The eta invariant and the Gromov–Lawson conjecture for elementary Abelian groups of odd order

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Abstract

Let $M$ be a compact connected spin manifold of dimension $m \geq 5$. Assume the fundamental group of $M$ is an elementary Abelian $p$ group of rank $k$ where $p$ is an odd prime. If $k = 2$ and $m$ is arbitrary or if $k = 3$ and $m$ is odd, we use the eta invariant to show that $M$ admits a metric of positive scalar curvature if and only if the $\hat{A}$-roof genus of $M$ vanishes. This establishes the Gromov–Lawson conjecture for these cases. © 1997 Elsevier Science B.V.

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1. Introduction

The Gromov–Lawson conjecture claims that a compact connected spin manifold $M$ of dimension $m \geq 5$ admits a metric of positive scalar curvature if and only if the generalized index $\alpha(M)$ of the Dirac operator on the manifold $M$ is zero. The precise statement of the conjecture depends on the fundamental group $G$ of the manifold in question.

The $\hat{A}$-genus is well defined on the bordism groups $\Omega^\text{Spin}_m$; it takes values in $\mathbb{Z}$ if $m \equiv 0 \mod 4$, it takes values in $\mathbb{Z}_2$ if $m \equiv 1, 2 \mod 8$, and it is zero otherwise. If $M$ admits a metric of positive scalar curvature, the Lichnerowicz formula shows $M$ has no harmonic spinors and hence the $\hat{A}$-genus vanishes. If $M$ is simply connected, Stolz [17] has shown the converse holds, i.e., $M$ admits a metric of positive scalar curvature if and only if $\hat{A}(M) = 0$. This establishes the Gromov–Lawson conjecture in this case.

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In this paper, we suppose that \( \pi_1(M) = G \) where \( G = (C_p)^k \) is an elementary Abelian \( p \)-group of rank \( k \) where \( p \) is an odd prime. For these groups, the invariant \( \alpha(M) \) is just the \( \hat{A} \)-genus. The Gromov–Lawson conjecture for the group \( G \) is that \( M \) admits a metric of positive scalar curvature if and only if \( \hat{A}(M) = 0 \). This assertion has been proved by Kwasik and Schultz [11] if the rank of \( G \) is 1 (see also [5] for a different proof using the \( \eta \) invariant); we understand that Schultz [16] has proved this result if the rank of \( G \) is 2. The following is the main result of this paper:

**Theorem 1.1.** Let \( p \) be an odd prime and let \( M \) be a compact connected spin manifold of dimension \( m \geq 5 \) with fundamental group \( G = (C_p)^k. \)

(a) If \( k = 2 \), then \( M \) admits a metric of positive scalar curvature if and only if \( \hat{A}(M) = 0. \)

(b) If \( k = 3 \) and \( m \) is odd, then \( M \) admits a metric of positive scalar curvature if and only if \( \hat{A}(M) = 0. \)

We now outline the proof of Theorem 1.1. Let \( \Omega^\text{Spin}_m(\cdot) \) be the spin bordism theory and let \( \Omega^\text{Spin}_m(\cdot) \) be the reduced theory. If \( M \) is a compact connected spin manifold with finite fundamental group \( G \), give \( M \) the canonical \( G \) structure to regard \( [M] \in \Omega^\text{Spin}_m(BG). \)

The work in [8,13–15] shows that if there exists a manifold \( M_1 \) which admits a metric of positive scalar curvature so that \( [M_1] = [M] \in \Omega^\text{Spin}_m(BG) \), then \( M \) admits a metric of positive scalar curvature. Thus the question of whether or not \( M \) admits a metric of positive scalar curvature reduces to a question in equivariant spin bordism; in light of the result of Stolz we can work with the reduced theory. Thus to prove Theorem 1.1, it suffices to prove that every element of \( \Omega^\text{Spin}_m(B(C_p)^k) \) can be represented by a manifold which admits a metric of positive scalar curvature if \( m \geq 5 \) and if \( k = 2 \) or if \( m = 5 \) is odd and if \( k = 3 \).

We can reduce the problem still further. We recall some results of Kreck and Stolz [10]. Let \( \mathbb{H}\mathbb{P}^2 \) be the quaternionic projective plane with the usual homogeneous metric of positive scalar curvature. If \( X \) is a topological space, let \( T_m(X) \) be the subgroup of the bordism group \( \Omega^\text{Spin}_m(X) \) consisting of bordism classes \( [(E^m, f \circ p)] \), where \( p: E^m \rightarrow B^{m-8} \) is a fiber bundle with fiber \( \mathbb{H}\mathbb{P}^2 \) and with transition functions belonging to the group of isometries \( PS\mathbb{P}(3) \) of \( \mathbb{H}\mathbb{P}^2 \). The functor \( X \rightarrow \Omega^\text{Spin}_m(X)/T_m(X) \) was studied by Kreck and Stolz [10]. We essentially use the fact from [10] that polynomial generators \( \zeta_{4k} \) of the ring

\[
\Omega_*^\text{Spin} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][\zeta_4, \zeta_8, \ldots, \zeta_{4k}, \ldots]
\]

