Groups with root system of type $B_\ell$, $C_\ell$ or $F_4$

F.G. Timmesfeld

Mathematisches Institut, Arndtstraße 2, 35392 Giessen, Germany

Received 13 May 2004
Available online 13 December 2004
Communicated by Gernot Stroth

1. Introduction

Let $\Phi$ be an irreducible, spherical, possibly nonreduced root system (i.e., $\Phi = BC_\ell$ is allowed) of rank $\ell \geq 2$ satisfying the crystallographic condition. If the group $G$ is generated by nonidentity subgroups $A_r$, $r \in \Phi$, satisfying:

1. $X_r = \langle A_r, A_{-r} \rangle$ is a rank one group with unipotent subgroups $A_r$ and $A_{-r}$ for $r \in \Phi$.
   (For definition and properties of rank one groups, which will be used throughout this paper, see [2, I].)

2. If $r, s \in \Phi$ with $s/\in\{-r, -2r\}$ and $r/\in\{-s, -2s\}$, then $[A_r, A_s] \leq \langle A_{\lambda r + \mu s} | \lambda, \mu \in \mathbb{N} \rangle$ with $\lambda r + \mu s \in \Phi$. Further $A_{2r} \leq A_r$ if $2r \in \Phi$. (Here we use the convention $\langle \emptyset \rangle = 1$, so that the condition $A_r' = 1$ if $2r/\in \Phi$ is included in (2)).

Then we say $G$ satisfies hypothesis $(S)$. (Condition $(S)$ can be seen as a generalization of the Steinberg presentation, whence the name $(S)$.)

Let now $B$ be an irreducible spherical Moufang building of rank $\ell \geq 2$, $A$ an apartment of $B$ and $\Phi$ the set of roots (half-apartments) of $A$. Then we call $\overline{G} = \langle \overline{A}_r | r \in \Phi \rangle \leq \text{Aut}(\overline{B})$, where $\overline{A}_r$ is the root group corresponding to $r$ on $B$ in the sense of Tits, the group of Lie type $B$. It has been shown in [2, II §5], see also [7], that we can enlarge $\Phi$ to some possibly nonreduced root system $\tilde{\Phi}$, such that the $\overline{A}_r$, $r \in \tilde{\Phi}$, satisfy (1) and (2). ($\Phi \neq \tilde{\Phi}$ only if $\tilde{\Phi}$ is of type $BC_\ell$ and $\Phi$ of type $B_\ell$ or $\tilde{\Phi}$ of type $^2F_4$.) Moreover the following additional condition is satisfied:

E-mail address: franz.timmesfeld@math.uni-giessen.de.

0021-8693/$-$ see front matter © 2004 Elsevier Inc. All rights reserved.
(3) If $r, s \in \tilde{\Phi}$ and $\pi_r \in \tilde{X}_r$ interchanging $A_r$ and $A_{-r}$ by conjugation, then

$$\tilde{X}_r' = \tilde{X}_{w_r},$$

where $w_r$ is the reflection along $r$ on $\tilde{\Phi}$.

Now by Theorem 1 of [3] the converse holds. I.e., if $G$ is a group satisfying (S) and (3), then there exists an irreducible spherical Moufang building $\mathcal{B}$ of rank $\ell$ with (extended) root system $\tilde{\Phi}$ and a surjective homomorphism

$$\varphi : G \rightarrow \biglangle \tilde{X}_r \mid r \in \tilde{\Phi} \big\rangle$$

mapping the $A_r$ with $r \neq 2s$ onto the corresponding $\tilde{X}_r$. Further $\ker \varphi \leq Z(G) \cap H$, where $H = \langle H_r \mid r \in \Phi \rangle$ and $H_r = N_{X_r}(A_r) \cap N_{X_r}(A_{-r})$. To abbreviate notation we say in this situation that $G$ is of type $B$.

Since condition (3) is very difficult to control it would be desirable to weaken it. For this purpose it has been shown in Theorem 2 of [3] that the conclusion of the above theorem (i.e., Theorem 1 of [3]) holds, if $G$ satisfies (S) and in addition:

\begin{itemize}
  \item[(3')] Always equality holds in condition (2).
\end{itemize}

Unfortunately (3') is also difficult to control and is actually false in Chevalley groups of type $C_2$ and $F_4$ in characteristic two and of type $G_2$ in characteristic three. On the other hand, if one just assumes (S), then all commutators in (2) might be trivial, whence $G$ is the central product of the $X_r$, $r \in \Phi$. So such a possibility has to appear in the statements.

Assume for the rest of the introduction that $G$ satisfies (S). If the Dynkin diagram of $\Phi$ has only single bonds, i.e., $\Phi$ of type $A_\ell, D_\ell$ or $E_\ell$, then it has been shown in [4] that the following hold:

\begin{itemize}
  \item[(i)] $\Phi = \bigcup \Psi_i$, $\Psi_i$ a root subsystem of $\Phi$ or of the form $\Psi_i = \{ \pm r_i \}$, $r_i \in \Phi$.
  \item[(ii)] $G$ is a central product of subgroups $G_i = \langle A_r \mid r \in \Psi_i \rangle$, which are either of type $B_i$, $B_i$ a building with root system $\Psi_i$ as defined above, or $G_i = X_{r_i}$ (in case $\Psi_i = \{ \pm r_i \}$).
\end{itemize}

Unfortunately this statement is no longer true when the Dynkin diagram has a double bond. The easiest example to see this is $\Phi = C_2 = \{ \pm r, \pm s, \pm (r+s), \pm (2r+s) \}$ and $G = X_s * G_1$ (central product) and $G_1 = \langle X_r, X_{r+s}, X_{2r+s} \rangle \cong SL_3$. (In this case $\{ \pm r, \pm (r+s), \pm (2r+s) \}$ can be considered as a root system of type $A_2$, but is certainly no subsystem of $\Phi$.)

In case $\Phi$ of type $B_\ell, C_\ell, F_4$ or $B C_\ell$ and characteristic different from two it has been shown in [5,6] that $G$ has a central factor (which might be equal to $G$) of type $B$, $B$ a spherical Moufang building, or a central factor $X_r$, $r \in \Phi$. The condition that the characteristic is different from two comes from [1], which treats the case $\Phi = C_2$ as induction basis. Here (i.e., in [1]) the condition that the characteristic is different from two is needed, to obtain central involutions in certain rank one groups. The existence of these central involutions plays a main role in the proof.
In this paper we generalize [5] to arbitrary characteristic. We show:

**Theorem 1.** Suppose $G$ satisfies $(S)$ with $\Phi$ of type $B_\ell$, $C_\ell$ or $F_4$. Then the following hold:

(a) $\Phi = \bigcup \Psi_i$, where either $\Psi_i$ carries the structure of a root system of one of the types: $A_n$, $B_n$, $C_n$, $D_n$ or $F_4$ or $\Psi_i = \{\pm r_i\}$.

(b) $G$ is the central product of subgroups $G_i = \langle A_r \mid r \in \Psi_i \rangle$, which are either of type $B_i$, $B_i$ a Moufang building with root system $\Psi_i$ or $G_i = X_{r_i}$.

If $\Phi$ is of type $B_\ell$ or $C_\ell$ then it can be shown that $n \leq \ell$ for $n$ as in (a). (See proof of [5].) But this is no longer true for $\Phi$ of type $F_4$, in which case one of the $G_i$ can be of type $A_5$.

Theorem 1 is advantageous to [5] because of:

1. It also holds in characteristic two, which case has not been treated in [5] (and [1]) at all.
2. It makes a precise statement about all central factors of $G$, while in [5] (and [1]) it has only been shown that there is one central factor, which is either of Lie-type or a rank one group.
3. The proof is more conceptual in so far, that is concentrates on a section of $G$, which is a finite group generated by $\{3, 4\}$-transpositions. This section plays the role of a Weyl-group.

Moreover the proof of Theorem 1 is independent of [5]. ([5] is only quoted to mention certain auxiliary lemmata, the proof of which we repeat in our more general situation for the convenience of the reader.)

