Symmetric Singular Linear Control Systems

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Abstract—Using the group inverse, an explicit solution to a symmetric singular system is described. The general explicit solution is derived when the symmetric singular system satisfies the regularity condition. Certain special properties of these singular systems are presented. Finally, a symmetric balanced realization is obtained from an equivalent standard control system. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND NOTATION

One technique for studying physical and engineering problems consists of the design of processing mathematical models. A common model in the design of circuit problems is a state-space symmetric system (see [1]). Symmetric systems in the standard case have been studied by different authors. For instance, structural properties for these systems have been analyzed in [2] using Gramian matrices. The special structure of the symmetric systems can be useful in the process to obtain the reduced model from the original system. In [3], some properties of the transfer matrix associated with a symmetric standard system have been given and different reduced systems have been presented as a realization of the transfer matrix. In [4], the model reduction problem for standard state-space symmetric systems has been studied. A usual technique for obtaining reduced models is provided in terms of the balanced realization. The idea of balancing standard systems was introduced by Moore in 1981 (see [5]).

In this paper, we introduce symmetric singular systems as a generalization of the symmetric standard systems. Singular systems present certain difficulties because the model involves singular matrices. The solution to these systems can be given by generalized inverses (see [6]). Some special properties of the matrices of the system permit us to obtain the solution in an easier form. The first step in this work is concerned with the explicit solutions to symmetric singular systems. We obtain these solutions by means of the group inverse. Using the special structure of coefficient matrices in the symmetric systems, the group inverse involved in the solution can easily be obtained. In addition, a specified symmetric system coincides with the reduced model by the residualization method. Using different methods, balanced realizations have proven to
be very useful to find reduced models. In [7], singular systems where the obtained closed-loop system is balanced through suitable feedback have been studied. In the last section of this paper, a symmetric balanced realization is obtained from an equivalent standard system.

Consider the symmetric singular control linear system

\[ Ex(k + 1) = Ax(k) + Bu(k), \]
\[ y(k) = Cx(k) + Du(k), \] (1)

where \( E^T = E, A^T = A, B^T = C, \) and \( D^T = D, \) \( x \in \mathbb{R}^n, y \in \mathbb{R}^p, u \in \mathbb{R}^m, E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}, \) where \( E \) is different from the zero matrix. We denote this system by \((E, A, B, C, D)\).

A singular system satisfies the regularity condition if there exists \( \alpha \in \mathbb{C} \) such that \( \det(\alpha E - A) \neq 0, \) or equivalently, if there exists \( \lambda \in \mathbb{C} \) such that \( \det(\lambda E + A) \neq 0. \) A system \((E, A, B, C, D)\) is called impulse-free if the pair \((E, A)\) is impulse-free, that is, if \( \deg(\det(\lambda E - A)) = \text{rank}(E). \)

In this research study, standard equivalence of linear systems is a useful tool. Further, we will use input-output equivalent systems, that is, systems with the same transfer matrix but not necessarily the same size. When the matrix \( E \) is invertible, system (1) is called a standard system. Symmetric standard systems have been studied in [8–10].

The Drazin inverse of a matrix \( A \in \mathbb{R}^{n \times n} \) is the matrix \( A^D \) satisfying

(i) \( A^DAA^D = A^D, \)

(ii) \( A^D = A^D A, \) and

(iii) \( A^{k+1}A^D = A^k, \)

where \( k = \text{ind}(A) \) is the smallest nonnegative integer such that \( \text{rank}(A^k) = \text{rank}(A^{k+1}). \)

If \( \text{ind}(A) \leq 1, \) then (iii) becomes \( A^D A^D A - A, \) and it is called the group inverse of \( A \) and is denoted by \( A^#. \) When \( A \) is symmetric, then \( A^# \) exists.

In Section 2, explicit solutions of symmetric singular systems are given, and in Section 3, a balanced symmetric realization of a symmetric singular system is constructed.