may be chosen so that \( \zeta_{4k} \in T_{4k}(pt) \), \( k \geq 2 \). In particular, \( \zeta_{4k} \) may be represented by manifolds with positive scalar curvature. Localized at odd prime \( p \) the ring \( (\Omega_*^\text{Spin})_p \) is isomorphic to the tensor product \( \mathbb{B}P_* \otimes P_* \), where \( \mathbb{B}P_* = \mathbb{Z}_p[v_1, v_2, \ldots] \) is the coefficient ring of the corresponding Brown–Peterson homology theory \( \mathbb{B}P_* \), and \( P_* = \mathbb{Z}_p[x_{4k} \mid 2k \neq p^l - 1] \). Even more, there is a splitting of the spectrum \( \mathbb{M}\mathbb{P}_*(p) \): \( \mathbb{M}\mathbb{P}_*(p) = \mathbb{B}P \wedge M(P_*) \), where \( M(P_*) \) is a generalized Moore spectrum (which is just a wedge of spheres in our case, so that \( \pi_*(M(P_*)) = P_* \)). We show that the generators \( x_{4k}, v_n \) for \( k, n \geq 2 \) may be chosen in such way that they are represented by manifolds
with positive scalar curvature. Let \( J \) be the ideal of \((\Omega^*_{\text{Spin}})_p\) generated by \( x_{4k}, u_n \), \( k, n \geq 2 \). Thus to prove Theorem 1.1 it is suffices to show that corresponding elements of the groups \( \Omega_*(B(C^k)) := \Omega^*_{\text{Spin}}(B(C^k))/J \) may be represented by manifolds with positive scalar curvature. Note that \( \Omega_*(B(C^k)) \) is a module over \( \Omega_* := \Omega^*_{\text{Spin}}/J \).

We reduce the problem still further by factorizing the group \( \Omega_{m}(BG) \) by the “topologically decomposable” elements. More precisely, let \( G = G_1 \oplus G_2 \) be a nontrivial decomposition of \( G \). Cartesian product defines a natural homomorphism \( \Omega_{m}(BG_1) \otimes \Omega_{m}(BG_2) \to \Omega_{m}(BG_1 \wedge BG_2) \to \Omega_{m}(BG) \); the last homomorphism is an inclusion of the direct summand. Let \( \mathcal{D}_m(BG) \) be the subgroup of \( \Omega^*_{\text{Spin}}(BG) \) which is generated by the image of all such Cartesian products. Let \( \mathcal{D}'_m(BG) \) be the subgroup of \( \Omega^*_{\text{Spin}}(BG) \) which is generated by the image of all such Cartesian products of the reduced groups \( \Omega^*_{\text{Spin}}(BG_1) \otimes \Omega^*_{\text{Spin}}(BG_1) \). Note that \( \mathcal{D}_m(BG) \) is not the projection of \( \mathcal{D}_{m}(BG) \) to the reduced theory. Let

\[
A_m(BG) := \Omega^*_{\text{Spin}}(BG)/[J + \mathcal{D}_m(BG)].
\]

The lens spaces generate \( \Omega^*_{\text{Spin}}(BC_p) \) as an \( \Omega^*_{\text{Spin}} \) module. We can use this observation and the solution of the Gromov–Lawson conjecture for \( k = 1 \) to see all the elements of \( \Omega^*_{\text{Spin}}(BC^k) \) can be represented by manifolds which admit metrics of positive scalar curvature if \( m \geq 2 \), see [5] for details. Thus every element of \( \mathcal{D}_m(B(C_p)^2) \) can be represented by a manifold that admits a metric of positive scalar curvature if \( m \geq 3 \). The proof of Theorem 1.1(a) will show that all the elements of \( \mathcal{D}'_m(B(C_p)^2) \) can be represented by manifolds which admit metrics of positive scalar curvature if \( m \geq 3 \) and consequently every element of \( \mathcal{D}_m(B(C_p)^3) \) can be represented by a manifold that admits a metric of positive scalar curvature if \( m \geq 4 \). Note that since we are not proving the Gromov–Lawson conjecture in even dimensions if \( k = 3 \), that the process stops at this point and we cannot proceed to discuss the case rank \( k = 4 \). This reduces the proof of Theorem 1.1 to showing the elements of \( A_m(B(C_p)^k) \) may be represented by manifolds which admit metrics of positive scalar curvature if \( m \geq 5 \) and if \( k = 2 \) or if \( m \geq 5 \) is odd and if \( k = 3 \).

In Section 2, we use results of Johnson and Wilson [9] on the structure of \( \mathbb{BP}_*(B(C_p)^k) \) to study \( \Omega_*(B(C_p)^k) \) as \( \Omega_* \) module. We will show that if \( k = 2 \) and if \( m \) is even, then \( A_m(B(C_p)^2) = 0 \); this completes the proof of Theorem 1.1 in this case. To summarize the discussion, if \( m \) even, we will show that the relevant classes are generated by Cartesian products or are the image of elements from \( \Omega^*_{\text{Spin}}(BC_p)^k \) and can be represented by manifolds that admit metrics of positive scalar curvature. If \( m \) is odd, the relevant classes are generated by \( \text{Tor} \) terms coming from the Küneth formula and require further discussion. If \( k = 3 \), we cannot handle the \( \text{Tor} \) terms which would arise in even dimensions and hence we restrict Theorem 1.1(b) to the case \( m \) odd.

