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# An inverse system approach to Menger manifolds

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**Abstract**

We define the concept of $\mu^k$-defining spectrum and give alternative proofs of some results of Menger manifolds, including the Characterization Theorem.

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## 1. Introduction

Bestvina [1] proved that a compactum is a $k$-dimensional Menger manifold (i.e., each point has a neighbourhood which is homeomorphic to $\mu^k$) if and only if it is a $k$-dimensional $LC^{k-1}$ compactum with the disjoint $k$-cell property ($DD^kP$). To prove this, he first introduced the notion of a triangulated $\mu^k$-manifold defined as the intersection of a sequence of compact PL manifolds, and proved that it does not depend on the choice of triangulations of these manifolds. Next, the Resolution Theorem was obtained, which asserts that each $LC^{k-1}$ compactum is a $UV^{k-1}$ image of a triangulated $\mu^k$-manifold. Finally, under the above three conditions, the $UV^{k-1}$ map was "resolved", i.e., it is approximated arbitrarily closely by a homeomorphism.

Further, as an application of his technique, Bestvina also proved [1, Appendix] that any $k$-connected map of a compact PL manifold with dimension $\geqslant 2k+1$ to a polyhedron is homotopic to a $UV^{k-1}$ map. This result was beautifully reproved by Ferry [6,7].

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The purpose of the present paper is to provide an approach to same results on Menger manifold theory, including the Characterization Theorem. Almost all results of this paper have been obtained in [1]. Motivated by the notion of triangulated $\mu^k$-manifolds, we introduce the notion of a $\mu^k$-spectrum which is an inverse sequence of compact PL manifolds and $\mathrm{UV}^{k-1}$ bonding maps. It is easy to construct a $\mu^k$-spectrum from any $\mu^k$-manifold defining sequence in the sense of Bestvina. Applying (a controlled version of) the result of Bestvina and Ferry mentioned above, we prove that any $\mathrm{UV}^{k-1}$ map between compact PL manifolds which are terms of two $\mu^k$-spectra can be approximately lifted to a homeomorphism between the limits of these spectra. This result can be used to obtain some basic theorems of Menger manifold theory. In particular, we can give another proof of the Characterization Theorem.

Since we need to use the partition argument of Bestvina to obtain the basic tool (Theorem 2.6), this paper does not provide an essentially simplified proof. However, the author hopes that it provides a viewpoint to the structure of this theory.

## 2. Preliminaries

**Definitions and notations 2.1.** (1) Let $f,g: X \to Y$ and $q: Y \to B$ be maps and let $\epsilon > 0$. The maps $f$ and $g$ are said to be *$\epsilon$-close*, denoted by $f =_\epsilon g$, if $d(f(x), g(x)) < \epsilon$ for each $x \in X$. The maps $f$ and $g$ are said to be *$q^{-1}(\epsilon)$-close* if $q \cdot f$ and $q \cdot g$ are $\epsilon$-close. The notions of *$\epsilon$-homotopy* and *$q^{-1}(\epsilon)$-homotopy* are similarly defined.

(2) Let $f: X \to Y$ and $q: Y \to B$ be maps and let $\delta > 0$.

(a) $\pi_i(f): \pi_i(X) \to \pi_i(Y)$ is said to *induce an epimorphism with a $\delta$-control with respect to $q$*, if for each $\beta: S^i \to Y$, there is an $\alpha: S^i \to X$ such that $f \cdot \alpha$ and $\beta$ are $q^{-1}(\delta)$-homotopic.

(b) $\pi_i(f)$ is said to *induce a monomorphism with a $\delta$-control with respect to $q$*, if for each $\alpha: S^i \to X$ such that $f \cdot \alpha \simeq 0$ by a homotopy $(H_t): S^i \to Y$, there is a homotopy $(K_t): S^i \to X$ such that $K_0 = \alpha$, $K_1 = \mathrm{const}$, and $f \cdot K_t$ is $q^{-1}(\delta)$-closed to $H_t$ for each $t$.

(c) The map $f$ is said to be *$k$-connected* (*with a $\delta$-control with respect to $q$* respectively) if $\pi_i(f)$ is an isomorphism for each $i = 0, 1, \ldots, k-1$ and $\pi_k(f)$ is an epimorphism (with a $\delta$-control with respect to $q$ respectively).

(3) Let $f: X \to Y$ be a map between $\mathrm{LC}^n$ compacta and $\epsilon > 0$.

(a) The map $f$ is called a *$\mathrm{UV}^n$ map* if for each point $y$ of $Y$ and for each neighbourhood $U$ of $y$, there is a neighbourhood $V$ of $y$ which is contained in $U$ such that the inclusion map $f^{-1}(V) \to f^{-1}(U)$ induces the zero homomorphism between $i$th homotopy groups for each $i = 0, \ldots, n$.

(b) The map $f$ is called an *$\mathrm{AL}^j(\epsilon)$ map* if each polyhedral pair $(P, Q)$ with $\dim P \leqslant j$ and for any pair of maps $\alpha: P \to Y$ and $a: Q \to X$ such that $f \cdot a = \alpha \mid Q$, there is an extension $\underline{\alpha}: P \to X$ of $a$ such that $f \cdot \underline{\alpha} =_\epsilon \alpha$.

