q}^q \leqslant C\alpha^{d-1}\log\alpha,$$

which implies (4.16).

For $d = 1$ the proof follows the same line argument and is somewhat simpler. We omit the details. □

Just as before, using an appropriate partition of unity one can deduce the following.

Corollary 4.7. *Let Λ and Ω be bounded admissible domains, and let $q \in (0,1]$. Then for any $\alpha \geqslant 2$,*

$$\left\| \chi_\Lambda P_{\Omega,\alpha}(1-\chi_\Lambda) \right\|_{\mathfrak{S}_q} \leqslant C_q \left(\alpha^{d-1} \log \alpha \right)^{\frac{1}{q}}. \tag{4.20}$$

The constant C_q may depend on the domains Λ, Ω.

Proof. The proof is similar to that of Corollary 4.4. Cover $\overline{\Lambda}$ with finitely open balls $B(\mathbf{z}_j, r)$, $j = 1, 2, \dots, J$ where r is chosen in such a way that for each j, $B(\mathbf{z}_j, 4r) \cap \Lambda = B(\mathbf{z}_j, 4r) \cap \Lambda_0$ with some basic domain $\Lambda_0 = \Lambda_0(j)$. Let $\{B(\boldsymbol{\mu}_k, r)\}$, $k = 1, 2, \dots, K$ be a covering of $\overline{\Omega}$ with the same properties. Let $\{\phi_k\}$ and $\{\psi_j\}$ be finite partitions of unity subordinate to the above coverings. Due to the triangle inequality (2.1) it suffices to obtain the bound (4.20) for the operators of the form

$$T_\alpha = \chi_\Lambda \operatorname{Op}_{\alpha,0}(b) P_{\Omega,\alpha}(1-\chi_\Lambda),$$

where $b(\mathbf{x}, \boldsymbol{\xi}) = \phi(\mathbf{x})\psi(\boldsymbol{\xi})$, and ϕ, ψ are elements of the partitions above supported in the balls $B(\mathbf{z}, r)$ and $B(\boldsymbol{\mu}, r)$. We omit the indices j, k for brevity. If Λ and Ω had been basic domains then the required bound would have followed from (4.16). Let Λ_0 and Ω_0 be basic domains such that

$$B(\mathbf{z}, 4r) \cap \Lambda = B(\mathbf{z}, 4r) \cap \Lambda_0, \qquad B(\boldsymbol{\mu}, 4r) \cap \Omega = B(\boldsymbol{\mu}, 4r) \cap \Omega_0. \tag{4.21}$$

By construction,

$$T_\alpha = \chi_{\Lambda_0} \operatorname{Op}_{\alpha,0}(b) P_{\Omega_0,\alpha}(I-\chi_\Lambda).$$

Now we show that the estimate (4.20) is preserved if one replaces Λ with Λ_0 in the last bracket. By (4.8),

$$\begin{aligned} \|T_\alpha\|_{\mathfrak{S}_q}^q &\leqslant \left\| \left[P_{\Omega_0,\alpha}, \operatorname{Op}_{\alpha,0}(b) \right] \right\|_{\mathfrak{S}_q}^q + \left\| \chi_{\Lambda_0} P_{\Omega_0,\alpha} \operatorname{Op}_{\alpha,0}(b)(I-\chi_\Lambda) \right\|_{\mathfrak{S}_q}^q \\ &\leqslant C\alpha^{d-1} + \left\| \chi_{\Lambda_0} P_{\Omega_0,\alpha} \operatorname{Op}_{\alpha,0}(b)(I-\chi_\Lambda) \right\|_{\mathfrak{S}_q}^q. \end{aligned} \tag{4.22}$$

In order to estimate the last term on the right-hand side let $\zeta \in \mathsf{C}^\infty(\mathbb{R}^d)$ be as defined in (2.20), and let $h(\mathbf{x}) = \zeta((|\mathbf{x}-\mathbf{z}|(4r)^{-1}))$, $\tilde{h} = 1-h$. Observe that the distance between the supports of ϕ and h is at least r. Thus by Theorem 3.2, for any $m \geqslant [dq^{-1}] + 1$ we have

$$\begin{aligned} \left\| \chi_{\Lambda_0} P_{\Omega_0,\alpha} \operatorname{Op}_{\alpha,0}(b)(I-\chi_\Lambda) \right\|_{\mathfrak{S}_q}^q &\leqslant \left\| \operatorname{Op}_{\alpha,0}(b)h \right\|_{\mathfrak{S}_q}^q + \left\| \chi_{\Lambda_0} P_{\Omega_0,\alpha} \operatorname{Op}_{\alpha,0}(b)\tilde{h}(I-\chi_\Lambda) \right\|_{\mathfrak{S}_q}^q \\ &\leqslant C_m(\alpha r)^{d-mq} + \left\| \chi_{\Lambda_0} P_{\Omega_0,\alpha} \operatorname{Op}_{\alpha,0}(b)\tilde{h}(I-\chi_{\Lambda_0}) \right\|_{\mathfrak{S}_q}^q. \end{aligned}$$

Here we have used (4.21). Reversing the argument for the last term on the right-hand side we arrive at the bound

$$\|T_\alpha\|_{\mathfrak{S}_q}^q \leqslant C\alpha^{d-1} + \big\|\chi_{\Lambda_0}\operatorname{Op}_{\alpha,0}(b)P_{\Omega_0,\alpha}(I-\chi_{\Lambda_0})\big\|_{\mathfrak{S}_q}^q.$$

Both domains Λ_0, Ω_0 are basic, and hence we can use (4.16) for the right-hand side. As explained earlier, this leads to (4.20). □

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