In Lemma 2.3, we will show that if \( k = 2 \) or if \( m \) is odd and if \( k = 3 \), then

\[
|A_m((C_p)^k)| \leq p^{\varepsilon(k, m)}
\]

where \( \varepsilon(k, m) \) is defined as follows:

\[
\varepsilon(k, m) = 0 \quad \text{if } m \text{ is even or if } m \leq 0,
\]
\[ \varepsilon(1, 4\nu + 1) = \varepsilon(1, 4\nu + 3) = \nu + 1, \quad \text{and} \]
\[ \varepsilon(k, m) = \varepsilon(k - 1, m) + \varepsilon(1, m) + \sum_{1 \leq t \leq p-1} \varepsilon(k - 1, m - 2t). \quad (2) \]

Let \( C_n \) be the cyclic group of order \( n \) where \( n = p^\ell \) is an odd prime power. Let \( U(\nu) \) be the unitary group. Let \( \tau : C_n \to U(\nu) \) be a fixed point free representation of the cyclic group \( C_n \) for \( n = p^\ell \) and let \( L^{2\nu-1}(n; \tau) := S^2^{2\nu-1}/\tau(C_n) \). The generalized lens space \( L^{2\nu-1}(n; \tau) \) admits a natural spin structure and a natural \( C_n \) structure. Let \( \tilde{L}^{2\nu-1}(n; \tau) \) be the natural projection of \( L^{2\nu-1}(n; \tau) \) to \( \mathcal{M}(BC_n) \). If \( \beta \) is a group homomorphism from \( C_n \) to \( (C_n)^k \), then \( \beta_* \) induces a homomorphism from \( \mathcal{A}_m(BC_n) \) to \( \mathcal{A}_m(B(C_n)^k) \). Let

\[ B_m(B(C_n)^k) := \text{span}_{Z}\{\beta_* \tilde{L}^m(n; \tau)\} \subset \mathcal{A}_m(B(C_n)^k), \quad (3) \]

where we use all embeddings \( \beta \) of \( C_n \) in \( (C_n)^k \) and all suitable lens spaces. We will show in Proposition 3.2 that \( |B_m(B(C_n)^k)| \geq n^{\varepsilon(k, m)} \). It then follows that \( B_m(B(C_p)^k) = \mathcal{A}_m(B(C_p)^k) \) if \( k = 2 \) or if \( m \) is odd and if \( k = 3 \). Since the lens spaces of dimension at least 3 admit metrics of positive scalar curvature, this will complete the proof of Theorem 1.1.

We can use these methods to generalize Theorem 1.1 slightly:

**Proposition 1.2.** Let \( n = p^\ell \) be an odd prime power and let \( M \) be a compact connected spin manifold of dimension \( m \geq 5 \) with fundamental group \( G = (C_n)^k \). Assume that \( m \leq 2(p - 1) \).

(a) If \( k = 2 \), then \( M \) admits a metric of positive scalar curvature if and only if \( \tilde{A}(M) = 0 \).

(b) If \( k = 3 \) and \( m \) is odd, then \( M \) admits a metric of positive scalar curvature if and only if \( \tilde{A}(M) = 0 \).

Thus if we fix the dimension \( m \), the Gromov–Lawson conjecture holds in this setting for \( p \) sufficiently large. We shall omit the proof of this generalization as it is straightforward. If \( \ell > 1 \), Proposition 2.3 fails if \( m > 2(p - 1) \) so our methods do not apply.

2. Spin bordism of elementary Abelian groups

Let \( p \) be an odd prime, and \( \text{BP}_*(\cdot) \) be the Brown–Peterson theory corresponding to the prime \( p \), \( \text{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \) with \( \text{deg } v_n = 2p^n - 2 \).

The reduced bordism groups \( \Omega_*^{\text{Spin}}(B(C_p)^k) \) are isomorphic to their localization at \( p \) since the space \( B(C_p)^k \) is already \( p \)-local. It is well known that the Thom spectrum \( 	ext{MSpin} \), localized at \( p \), splits into the wedge of suspensions of the spectrum \( \text{BP} \).

**Proposition 2.1.** Let \( p \) be an odd prime. There is a \( \text{BP}_* \)-module isomorphism

\[ (\Omega_*^{\text{Spin}})_{(p)} \cong \text{BP}_* \otimes \text{P} \quad \text{where } \text{P} := \mathbb{Z}_{(p)}[x_{4i} \mid 2i \neq p^i - 1], \quad (4) \]
where the generators $v_2, v_3, \ldots$ of $BP_* = Z_{(p)}[v_1, v_2, \ldots]$, deg $v_n = 2p^n - 2$, and the generators $x_{4i}, i \geq 2, 2i \neq p^j - 1$ may be chosen in such way that they are represented by manifolds which admit metrics of positive scalar curvature.

**Proof.** Recall that the natural homomorphism $\Omega^*_{\text{Spin}} \to \Omega^*_{\text{SO}}$ induces an isomorphism

$$\Omega^*_{\text{Spin}} \otimes Z[\tfrac{1}{2}] \cong \Omega^*_{\text{SO}} \otimes Z[\tfrac{1}{2}],$$

Polynomial generators $\zeta_{4m}$ of the ring

$$\Omega^*_{\text{Spin}} \otimes Z[\tfrac{1}{2}] \cong Z[\tfrac{1}{2}][\zeta_4, \zeta_8, \ldots, \zeta_{4m}, \ldots]$$

may be chosen such that $\zeta_{4m} \in T_{4m}$ for $m \geq 2$; see [10, Proposition 4.21, thus the generators $\zeta_8, \ldots, \zeta_{4m}$ are represented by manifolds with positive scalar curvature. Natural homomorphisms

$$\Omega^*_{\text{Spin}} \xrightarrow{\mu_{\text{Spin}}} \Omega^*_{\text{SO}} \xrightarrow{\mu_{\text{SO}}} \Omega^*_{\text{U}}$$

induce

$$(\Omega^*_{\text{Spin}})_{(p)} \xrightarrow{\cong} (\Omega^*_{\text{SO}})_{(p)} \xrightarrow{\mu_{\text{SO}}} BP_* \otimes R_*$$

where $\mu_{\text{SO}}$ is a surjection, $R_* = Z_{(p)}[y_{2j} \mid j \neq p^f - 1]$, and $\mu_{\text{SO}}$ restricted on $BP_* \otimes Z_{(p)}[y_{2q} \mid 2q \neq p^f - 1]$ is an isomorphism, see [18]. In particular, a splitting $\text{MSpin}_{(p)} = BP \wedge M(P)$ (where $M(P)$ is a generalized Moore spectrum, $\pi_*(M(P)) = P_*$) may be chosen so that $BP_* \otimes 1 = \mu_{\text{SO}}(BP_*)$, and $P_* = \mu_{\text{SO}}(R_*)$.