Similarly as in [5] the proof of Theorem 1 depends on the treatment of the $B_2 = C_2$ case as induction basis. For this purpose we have to extend the results of [1] to characteristic 2. We will prove in Section 3:

**Proposition 2.** Suppose $G$ satisfies $(S)$ with $\Phi = C_2$. Then one of the following holds:

1. $G$ is of type $B$, $B$ a Moufang building of type $C_2$.
2. There exists a long root $s \in \Phi$ such that $X_s \lhd G$. Further $X_r \leq C(X_s)$ for all $r \in \Phi - \{\pm s\}$.

From Proposition 2 we obtain as a corollary:

**Corollary 3.** Suppose $G$ satisfies $(S)$ with $\Phi = C_2$. Then one of the following holds:

1. $G$ is the central product of the $X_r$, $r \in \Phi$.
2. $G$ is of type $B$, $B$ a Moufang building with root system $\Phi$. Moreover for all $r, s \in \Phi$ and $n_r \in X_r$ interchanging $A_r$ and $A_{-r}$ we have $A^n_{sr} = A_{sr}$, $w_r$ the reflection on $\Phi$ along $r$.
3. There exists a long root $s \in \Phi$ with $X_s \lhd G$. Moreover, if $\Delta = \Phi - \{\pm s\}$, then $\Delta$ carries the structure of a root system of type $A_2$, $G_1 = G(\Delta) = \langle X_r \mid r \in \Delta \rangle$ is of
type $B$, $B$ a Moufang plane with root system $\Delta$, and $G = X_\times * G_1$. Further, for all $\alpha, \beta \in \Delta$ we have $A^\alpha_\beta = A^\beta_{\alpha\nu}, n_\alpha, w_\alpha$ similar as in (2).

The proof of Proposition 2 depends on [1] and Theorem 2 of [3]. Namely the main result of [1] says, that if the $A_r, r \in \Phi$, are not elementary abelian 2-groups, either case (2) of Proposition 2 holds or we have equality in all commutator relations of (5)(2). In the second case $G$ is by Theorem 2 of [3] of Lie-type $B$, $B$ a Moufang building of type $C_2$. Thus, to prove Proposition 2, we may assume that all $A_r, r \in \Phi$, are elementary abelian 2-groups. In this situation the existence of a central involution in $X_r$ will be replaced by the following:

**Observation.** Suppose $X = \langle A, B \rangle$ is a rank one group with $A$ and $B$ elementary abelian 2-groups. Then $X$ is special. (i.e., for $a \in A^\#$ and $b \in B$ with $A^b = B^a$ we have $a^b = b^a$)

The proof of this observation, which is very easy, will be given in Section 2. Now, to prove Theorem 1 assume that the proposition and its corollary hold. Then, by the conjugation condition in (2) and (3) of the corollary, the $H_\alpha, \alpha \in \Phi$, normalize each other and, if $H = \prod H_\alpha, \alpha \in \Phi$, then $N = \langle H, n_\alpha \mid \alpha \in \Phi \rangle$ normalizes $H$ and it can be shown that $\{n_\alpha \mid \alpha \in \Phi \}$ is a set of $[3, 4]$-transpositions of $\overline{N} = N / H$, which satisfies the conditions (a) and (b) of the theorem of [8]. Hence by [8], $\overline{N}$ is an image of a (possibly nonreducible) spherical Weyl-group and we show that $G$ is a central product of subgroups corresponding to the irreducible factors of $\overline{N}$.

In the meantime I have also proved the corresponding version of Theorem 1 for $\Phi$ of type $BC_\ell, \ell \geq 2$, so that, together with [4], we have a uniform theorem for all irreducible, spherical, possibly nonreduced root-systems and Lie-type groups different from $G_2$ and $^2F_4$.

### 2. Preliminary results

We first prove the observation on rank one groups made in the introduction.

**2.1. Lemma.** Let $X = \langle A, B \rangle$ be a rank one group such that the unipotent subgroups $A$ and $B$ are elementary abelian 2-groups. Then $X$ is special, i.e., for $a \in A^\#$ and $b = b(a) \in B$ with $A^b = B^a$ we have $a^b = b^{-a}$.

**Proof.** Suppose there exists an $a \in A^\#$ such that for $b = b(a)$ we have $a^b \neq b^{-a} = b^a$. Then there exists a $b \neq b \in B^\#$ with $a^b = b^\#$. Hence

$$bab = abba \quad \text{and thus} \quad ababa = b.$$

This implies $b = ab a^\#$ and thus

$$a^b a^\# = b(ab a^\#) = b b^\# \in B^\#.$$
Now $A^b = B^a$ and thus $1 \neq a^b b^a \in B^a \cap B$. Hence $B = B^a$, a contradiction to $A \cap N(B) = 1$. □

Next a lemma on $\{3, 4\}$-transpositions. Here a set $D$ of involutions generating the group $G$, is called a set of $\{3, 4\}$-transpositions of $G$, if the following hold:

(i) $D^g \subseteq D$ for all $g \in G$.
(ii) $o(de) \leq 4$ for all $d, e \in D$.

Notice that $G$ is finite if $|D|$ is finite. Namely, as $\langle D \rangle = G$, the kernel of the action of $G$ on $D$ is $Z(G)$. Hence $G/Z(G)$ is finite and thus also $G$. With this notation we have:

2.2. Lemma. Suppose the group $X$ is generated by the involutions $a, b, \ldots, e$ satisfying:

\begin{center}
\begin{tabular}{ccc}
a & b & d \\
\end{tabular}
\end{center}

and

\begin{enumerate}
\item $E = a^X$ is a finite set of $\{3, 4\}$-transpositions of $X$.
\item If $x, y, z \in E$, then $\langle x, y, z \rangle$ is an image of a finite Weyl-group.
\end{enumerate}

Then $X/N \simeq \Sigma_3$, where $N = O_2(X)$, and $|E| = 3 \cdot 2^n$, where $2^n = |N : C_N(a)|$. Moreover $X$ is a factor group of $W(D_4)$ if $n \leq 2$.

Proof. Let $\mathcal{F}(E)$ be the graph with vertex set $E$ and edges $(x, y)$, where $x, y \in E$ with $o(xy) = 3$. Then $\mathcal{F}(E)$ is connected, since $a, b, \ldots, e$ normalize the connectivity component of $\mathcal{F}(E)$ containing $\{a, b, \ldots, e\}$. We show

(1) $E$ is a class of $3$-transpositions of $X$. (This has already been proved in (2.2) of [8]. But since there the hypothesis is different, we repeat the proof for the convenience of the reader.)

Suppose (1) is false and under all pairs $e, f \in E$ with $o(ef) = 4$, choose $e, f$ so that $e$ and $f$ are connected by a path of minimal length in $\mathcal{F}(E)$. Let $e_i \in E$, $i = 0, \ldots, n$, such that $e_0 = e$, $e_n = f$ and $(e_0, e_1, \ldots, e_n)$ is such a minimal path from $e$ to $f$ in $\mathcal{F}(E)$. Then, by minimality of $d(e, f)$ ($d$ denotes the distance in $\mathcal{F}(E)$) we have $o(e_i e_j) \neq 4$ for all $i, j \leq n$ with $(i, j) \neq (0, n)$.

Suppose $n = 2$. Then we have

\begin{center}
\begin{tabular}{ccc}
e_0 & e_1 & e_2 \\
\end{tabular}
\end{center}
But by hypothesis (2), \( \langle e_0, e_1, e_2 \rangle \) is an image of a finite Weyl-group (of rank 3), a contradiction, since \( W(A_3) \simeq \Sigma_4 \) and \( W(B_3) \) do not contain 3 reflections, satisfying these relations.

Hence \( n \geq 3 \). Now, by the minimality of the path from \( e \) to \( f \), we have \( o(e_i e_j) \neq 3 \) for all \( i, j \leq n \) with \( j \neq i \pm 1 \). Thus we have

\[
\begin{array}{c}
e_0 \quad e_1 \quad e_2 \\
\vdots \\
\end{array}
\]

and \( Y = \langle e_1, \ldots, e_{n-1} \rangle \simeq \Sigma_n \) and \( \langle e, Y \rangle \simeq \Sigma_{n+1} \simeq \langle Y, f \rangle \). Let now \( a \) be the highest reflection in \( Y \). Then, by the structure of \( \Sigma_{n+1} \), we have \( o(ea) = 3 = o(af) \), a contradiction to \( n \neq 2 \). This proves (1).