**Remarks.** (1) If a map $f: K \to L$ is $k$-connected with a $\delta$-control with respect to a map $q: L \to B$, then the following condition holds:

For each $i \leqslant k$, and for each pair of maps $\alpha: D^i \to L$ and $a: S^{i-1} \to K$ such that $f \cdot a = \alpha \mid S^{i-1}$, there is a homotopy $(\alpha_t): D^i \to L$ and a map $\alpha: D^i \to K$ such that

(a) $\alpha_0 = \alpha$, $\alpha_1 = f \cdot \underline{\alpha}$, and $\alpha_t \mid S^{i-1} = f \cdot a$,

(b) $(\alpha_t)$ is a $q^{-1}(\delta)$-homotopy.

(2) The above definition of $\mathrm{UV}^n$ maps is known to coincide with the usual one (see [1, 2.1.3]).

(3) It is known that a map is a $\mathrm{UV}^n$ map if and only if it is an $\mathrm{AL}^{n+1}(\epsilon)$-map for each $\epsilon > 0$, and such a map is $(n+1)$-connected [9].

**Definition 2.2.** An inverse sequence $M = (M_j, p_{ij}: M_j \to M_i)$ is called a $\mu^k$*-defining spectrum* if:

(1) Each $M_j$ is a compact PL manifold and dim $M_i \geqslant 2k + 1$.

(2) Each bonding map $p_{i,i+1}$ is a $\mathrm{UV}^{k-1}$ map.

(3) For each $i$, there exist a $k$-dimensional polyhedron $K_i$ and a PL $\mathrm{UV}^{k-1}$ map $r_i: M_i \to K_i$ satisfying the following condition:

For each $\epsilon > 0$ and for each $i$, there are a $j > i$ and a map $s_{ij}: K_j \to M_i$ such that $d(p_{ij}, s_{ij} \cdot r_j) < \epsilon$.

The limit $M = \varprojlim M$ is called a $\mu^k$*-manifold defined by a* $\mu^k$*-defining spectrum* $M$. For $i \geqslant 1$, $p_i: M \to M_i$ denotes the projection.

**Example 2.3.** Let $(M_i)$ be a $\mu^k$-manifold defining sequence in the sense of Bestvina [1, 1.2.1–3]. Each $M_i$ is a compact PL manifold with $m_i = \dim M_i \geqslant 2k + 1$, and $M_{i+1}$ is obtained from $M_i$ by digging out a regular neighbourhood of a $(m_i - k - 1)$-skeleton of a triangulation of $M_i$. Let $Q_i: M_i \to M_i$ be a PL CE map which retracts the above regular neighbourhood onto the $(m_i - k - 1)$-skeleton and define $q_{i,i+1}: M_{i+1} \to M_i$ by the restriction of $Q_i$. Then it is a $\mathrm{UV}^{k-1}$ map. By the above construction, each $M_i$ is a regular neighbourhood of the dual $k$-skeleton $K_i$. Thus it admits a PL CE retraction $r_i: M_i \to K_i$. Since the triangulations of the $M_i$ must be finer and finer, we see that $d(r_i, \mathrm{id}) \to 0$ as $i \to \infty$. Therefore, the $M_i$, $q_{i,i+1}$ and $r_i$ form a $\mu^k$-defining spectrum.

**Definition 2.4.** A compactum $X$ is said to have the *disjoint k-cell property* ($\mathrm{DD}^k\mathrm{P}$) if, for each pair of maps $\alpha, \beta: I^k \to X$ and for each $\epsilon > 0$, there are two maps $\alpha', \beta': I^k \to X$ such that $d(\alpha, \alpha') < \epsilon$, $d(\beta, \beta') < \epsilon$ and $\alpha'(I^k) \cap \beta'(I^k) = \emptyset$.

**Theorem 2.5.** *Any* $\mu^k$*-manifold* $M = \varprojlim M$ *defined by a* $\mu^k$*-defining spectrum* $M = (M_i, p_{ij}: M_j \to M_i)$ *is a k-dimensional* $\mathrm{LC}^{k-1}$ *compactum with the disjoint k-cell property.*

**Sketch of proof.** By condition (2) and [5, Proposition 5.6], $M$ is $\mathrm{LC}^{k-1}$. The dimension condition (1) and the lifting property of $\mathrm{UV}^{k-1}$ maps guarantee the

$\mathrm{DD}^k\mathrm{P}$. Finally, condition (3) implies that, for each $\epsilon > 0$, there is a $j$ such that the map $r_j \cdot p_j$ is an $\epsilon$-map of $M$ onto a $k$-dimensional polyhedron $K_j$. □

A noncontrolled version of the next result has been obtained in [1, Appendix]. In [6; 7, Theorem 3.1], an alternative proof has been given.