Let $v_1, v_2, \ldots$ be the standard polynomial generators of the $BP_* \otimes 1 \subset (\Omega^*_{\text{Spin}})_{(p)}$. We consider two cases: (1) $p = 3$, (2) $p > 3$.

**Case 1.** Let $p = 3$, then $v_1$ has a degree 4, so $v_1 = \lambda_4$, where $\lambda \in Z_{(p)}$. We let $v_1 = \zeta_4$. The element $\zeta_4$ may be represented by a spin manifold $M^4$, so that $A(M^4) = 2$. Now we decompose a generator $v_2$ (of degree 16) as a polynomial on $v_1$:

$$v_2 = \mu_2 \zeta_4 + Q_0(\zeta_4) + v_1 Q_1(\zeta_4) + \cdots + v_1^3 Q_3(\zeta_4) + \lambda_2 \bar{v}_1^4,$$

where $i = 2, 3, 4.$ Then the generator $v_2 = v_2 - \lambda_2 \bar{v}_1^4$ is represented by manifold with positive scalar curvature since $\hat{A}(v_2) = 0$ [17]. Evident induction completes the argument.

**Case 2.** Let $p > 3$. It is easy to see that $\hat{A}(v_1) = a \neq 0, a \in Z_{(p)}$, for any choice of the generator $v_1$ of the $BP_* \otimes 1 \subset (\Omega^*_{\text{Spin}})_{(p)}$. Let $\bar{v}_1 = v_1$. We note that the element

$$z = \bar{v}_1 - \bar{a} \zeta_4^{(p-1)/2}, \quad \bar{a} = a/2^{(p-1)/2},$$

is represented by a manifold with positive scalar curvature since $\hat{A}(z) = 0$. Now we decompose the element $v_2$ as a polynomial on $\bar{v}_1, \zeta_4$:

$$v_2 = Q_0(\zeta_4, i \neq 4) + \sum_{i,j} \zeta_i \bar{v}_1^j Q_{i,j}(\zeta_4, i \neq 4) + Q(\zeta_4, \bar{v}_1)$$

where $2i + j(p - 1) < p^2 - 1$, and the polynomial $Q(\zeta_4, \bar{v}_1)$ is a sum of monomials $\lambda_{i,j} \zeta_4^2 \bar{v}_1^j$, $2i + j(p - 1) = p^2 - 1, \lambda_{ij} \in Z_{(p)}$. The polynomials $Q_{i,j}(\zeta_4, i \neq 4), 2i + j(p - 1) < p^2 - 1$ may be represented by manifolds with positive scalar curvature. It is clear that there is
a number $\mu \in \mathbb{Z}_p$ so that $\tilde{A}(Q(\zeta_4, v_1) - \mu \tilde{v}_1^{p+1}) = 0$, so the element $\tilde{v}_2 = v_2 - \mu \tilde{v}_1^{p+1}$ is represented by a manifold with positive scalar curvature [17]. Induction completes the argument. \hfill \Box

Remark 2.2. Let $X$ be a $(-1)$-connected $p$-local ring spectrum. Recall from [4] that $X$ is a free-free spectrum if the homology $H_n(X; \mathbb{Z})$ and homotopy $\pi_n(X)$ groups are free finitely generated modules over $\mathbb{Z}_p$ for all $n$. A free-free ring spectrum $X$ defines a cohomology theory $X_*(\cdot)$, so that there exists the first Chern class $c^X$. The formal group theory implies that there is a map of ring spectra $j : X \to BP$ classifying the Chern class $c^X$. In particular, the coefficient ring $X_*$ becomes a free $BP_*$-module. One may construct the homology theory $E(Y) = X_\* \otimes_{BP_*} BP_*(Y)$. The spectrum $E$ may be considered as $BP$-algebra. Boardman [4, Theorem B] shows that the map $j$ induces isomorphism ring spectra $X \to E$.

In the case of the spectrum $MSpin(p)$, the coefficient ring $(\Omega^*_\text{Spin}(p))$ is a free $BP_*$-module, as it follows from Proposition 2.1. We obtain the isomorphism of $BP_*$-modules: $(\Omega^*_\text{Spin}(Y))_p \cong (\Omega^*_\text{Spin})_p \otimes_{BP_*} BP_*(Y)$.

Now we review some known facts on $\Omega^*_\text{Spin}(B(C^k_p))$. Since

$$\Omega^*_\text{Spin}(X \times Y) \cong \Omega^*_\text{Spin}(X \wedge Y) \oplus \Omega^*_\text{Spin}(X) \otimes \Omega^*_\text{Spin}(Y)$$

for any pointed spaces $X, Y$, to calculate $\Omega^*_\text{Spin}(B(C^k_p))$ is enough to know $\Omega^*_\text{Spin}(B(C^k_p)^\wedge k)$. Let $P_*$ be the polynomial ring from Eq. (4), considered as a free $BP$-module. At the odd prime $p$, we have that

$$\Omega^*_\text{Spin}(B(C^k_p)^\wedge k) \cong BP_* (B(C^k_p)^\wedge k) \otimes_{BP_*} P_*.$$