Now, as \( E \) is finite, \( X \) is a finite image of \( W(D_4) \). Hence \( X \) is solvable. Let \( E_a = \{ f \in E \mid C_E(a) = C_E(f) \} \). Then

(2) \( Q(E_a) := \langle f \cdot h \mid f, h \in E_a \rangle \leq N = O_2(X) \). ((2) is one of the well-known little facts about 3-transposition groups, the proof of which we will, also for the convenience of the reader, repeat here.)

Namely let \( y \in E_a \neq E_a \). By definition of \( E_a \) we have \( o(f \cdot h) = 2 \) (or \( f = h \)). Hence, also by definition of \( E_a \) either \( o(yf) = 2 = o(yh) \) or

\[
\begin{array}{c}
f \\
y \quad h
\end{array}
\]

In any case by the structure of \( \Sigma_4 \), we obtain \( yf \in E_a \), since \( E_a \) is a TI-subset of \( E \). This shows that \( f \) normalizes all \( E_a, g \in X \), and thus the subgroups \( Q(E_a), g \in X \), normalize each other. Since \( \langle E_a \rangle \) is an elementary abelian 2-group, this shows that

\[
\langle Q(E_a) \mid g \in X \rangle
\]

is a normal 2-subgroup, which proves (2).

We claim \( \{a, c, d, e\} \subseteq E_a \). Namely if this claim holds, then by (2), \( X/N = \langle a, b \rangle N \) / \( N \simeq \Sigma_3 \).

To prove our claim suppose \( f \in C_E(a) = C_E(c) \). If \( f \in C(b) \) we have

\[
\begin{array}{c}
a \\
b \quad c \\
f
\end{array}
\]

and \( \langle a, b, c, f \rangle \simeq \Sigma_5 \), a contradiction to the solvability of \( X \). If \( o(bf) = 3 \), then we have

\[
\begin{array}{c}
b \\
\quad c
\end{array}
\]

and \( f \).

\[
\begin{array}{c}
f
\end{array}
\]
But by condition (2), \( \langle b, c, f \rangle \) is a factor group of a spherical Weyl-group and whence a factor group of \( \Sigma_4 \). But then \( \langle b, c, f \rangle \simeq \Sigma_4 \), since clearly \( f \neq b^c \). Hence we obtain

![Diagram](attachment:image.png)

again a contradiction to the solvability of \( X \). This shows \( C_E(a) \subseteq C_E(c) \) and with symmetry \( C_E(a) = C_E(c) \). Since for \( d \) and \( e \) the same proof applies, this proves our claim. The additional statements in 2.2 are easy exercises. □

2.3. Notation. If \( \Phi \) is an irreducible, spherical root system and \( r, s \in \Phi \), let \( \langle r, s \rangle \) be the root subsystem spanned by \( r \) and \( s \). If \( G \) is a group satisfying (S) with respect to \( \Phi \) and if \( r \in \Phi \) let \( n_r \) be an arbitrary element of \( X_r \) interchanging \( A_r \) and \( A_{-r} \). Notice that

\[
H_r n_r = \{ n \in X_r \mid A_n^r = A_{-r}, \ A_{n}^{-r} = A_r \} \quad \text{and} \quad H_r = \{ n \cdot n' \mid n, n' \in X_r \ \text{interchanging} \ A_r \ \text{and} \ A_{-r} \}.
\]

2.4. Lemma. Suppose \( G \) satisfies (S) with respect to \( \Phi \) and let \( r, s \in \Phi \). Let \( Y = \langle X_a \mid \alpha \in \langle r, s \rangle \rangle \), where \( \langle r, s \rangle \) is the root subsystem of \( \Phi \) spanned by \( r \) and \( s \). Then we have

1. If \( \langle r, s \rangle \) is of type \( A_1 \times A_1 \) then \( [X_r, X_s] = 1 \) and \( Y = X_r \ast X_s \).
2. If \( \langle r, s \rangle \) is of type \( A_2 \), then one of the following holds:
   i. \( Y \) is the central product of the \( X_a \), \( \alpha \in \langle r, s \rangle \).
   ii. \( Y \) is of type \( A_2 \) and if \( \alpha, \beta \) and \( \alpha + \beta \in \langle r, s \rangle \), then \( A_{\beta} = A_{\alpha + \beta} = A_{\alpha}^\beta \).

Proof. (1) is obvious. (2) is just the main result of [4] restated for the \( A_2 \) case. (See [5, (2.6)].) □

We now restate Corollary 3 for root-subsystems \( \langle r, s \rangle ; r, s \in \Phi \).

2.5. Lemma. Suppose \( G \) satisfies (S) with respect to \( \Phi \) and the conclusion of Corollary 3 holds. Then, if \( r, s \in \Phi \) and \( \langle r, s \rangle \) is of type \( B_2 \) \((= C_2)\) one of the following holds for \( Y = \langle X_a \mid \alpha \in \langle r, s \rangle \rangle \):

1. \( Y \) is the central product of the \( X_a \), \( \alpha \in \langle r, s \rangle \).
2. \( Y \) is of type \( B_2 \). Moreover for all \( \alpha, \beta \in \langle r, s \rangle \) we have \( A_{\beta} = A_{\alpha + \beta} = A_{\alpha}^\beta \), where \( w_{\alpha} \) is the reflection along \( \alpha \) on \( \langle \alpha, \beta \rangle \).
3. There exists a long root \( \alpha \in \langle r, s \rangle \) such that \( X_\alpha \triangleleft Y \). Moreover if \( \Delta = \langle r, s \rangle - \{ \pm \alpha \} \) and \( R = \{ X_\beta \mid \beta \in \Delta \} \), then \( R \) is of type \( A_2 \) and \( A_R^\beta = A_{r, w_{\beta}} \), where \( w_{\beta} \) is the reflection along \( \beta \) on \( \Delta \). (\( \Delta \) considered as root system of type \( A_2 \).)

2.6. Notation. Suppose \( G \) satisfies (S) with respect to \( \Phi \), and \( \Phi \) is a reduced, irreducible, spherical root system satisfying the crystallographic condition, different from \( G_2 \). (i.e., \( \Phi \) is of type \( A_1, B_1, C_1, D_1, E_1 \) or \( F_3 \).) Suppose further that the conclusion of the corollary holds. Then for all \( r, s \in \Phi \) one of Lemmas 2.3–2.6 holds for \( \langle r, s \rangle \) and \( Y = \langle X_a \mid \alpha \in \langle r, s \rangle \rangle \).
We do not have, as in the single bond case in [4], that $\langle n\rangle \cap \langle m'\rangle = \emptyset$. Remark.

Suppose further that the conclusion of Corollary 3 holds and use the notation introduced in 2.7. Then we have:

1. $H$ normalizes $\Lambda_0, X_\alpha, H_\alpha$ and $H_\alpha(n_\alpha)$ for each $\alpha \in \Phi$.
2. $N$ acts on $\{A_\alpha \mid \alpha \in \Phi\}$, $X_\alpha$ and $\{H_\alpha(n_\alpha) \mid \alpha \in \Phi\}$.
3. $\overline{D} = \langle \pi_\alpha \mid \alpha \in \Phi \rangle$ is a finite set of $(3, 4)$-transpositions of $N$ (and whence $\overline{N}$ is finite).

Proof. Since each $H_\alpha$ normalizes $\Lambda_0$ for each $\alpha \in \Phi$, (1) follows directly from the definition of $H$. By 2.4–2.6 we have $A_\alpha^{n_\alpha} = A_\beta^{n_\beta}$ for all $\alpha, \beta \in \Phi$ and some $\beta' \in \langle \alpha, \beta \rangle$. This implies (2).

By (2) we have $\overline{D} \subseteq \overline{D}$ for each $\pi \in \overline{N}$ and by definition of $\overline{N}$ we have $\overline{N} = \langle \overline{D} \rangle$. Clearly $|\overline{D}| < |\Phi| < \infty$. So it remains to show $o(\pi_\alpha \pi_\beta) \leq 4$ for all $\alpha, \beta \in \Phi$ to prove (3). Now by 2.4–2.6, $(n_\alpha, n_\beta)$ normalizes $H_\alpha H_\beta$ and $o(n_\alpha n_\beta) \leq 4 \text{ mod } H_\alpha H_\beta$. Since $H_\alpha H_\beta \leq H$ this proves (3). \qed

2.9. Proposition. Suppose $G$ satisfies (S) with respect to $\Phi$ and $\Phi$ is a reduced, irreducible, spherical root system different from $G_2$, satisfying the crystallographic condition. Suppose further that the conclusion of Corollary 3 holds and use the notation introduced in 2.6. Suppose further that $\Delta \subseteq \Phi$ satisfies:

(a) $\overline{\Delta}(\Delta) = \langle \pi_r \mid r \in \Delta \rangle$ is a center-factor group of an irreducible spherical Weyl-group of rank $\geq 2$ and $[\pi_r \mid r \in \Delta]$ is the set of reflections of $\overline{\Delta}(\Delta)$.
(b) If $\pi \in \overline{\Delta}(\Delta)$ and $r \in \Delta$ with $A_s^{\pi} = A_r$, then $s \in \Delta$.