**Theorem 2.6.** *Let $N$ be a compact PL manifold and suppose that a* $\mathrm{UV}^{k-1}$ *map $q : N \to B$ onto a polyhedron $B$ is given. Then for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon, q) > 0$ such that for each map $f : M \to N$ of any compact PL manifold $M$ with* $\dim M \geqslant 2k + 1$ *which is $k$-connected with a $\delta$-control with respect to $q$, there is a* $\mathrm{UV}^{k-1}$ *map $g : M \to N$ which is $q^{-1}(\epsilon)$-close to $f$.*

**Sketch of proof.** The proof is a straightforward modification of [1, Appendix]. Take a handlebody decomposition $Q$ of $N$ such that mesh $q(Q) < \epsilon$. Take a sufficiently small $\delta > 0$ and suppose that $f$ is a $k$-connected map with a $\delta$-control with respect to $q$. Following the argument of [1, 2.4], we can define a partition $P$ of $M$ and an intersection preserving function $T : P \to Q$ such that for each member $p$ of $P$, $f(p) \subset T(p)$. Using the $\mathrm{UV}^{k-1}$ property, we can replace $P$ by another partition $P'$ and $T$ by an intersection preserving bijection $T' : P' \to Q$ which is "$q^{-1}(\delta)$-close" to $T$ (See [1, 2.4.7–9]).

Then we can proceed in exactly the same way as in [1] to define the desired $\mathrm{UV}^{k-1}$ map. In these construction, it is easy to see that we can take the above $\delta$ so as to depend only on $\epsilon$ and $q$. □

**Remark 2.7.** The proof of Ferry, based on the Černavskii's theorem on the existence of $\mathrm{UV}^{k-1}$ maps of a compact manifold onto its product with any cell, is simpler than the original proof of Bestvina. Unfortunately, it is not clear to the author whether Ferry's proof can be modified to obtain the controlled version as above. What is important to us is that the above $\delta$ does not depend on the domain.

We need "controlled versions" of some results of [1]. Most of the proofs are straightforward modifications of his arguments. To simplify the estimation, the following notation is convenient in the sequel. A collection $\mathscr{D}$ is defined as

$$\mathscr{D} = \{f : (0, \infty) \to (0, \infty) \mid f \text{ is a (not necessarily continuous) function such that } f(t) < t \text{ for each } t \text{ and } \lim_{t \to 0} f(t) = 0\}.$$

**Lemma 2.8** (cf. [1, Lemma 2.8.7]). *Let $X$, $Y$ and $B$ be polyhedra. There is a function $u$ in $\mathscr{D}$ satisfying the following:*

*Let $q : Y \to B$ be a* $\mathrm{UV}^{k-1}$ *map onto $B$. For each map $f : X \to Y$ so that $\pi_i(f)$ is an isomorphism with a $\delta$-control with respect to $q$, for each $i = 0, \ldots, k-1$, then the following holds:*

*For each map $\beta : P \to Y$ of a $k$-dimensional compact polyhedron $P$ to $Y$, there is a map $\alpha : P \to X$ such that $f \cdot \alpha$ is $q^{-1}(u(\delta))$-close to $\beta$.*

**Sketch of proof.** By the same way as the theorem cited above, we can construct a map $\alpha_{k-1}: P^{k-1} \to X$ such that $f \cdot \alpha_{k-1}$ is homotopic to $\beta | P^{k-1}$. Using the $\delta$-control for $\pi_{k-1}(f)$, we can extend $\alpha_{k-1}$ to $\alpha: P \to X$ such that $f \cdot \alpha$ is close to $\beta$. From the construction, it can be seen that the closeness depends only on the control for $\pi_i(f)$, $i = 0, \ldots, k-1$. Thus, we can find a function $u$ of $\mathscr{D}$ satisfying the desired condition. □

**Proposition 2.9** (cf. [1, Proposition 2.1.4]). *Let $(K, L)$ be a polyhedral pair with $\dim K \leqslant k$ and let $q: M \to B$ be a $\mathrm{UV}^{k-1}$ map onto a $\mathrm{LC}^{k-1}$ compactum $B$. There is a function $v$ in $\mathscr{D}$ which satisfies the following:*

*Let $f, g: L \to M$ be a pair of maps which are $q^{-1}(\delta)$-homotopic. If $f$ has an extension $F: K \to M$, then $g$ also has an extension $G: K \to M$ such that $G$ and $F$ are $q^{-1}(v(\delta))$-homotopic.*

In the final section, we need the following result.

**Definition 2.10.** A compactum $K$ in $\mathbb{R}^n$ is said to be *pseudo-polyhedral* if there are sequences $(N_i)$ and $(K_i)$ of rectlinear polyhedra of $\mathbb{R}^n$ such that

(1) $\dim K_i \leqslant \dim K$ and $N_i$ is a regular neighbourhood of $K_i$,

(2) $X \subset \operatorname{int} N_i \subset N_i \subset U(X, 1/2^i)$, where $U(X, 1/2^i)$ denotes the $1/2^i$-neighbourhood of $X$.

**Theorem 2.11** [8,11]. *Any $k$-dimensional compactum can be embedded into $\mathbb{R}^{2k+1}$ so that its image is pseudo-polyhedral.*

## 3. Properties of $\mu^k$-manifolds defined by $\mu^k$-spectra

In this section, we prove the following results.