Let $J$ be an ideal of $\Omega^*_\text{Spin}$ generated by the elements $\tilde{v}_2, \tilde{v}_3, \ldots, x_8, \ldots, x_{4k}, \ldots$ where $2k \neq p^i - 1$. Note that all these elements are represented by manifolds with positive scalar curvature. Denote by $\text{to}_*(B(C^k_p)^\wedge k) = \Omega^*_\text{Spin}(B(C^k_p)^\wedge k)/J$. We use results of Johnson and Wilson [9] on the structure of $BP_* (B(C^k_p)^\wedge k)$ as a $BP_*$ module to describe $\text{to}_*(B(C^k_p)^k)$. Let $\text{to}_* = \mathbb{Z}(p)[\hat{x}_4]$, where $\hat{x}_4$ is the projection of the element $v_1$ (when $p = 3$) and of the element $x_4$ (when $p > 3$). Let $L_k$ be a free $\text{to}_*$-module on generators of degree $2i$, $0 < i < p^k$. For $\text{to}_*$-modules $M, N$ we denote $M \otimes_{\text{to}_*} N$ by $M \otimes N$, and $M \otimes_{\text{to}_*} \cdots \otimes_{\text{to}_*} N$ by $M^{\otimes k}$. We adopt the convention that $M^0$ is a free $\text{to}_*$-module on a generator of degree 0. The following proposition is an immediate consequence of [9, Theorem 5.1].

Proposition 2.3. There is a filtration of $\text{to}_*((B(C^k_p)^\wedge k)$, localized at the prime $p$, such that the associated graded module $\text{to}_*((B(C^k_p)^\wedge k)$ is given by

$$\text{to}_*((B(C^k_p)^\wedge k) \cong \bigoplus_{i_1 + \cdots + i_k = n-k} L_1^{\otimes i_1} \otimes \cdots \otimes L_k^{\otimes i_k} \otimes \text{to}_*(B(C^p)^\wedge k).$$

Let $\mathcal{A}_m(BG)$ and $\varepsilon(k, m)$ be as defined in Eqs. (1) and (2). The following estimate is an immediate consequence of Proposition 2.3.

Lemma 2.4. We have $|\mathcal{A}_m(B(C^k_p)^k)| \leq p^{\varepsilon(k, m)}$ if $k = 2$ or if $m$ is odd and if $k = 3$. 
Remark 2.5. To prove Proposition 1.2, we need a similar estimate for the order of the groups $\mathcal{A}_m(B(C_n)^k)$ where $n - p^e$ is a prime power. Proposition 2.3 generalizes immediately, but the free modules $L_k(\ell)$ have more generators. By restricting $m \leq 2(p - 1)$, we truncate these modules and can replace $L_k(\ell)$ by $L_k(1)$ and obtain the estimate

$$|\mathcal{A}_m(B(C_n)^k)| \leq p^{r(k,m)} \quad \text{if } m \leq 2(p - 1). \quad (5)$$

3. The eta invariant

If $M$ is an odd dimensional spin manifold with a $G$ structure and if $\sigma$ is a unitary representation of $G$, let $\eta(M, \sigma) \in \mathbb{R}/\mathbb{Z}$ be the eta invariant of the tangential operator of the spin complex with coefficients in the flat bundle defined by $\sigma$; we refer to [2, p. 414] for details. The eta invariant is additive with respect to direct sums so the map $\sigma \rightarrow \eta(M, \sigma)$ extends to the augmentation ideal $R_0(G)$ of the group representation ring $R(G)$. If $A$ is an Abelian group, the dual group $A^*$ is the group of homomorphisms from $A$ to $\mathbb{R}/\mathbb{Z}$. Define $\eta^*(M) \subset R_0(G)^*$ by $\eta^*(M)(\sigma) = \eta(M, \sigma) \in \mathbb{R}/\mathbb{Z}$. Let $\mathcal{A}_m(BG)$ be as defined in Eq. (1).

Lemma 3.1. Let $m$ be odd and let $G$ be a finite Abelian group. Then the map $\sigma : M \rightarrow \eta^*(M)$ extends to a homomorphism $\eta^*$ from $\mathcal{A}_m(BG)$ to $R_0(G)^*$. 

Proof. If $\sigma \in R_0(G)$, let $\eta_\sigma(M) = \eta(M, \sigma)$. We use the index theorem for manifolds with boundary [2, Theorem 3.3] to see that the eta invariant extends to a map in bordism $\eta_\sigma$ from $\tilde{\mathcal{A}}_m(BG)$ to $\mathbb{R}/\mathbb{Z}$. We proved in [6, Lemma 3.2] that $\eta_\sigma(M) = 0$ if $M \in \tilde{\mathcal{T}}_m(BG)$. To show that $\eta^*$ extends to $\mathcal{A}_m(BG)$, we must show that $\eta(M, \sigma) = 0$ for $[M] \in \tilde{\mathcal{T}}_m(BG)$. The tangential operator of the spin complex is multiplicative with respect to Cartesian product in a suitable sense. If we decompose $G = G_1 \oplus G_2$, let $M := M_1 \times M_2$ where $[M_i] \in \tilde{\mathcal{A}}_{m_i}(BG_i)$ and $m_1 + m_2 = m$. Since $R(G) = R(G_1) \otimes R(G_2)$, we may assume without loss of generality that $\sigma = \sigma_1 \otimes \sigma_2$. Assume $m_1$ is even and $m_2$ is odd and use [3, last equation on p. 84] to see that

$$\eta(M, \sigma) = \dim(\sigma_1)\tilde{A}(M_1)\eta(M_2, \sigma_2).$$

Since $M_1 \in \tilde{\mathcal{A}}_{m_1}(BG)$, $\tilde{A}(M_1) = 0$. \qed

Let $n = p^e$. Let $C_n := \{ \lambda \in \mathbb{C}: \lambda^n = 1 \}$, and let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = C_n^*$. Let $\varepsilon(k, m)$ and $B_m(B(C_n)^k)$ be as defined in Eqs. (3) and (2). The remainder of this section is devoted to the proof of the following result:

Proposition 3.2. We have that $|\eta^*B_m(B(C_n)^k)| \geq n^{\varepsilon(k,m)}$. 