Let $G(\Delta) = \langle X_r \mid r \in \Delta \rangle$. Then the following hold:

1. $\Delta$ carries the structure of an irreducible spherical root system of one of the types $A_\ell, B_\ell, C_\ell, D_\ell, E_\ell$ or $F_\ell$, $\ell \geq 2$, and $\overline{\Delta}(\Delta)$ is the Weyl-group of this root system.
(2) $G(\Delta)$ is of type $B$ for some irreducible spherical Moufang building with root system $\Delta$.

**Proof.** 2.9 is essentially (2.13) of [5], the only difference being that we also admit the characteristic two case, which does not affect the proof much. But for the convenience of the reader we repeat it.

The aim of the proof is to introduce the structure of a root system on $\Delta$ and then apply Theorem 1 of [3] to $G(\Delta)$.

First notice that by 2.8, $N(\Delta)$ acts on $\Phi$ by $\lambda, r := (A_r)\lambda$ for $r \in \Phi$, $n \in N(\Delta)$. Hence $N(\Delta)$ acts on $\Delta$ by hypothesis (b). Next we need to show that $N(\Delta)$ is actually isomorphic to the corresponding Weyl-group. For this it suffices by hypothesis (a) to show that $Z(N(\Delta)) \neq 1$, if the center of the corresponding Weyl-group is different from 1. If now $1 \neq z \in Z(W(\Delta))$, then $z = w_{r_1} \cdots w_{r_m}$ with pairwise commuting different reflections $w_{r_i}$ and $r_i \in \Delta$. (This is a well-known fact about spherical Weyl-groups.) Hence, if we set $n = n_{r_1} \cdots n_{r_m}$, then $n_{r_i}n_{r_j} = n_{r_j}n_{r_i}$ for $i \neq j$. Thus by 2.4, 2.5, $r_i n = r_i$ for $i \leq m$. Since $z$ is mapped onto $n$ by the natural homomorphism $W(\Delta) \to N(\Delta)$, this implies $1 \neq n \in Z(N(\Delta))$, which is to show.

Next set $\Delta^+ = \Delta \cap \Phi^+$. Then $\Delta = \Delta^+ \cup \Delta^-$ with obvious notation. Let $r_1, \ldots, r_k \in \Delta^+$, such that $n_{r_1}, \ldots, n_{r_k}$ is a set of fundamental reflections of $N(\Delta)$, and let $V$ be a (formal) $k$-dimensional $\mathbb{R}$-vectorspace with basis $(r_1, \ldots, r_k)$. Define a scalar product on $V$ by $(r_i, r_j) = -\cos(\pi/\omega(n_{r_i}n_{r_j}))$ for $1 \leq i, j \leq m$ and linear extension. Then it is well known from the theory of finite Coxeter-groups that $V$ is with $(\cdot, \cdot)$ an euclidean space and $N(\Delta)$ acts on $V$ such that the $n_{r_i}$ acts as a reflection along $r_i$. Now identify $\Delta$ with the images of the $r_i$ (in $V$) under $N(\Delta)$. (This is the so-called geometric realization of $N(\Delta) \simeq W(\Delta)$. Notice that in $\Delta$ all roots have length one so far!)

In this way $\Delta$ becomes a root-system with Weyl-group $N(\Delta)$. For $r, s \in \Delta$ let $(r, s)_{\Delta}$ be the root-subsystem of $\Delta$ spanned by $r$ and $s$. Now, to obtain the crystallographic condition, we must choose the length of the roots in $\Delta$ accordingly. In the single bond case (i.e., $o(n_{r_i}n_{r_j}) \leq 3$ for all $i, j \leq m$) simply let $|r| = 1$ for all $r \in \Delta$. If $o(n_{r_i}n_{r_j}) = 4$ then by 2.5, $(r_i, r_j)$ and $(r_i, r_j)_{\Delta}$ are both of type $B_2$ (i.e., $C_2$). If now $|r_i| > |r_j|$ in $\Phi$ we let $|r_i| = 1$ and replace $r_i$ by $\sqrt{2}|r_i|$ and extend this definition of length to all roots of $\Delta$ by the action of $N(\Delta)$. (Notice that this definition of length is well defined, since $N(\Delta)$ is a spherical Weyl-group and whence each root is either conjugate to a long fundamental root or to a short fundamental root. The length of the roots in $\Delta$ is not necessarily the same as in $\Phi$, since for example if $|r_i| > |r_j|$ in $\Phi$ it might by 2.5 happen that $(r_i, r_j)_{\Delta}$ is of type $A_2$ and whence $|r_i| = |r_j|$ in $\Delta$.) In this way $\Delta$ becomes a spherical root system of one of the types $A_\ell, B_\ell, C_\ell, D_\ell, E_\ell$ or $F_4$. If now $\alpha = \pm \beta \in (r, s)_{\Delta}$ then we obtain by 2.4, 2.5:

\begin{enumerate}
\item $[A_\alpha, A_\beta] = \langle A_{2\alpha+\mu_\beta} | \lambda, \mu \in \mathbb{N} \text{ and } \lambda\alpha + \mu\beta \in (r, s)_{\Delta} \rangle \leq \langle A_{2\alpha+\mu_\beta} | \lambda, \mu \in \mathbb{N} \text{ and } \lambda\alpha + \mu\beta \in \Delta \rangle$,
\item $A_{\beta_{w\alpha}}^\Phi = A_{\beta_{w\alpha}}$, where $w_{\alpha}$ is the reflection along $\alpha$ on $(r, s)_{\Delta}$.
\end{enumerate}

Since (ii) holds for all $r, s \in \Delta$ this implies

\begin{enumerate}
\setcounter{enumi}{2}
\item $A_{\beta_{w\alpha}}^\Phi = A_{\beta_{w\alpha}}$, where $w_{\alpha}$ is the reflection along $\alpha$ on $\Delta$ for all $\alpha, \beta \in \Delta$.
\end{enumerate}
Now (i) and (iii) show that $G(\Delta)$ satisfies the hypothesis of Theorem 1 of [3] with respect to $\Delta$. Hence by that Theorem 2 holds. □

3. Proof of the proposition

We assume in this section that hypothesis $(S)$ holds with

$$
\Phi : \begin{array}{ccc}
s & r+s & 2r+s \\
-r & r & -s \\
-2r-s & -r-s & -s \\
\end{array}
$$

We also use the notation of §2 of [3], that is for $\alpha \in \Phi$ the subgroups $U_\alpha$, $X_\alpha$ and $H_\alpha$ are defined. Assuming that case (2) of the proposition does not hold, we have by [3, (2.4)] for $U_s = U_s/A_{2r+s}:

$$
\begin{align*}
\overline{U}_s &= \overline{X}_{r+s} \times \overline{X}_s, \\
\overline{X}_{r+s} &= C_{\overline{U}_s}(\overline{X}_s) = [\overline{U}_s, \overline{X}_s], \text{ and} \\
\overline{A}_r &= C_{\overline{U}_s}(\overline{A}_s) = [\overline{U}_s, \overline{A}_s] \\
\end{align*}
$$

and the same statement holds for all long roots $\alpha \in \Phi$. Since $X_s = (A_{-s}, a) = (A_s, b)$ for each $a \in A^\#, b \in A^\#_{-s}$ we obtain as an immediate consequence of (*):

$$(3.1) \overline{X}_{r+s} = [\overline{U}_s, a] = [\overline{A}_r, a] = C_{\overline{U}_s}(a) \text{ for each } a \in A^\# \text{ and } \overline{A}_r = C_{\overline{U}_s}(\overline{A}_s) = [\overline{U}_s, \overline{A}_s] \text{ for each } b \in A^\#_{-s} \text{ and the same statements hold for all long roots } \alpha \in \Phi.
$$