**Theorem 3.1** [1, Theorem 2.8.9]. *Let $M$ and $N$ be $\mu^k$-manifolds defined by $\mu^k$-defining spectra $M = (M_j, p_{ij}: M_j \to M_i)$ and $N = (N_i, q_{ij}: N_j \to N_i)$ respectively. Let $p_i: M \to M_i$ and $q_i: N \to N_i$ be the projections. For any $\mathrm{UV}^{k-1}$ map $f: M_1 \to N_1$ and for any $\epsilon > 0$, there exists a homeomorphism $h: M \to N$ such that $q_1 \cdot h$ is $\epsilon$-close to $f_1 \cdot p_1$.*

*In particular, if $M_1$ and $N_1$ are homeomorphic, then $M$ and $N$ are homeomorphic.*

**Theorem 3.2** [1, Theorem 4.3.1]. *Let $M$ and $N$ be $\mu^k$-manifolds defined by $\mu^k$-defining spectra. A map between $M$ and $N$ is a near-homeomorphism if and only if it is a $\mathrm{UV}^{k-1}$ map.*

**Theorem 3.3** [1, Theorem 2.8.6]. *Let $f: M \to N$ be a map between $\mu^k$-manifolds defined by $\mu^k$-defining spectra and $u: N \to B$ be a $\mathrm{UV}^{k-1}$ map onto a $\mathrm{LC}^{k-1}$*

*compactum B. Suppose that, for each* $i = 0, 1, \ldots, k-1$, $\pi_i(f)$ *is an isomorphism with a* $\delta$*-control with respect to u. Then, for each* $\epsilon > 0$, *there is a homeomorphism* $h: M \to N$ *which is* $u^{-1}(\delta + \epsilon)$ *close to f.*

**Corollary 3.4** [1, Remark 5.1.1]. *Let* $f: M \to B$ *and* $g: N \to B$ *be* $UV^{k-1}$ *maps from* $\mu^k$*-manifolds M and N defined by* $\mu^k$*-defining spectra onto a* $\mathrm{LC}^{k-1}$ *compactum B. For each* $\epsilon > 0$ *there is a homeomorphism* $h: M \to N$ *such that* $g \cdot h$ *is* $\epsilon$*-close to f.*

As was mentioned in the introduction, these have been proved by Bestvina for triangulated $\mu^k$-manifolds by partitioning techniques. Employing Theorem 2.6, we give alternative proofs.

**Proposition 3.5** (cf. [1, Theorem 2.8.6, step 1 of the proof]). *Let* $\boldsymbol{M} = (M_i, p_{ij}: M_j \to M_i)$ *and* $\boldsymbol{N} = (N_i, q_{ij}: N_j \to N_i)$ *be* $\mu^k$*-defining spectra and* $q: N_1 \to B$ *be a* $\mathrm{UV}^{k-1}$ *map onto a* $\mathrm{LC}^{k-1}$ *compactum B. For each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for each map* $f_1: M_1 \to N_1$ *which induces an isomorphism between ith homotopy groups with a* $\delta$*-control with respect to q, for* $i = 0, \ldots, k-1$, *there are* $j > 1$ *and a map* $f_j: M_j \to N_1$ *such that*

(a) $f_j$ *is* $q^{-1}(\epsilon)$*-close to* $f_1 \cdot p_{1j}$,

(b) $\pi_\lambda(f_j)$ *is an isomorphism for* $\lambda = 0, \ldots, k-1$, *and an epimorphism for* $\lambda = k$, *with an* $\epsilon$*-control with respect to q.*

**Proof.** Let $M = \varprojlim \boldsymbol{M}$ and $N = \varprojlim \boldsymbol{N}$. For any $a > 0$, we can take a fine triangulation of $N_1$ such that the inclusion $\beta: P = N_1^k \to N$ of its $k$-skeleton $P$ into $N_1$ is a $k$-connected map with an $a$-control with respect to $\mathrm{id}_{N_1}$. By Lemma 2.8, there is a map $\alpha: P \to M_1$ such that

(1) $f_1 \cdot \alpha$ is $q^{-1}(u(a))$-close to $\beta$, where $u$ is a function in $\mathscr{D}$ as in Lemma 2.8.

Since $\dim M_1 \geqslant 2k+1$, we may assume that $\alpha$ is a PL embedding. Using the $\mathrm{UV}^{k-1}$ property of $p_1: M \to M_1$ and the $\mathrm{DD}^k\mathrm{P}$ of $M$, we can take, for each $b > 0$, an embedding $\underline{\alpha}: P \to M$ such that

(2) $p_1 \cdot \underline{\alpha} =_b \alpha$.

By condition (3) of Definition 2.2, for each $c$, $d > 0$, there are a $j > 1$ and a map $s_{1j}: K_j \to M_1$ such that

(3) $p_{1j} =_c s_{1j} \cdot r_j$, and

(4) each of the fibres of the map $\alpha_j := r_j \cdot p_j \cdot \underline{\alpha}: P \to K_j$ has a diameter less than $d$.

Replacing $\alpha_j$ with a PL approximation, we may assume that

(5) $\alpha_j(P)$ is a subpolyhedron of $K_j$.

Making use of (5) and (4), we can define a map $k_j: \alpha_j(P) \to P$ such that

(6) $k_j \cdot \alpha_j =_{c+d} \mathrm{id}_P$.

Because of the continuity of $\alpha$, we can take a function $e$ in $\mathscr{D}$ such that if $d(x, y) < \nu$, then $d(\alpha(x), \alpha(y)) < e(\nu)$. By (6), we have $\alpha =_{e(c+d)} \alpha \cdot k_j \cdot \alpha_j$. Then

(7) $s_{1j} \cdot \alpha_j = s_{1j} \cdot r_j \cdot p_j \cdot \underline{\alpha} =_c p_{1j} \cdot p_j \cdot \underline{\alpha} = p_1 \cdot \underline{\alpha} =_b \alpha =_{e(c+d)} \alpha \cdot k_j \cdot \alpha_j$.