We begin the proof of Proposition 3.2 by recalling some combinatorial facts concerning the eta invariant. Let $\rho_s(\lambda) = \lambda^s$ for $s \in \mathbb{Z}_n$ and $\lambda \in C_n$ be the irreducible unitary representations of $C_n$. If $\vec{s} = (s_1, \ldots, s_v)$ is a collection of integers coprime to $p$ with $s_1 + \cdots + s_v$ even, let $\tau(\vec{s}) := \rho_{s_1} \oplus \cdots \oplus \rho_{s_v}$ define $L^{2u-1}(n; \vec{s})$. 


Lemma 3.3. Let $\psi(\vec{s}) := \rho(s_1, \ldots, s_\mu)/2 \det(I - \tau(\vec{s})) \in R_0(C_n)$.
(a) If $\sigma \in R_0(C_n)$, then $\eta(\vec{L}^{2\nu-1}(n; \vec{s}), \sigma) = n^{-1} \sum_{\lambda \neq 1} \lambda^{\nu} \det(\lambda) \{\psi(\vec{\lambda})(\lambda)\}^{-1}$.
(b) We have $\eta(\vec{L}^{2\nu-1}(n; \vec{s}), \psi(\vec{s})) = (n-1)/n$.
(c) If $\sigma_G \in R_0(C_n^k)$ and $\beta : C_n \to (C_n)_k$, then we have that
$$\eta(\beta_*\vec{L}^{2\nu-1}(n; \vec{s}), \sigma_G) = \eta(\vec{L}^{2\nu-1}(n; \vec{s}), \beta^* \sigma_G).$$
(d) If $\sigma \in R_0(C_n)^{\nu+1}$, then $\eta(\vec{L}^{2\nu-1}(n; \vec{s}), \sigma) = 0$.

Proof. The first assertion follows from work of Donnelly [7]. The second assertion follows from the first and the third assertion is immediate. Let $\sigma \in R_0(C_n)^{\nu+1}$. Since $\psi(\vec{s}) R_0(C_n) = R_0(C_n)^{\nu+1}$, there exists $\delta \in R_0(C_n)$ so that $\sigma = \psi(\vec{s}) \cdot \delta$. We use the orthogonality relations and the observation $\text{Tr}(\delta(1)) = 0$ to see that we have
$$\eta(\vec{L}^{2\nu-1}(n; \vec{s}), \sigma) = n^{-1} \sum_{\lambda \neq 1, \lambda^{\nu+1}} \text{Tr}(\delta(\lambda)) = n^{-1} \sum_{\lambda \neq 1} \text{Tr}(\delta(\lambda)) \in \mathbb{Z}. \quad \Box$$

Lemma 3.4. In the free Abelian group generated by the reduced lens spaces $\vec{L}$, define
$$\mathcal{B} \vec{L}^m(n; \vec{s}) := 4 \vec{L}^{m+4}(n; \vec{s}, 2, -2) - \vec{L}^{m+4}(n; \vec{s}, 1, -1),$$
$$K^{4\mu+1} := \mathcal{B} \vec{L}^1(n; 2), \quad \text{and} \quad K^{4\mu+3} := \mathcal{B} \vec{L}^3(1; n, -1).$$
Let $\sigma \in R_0(C_n)$.
(a) We have $\eta(\vec{L}^{m+4}(n; \vec{s}, 1, -1), \psi(1, -1) \sigma) = \eta(\vec{L}^m(n; \vec{s}), \sigma)$.
(b) We have $\eta(\mathcal{B} \vec{L}^m(n; \vec{s}), \sigma) = \eta(\vec{L}^{m+4}(n; \vec{s}, 2, -2), \sigma \psi(1, -1))$.
(c) If $\sigma \in R_0(C_n)^3$, then $\eta(K^m, \sigma) = 0$.
(d) There exists $\sigma_m \in R_0(C_n)$ so that $\eta(K^m, \sigma_m) = (n-1)/n$.

Proof. We use Lemma 3.3. If $\vec{s} = (s_1, \ldots, s_\nu)$ and $\vec{t} = (s_1, \ldots, s_\nu, 1, -1)$, then we have $\psi(\vec{t}) = \psi(\vec{s}) \psi(1, -1)$ and the first assertion now follows. The second assertion follows from the identity $4 \psi(1, -1) - \psi(2, -2) = \psi(1, -1)^2$. If $\sigma \in R_0(C_n)^2$, then $\sigma \psi(1, -1) \mu \in R_0(C_n)^{2\mu+2}$ so that
$$\eta(K^{4\mu+1}, \sigma) = \eta(\mathcal{B} \vec{L}^1(n; 2), \sigma) = \eta(\vec{L}^{4\mu+1}(n; 2, 2, -2, \ldots), \sigma \psi(1, -1) \mu) = 0.$$ Since $\psi(1, -1) \mu R_0(C_n) = R_0(\mathbb{Z})^{2\mu+1}$, we can choose $\sigma_{4\mu+1} \in R_0(C_n)$ so that we have $\sigma_{4\mu+1} \psi(1, -1) \mu = \psi(2, 2, -2, \ldots)$. Then $\eta(K^{4\mu+1}, \sigma) = (n-1)/n$. This completes the proof if $m \equiv 1 \mod 4$; the remaining case is similar. $\Box$