Proof. As $X_s = (A_{-s}, a)$ for $a \in A^\#$ we obtain that $\overline{N} = \overline{A}_r[\overline{A}_r, a]$ is $X_s$-invariant by (*). Hence

$$
\overline{X}_{r+s} = [\overline{A}_r, A_s] = [\overline{U}_s, A_s] \subseteq \overline{N}
$$

and $\overline{N} = \overline{U}_s$. Thus by (*), $\overline{A}_{r+s} = [\overline{A}_r, a]$. If now $C_{\overline{U}_s}(a) > \overline{A}_{r+s}$, then by (*), $1 \neq \overline{Z} := \overline{A}_r \cap C_{\overline{U}_s}(X_s)$. Hence $\overline{A}_r \cap \overline{A}_{r+s} \geq \overline{Z}$, since by (*) $\overline{A}_r^{n_s} = \overline{A}_{r+s}$ for $n_s \in X_s$ interchanging $A_s$ and $A_{-s}$. This contradiction proves (3.1). □

Now, to prove that the proposition holds, we may and will, as shown in the introduction, by [1] and Theorem 2 of [3] assume that all $A_\alpha$, $\alpha \in \Phi$, are elementary abelian 2-groups.

Our aim is to show, that for all $\alpha, \beta \in \Phi$ and $n_\alpha \in X_\alpha$ interchanging $A_\alpha$ and $A_{-\alpha}$ we have

$$(A_\beta)^{n_\alpha} = A_{\beta^{n_\alpha}}, \quad \text{where } \omega_\alpha \text{ is the reflection along } \alpha \text{ on } \Phi.
$$

(+)

Notice that by [1, (3.4)], (+) already holds if \( \alpha \) is long. So it remains to prove (+) for short roots \( \alpha \). If (+) is proved, then the proposition is a consequence of Theorem 1 of [3].

We first show:

**3.2. Lemma.** Let \( x \in A^\#_r \) and \( y \in A^\#_r \) such that \( x^y = y^x \) (i.e., \( x \) and \( y \) are determined as element of \( A \)).

But since \( \alpha \in \) assumption.

Thus \( x \) and \( y \) satisfy the equation \( x^y = y^x \).

To prove 3.2 assume first \( (A_1 A_{r^{-1}})^n \not\subseteq A_{r+s} A_{2r+s} \). Then we have by (3.1) for \( a \in (A_1 A_{r^{-1}})^n \) that:

\[ [a, A_r] A_{2r+s} = A_{r+s} A_{2r+s}. \]

But since \( [A_1 A_{r^{-1}}, A_{r^{-1}}, A_{r^{-1}}] = 1 \) we get

\[ [(A_1 A_{r^{-1}})^n, A_r, A_r] = 1 \]

(since \( n \) interchanges \( A_r \) and \( A_{r^{-1}} \)) and thus

\[ A_{s^{-1}} A_{s} A_{2r+s} \leq [(A_1 A_{r^{-1}})^n, A_r] A_{s^{-1}} A_{2r+s} \leq C(A_r). \]

Hence \( A_{s^{-1}} A_{s} A_{2r+s} = C_{U_{s^{-1}}}(A_r) \) and \( [U_{s^{-1}}, A_r, A_r] = 1 \). Whence also \( [U_{s^{-1}}, A_{r^{-1}}, A_{r^{-1}}] = 1 \) and thus, since by \((*)\) applied to the action of \( X_{2r+s} \) on \( U_{2r-s} \) we have \( A_{s^{-1}} A_{s} A_{2r+s} \leq A_{s^{-1}} A_{s} A_{2r+s} \), we obtain \( A_{s^{-1}} A_{s} A_{2r+s} = C_{U_{s^{-1}}}(A_r) \). In particular \( A_{s^{-1}} A_{s} A_{2r+s} \) is normalized by \( r \) and \( A_1 A_{r^{-1}} \) and \( A_{r^{-1}} A_{2r+s} \) are interchanged by \( n \), a contradiction to the assumption.

Thus \( (A_1 A_{r^{-1}})^n \subseteq A_{r+s} A_{2r+s} \) and therefore, since \( n^2 = 1 \), we obtain \( A_1 A_{r^{-1}} \subseteq (A_{r^{-1}} A_{r+s})^n \). But by symmetry (since \( n_r(x) = n_r(y) \)) also \( (A_{r+s} A_{2r+s})^n \subseteq A_{r+s} A_{2r+s} \), which proves 3.2 since \( A_{r+s} = A_{r+s} \cap A_{s^{-1}} A_{s} A_{2r+s} \).

**Proof.** Since \( n \) is an involution acting on the elementary abelian 2-group \( A_{r+s} \) clearly \( C_{A_{r+s}}(n) \neq 1 \). So it remains to show \( C_{A_{r+s}}(X) \neq C_{A_{r+s}}(n) \). For this pick \( 1 \neq a \in C_{A_{r+s}}(n) \). Then \( a^x = a[a, x] \in C(Y) \) since \( n^x = y \). Suppose \([a, x] \neq 1 \). Then

\[ a^x = a[a, x] \notin A_{s} A_{s} A_{2r+s}. \]

since \( A_{s} A_{s} A_{2r+s} = 1 \) by [3, (2.1)]. But \( C_{U_{s^{-1}}}(y) \subseteq A_{s} A_{s} A_{2r+s} \) by (3.1) applied to the action of \( X_{2r+s} \) on \( U_{s^{-1}} A_{s} A_{2r-s} \). Hence \([a, x] = 1 \) and \( a \in C(X) \), which proves 3.3.

**Proof.** Let \( x \) and \( y \) be in 3.2. Pick by 3.3, \( 1 \neq a \in C_{A_{r+s}}(X) \) and let \( b \in A_{r^{-1}} A_{s} \) with \( a^b = b^a \) and \( Y = \langle a, b \rangle \). (\( b \) exists by 2.1!) Then \([X, Y] = 1 \).
Proof. Since $a^b \in X_{r+s}$ interchanging $A_{r+s}$ and $A_{r-s}$, $a^b$ normalizes $A_r$ by 3.2 applied to the action of $X_{r+s}$ on $U_{r+s}$. Suppose $x \notin C_{A_r}(a^b)$. Then we get the relations:

$$x \quad a^b \quad a.$$ 

By the structure of $W(C_3) \cong \Sigma_4 \times \mathbb{Z}_2$ we have $z = xx^ba^b \in C((a, b))$. Since by (3.1) applied to the action of $X_s$ on $U_s$ and $X_{2r+s}$ on $U_{2r+s}/A_{r-s}$ we have $C_{U_{r+s}}((a, b)) \leq A_r$, it follows $z \in A_r$. But then, since also $x^b = x^b \in A_r$, we obtain $x^b \in A_r$. On the other hand $[x, b] \in [A_r, A_{r-s}] \leq A_{r-s}$, so that $[x, b] \in A_r \cap A_{r-s}$ and $x \in C((a, b))$. Hence $x \in C((a, b))$. Since by the same argument also $y \in C((a, b))$, this proves (3.4). $\square$

(3.5) Each $n_r(x), x \in A_r$, interchanges $A_r$ and $A_{2r+s}$ by conjugation.

Proof. Let $n = n_r(x)$ and let $X = (x, y)$ and $Y = (a, b)$ be as in (3.4). Then $\langle A_r^x \rangle \leq U_{2r-s}$ and thus is an elementary abelian 2-group. Hence also

$$R = \{ (A_r^x)^n \} = \{ A_r^x \}^n$$

is elementary abelian, since $[n, Y] = 1$ by (3.4). By 3.2, $A_r^n \leq A_{r+s}A_{2r+s} \leq A_{r+s}U_{r+s}$. Suppose $A_r^n \notin U_{r+s}$. Let $F = X_{r+s}U_{r+s}$ and $\tilde{F} = F/U_{r+s}$. Then $\tilde{F}$ is a rank one group with unipotent subgroups $\tilde{A}_{r+s}$ and $\tilde{A}_{r-s}$. Since $1 \neq A_r^n \leq \tilde{A}_{r+s}$ this implies that $\langle \tilde{A}_r^n, (\tilde{A}_r^n)^b \rangle$ is not elementary abelian (see for example 2.1 applied to $\tilde{F}$), a contradiction to the above. This shows $A_r^n \leq U_{r+s} \cap A_{r+s}A_{2r+s} \leq A_{2r+s}$ by [3, (2.1)]. Symmetry now implies (3.5). $\square$

(3.6) For each $n_r \in X_r$ interchanging $A_r$ and $A_{r-s}$ we have $A_n^{n_r} = A_{r+s}$ and $n_r$ interchanges $A_{r+s}$ and $A_{2r+s}$ by conjugation.

