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The Number of Meets between Two Subsets of a Lattice

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Let L be a lattice of divisors of an integer (isomorphically, a direct product of chains). We prove $|A||B| \leqslant |L||A \wedge B|$ for any A, $B \subset L$, where $|\cdot|$ denotes cardinality and $A \wedge B = \{a \wedge b : a \in A,\ b \in B\}$. $|A \wedge B|$ attains its minimum for fixed $|A|$, $|B|$ when A and B are ideals. $|\cdot|$ can be replaced by certain other weight functions. When the n chains are of equal size k, the elements may be viewed as n-digit k-ary numbers. Then for fixed $|A|$, $|B|$, $|A \wedge B|$ is minimized when A and B are the $|A|$ and $|B|$ smallest n-digit k-ary numbers written backwards and forwards, respectively. $|A \wedge B|$ for these sets is determined and bounded. Related results are given, and conjectures are made.

1. INTRODUCTION

A lattice L is a partially ordered set (poset) where any two elements a, b have a least upper bound $a \vee b$ ("a join b") and a greatest lower bound $a \wedge b$ ("a meet b"). An ideal (resp. filter) is a subset A of a lattice (or poset) which is closed downward (resp. upward). That is, if $a \in A$, $b \in L$, $b \leqslant a$ (resp. $b \geqslant a$) then $b \in A$. Note that the complement of an ideal is a filter. If A, $B \subset L$ we write $A \wedge B$ for the set $\{a \wedge b: a \in A, b \in B\}$. Trivially, if B is an ideal, so is $A \wedge B$. If A, B are both ideals, $A \wedge B = A \cap B$. The cardinality of a set C is denoted by $|C|$ and its complement by $\bar{C}$. $\lfloor x \rfloor$ is the greatest integer $\leqslant x$, and $\lceil x \rceil$ is the least integer $\geqslant x$.

Our objective is to study $|A \wedge B|$. Let $s(L)$, $t(L)$ be the smallest real numbers s, t such that

$$|A||B| \leqslant s|A \cap B| \qquad \text{for all ideals} \quad A, B \subset L$$
$$|A||B| \leqslant t|A \wedge B| \qquad \text{for all subsets} \quad A, B \subset L.$$

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For example, suppose L is the lattice on $\{a, b, c, d, e, f, g\}$ where $a < b$ and each of $(c, d, e, f\}$ is greater than b and less than g. Then $s(L) = 8$ and $t(L) = 9$, achieved by the ideals $A = \{a, b, c, d\}$, $B = \{a, b, e, f\}$, $A \cap B = \{a, b\}$ and the subsets $A = \{b, c, d\}$, $B = \{b, e, f\}$, $A \wedge B = \{b\}$. Clearly we always have

$$|L| \leqslant s(L) \leqslant t(L).$$

These functions have their origins in [7], where it was proved that if L is a lattice of subsets of a finite set, then

$$|\bar{A} \cap B| \, |L| \leqslant |\bar{A}| \, |B| \qquad \text{for all ideals } A, B \subset L \tag{1}$$

Trivially, if $|\bar{A} \cap B|$ is written as $|B| - |A \cap B|$, (1) is equivalent to

$$|A| \, |B| \leqslant |L| \, |A \cap B| \qquad \text{for all ideals } A, B \subset L. \tag{2}$$

The function $s(L)$ was first defined in [2], and it has been shown that (1) holds in an arbitrary lattice iff $s(L) = |L|$. That (1) holds for the divisors of an integer was proved by Anderson [1] and independently by Greene and Kleitman [5]. As remarked by Welsh [8], the *FKG* inequality easily shows that (2) holds when L is a distributive lattice, so then $s(L) = |L|$.

Most of our analysis applies to products of chains (linearly ordered sets). The size of a chain C is $|C|$, its length is $|C| - 1$. The direct product of n chains of sizes $k_1, \ldots, k_n$ is a lattice of cardinality $k_1 k_2, \ldots, k_n$. There are three equivalent ways we may view the elements of such a lattice. They may be considered divisors of an integer $N = p_1^{k_1 - 1} \cdots p_n^{k_n - 1}$, where p_i are distinct primes. In this case $a \leqslant b$ in the lattice iff a divides b (written $a \mid b$). Or, they are multisets of a set, where the ith element appears at most $k_i - 1$ times. Most often we will view an element a of the lattice as an n-tuple $(a_1, a_2, \ldots, a_n)$ of integers where $0 \leqslant a_i \leqslant k_i - 1$ for all $1 \leqslant i \leqslant n$. The lattice consists of all such n-tuples. Two elements $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are related, $a \leqslant b$, iff $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant n$. Note that $(a \wedge b)_i = \min\{a_i, b_i\}$ and $(a \vee b)_i = \max\{a_i, b_i\}$.

In section 2 we show that for a product of chains, $t(L) = |L|$ and $s(L) = t(L)$, and we extend the result to slightly more general weight functions than cardinality. In section 3 we find sets A and B that minimize $|A \wedge B|$ for fixed $|A|$, $|B|$ and discuss the value of this minimum. In section 4 we present two related results that depend primarily on (1) and (2).