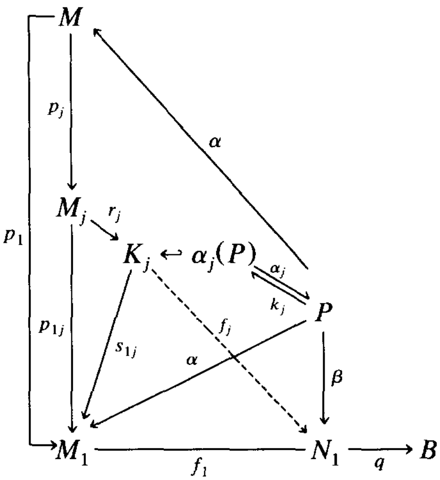


Fig. 1.

By the continuity of $f_1$, we can take a function $\phi$ in $\mathscr{D}$ such that

(8) if $d(x', y') < \nu$, then $d(f_1(x'), f_1(y')) < \phi(\nu)$.

By (7), $s_{1j} | \alpha_j(P) =_{c+b+e(c+d)} \alpha \cdot k_j | \alpha_j(P)$. Hence,

$$(9)\ \beta \cdot k_j | \alpha_j(P) =_{q^{-1}(u(a))} f_1 \cdot \alpha \cdot k_j | \alpha_j(P) \quad \text{(by (1))}$$
$$=_{\phi(c+b+e(c+d))} f_1 \cdot s_{1j} | \alpha_j(P) \quad \text{(by (8) and the above).}$$

By the continuity of $q$, we can choose a function $\theta \in \mathscr{D}$ such that if $d(x'', y'') < \nu$, then $d(q(x''), q(y'')) < \theta(\nu)$. Then,

$$\beta \cdot j_j | \alpha_j(P) =_{q^{-1}(u(a)+\theta\phi(b+c+e(c+d)))} f_1 \cdot s_{1j} | \alpha_j(P).$$

Let $x = x(a, b, c, d) = u(a) + \theta\phi(b + c + e(c + d))$, then $x \to 0$ if all variables converge to 0.

By Proposition 2.9, there is an extension $\underline{f}_j : M_j \to N_1$ of $\beta \cdot k_j$ such that $\underline{f}_j =_{q^{-1}(v(x))} f_1 \cdot s_{1j} | K_j$, where $v$ is a function in $\mathscr{D}$ as in Proposition 2.9. Finally define $f_j = \underline{f}_j \cdot r_j$. Then,

(10) $f_1 \cdot p_{1j} =_{\phi(c)} f_1 \cdot s_{1j} \cdot r_j =_{q^{-1}(v(x))} \underline{f}_j \cdot r_j = f_j$ and $f_j | \alpha_j(P) = \beta \cdot k_j$.

Noticing that all functions in $\mathscr{D}$ converge to 0 if their variables converge to 0, the conclusion is easily obtained from (10). See Fig. 1. □

We are ready to prove Theorems 3.1–3.4.

**Proof of Theorem 3.1.** Let $f_1 : M_1 \to N_1$ be a UV$^{k-1}$ map and take any $\epsilon > 0$. Let $m_1 = n_1 = 1$ and take $\eta > 0$ for $\epsilon/2$ as in Theorem 2.6. We may assume that $\eta$ is less than $\epsilon/2$. Next choose $2\delta > 0$ for $\eta/2$ as in Proposition 3.5. Since dim $N = k$, there is a map $g_1' : N \to N_{m_1}$ such that $f_1 \cdot g_1' =_\delta q_{m_1}$. Factorizing $g_1'$ through $N_i$, for

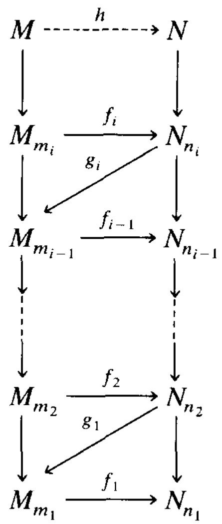


Fig. 2.

large $i$, we have a map $g_1^*: N_i \to M_{m_1}$ such that $f_1 \cdot g_1^* =_\delta q_{m_1 i}$. Using the facts that $f_1$ and $q_{m_1 i}$ are $UV^{k-1}$ maps, it is easy to see that

(1) $\pi_i(g_1^*)$ is an isomorphism for $i = 0, \ldots, k-1$ with a $2\delta$-control with respect to $f_1$.

By Proposition 3.5, we have a large $n_2 > i$ and a map $g_1^\wedge : N_{n_2} \to M_{m_1}$ such that

(2) $f_1 \cdot g_1^\wedge =_{\eta/2} q_{n_1 n_2}$, and

(3) $\pi_i(g_1^\wedge)$ is an isomorphism for $i = 0, \ldots, k-1$, and an epimorphism for $i = k$, with an $\eta/2$-control with respect to $f_1$.

Apply Theorem 2.6 to $g_1^\wedge$ to obtain a $UV^{k-1}$ map $g_1 : N_{n_2} \to M_1$ which is $f_1^{-1}(\epsilon/2)$-close to $g_1^\wedge$. By (2), we have that

(4) $f_1 \cdot g_1 =_{\epsilon/2} f_1 \cdot g_1^\wedge =_{\epsilon/2} q_{n_1 n_2}$.