### 2. EXPLICIT SOLUTIONS

In this section, two explicit solutions are derived for singular systems. The technique used here is based on the construction of equivalent (symmetric) systems that are easier to solve or smaller than the original. We obtain an equivalent orthogonal symmetric singular system from which we construct a singular system whose explicit solution is derived.

Without loss of generality, we can assume that \( D \) is the null matrix in system (1). Let \( P \) be the orthogonal matrix diagonalizing the symmetric matrix \( E, \) then \( P^T EP = \text{diag}(D_{n_1}, 0), \) where \( n_1 = \text{rank}(E) \) and \( D_{n_1} \) is an invertible diagonal matrix. Let us consider the transformation \( x(k) = P\hat{x}(k). \) Then we obtain the similar system

\[ \hat{E}\hat{x}(k + 1) = \hat{A}\hat{x}(k) + \hat{B}u(k), \]
\[ y(k) = \hat{C}\hat{x}(k), \] (2)

where, write by blocks,

\[ \hat{x}(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}, \]

with \( \hat{x}_1(k) \in \mathbb{R}^{1 \times n_1}, \hat{x}_2(k) \in \mathbb{R}^{1 \times (n - n_1)} \) and written by blocks of compatible sizes, we have

\[ \hat{E} = P^T EP, \quad \hat{A} = P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \]
\[ \hat{B} = P^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \hat{C} = CP = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \]
This system is also symmetric. Next, applying \( \dot{x}(k) = T\tilde{x}(k) \), with \( T = \text{diag}(I_{n_1}, A_{22}^\# A_{22}) \), to system (2), using that \( A_{22} A_{22}^\# A_{22} = A_{22} \) and setting \( \tilde{x}(k) = [\tilde{x}_1(k) \ \tilde{x}_2(k)] \), we obtain the system

\[
\begin{align*}
D_{n_1} \tilde{x}_1(k + 1) &= A_{11} \tilde{x}_1(k) + A_{12} A_{22}^\# A_{22} \tilde{x}_2(k) + B_1 u(k), \\
O &= A_{21} \tilde{x}_1(k) + A_{22} \tilde{x}_2(k) + B_2 u(k), \\
y(k) &= C_1 \tilde{x}_1(k) + C_2 A_{22}^\# A_{22} \tilde{x}_2(k).
\end{align*}
\]

Then, we can consider the subsystem \((I, \bar{\tilde{A}}, \bar{\tilde{B}}, \bar{\tilde{C}}, \bar{\tilde{D}})\) with \( \bar{\tilde{A}} = D_{n_1}^{-1}(A_{11} - A_{12} A_{22} A_{22}), \bar{\tilde{B}} = D_{n_1}^{-1}(B_1 - A_{12} A_{22}^\# B_2), \bar{\tilde{C}} = C_1 - C_2 A_{22}^\# A_{21}, \) and \( \bar{\tilde{D}} = -C_2 A_{22}^\# B_2 \), which is a standard system.

Now, we are ready to give an explicit solution to system (1). Since the singular linear systems (1) and (2) are equivalent, we will obtain a solution to the second system. Further, the solution to the standard linear subsystem \((I, \bar{\tilde{A}}, \bar{\tilde{B}}, \bar{\tilde{C}}, \bar{\tilde{D}})\) is \( \tilde{x}_1(k) = \bar{\tilde{A}}^k \tilde{x}_1(0) + \sum_{i=0}^{k-1} \bar{\tilde{A}}^i \bar{\tilde{B}} u(k - i - 1). \) In addition, the solution to the algebraic linear system \( A_{22} \tilde{x}_2(k) = -A_{21} \tilde{x}_1(k) - B_2 u(k), k \in \mathbb{Z} \) is given (see [11]) by \( \tilde{x}_2(k) = A_{22}^\# (-A_{21} \tilde{x}_1(k) - B_2 u(k)) + (I_n - A_{22}^\# A_{22}) w(k), \) \( w(k) \) being an arbitrary vector. Applying transformations \( T \) and \( P \), we obtain the desired solution. The above results are summarized in the following theorem.