Proof. First notice that $n_r \in H_r n$ with $n = x^y$ in the notation of 3.2. Now by (*) we have

$$A_r A_{r+s} = C_{U_{r-s}}(A_{r-s})[U_{r-s}, A_{r-s}]$$

and

$$A_{r+s} A_{2r+s} = C_{U_{r-s}}(A_r)[U_r, A_r].$$

Hence $H_r$ and $n_r$ normalize $A_{r+s} = A_r A_{r+s} \cap A_{r+s} A_{2r+s}$ by 3.2. By the same argument applied to the action of $X_r$ on $U_r$, we also get that $H_r$ normalizes $A_{r-s} A_{r-s} = C_{U_{r-s}}(A_{r-s})[U_{r-s}, A_{r-s}]$. Hence $H_r$ normalizes $U_{r-s} = A_{r-s} A_{r-s} A_{r-s}$. This implies $\langle A_r^n \rangle \leq N(U_{r-s})$. But clearly $N_{A_r}(U_{r-s}) = 1$, since $\langle A_{r-s}, u \rangle = X_{r+s}$ for each $1 \neq u \in A_{r-s}$. Hence $N_{A_r}(U_{r-s}) = A_r$ and thus by the above

$$\langle A_r^n \rangle \leq N_{A_r}(U_{r-s}) A_r = A_r.$$ 

Since $A_r^n = A_{2r+s}$, this shows that $H_r$ normalizes $A_r$ and $A_{2r+s}$. As $n_r \in H_r n$, (3.5) now implies (3.6). $\square$
3.7. Proof of the proposition. Since the denotation of the roots is arbitrary, (3.6) shows that either case (2) of the proposition holds or
\[ A_{\beta}^{n} = A_{\beta + n} \]
for all short roots \( \alpha \in \Phi \) and all \( n_{\alpha} \in X_{\alpha} \) interchanging \( A_{\alpha} \) and \( A_{-\alpha} \). Since the same holds by [1, (3.4)] for the long roots, the proposition is now a direct consequence of Theorem 1 of [3]. \( \square \)

3.8. Proof of the corollary. To prove the corollary we may assume that case (1) of the proposition holds. Choosing the denotation of the roots appropriately we may further assume that
\[ X_{s} \cong G \]
and, if \( \Delta = \Phi - \{ \pm s \} \), then \( R = \langle X_{\beta} | \beta \in \Delta \rangle \subseteq C(X_{s}) \).

Set \( \beta = r, \gamma = r + s \). Then \( \Delta = \{ \pm \beta, \pm \gamma, \pm (\beta + \gamma) \} \) and we have \( [A_{\beta}, A_{-\gamma}] \subseteq A_{-r} \cap C(X_{s}) = 1 \) and also by the same argument \( [A_{-\beta}, A_{\gamma}] = 1 \). Moreover
\[ [A_{\beta}, A_{-\beta - \gamma}] \subseteq A_{-r - s} \cap C(X_{s}) \subseteq A_{-r - s} = A_{-\gamma} \]
and similarly
\[ [A_{\gamma}, A_{-\beta - \gamma}] \subseteq A_{-\beta} \]
Since by definition of \( \beta \) and \( \gamma \) we have \( [A_{\beta}, A_{\gamma}] \subseteq A_{\beta + \gamma} \) and \( [A_{-\beta}, A_{-\gamma}] \subseteq A_{-\beta - \gamma} \), \( R \) satisfies the hypothesis \((H)\) of [4], with \( \Delta \) a root system of type \( A_{2} \). Hence by the theorem of [4] either \( R \) is the central product of the rank one groups \( X_{\alpha}, \alpha \in \Delta \) and case (1) of the corollary holds or we have
\[ [A_{\beta}, A_{\psi}] = A_{\beta + \psi} \]
for all \( \delta, \psi \in \Delta \) with \( \delta + \psi \in \Delta \).

But in the latter case by Theorem 2 of [3] \( R \) is of type \( A_{2} \) and \( A_{\beta}^{n_{\varphi}} = A_{\beta + n_{\varphi}} \) for all \( \delta, \varphi \in \Delta \) and \( n_{\alpha} \in X_{\varphi} \) interchanging \( A_{\varphi} \) and \( A_{-\varphi} \), where \( \omega_{\varphi} \) is the reflection along \( \varphi \) on \( \Delta \). (Considering \( \Delta \) as root system of type \( A_{2} \).) Hence case (3) of the corollary holds. \( \square \)

4. Proof of Theorem 1

We assume in this section that \( G \) and \( \Phi \) satisfy the hypothesis of Theorem 1 and we use the notation introduced in 2.7. Then by 2.8, \( \overline{D} \) is a set of \( \{3, 4\} \)-transpositions of \( \overline{N} \). We first show that \( \overline{D} \) satisfies the hypothesis (a) and (b) of the main theorem of [8], which then shows that \( \overline{N} \) is an image of a finite Weyl-group and \( \overline{D} \) is the image of the reflections.

4.1. Lemma. \( \langle \overline{n}_{r_{1}}, \overline{n}_{r_{2}}, \overline{n}_{r_{3}} \rangle \) is an image of a finite Weyl-group for all \( r_{1}, r_{2}, r_{3} \in \Phi \).

Proof. Let \( \Psi = \langle r_{1}, r_{2}, r_{3} \rangle \). If \( \Psi \) is decomposable of rank 3, then \( \Psi \) is of type \( A_{1} \times A_{1} \times A_{1} \), \( A_{2} \times A_{1} \) or \( C_{2} \times A_{1} \). Hence 4.1 is a consequence of 2.4–2.6. The same lemmata also imply 4.1 if rank \( \Psi \leq 2 \). So we may assume that \( \Psi \) is irreducible of rank 3.

Then \( \Psi \) is of type \( A_{3}, B_{3} \) or \( C_{3} \), since \( \Psi \) is a root subsystem of \( \Phi \). If \( \Psi \) is of type \( A_{3}, B_{3} \) or \( C_{3} \), then \( \Psi \) is of type \( A_{3}, B_{3} \) or \( C_{3} \). Hence we may assume that \( \Psi \) is of type \( B_{3} \) or \( C_{3} \).
Assume first $o(n_{ri}n_{rj}) = 4$ for $1 \leq i \neq j \leq 3$. Then by 2.5 also $o(w_rw_{r'}) = 4$ for $1 \leq i \neq j \leq 3$, where $w_r$ denotes the reflection along $r$ on $\Phi$ for $r \in \Phi$. But in a root system of type $B_3$ or $C_3$, there do not exist 3-reflections with this property. Next suppose we have

$$\overline{n_{r_1}} \overline{n_{r_2}} \overline{n_{r_3}}.$$

Since $\Psi$ is of type $B_3$ or $C_3$ we cannot have

$$\overline{w_{r_1}} \overline{w_{r_2}} \overline{w_{r_3}}.$$

(If so, then rank $(r_1, r_2, r_3) = 2$!) Hence we obtain

$$w_{r_1} \overline{w_{r_2}} \overline{w_{r_3}},$$

But then, by the structure of $W(B_3) \simeq \Sigma_4 \times \mathbb{Z}_2$ we must have $o(w_{r_1}w_{r_2}w_{r_3}) = 2$. But this is a contradiction to $o(n_{r_1}n_{r_2}n_{r_3}) = 4$. $(w_{r_2}w_{r_3} = w_s$, where $s$ is the only root in $(r_2, r_3)$ of the same length as $r_2$ but different from $r_2$. Hence $(r_1, s)$ is of type $A_1 \times A_1$). So the only cases which we need to consider seriously are:

I.  
II.  
III.  

(Up to change of enumeration of the roots. In all other cases the diagram of $n_{r_1}, n_{r_2}, n_{r_3}$ is already spherical, whence $(n_{r_1}, n_{r_2}, n_{r_3})$ must be the image of a spherical Weyl-group.)

