Remark on Wielandt's Visibility Theorem

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To H. Wielandt, on the occasion of his 75th birthday.

Submitted by Karl Hadeler

ABSTRACT

A short proof is given of Wielandt's visibility theorem, using a special case of a theorem of Rédei, which was proved in an elementary way by Lóvasz and Schriver.

1. INTRODUCTION

During the Conference on Groups and Geometries, May 1972, in Oberwolfach, Wielandt asked the audience of his lecture to provide a more direct geometric proof for his visibility theorem. This theorem had been proved by complicated arguments in Wielandt's lecture notes on permutation groups [3], during the classification of groups of degree p^2. In the present note we will give a short proof of Wielandt's result, using a theorem by Lovász and Schrijver [1], which is equivalent to a theorem of Rédei [2].

2. THE THEOREM

For any prime p, let AG$(2, p)$ denote the affine plane over the field of p elements. Let G be a permutation group acting on the points of AG$(2, p)$, such that G contains all translations of AG$(2, p)$.

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Let $G_0 \subset G$ denote the stabilizer in G of the origin in AG$(2, p)$.

THEOREM 1 (Wielandt [3]). *Let S be a collection of k lines through the origin in* AG$(2,p)$, *with p prime and* $1 \leqslant k \leqslant \frac{1}{2}(p+1)$. *If* G_0 *fixes S as a point set, then any* $g \in G$ *maps any line of S into a line of* AG$(2,p)$.

In other words, the theorem says that if G_0 fixes S as a point set, then it permutes the lines of S as wholes.

3. THE PROOF

The proof of Theorem 1 is a direct consequence of the following version of a theorem by Rédei [2, Satz 24[1]], which was proved in an elementary way by Lovász and Schrijver [1].

THEOREM 2 (Rédei; Lovász and Schrijver). *Let p be a prime, and let X be a subset of* AG$(2,p)$ *with* $|X| = p$. *Then either X is a line, or X determines at least* $\frac{1}{2}(p+3)$ *directions* (*of lines intersecting X in at least two points*).

To see that Theorem 2 implies Theorem 1, we consider the set of the k directions of the lines of the collection S. Let $g \in G$, and let $t(y)$ denote the translation over y. Then $t(-g(y))gt(y)$ maps $x-y$ onto $g(x)-g(y)$. If l is a line in S, then the set $g(l)$ is a set of p points which determines at most k directions. Since $k < \frac{1}{2}(p+3)$, Theorem 2 implies that $g(l)$ is a line. This proves the assertion.

REFERENCES

1 L. Lovász and A. Schrijver, Remarks on a theorem of Rédei, *Studia Sci. Math. Hungar.* 16:449–454 (1981).
2 L. Rédei, *Lückenhafte Polynome über endlichen Körpern*, Birkhäuser, 1970.
3 H. Wielandt, *Permutation Groups through Invariant Relations and Invariant Functions*, Ohio State Univ. Lecture Notes, 1969.

Received 9 September 1984; revised 18 September 1984