**Theorem 1.** With the matrices previously defined, an explicit solution to the singular system \((E, A, B, C)\) is given by

\[
x(k) = P T \begin{bmatrix} \bar{\tilde{A}}^k \tilde{x}_1(0) + \sum_{i=0}^{k-1} \bar{\tilde{A}}^i \bar{\tilde{B}} u(k - i - 1) \\
A_{22}^\# (-A_{21} \tilde{x}_1(k) - B_2 u(k)) + (I_n - A_{22}^\# A_{22}) w(k) \end{bmatrix}.
\]

**Remark** (i). Notice that the obtained solution may not be the general solution to system (1) since the matrix \( T \) could be nonsingular.

**Remark** (ii). When \( D_{n_1} \) is the identity matrix, then the subsystem \((I, \bar{\tilde{A}}, \bar{\tilde{B}}, \bar{\tilde{C}}, \bar{\tilde{D}})\) is symmetric. In addition, we observe that if \( A_{22} \) is invertible, that subsystem coincides with the reduced model obtained by the residualization method (see [12]).

**Remark** (iii). Observing the blocks of the matrix \( T \), we see that the projector \( A_{22}^\# A_{22} \) leaves the vector \( \tilde{x}_1(k) \) invariant and projects the vector \( \tilde{x}_2(k) \) onto \( \text{Im}(A_{22}) \) along \( \text{Ker}(A_{22}) \).

Next, we will construct a new equivalent system of (1), which will be used for obtaining the general solution to a symmetric singular system.

Hereafter, in all considered singular systems, we will suppose that the regularity condition is fulfilled.

Consider the symmetric system \((E, A, B, C)\). Since \( \det(\alpha E + A) \neq 0 \) for some \( \alpha \in \mathbb{C} \), in a similar way as in [13], we have \( \tilde{A} = I - \alpha \bar{E} \) defining \( \bar{E} = (\alpha E + A)^{-1} E \) and \( \tilde{A} = (\alpha E + A)^{-1} A. \) When the condition \( EA = AE \) holds, it is easy to verify that \( \bar{E} \) is a symmetric matrix, and so there exists an orthogonal matrix \( T \) such that \( T \bar{E} T^T = \text{diag}(D_{n_1}, O) \), where \( n_1 = \text{rank}(E) \) and \( D_{n_1} \) is diagonal and nonsingular. Let us define \( Q = \text{diag}(D_{n_1}^{-1}, I_{n_1}, I_{n-n_1})T(\alpha F + A)^{-1} \) and \( P = T^{-1} \). Using the transformation \( \bar{x}(k) = P \tilde{x}(k) \) in system (1), we obtain the equivalent system

\[
\begin{align*}
\dot{\tilde{x}}(k + 1) &= \tilde{A} \tilde{x}(k) + \bar{B} u(k), \\
y(k) &= \bar{C} \tilde{x}(k),
\end{align*}
\]

where \( \tilde{E} = QEP = \text{diag}(I_{n_1}, O), \tilde{A} = QAP = \text{diag}(D_{n_1}^{-1} - \alpha I_{n_1}, I_{n-n_1}), \bar{B} = QB = [B_1 \ B_2], \) and \( \bar{C} = CP = [C_1 \ C_2]. \)

Now, we are able to provide the general explicit solution to system (1). First, define the following matrices: \( E = (\beta \bar{E} \quad \tilde{A})^{-1} \bar{E}, \tilde{A} = (\beta \bar{E} \quad \tilde{A})^{-1} \bar{A}, \bar{B} = (\beta \bar{E} \quad \tilde{A})^{-1} \bar{B}, \) and \( \bar{C} = \bar{C}. \)
THEOREM 2. Let \((E,A,B,C)\) be the symmetric singular system (1) satisfying the regularity condition and \(EA = AE\). Then its general explicit solution, for an admissible initial condition \(\dot{x}_0 = P^{-1}x_0\), is given by

\[
x(k) = T^{-1} \left[ (E\#A)^k \dot{E} \dot{x}_0 + \sum_{i=0}^{k-1} (E\#A)^{k-i-1} B u(i) - (I - \dot{E}) \dot{A}\#B \dot{u}(k) \right],
\]

where all matrices are previously defined.