Repeating the above argument for $g_1$, we have an index $m_2$ and a $UV^{k-1}$ map $f_2 : M_{m_2} \to N_{n_2}$ such that $f_2 \cdot g_1$ is close to $p_{m_1 m_2}$. Continuing this process, we have the diagram of Fig. 2 each of whose triangles is "approximately commutative". Thus it induces the required homeomorphism $h : M \to N$. (See [2,10] for example.)

This completes the proof. □

**Proof of Theorem 3.2.** By the same proof as [3, Proposition 11, p. 32], we can see that any near-homeomorphism between $LC^{k-1}$ compacta is a $UV^{k-1}$ map.

To prove the reverse implication, assume that $f: M \to N$ is a $UV^{k-1}$ map between $\mu^k$-manifolds $M = \varprojlim \boldsymbol{M}$ and $N = \varprojlim \boldsymbol{N}$ defined by $\mu^k$-defining spectra $\boldsymbol{M} = (M_k, p_{ij})$ and $\boldsymbol{N} = (N_i, q_{ij})$ respectively. For any $\epsilon > 0$ take a large $n$ so that the projection $q_n : N \to N_n$ is an $\epsilon/2$-map. Choose a $\delta > 0$ as in Theorem 2.6 for $N_n$ and $\epsilon/4$, and take a sufficiently large $m$ and a map $f_{mn}: M_m \to N_n$ such that $q_n \cdot f =_{\delta/2} f_{mn} \cdot p_m$. From the $UV^{k-1}$ property of $p_m$ and $q_n$, it is easy to see that

$f_{mn}$ induces an isomorphism between the $i$th homotopy group for $i=0,\dots,k-1$, and an epimorphism for $i=k$, with a $\delta$-control with respect to $\mathrm{id}_{N_n}$. Apply Theorem 2.6 to $f_{mn}$ and replace it by a $\mathrm{UV}^{k-1}$ map $\phi: M_m \to N_n$ which is $\epsilon/2$-close to $f_{mn}$. By Theorem 3.1, we can find a homeomorphism $h: M \to N$ which satisfies $q_n \cdot h =_{\epsilon/4} \phi \cdot p_m$. Then $h$ is $\epsilon$-close to $f$.

This completes the proof. □

**Proof of Theorem 3.3.** Let $f: M \to N$ be a map between $\mu^k$-manifolds $M = \varprojlim \boldsymbol{M}$ and $N = \varprojlim \boldsymbol{N}$ defined by $\mu^k$-defining spectra $\boldsymbol{M} = (M_i, p_{ij})$ and $\boldsymbol{N} = (N_i, q_{ij})$ respectively. And suppose that $\pi_i(f)$ is an isomorphism with a $\delta$-control with respect to a $\mathrm{UV}^{k-1}$ map $u: N \to B$ onto a $\mathrm{LC}^{k-1}$ compactum, for $i=0, 1,\dots,k-1$.

For any $\epsilon > 0$, take indexes $m$, $n$ and maps $u_n: N_n \to B$ and $f_{mn}: M_m \to N_n$ such that $u_n \cdot q_n =_{\epsilon/4} u$ and $f_{mn} \cdot p_m =_{\epsilon/4} q_n \cdot f$. Although $q_n$ may not be a $\mathrm{UV}^{k-1}$ map, it is an $\mathrm{AL}^k(\epsilon/4)$ map. (See Definition 2.1.) And $\pi_i(f_{mn})$ is an isomorphism with an $\epsilon/4$-control with respect to $u_n$, for $i=0,\dots,k-1$.

Using the same proof as that of Proposition 3.5, we can find an index $m' > m$ and a map $f_{m'n}: M_{m'} \to N_n$ such that

(1) $f_{m'n}$ is close to $f_{mn} \cdot p_{m'm}$, and

(2) $f_{m'n}$ is a $k$-connected map with a $(\delta + \rho(\epsilon))$-control with respect to $u_n$, where $\rho$ is a function in $\mathscr{D}$.

Apply Theorem 2.6 to replace $f_{m'n}$ by a $\mathrm{UV}^{k-1}$ map which is "$u_n^{-1}$-close" to $f_{m'n}$. By Theorem 3.1, we can find a homeomorphism $h: M \to N$ and it has the desired property. □

**Remark.** If $B$ is a polyhedron, we can apply [7, Theorem 8.1] to approximate $u_n$ by a $\mathrm{UV}^{k-1}$ map, which enables us to use Proposition 3.5 directly.

Corollary 3.4 is clear from Theorem 3.3.

## 4. Characterization Theorem

This section is devoted to give another proof of the following Characterization Theorem of Menger manifolds.

**Theorem 4.1** [1, Theorem 5.2.1]. *A compact metric space is a $\mu^k$-manifold defined by a $\mu^k$-defining spectrum if and only if it is a $k$-dimensional* $\mathrm{LC}^{k-1}$ *compactum which has the* $\mathrm{DD}^k\mathrm{P}$.