We can embed the group of units $U(\mathbb{Z}_p)$ of the field $\mathbb{Z}_p$ in the group of units $U(\mathbb{Z}_n)$ of the ring $\mathbb{Z}_n$ by requiring that $\alpha \in U(\mathbb{Z}_p)$ satisfy $\alpha^{p-1} \equiv 1 \mod n$. Let $\gamma(\alpha)(\lambda) = \lambda^\alpha$ and $\gamma(\alpha)(\rho_\alpha) = \rho_\alpha \alpha$ define adjoint representations of $U(\mathbb{Z}_p)$ on $C_n$ and on $R_0(C_n)$. We note that $R_0(C_n)/R_0(C_n)^3$ is a finite Abelian group of order $n^2$: we work in this quotient henceforth where $j$ is chosen to be large. For $0 < t < p-1$, we define the projection
$$\pi_t := (p-1)^{-1} \sum_{\alpha \in U(\mathbb{Z}_p)} \alpha^{-t} \gamma(\alpha).$$
The $\pi_t$ are an orthogonal family of projections whose range is the eigenspace of the action of $U(\mathbb{Z}_p)$; $\gamma(\alpha)\pi_t = \alpha^t\pi_t$. We have localized at the prime $p$ to define $(p - 1)^{-1}$ and $\alpha^{-1}$; all the torsion we shall be considering is $p$ torsion so this does no harm.

**Lemma 3.5.** Let $\xi := \pi_1(\rho_1 - \rho_0)$. We have $\pi_s\xi^t = \delta_{s,t}\xi^t$ in $R_0(C_n)/R_0(C_n)^2$ and $\xi^t R(C_n)/R_0(C_n)^{\nu+1} = R_0(C_n)^t / R_0(C_n)^{\nu+1}$.

**Proof.** Since $\gamma$ is a ring homomorphism, $\gamma(\alpha)(\xi^t) = (\gamma(\alpha)\xi)^t = \alpha^t\xi^t$ and the first identity follows. The second identity will follow for arbitrary $t$ from the corresponding assertion for $t = 1$. We may expand $\xi = c(\rho_1 - \rho_0) + x$ for $c \in R_0(\mathbb{Z}_p)$; since $R_0(C_n) = (\rho_1 - \rho_0)R(C_n)$, it suffices to show $c$ is coprime to $p$. We reduce mod $p$ to take $n - p$ and evaluate the eta invariant on the circle; by Lemma 3.3, $\eta(S^1, \rho_0 - \rho_0) = \alpha/p$ in $\mathbb{R}/\mathbb{Z}$. Thus

$$c/p = \eta(S^1, c(\rho_1 - \rho_0)) = \eta(S^1, \xi) = (p - 1)^{-1} \sum_{\alpha \in \Omega(\mathbb{Z}_p)} \alpha^{-1} \eta(S^1, \rho_0 - \rho_0) = 1/p. \quad \square$$

Let $R_m(k, n, \ell)$ be the subgroup of $R_0((C_n)^k)^*$ which is generated by the maps $\psi_M \mapsto \eta(M, \beta^* \sigma_G \xi^t)$ where $\beta$ ranges over all embeddings of $C_n$ in $G = (C_n)^k$ and where $M$ ranges over all lens spaces of dimension $m$. Since we have that $R_m(k, n, 0) = \eta^* B_m(BG)$, Proposition 3.2 will follow from the following lemma:

**Lemma 3.6.** We have that

(a) $|R_m(1, n, \ell)| \geq n\varepsilon(1, m - 2\ell)$.

(b) $|R_m(k, n, \ell)| \geq |R_m(k - 1, n, \ell)| / |R_m(1, n, \ell)| \cdot \prod_{1 \leq t \leq p-1} |R_m(k - 1, n, \ell + t)|$.

**Proof.** When $k = 1$, we take $\beta$ to be the identity map. We first prove (a) with $\ell = 0$. If $m = 1$ or $m = 3$, then $|\eta^*(B_m(BC_n))| = n = n\varepsilon(1, m)$ since $\eta(\tilde{L}^m(n; \tilde{s}), \psi(\tilde{s})) = (n - 1)/n$. We use induction on $m$. The map $\sigma$ goes to $\psi(1, -1)\sigma$ induces a dual map $\psi^*$ from $R_0(C_n)^*$ to $R_0(C_n)^*$. Since $\eta^*(B_m(BC_n))$ is a subset of $\psi^*(\eta^*(B_{m+4}(BC_n)))$. Since $\sigma(\psi(1, -1)) \in R_0(C_n)^3$, $\eta^*(B_{m+4}(\psi(1, -1)\sigma)) = 0$ and thus $\psi^* \eta^* K^{m+4} = 0$ so $\eta^* K^{m+4} \in \ker(\psi^*)$. Since we can choose $\sigma_m$ so $\eta^* K^{m+4}$ has order $n$ in $\mathbb{R}/\mathbb{Z}$, $\eta^* K^{m+4}$ is an element of order at least $n$. We prove assertion (a) if $\ell = 0$ by computing:

$$|\eta^* B_{m+4}(C_n)| = |\psi^* \eta^* B_{m+4}(C_n)| \cdot |\ker(\psi^*) \cap \eta^* B_{m+4}(C_n)|$$

$$\geq |\eta^* B_m(C_n)| \cdot n \geq n\varepsilon(1, m + 1) = n\varepsilon(1, m + 4).$$