PROOF. The system \((E,A,B,C)\) with \(EA = AE\), is equivalent to (3). As \((\dot{E},\dot{A},\dot{B},\dot{C})\) satisfies the regularity condition \(\det(\beta \dot{E} - \dot{A}) = (-1)^{n-n_1} \prod_{i=1}^{n_1} (\beta - d_i) 
\neq 0\) for some \(\beta \in \mathbb{C}, \beta \neq d_i, i = 1,2,\ldots,n_1, d_i\) being the diagonal elements of the matrix \(D_{n_1} - \alpha I_{n_1}\), then the system is given by \((\dot{E},\dot{A},\dot{B},\dot{C})\), where \(\dot{E} = (\beta \dot{E} - \dot{A})^{-1} \dot{E}, \dot{A} = (\beta \dot{E} - \dot{A})^{-1} \dot{A}, \dot{B} = (\beta \dot{E} - \dot{A})^{-1} \dot{B},\) and \(\dot{C} = \dot{C}\). Thus, we have (see [6]) \(\dot{x}(k) - (\dot{E} D\dot{A})^k \dot{E} \dot{x}_0 + \sum_{i=0}^{k-1} \dot{E} D\dot{A}^k \dot{E} \dot{x}_0 = \ln(x) - (I - \dot{E} \dot{E} D) \dot{E} \dot{A} D \dot{B} \dot{u}(k + i),\) where \(q = \text{ind}(\dot{E})\) and \(\dot{x}_0\) is an admissible initial condition. Since \(\dot{E} \) and \(\dot{A}\) have index one, \(\dot{E} \dot{D}\) and \(\dot{A} \dot{D}\) are both group inverses. By the structure of \(\dot{E}\), we have \(\dot{E} = \dot{E}\), that is, \(\dot{E}\# = \dot{E}\), i.e., \(\dot{E}\) is a group involutory matrix (see [14]). Then \(\dot{E}\# = \dot{E}(\beta \dot{E} - \dot{A}) = \text{diag}(\beta \dot{I}_{n_1} - \dot{D}_n,O)\). By the structure of \(\dot{A}\), \(\text{rank}(\dot{A}\#) = \text{rank}(\dot{A})\), and so \(\text{ind}(\dot{A}) = 0\) or 1. If \(\dot{A}\) is invertible, \(\dot{A}\# = \text{diag}(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_n)\) with \(d_i = 1/d_i\) if \(d_i \neq 0\) and \(d_i = 0\) if \(d_i = 0\), where \(d_i\) are the diagonal elements of \(\dot{A}\), and hence, \(\dot{A}\#\) is diagonal. Then, for an admissible initial condition \(\dot{x}_0\), we have

\[
\dot{x}(k) = (E\#A)^k \dot{E} \dot{x}_0 + \sum_{i=0}^{k-1} (E\#A)^{k-i-1} B u(i) - (I - \dot{E}) \dot{A}\#B \dot{u}(k).
\]

Restating the transformation \(P\), we obtain the explicit solution. \(\blacksquare\)

3. BALANCED SYMMETRIC REALIZATION

In this section, we construct a balanced symmetric realization of a symmetric singular system using the following input-output equivalence.

PROPOSITION 1. Consider the impulse-free symmetric singular system \((E,A,B,C)\) satisfying the regularity condition. Then, this system is input-output equivalent to a system of the type \((I,A,\dot{A},\dot{B},\dot{C},\dot{D})\).