Let now $Y = \langle n_{r_1}, n_{r_2}, n_{r_3} \rangle$ and $E = n_{r_1}^Y \cup n_{r_2}^Y \cup n_{r_3}^Y$. Then $\overline{E} \subseteq D \cap \{n_r \mid r \in \Psi^+\}$. Since $|\Psi^+| = 9$, we have $|E| \leq 9$.

Suppose first that we are in case I. Then, as $Y$ is an image of $\widehat{W(A_2)} \simeq (\mathbb{Z} \times \mathbb{Z})\Sigma_3$, it is easy to see that either $Y$ is an image of $\Sigma_4 = W(A_3)$ and (4.1) holds or $|Y| = 3^3 \cdot 2$ or $3^2 \cdot 2$. In the second case $|E| = 9$ and all different elements of $E$ do not commute. But this is impossible, since either $\Psi$ has a root subsystem of type $D_3 = A_3$ in case $\Psi = B_3$ or of type $A_1 \times A_1 \times A_1$ in case $\Psi = C_3$.

Next assume that we are in case II. Then without loss $r_1$ is short and $r_3$ is long. If $\Psi = C_3$ then $r_2$ is also short and we obtain

$$w_{r_1} \overline{w_{r_2}} \overline{w_{r_3}}.$$
Hence \( w_{r_1^2} = w_{r_1+r_3} \) and \( o(w_{r_1^2}, w_{r_1}) = 2 \). Thus \( \overline{\pi_{r_1}} = \overline{\pi_{r_1+r_3}} \) and \( o(\overline{\pi_{r_1} \pi_{r_1}}) = 2 \). Hence \( Y \) is an image of \( W(C_3) \) and 4.1 holds.

So we may assume \( \Psi = B_3 \) in case II. If \( r_2 \) is long, then \( o(w_{r_2} w_{r_3}) = 3 \) and \( o(w_{r_1} w_{r_2}) = 4 \) by 2.5. Hence we have

\[
\begin{array}{c}
\text{w}_{r_2} \quad \text{w}_{r_3} \\
\text{w}_{r_1}
\end{array}
\]

But then, as before, \( w_{r_2} w_{r_3} = w_{r_2+r_3} \) and \( o(w_{r_2} w_{r_1}) = 2 \). Hence we obtain

\[
\begin{array}{c}
\overline{n}_{r_2} \quad \overline{n}_{r_3} \quad \overline{n}_{r_1}
\end{array}
\]

and \( Y \) is an image of \( W(B_3) \).

So we may finally in case II assume that \( \Psi = B_3 \) and \( r_2 \) is short. Hence \( o(w_{r_2} w_{r_3}) = 4 \) and we obtain by 2.6 one of the following two possibilities:

(i) \[
\begin{array}{c}
\text{w}_{r_1} \quad \text{w}_{r_2} \\
\text{w}_{r_3}
\end{array}
\]

or (ii) \[
\begin{array}{c}
\text{w}_{r_1} \quad \text{w}_{r_2} \\
\text{w}_{r_3}
\end{array}
\]

But in case (ii) rank \( \Psi \leq 2 \), while in case (i) \( w_{r_1^2} = w_{r_1+r_2} \) and \( o(w_{r_1^2}, w_{r_1}) = 2 \). Hence \( \langle r_1 + r_2, r_3 \rangle \) is of type \( A_1 \times A_1 \) and we obtain

\[
\begin{array}{c}
\overline{n}_{r_1} \quad \overline{n}_{r_1} \quad \overline{n}_{r_3}
\end{array}
\]

and again \( Y \cong W(B_3) \).

So we may finally assume that we are in case III. Then \( r_1, r_2 \) are short and \( r_3 \) is long or vice versa. Since, if

\[
\begin{array}{c}
\text{w}_{r_1} \quad \text{w}_{r_3} \quad \text{w}_{r_2}
\end{array}
\]

then rank \( \Psi \leq 2 \) as before, we obtain in any case

\[
\begin{array}{c}
\text{w}_{r_2} \quad \text{w}_{r_3}
\end{array}
\]

But then, also as before, \( o(\overline{\pi_{r_3} \pi_{r_1}^2}) = 2 \) and \( Y \) is an image of \( W(B_3) \). This proves 4.1. \( \square \)
4.2. Lemma. Let \( r_1, \ldots, r_5 \in \Phi \) with

\[
\begin{array}{c}
\overline{n}_{r_1} \\
\overline{n}_{r_2} \\
\overline{n}_{r_3} \\
\overline{n}_{r_4} \\
\overline{n}_{r_5}
\end{array}
\]

Then \( Y = \langle \overline{n}_{r_1}, \ldots, \overline{n}_{r_5} \rangle \) is an image of \( W(D_4) \).

Proof. By 2.2 and 4.1, \( Y/M \cong \Sigma_3 \) where \( M = O_2(Y) \) and \( |E| = 2^n \cdot 3 \), where \( E = \pi_Y^r \) and \( 2^n = |M : C_M(\pi_{r_1})| \). It remains to show \( n \leq 2 \).

So assume \( n \geq 3 \). Then \( |E| \geq 24 \). Let \( \Psi = \langle r_1, \ldots, r_5 \rangle \). Since by 2.4–2.6 \( \langle r_i, r_5 \rangle \) is of type \( A_2 \) or \( C_2 \) for \( i = 1, \ldots, 4 \), \( \Psi \) must be irreducible. We have the cases

I. rank \( \Psi = 5 \).
II. rank \( \Psi \leq 4 \).

Suppose first that case I holds. If \( \Psi \) is of type \( A_5 \), then by [4], \( Y \) must be a subgroup of \( W(A_5) \cong \Sigma_6 \), which is not the case. Hence \( \Psi \) is of type \( B_5 \) or \( C_5 \). In any case \( |\Psi^+| = 25 \) and thus \( n = 3 \) and \( |E| = 24 \). Now by the structure of \( Y \) we have

\[
C_{\overline{E}}(x, y) = \emptyset \quad \text{for} \; x, y \in \overline{E} \; \text{with} \; o(xy) = 3. \quad (\ast)
\]

If now \( \Psi \) is of type \( B_5 \), it contains a subsystem \( \Omega \) of type \( D_5 \) consisting of long roots. Hence, applying the main theorem of [4] to \( \Omega \), we obtain that \( \langle \overline{n}_{r} \mid r \in \Omega \rangle \) is a central product of Weyl-groups corresponding to root subsystems of \( D_5 \). As \( |\langle \overline{n}_{r} \mid r \in \Omega \rangle - \overline{E}| \leq 1 \), \( (\ast) \) implies \( \langle \overline{n}_{r} \mid r \in \Omega \rangle \cong W(D_5) \). But clearly \( \langle \overline{n}_{r} \mid r \in \Omega \rangle = \langle \overline{n}_{r} \mid r \in \Omega \rangle \cap \overline{E} \), which is impossible since \( W(D_5) \) is not a subgroup of \( Y \). (\( Y \) is solvable, but \( W(D_5) \) not.)

If \( \Psi \) is of type \( C_5 \) then it contains a root subsystem \( \Lambda \) of type \( A_4 \) consisting of short roots. Hence again by \( (\ast) \) and [4]

\[
\langle \overline{n}_{r} \in \overline{E} \mid r \in \Lambda \rangle
\]

is an elementary abelian 2-group. Thus \( \langle \overline{n}_{r} \mid r \in \Lambda \rangle \) consists of at least 9 pairwise commuting involutions, a contradiction since a maximal set of pairwise commuting involutions of \( \overline{E} \) has 8 elements.