Let $m = 2i - 1$; we assume $m - 2\ell > 0$ in the proof of assertion (a) as otherwise the inequality is vacuous. If $\sigma \in R_0(C_n)^{i+1}$, then $\eta(M, \sigma) = 0$ for any lens space $M$ of dimension $m$. Thus we may work modulo the ideal $R_0(C_n)^{i+1}$. By Lemma 3.5,
$R_0(C_n)^\ell = \xi^\ell R(C_n) + R_0(C_n)^{i+1}$. We may therefore choose $y \in R(C_n)$ so that $y\xi^\ell - \psi(2)^{\ell} \in R_0(C_n)^{i+1}$. Let $|\tilde{\ell}| = i - \ell > 0$. By Lemma 3.4,

$$
\eta(\tilde{L}^{-2\ell}(n; \tilde{\ell}), \sigma) = \eta(\tilde{L}^{2i-1}(n; \tilde{\ell}, 2, \ldots, 2), \psi(2)^{\ell} \sigma) = \eta(\tilde{L}^{2i-1}(n; \tilde{\ell}, 2, \ldots, 2), y\xi^\ell \sigma).
$$

Multiplication by $y$ induces a natural map $y^*: R_0(C_n)^* \to R_0(C_n)^*$; the desired inequality in general now follows from the case $\ell = 0$ since

$$
R_{m-2\ell}(1, n, 0) \subset y^* R_m(1, n, \ell) \quad \text{so} \quad |R_{m-2\ell}(1, n, 0)| \leq |R_m(1, n, \ell)|.
$$

We will use the following lemma to prove Lemma 3.6(b).

**Lemma 3.7.** Let $k \geq 2$. Decompose $G - (C_n)^k = H \oplus C_n$ where $H = (C_n)^{k-1}$. Let $\sigma_H \in R_0(H)$, $\sigma_n \in R_0(C_n)$ and $M = \tilde{L}^m(n; \tilde{s})$. If $\beta$ embeds $C_n$ in $H$ and $1 \leq s \leq p - 1$, let

$$
\beta_s := (p - 1)^{-1} \sum_{\alpha \in U(\mathbb{Z}_p)} \alpha^{-s} \{(\beta \circ \gamma(\alpha)) - \beta \oplus 0 - (0 \circ \gamma(\alpha))\}
$$

be a virtual embedding where we localize at the prime $p$. Then we have that:

(a) $\eta(M, (\beta \oplus 0)^*(1 \otimes \sigma_n)\xi^\ell) = 0$,

(b) $\eta(M, (\beta \oplus 0)^*(\sigma_H \otimes \xi^\ell)\xi^\ell) = 0$,

(c) $\eta(M, (\beta \oplus 0)^*(\sigma_H \otimes 1)\xi^\ell) = \eta(M, \sigma_n\xi^\ell)$,

Proof. Since $(\beta \oplus 0)^*(1 \otimes \sigma_n) = 0$, $(\beta \oplus 0)^*(\sigma_H \otimes \xi^\ell) = 0$, and $(\beta \oplus 0)^*(\sigma_H \otimes 1) = \beta^*(\sigma_H)$, (a) follows; the proof of (b) is similar. Since $\beta_s^*(1 \otimes \sigma_n) = 0$ and $\beta_s^*(\sigma_H \otimes 1) = 0$, two of the vanishing assertions of (c) also follow. Since $(\beta \oplus 0)^*(\sigma_H \otimes \xi^\ell) = 0$ and $(0 \oplus \text{id})^*(\sigma_H \otimes \xi^\ell) = 0$ we may replace $\beta_s$ by $\chi_s := (p - 1)^{-1} \sum_{\alpha} \alpha^{-s}(\beta \oplus \gamma(\alpha))$ in the proof of the final assertion of (c). We complete the proof by observing $\chi_s^*(\sigma_H \otimes \xi^\ell) = \beta^*(\sigma_H) \pi_s(\xi^\ell)$ and $\pi_s(\xi^\ell) = \delta_s, s \xi^\ell$.

We have inclusions and dual projections

$$
i_H : R_0(H) \to R_0(H) \otimes 1, \quad i_H^* : R_0(G)^* \to R_0(H)^*,$$

$$
i_C : R_0(C_n) \to 1 \otimes R_0(C_n), \quad i_C^* : R_0(G)^* \to R_0(C_n)^*,$$

$$
i_t : R_0(H) \to R_0(H) \otimes \xi^t, \quad i_t^* : R_0(G)^* \to R_0(H)^*.$$
Let $S_m(H)$ (respectively $S_m(C_n)$ and $S_m(t)$) be the elements of $\mathcal{R}_m(k, n, \ell)$ generated the eta invariant of lens spaces using embeddings of the form $\beta \oplus 0$ (respectively $0 \oplus \text{id}$ and $\beta$). Lemma 3.7 shows $i^*_H(S_m(H)) = 0$, $i^*_C(S_m(H)) = 0$, $i^*_H(S_m(C_n)) = 0$, $i^*_C(S_m(C_n)) = 0$, $i^*_H(S_m(t)) = 0$, $i^*_C(S_m(t)) = 0$, and $i^*_S(S_m(t)) = 0$ for $s \neq t$. Thus these eta invariants are supported on disjoint subgroups of $R_0(G)$ and
\[
|\mathcal{R}_m(k, n, \ell)| \geq |i^*_H S_m(H)| \cdot |i^*_C S_m(C_n)| \cdot \prod_{1 \leq t \leq p-1} |i^*_t S_m(t)|.
\]
We use Lemma 3.7 to see $i^*_H S_m(H) = \mathcal{R}_m(k - 1, n, \ell)$, $i^*_C S_m(C_n) = \mathcal{R}_m(1, n, \ell)$, and $i^*_S S_m(t) = \mathcal{R}_m(k - 1, n, \ell + t)$; Lemma 3.6(b) now follows. □

References