PROOF. Since the original system is similar to system (2) then \(\det(zE - A) = \det(z\dot{E} - \dot{A})\) and \(\text{deg}\det(zE - A) = n_1 = \text{deg}\det(zD_{n_1} - A_{11})\). The leading coefficient of that determinant is \(\det(A_{22})\). Therefore, the submatrix \(A_{22}\) is nonsingular. Finally, it suffices to prove that this last system and the system \((I,\dot{A},\dot{B},\dot{C},\dot{D})\) have the same transfer matrix. Since

\[
\left[ \begin{array}{cc} zD_{n_1} - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{array} \right]^{-1} = \left[ \begin{array}{cc} Z & -ZA_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}Z & -A_{22}^{-1} + A_{22}^{-1}A_{21}ZA_{12}A_{22}^{-1} \end{array} \right],
\]

where \(Z = (zD_{n_1} - A_{11} + A_{12}A_{22}^{-1} A_{21})^{-1}\), and by \(A_{22} \# = A_{22}^{-1}\), one can obtain \(\dot{G}(z) = \dot{C}(zI_{n_1} - \dot{A})^{-1} \dot{B} + \dot{D}\), which is the transfer matrix \(\dot{G}(z)\). \(\blacksquare\)

THEOREM 3. Let \(G(z)\) be the transfer matrix of the impulse-free symmetric singular control system \((E,A,B,C,D)\) satisfying the regularity condition. Assume that the matrix \(E\) is a projector. Then we have the following.

(i) \(G(z)\) has a symmetric realization \((I,\dot{A},\dot{B},\dot{C},\dot{D})\) of less size with \(\dot{A}\) diagonal.

(ii) In addition, if the system \((E,A,B,C,D)\) is minimal, the matrix \(G(z)\) admits a balanced symmetric realization \((I,A_0,B_0,C_0,D_0)\) of less size.
PROOF. By the above proposition, the system \((E, A, B, C, D)\) is input-output equivalent to the system \((I, \bar{A}, \bar{B}, \bar{C}, \bar{D})\).

(i) Since the eigenvalues of \(E\) are 0 or 1, then \(D_{n_1} = I_{n_1}\), and hence, \((I, \bar{A}, \bar{B}, \bar{C}, \bar{D})\) is symmetric. Therefore, there exists an orthogonal matrix \(Q\) satisfying \(Q\bar{A}Q^T = \bar{D}_{n_2}\) such that the realization \((I, \bar{D}_{n_2}, \bar{B}, \bar{C}, \bar{D})\) defined by \(Q\bar{A}Q^T = \bar{D}_{n_2}\), \(Q\bar{B} = \bar{B}\), \(\bar{C}Q^T = \bar{C}\), and \(\bar{D} = \bar{D}\) is symmetric.

(ii) Let us denote by \(W_r\) and \(W_o\) the observability and reachability Gramian matrices of the system \((I, \bar{A}, \bar{B}, \bar{C}, \bar{D})\). There exists a nonsingular matrix \(T\) such that the system \((I, A_b, B_b, C_b, D_b)\), with \(A_b = TAT^{-1}\), \(B_b = TB\), \(C_b = CT^{-1}\), and \(D_b = \bar{D}\), determines a balanced realization. The relationships between the Gramian matrices \(W_r\) and \(W_o\) of the system \((I, A, B, C, D)\) and those of the balanced system \(W_r^b\) and \(W_o^b\) are \(W_r^b = TW_rT^{-1}\) and \(W_o^b = T^{-TW_oT^{-1}}\), where \(W_r^b = W_o^b = D\), being \(D\) invertible and diagonal. Then \(D = TT^T\) \(DTT^T = TT^TD\) and so \(T\) is orthogonal and it is easy to verify that the system \((A_b, B_b, C_b, D_b)\) is symmetric. Finally, \(G(z) = C_b(zI - A_b)^{-1}B_b + D_b\) implies that \((A_b, B_b, C_b, D_b)\) is a realization of \(G(z)\).

REFERENCES