In case II \( \Psi \) is of type \( F_4 \), since

\[
|\Psi^+| \geq |\overline{E}| \geq 24.
\]
Hence \( E = \{ n_r | r \in \Psi \} \). Now \( \Psi \) contains a root subsystem \( \Omega \) of type \( D_4 \) consisting only of long roots. Hence [4] and (\ast) imply that

\[
\{ n_r | r \in \Omega \} \simeq W(D_4) \text{ or } W^+(D_4),
\]

since \( \{ n_r | r \in \Omega \} \) cannot be elementary abelian as shown above. But there exist roots \( \alpha, \beta \in \Omega \) and \( s \in \Psi - \Omega \) such that \( (\alpha, \beta, s) \) is of type \( A_2 \times A_1 \). Hence (\ast) implies \( \langle n_\alpha, n_\beta, n_s \rangle \simeq \Sigma_3 \times \mathbb{Z}_2 \), a contradiction to (\ast). This proves 4.2. \( \square \)

4.1, 4.2 and the main theorem of [7] show that \( \overline{N} \) is an image of a finite Weyl-group. We next show that no \( \overline{D} \)-subgroup of \( \overline{N} \) is isomorphic to \( W(E_\ell) \), \( 6 \leq \ell \leq 8 \), or to a center factor group of \( W(E_\ell) \).

4.3. Lemma. There exists no subset \( \overline{E} \) of \( \overline{D} \) with \( \langle \overline{E} \rangle \simeq W(E_\ell) \) or \( W^+(E_\ell) = W(E_\ell)/Z(W(E_\ell)) \) for \( 6 \leq \ell \leq 8 \).

Proof. Suppose false. Then there exist \( r_1, \ldots, r_6 \in \Phi \) with

\[
\begin{align*}
\overline{n}_{r_1} & \quad \overline{n}_{r_2} & \quad \overline{n}_{r_3} & \quad \overline{n}_{r_4} & \quad \overline{n}_{r_5} & \quad \overline{n}_{r_6} \\
\end{align*}
\]

Let \( \Psi = \langle r_1, \ldots, r_6 \rangle \), \( Y = \langle \overline{n}_{r_1}, \ldots, \overline{n}_{r_6} \rangle \) and \( \overline{E} = \overline{n}_{r_1}^Y \). Since by 2.4–2.6, \( \langle r_i, r_{i+1} \rangle \), \( i = 1, \ldots, 4 \) and \( \langle r_3, r_6 \rangle \) are either of type \( A_2 \) or of type \( B_2 \) or \( C_2 \). \( \Psi \) is connected. By 2.8, \( Y \) acts on \( \Psi \). Hence 36 = \( |\overline{E}| \leq |\Psi^+| \). Since rank \( \Psi \leq 6 \) this shows that \( \Psi \) is of type \( B_6 \) or \( C_6 \). (\( \Psi \) is a root subsystem of \( \Phi \)!

Suppose first \( \Psi \) is of type \( B_6 \). Then \( A = \{ r \in \Psi | r \text{ long} \} \) is a root subsystem of type \( D_6 \). Hence by [4], \( R = \langle \overline{n}_r | r \in A \rangle \) is a central product of Weyl-groups corresponding to irreducible root subsystems of \( A \). But as \( |A^+| = 30 \) it is easy to see that \( R = Y \), a contradiction.

Next suppose \( \Psi \) is of type \( C_6 \). Then \( A = \{ r \in \Psi | r \text{ long} \} \) is of type \( A_1 \times \cdots \times A_1 \) (6-times). Hence \( \langle \overline{n}_r | r \in A \rangle \) consists of 6 pairwise commuting involutions. Since \( |\overline{E}| = |\Psi^+| \) all these involutions lie in \( \overline{E} \), a contradiction since a maximal set of pairwise commuting reflections of \( W(E_6) \) contains only 4 elements. \( \square \)

4.4. Lemma. Let \( F(\overline{D}) \) be the graph with vertex set \( \overline{D} \) and edges pairs \( \overline{n}_\alpha, \overline{n}_\beta \in \overline{D} \) with \( o(\overline{n}_\alpha, \overline{n}_\beta) = 3 \) or 4. Then \( [X_\alpha, X_\beta] = 1 \) if \( \overline{n}_\alpha \) and \( \overline{n}_\beta \) are in different connectivity components of \( F(\overline{D}) \).

Proof. Suppose false. Then by 2.4–2.6, \( o(\overline{n}_\alpha, \overline{n}_\beta) = 2 \), \( (\alpha, \beta) \) is of type \( B_2 \) (\( = C_2 \)) and \( \alpha \) and \( \beta \) are both short. Moreover by 2.6, \( Y = \langle X_r | r \in (\alpha, \beta) \rangle \) is of type \( B_2 \), since in case 2.6(1), \( [X_\alpha, X_\beta] = 1 \), while in case 2.6(3), \( o(\overline{n}_\alpha, \overline{n}_\beta) \neq 2 \). Hence if \( s \in (\alpha, \beta) \) is long, then \( o(\overline{n}_\alpha, \overline{n}_\beta) = 4 = o(\overline{n}_\beta, \overline{n}_\beta) \), a contradiction since \( \overline{n}_\alpha \) and \( \overline{n}_\beta \) are in different connectivity components of \( F(\overline{D}) \). \( \square \)
4.5 Proof of Theorem 1. Let $\mathcal{E} \subseteq \mathcal{D}$ be a connectivity component of $\mathcal{F}(\mathcal{D})$ and let $Y = \langle \mathcal{E} \rangle$. Then by the proof of the main theorem of [7] $Y$ is an epimorphic image of an irreducible spherical Weyl-group and $\mathcal{E}$ is the image of the reflections. We need to show that $Y$ is a center factor group of a Weyl-group. If not, then by the structure of the finite Weyl-groups (different from $W(G_2)$) we are in one of the cases:

(i) $W(D_\ell), \ell \geq 4$,
(ii) $W(B_\ell), \ell \geq 2$, or
(iii) $W(F_4)$.

(If, for example $Y \simeq \Sigma_3$, as an image of $W(A_3)$, then $Y \simeq W(A_2)$ which is to show.)

If in case (i), $Y$ is not isomorphic to $W(D_\ell)$ or $W(D_\ell)/Z(W(D_\ell))$ then $Y \simeq \Sigma_\ell \simeq W(A_{\ell-1})$. In case (ii) the only possibility different from $W(B_\ell), W(B_\ell)/Z(W(B_\ell))$ and $\Sigma_\ell = W(A_{\ell-1})$ is $Y \simeq \Sigma_\ell \times \mathbb{Z}_2$. But then there exists a central involution in $\mathcal{E}$, a contradiction to the connectivity of $\mathcal{F}(\mathcal{E})$.

So the case $W = W(F_4) \simeq M(\Sigma_3 \times \Sigma_3), M = O_2(W(F_4))$ remains to be treated. Now $M$ is extraspecial of order $2^{1+4}$ and $\Sigma_3 \times \Sigma_3$ is irreducible on $M/Z(M)$. Thus, if $W/F \simeq Y$ and $F \cap M \not\subseteq Z(M)$, then $M \leq F$. Hence either $Y \simeq \Sigma_3 \times \Sigma_3$ or $Y \simeq \Sigma_3$. The first case contradicts the connectivity of $\mathcal{F}(\mathcal{E})$, while in the second case $Y \simeq W(A_2)$. Thus we may assume $F \cap M \leq Z(M)$. But since $[F, M] \leq M$ this implies $F \leq Z(M)$ and $Y$ is a center factor group of $W(F_4)$.

We have shown that $Y$ is a center-factor group of an irreducible spherical Weyl-group. Let $\Delta = \{ r \in \Phi \mid \pi_r \in \mathcal{E} \}$. Then $Y = \mathcal{N}(\Delta)$ and $\mathcal{N}(\Delta), G(\Delta)$ satisfy the hypothesis of 2.8. Hence by 2.8, $G(\Delta)$ is of type $B, C$ an irreducible spherical Moufang building with root system $\Delta$ of type $A_\ell, B_\ell, C_\ell, D_\ell$ or $E_\ell$. By 4.3 type $E_\ell$ does not occur, so that $G(\Delta)$ satisfies the conditions of the central factors of $G$ in Theorem 1.

Let now $\mathcal{D} = \bigcup \mathcal{E}_i$, $\mathcal{E}_i$ the connectivity components of $\mathcal{F}(\mathcal{D})$ and let $\Delta_i = \{ r \in \Phi \mid \pi_r \in \mathcal{E}_i \}$. Then by 4.4, $G$ is the central product of the $G(\Delta_i)$ and by the above each $G(\Delta_i)$ is either of Lie type $B_\ell$ with root system $\Delta_i$ or $E_\ell = \{ \pm r_i \}, E_i = \{ \pi_r_i \}$ and $G(\Delta_i) = X_\ell$. This proves Theorem 1. $\square$

References