In the sequel, we assume that $X$ is a $k$-dimensional $\mathrm{LC}^{k-1}$ compactum with $\mathrm{DD}^k\mathrm{P}$. Embed $X$ in $\mathbb{R}^{2k+1}$ as in Theorem 2.11. Take sequences $(N_i)$ and $(K_i)$

such that

$K_i$ is a $k$-dimensional rectlinear polyhedron and $N_i$ is its regular neighbourhood so that $X \subset \text{int } N_i \subset N_i \subset U(X, 1/2^i)$. In particular, there is a PL CE retraction $r_i: N_i \to K_i$ such that $d(r_i, \text{id}) < 1/2^i$. (*)

Clearly, $X = \bigcap_{i=1}^{\infty} N_i$. By the $\text{LC}^{k-1}$ property of $X$, we have that

there is a function $\rho$ in $\mathscr{D}$ such that, for each polyhedral pair $(P, Q)$ with $\dim P \leqslant k$ and for each map $a:(P, Q) \to (U_\delta(X) \cap N_j, X)$, there is a map $b: P \to X$ such that $b \mid Q = a \mid Q$ and $b$ is $\rho(\delta)$-homotopic to $a$ in $N_j$. (**)

Further, we may assume that $\Sigma_i \rho(1/2^i) < \infty$.

Let $N = (N_i, q_{ij}: N_j \to N_i)$ be the inverse sequence consisting of the above $N_i$ and inclusions $q_{ij}$. Obviously, $X = \varprojlim N$. Starting with $M_1 = N_1$, define a $\mu^k$-manifold defining sequence $(M_i)$ in the sense of Bestvina. Then as in Example 2.3, we obtain a $\mu^k$-defining spectrum $(M_i, p_{ij}: M_j \to M_i)$. Let $M$ be the limit of this spectrum, and we will prove that $M$ is homeomorphic to $X$. The proof proceeds as in Proposition 3.5 and Theorem 3.1. We will define sequences $(f_i: M_{m_i} \to N_{n_i})$ and $(g_i: N_{n_{i+1}} \to M_{m_i})$ of $\text{UV}^{k-1}$ maps such that $f_i \cdot g_i$ and $q_{n_i n_{i+1}}$, $g_i \cdot f_{i+1}$ and $p_{m_i m_{i+1}}$ are close enough respectively, so as to induce a homeomorphism between $M$ and $X$. The main difference from Theorem 3.1 is, because of the lack of the approximate lifting property of $q_{ij}$, the commutativity of each triangle in Fig. 1 may not be arbitrarily close.

**Construction.** Start with $f_1 = \text{id}_{M_1}: M_1 \to N_1$. To describe the general procedure, we consider two cases.

*Case* 1: *Suppose that we are given a* $\text{UV}^{k-1}$ *map* $g_i: N_{n_{i+1}} \to M_{m_i}$. Then by exactly the same way as in Theorem 3.1 and Proposition 3.5, we can define a map $f_i: M_{m_i} \to N_{n_i}$ such that $g_i \cdot f_i$ is close to $p_{m_i m_{i+1}}$. Recall that $(M_i, p_{ij})$ is a $\mu^k$-defining spectrum.

*Case* 2: *Suppose that we are given a* $\text{UV}^{k-1}$ *map* $f_i: M_{m_i} \to N_{n_i}$. We construct another $\text{UV}^{k-1}$ map $g_i: N_{n_{i+1}} \to M_{m_i}$. To simplify the description, we use the notation $f \doteqdot g$ to mean that "$f$ and $g$ can be arbitrarily close if we choose a suitable index or a triangulation etc.".

Since $f_i$ is a $\text{UV}^{k-1}$ map and $\dim X = k$, there is a map $g_i': X \to M_{m_i}$ such that $f_i \cdot g_i' \doteqdot q_i$. (According to the above convention, this means that, for each $\epsilon > 0$, there is a map $g_i': X \to M_{m_i}$ such that $d(f_i \cdot g_i', q_i) < \epsilon$.) Take a neighbourhood extension $g_i''$ of $g_i'$ and choose a sufficiently large $t > i$ so that $N_t$ is contained in the domain of $g_i''$. By the continuity of $g_i''$, there is a $\gamma$ in $\mathscr{D}$ such that

$$\text{if } d(x, y) < \nu, \text{ then } d(g_i''(x), g_i''(y)) < \gamma(\nu).$$

Define $g_i^*: N_t \to N_{m_i}$ as the restriction of $g_i''$ to $N_t$. Suppose that

(1) $f_i \cdot g_i^* \simeq_{\epsilon_i} q_{it}$ in $N_i$. We may suppose that $\epsilon_i < \rho(1/2^i)$. By (1) and (**), we prove that

(2) $\pi_\lambda(g_i^*)$ is an isomorphism with a $2\rho(1/2^i)$-control with respect to $f_i$, for each $\lambda = 0, 1, \ldots, k-1$.

**Proof of (2).** Suppose that a map $\alpha : S^\lambda \to N_t$ satisfies $g_i^* \cdot \alpha \simeq 0$. Let $H$ be the homotopy such that $H_0 = g_i^* \cdot \alpha$ and $H_1 = \text{const}$ and suppose that $f_i\alpha(S^\lambda) \subset U(X, d_\alpha)$. By $(**)$, there is a map $\underline{\alpha} : S^\lambda \to X$ such that $\alpha$ is $\rho(d_\alpha)$-homotopic to $\underline{\alpha}$ in $N_t$. Let $K$ be the $\rho(d_\alpha)$-homotopy such that $K_0 = \alpha$ and $K_1 = \underline{\alpha}$. By (1), $f_i \cdot g_i^* \cdot \underline{\alpha} \simeq_{\epsilon_i} q_{n_i t} \cdot \underline{\alpha} = \underline{\alpha}$ in $N_i$. Suppose that $L$ is the $\epsilon_i$-homotopy such that $L_0 = \underline{\alpha}$ and $L_1 = f_i \cdot g_i^* \cdot \underline{\alpha}$. Then $M = K\#L\#(f_i \cdot H)$ gives a homotopy from $\alpha$ to the constant map in $N_i$. As $K\#L$ is a $(\rho(d_\alpha) + \epsilon_i)$-homotopy, it can be seen that $g_i^* \cdot M$ is $f_i^{-1}(\rho(d_\alpha) + \epsilon_i)$-close to $H$. Since $d_\alpha < 1/2^i$ and $\epsilon_i < \rho(1/2^i)$, we have that $g_i^*$ is a monomorphism with a $2\rho(1/2^i)$-control with respect to $f_i$.

Similarly, we can prove that $g_i^*$ is an epimorphism with a $2\rho(1/2^i)$-control with respect to $f_i$ (here, we use the $\mathrm{UV}^{k-1}$ property of $f_i$). This completes the proof of (2).

As in the proof of Proposition 3.5, we prove that there is a function $u$ in $\mathscr{D}$, an index $n_{i+1} > t$ and a map $g_i^\wedge : N_{n_{i+1}} \to M_i$ such that $g_i^\wedge$ is $k$-connected with a $u(2\rho(1/2^i))$-control with respect to $f_i$. The following is the procedure of the construction.

*Step* 1. Find a map $\beta : P^k \to M_{m_i}$ of a $k$-dimensional polyhedron $P$ which is $k$-connected with an arbitrarily small control with respect to $\mathrm{id}_{M_{m_i}}$.

*Step* 2. Apply Lemma 2.8 and the general position argument to get an embedding $\alpha : P \to N_t$ such that $g_i^* \cdot \alpha$ is $f_i^{-1}(2\rho(1/2^i))$ close to $\beta$.

*Step* 3. Use $(**)$ and $\mathrm{DD}^k\mathrm{P}$ of $X$ to obtain an embedding $\underline{\alpha} : P \to X$ such that $q_t \cdot \underline{\alpha}$ is $\rho(1/2^t)$-close to $\alpha$.

*Step* 4. Take a sufficiently large $j > t$ and choose a PL approximation $\alpha_j : P \to K_j$ of $r_j \cdot q_j \cdot \underline{\alpha}$ so that the diameter of each fiber of $\alpha_j$ is (arbitrarily) small.

*Step* 5. Making use of Step 4, define a map $k_j : \alpha_j(P) \to P$ such that $k_j \cdot \alpha_j \doteqdot \mathrm{id}$.

*Step* 6. Let $s_{tj} = q_{tj} \mid K_j : K_j \to N_t$, then we have

$$s_{tj} \cdot \alpha_j \doteqdot s_{tj} \cdot r_j \cdot q_j \cdot \underline{\alpha} \doteqdot q_t \cdot \underline{\alpha} =_{\rho(1/2^t)} = \alpha \doteqdot \alpha \cdot k_j \cdot \alpha_j.$$

*Step* 7. From the continuity of $g_i^*$, we have that

$$g_i^* \cdot s_{tj} \cdot \alpha =_{\gamma(\rho(1/2^t))} g_i^* \cdot \alpha =_{f_i^{-1}(2\rho(1/2^i))} \beta \doteqdot \beta \cdot k_j \cdot \alpha_j.$$

Taking a sufficiently large $t$, we may assume that

$$g_i^* \cdot s_{tj} \cdot \alpha_j =_{(2\rho(1/2^i))} \beta \cdot k_j \cdot \alpha_j.$$

(Notice that $\gamma$ is defined by the continuity of $g_i''$, thus is independent of the choice of $t$.)

*Step* 8. By Proposition 2.9, there is an extension $\underline{g}_j : K_j \to M_{M_i}$ of $\beta \cdot k_j \mid \alpha_j(P)$ such that

$$\underline{g}_j =_{f_i^{-1}(2\rho(1/2^i))} g_i^* \cdot s_{tj}.$$

*Step* 9. Define $g_j = \underline{g}_j \cdot r_j$, then we have that $f_i \cdot g_j =_{\epsilon_i + 2\rho(1/2^i)} q_{n_i j}$. The map $g_j$ is a $k$-connected map with a $\eta_i = (\epsilon_i + 2\rho(1/2^i))$-control with respect to $f_i$. Let $n_{i+1} = j$.

*Step* 10. Finally, apply Theorem 2.6 to $g_j$ to find a $\mathrm{UV}^{k-1}$ map $g_i : N_{n_{i+1}} \to M_{m_i}$ such that $f_i \cdot g_i =_{\eta_i} q_{n_i n_{i+1}}$.

In the above construction, we may assume that $\Sigma \eta_j < \infty$. Hence, from the proof of [10], it follows that the above sequences of maps induce a homeomorphism between $M$ and $X$.

This completes the proof. □

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