2. General Results Concerning $A \wedge B$

Our first result is an inequality for $|A \wedge B|$.

THEOREM 1. *If a lattice L is a product of chains and A, B are any subsets, then*

$$|A|\,|B| \leqslant |L|\,|A \wedge B|. \tag{3}$$

By complementation, we also have $|A|\,|V| \leqslant |L|\,|A \vee B|$. Phrased in terms of divisors of an integer, theorem 1 becomes: If A and B are families of divisors of an integer N, the proportion of divisors of N that are greatest common divisors (or least common multiples) of a member of A with a member of B is not less than the product of the like proportions for A and B.

Theorem 1 can be proved independently, but we omit the proof for several reasons. Phrased in the terminology of the introduction, theorem 1 is equivalent to $t(L) = |L|$ for a product of chains. Since theorem 2 will show $s(L) = t(L)$ in this case, and since $s(L) = |L|$ for any distributive lattice [8], theorem 2 will imply theorem 1. Also, theorem 1 is a special case of theorem 3. Finally, since the writing of this paper, one of the authors has obtained a stronger result than theorem 1 by a different method of proof. In [3] we find that a lattice is distributive iff $|A|\,|B| \leqslant |A \vee B|\,|A \wedge B|$. Since $|A \vee B| \leqslant |L|$ and a product of chains is a distributive lattice, theorem 1 follows again as a corollary. Using the result in [3] instead of theorem 1 does not significantly strengthen our later results concerning error bounds, so we will continue to use the bound $|A \wedge B| \geqslant |A|\,|B|/|L|$ given by theorem 1. As we will see, it is sometimes strict.

THEOREM 2. *If a lattice L is a product of chains and A, B are any subsets with $|A| = \alpha$, $|B| = \beta$, then $\min |A \wedge B|$ can be attained when A and B are ideals.*

Proof. Without changing the size of A or B, we will transform them to make the first component of their elements as small as possible. In doing so we will never increase $|A \wedge B|$. Doing this successively on each component, we will transform A and B into ideals, which suffices to prove the theorem.

We now define operators to formalize this transformation. For $c \in L$, define $\delta_p c$ by

$$(\delta_p c)_i = \begin{cases} c_i; & \text{for } \ i \neq p \\ 0; & \text{for } \ i = p, \quad c_i = 0 \\ c_i - 1; & \text{for } \ i = p, \quad c_i > 0. \end{cases}$$

For $C \subset L$, let $f_p C = \{c : c \in C,\ \delta_p c \in C\} \cup \{\delta_p c : c \in C,\ \delta_p c \notin C\}$ and $\delta_p C = \lim_{m\to\infty} f_p{}^m(C)$. Clearly f^m attains a limit and δ_p is well-defined. δ_p wherever possible replaces an element c of a set C with a lattice element $c' < c$ not in C which agrees with c everywhere but in the pth component. Viewing the lattice as divisors of an integer, δ_p takes a subset of divisors and divides out the pth prime as much as possible without producing equality of elements. We have

$$c \in \delta_p C \text{ only if } (c_1, ..., c_{p-1}, j, c_{p+1}, ..., c_n) \in \delta_p C \text{ for all } 0 \leqslant j \leqslant c_p \tag{4}$$

and

$$|\delta_p C| = |C|. \tag{5}$$

We claim

$$(\delta_p A \wedge \delta_p B) \subset \delta_p(A \wedge B). \tag{6}$$

Suppose $c \in (\delta_p A \wedge \delta_p B)$ with $c_p = j$. Take $a \in \delta_p A$, $b \in \delta_p B$ such that $a \wedge b = c$. $\min\{a_p, b_p\} = j$, so (4) guarantees that $\delta_p A$ and therefore A has at least $j + 1$ elements that agree with a in everything but a_p, similarly for b in B. Call these sets of elements $\tilde{a}$ and $\tilde{b}$. Take the element in $\tilde{a} \cup \tilde{b}$ with largest pth component; we may assume it is a. Form $\{a\} \wedge \tilde{b}$. This yields at least $j + 1$ elements in $A \wedge B$ which agree with c in the other components. (4) then ensures $c \in \delta_p(A \wedge B)$, which establishes (6).

By (5) and (6) we may replace A, B by $\delta_p A$, $\delta_p B$ and have $|\delta_p A \wedge \delta_p B| \leqslant |\delta_p(A \wedge B)| = |A \wedge B|$.

For any $C \subset L$, let us now define a sequence of sets C_i generated by pushing down on successive components. That is, let $C_0 = C$, $C_i = \delta_i C_{i-1}$ for $1 \leqslant i \leqslant n$. We wish to show $C_n = C^*$ is an ideal. Suppose $c \in C^*$, $c' \leqslant c$. If the first component where c and c' differ is the $(n - i)$th, (4) and the fact that δ_i changes nothing outside the lth component establishes a simple induction on i to show $c' \in C^*$. Applying this to any A and B, A^* and B^* will be ideals having the same cardinality, such that $|A^* \wedge B^*| \leqslant |A \wedge B|$. ∎

By a complementary argument, we could construct filters A^*, B^* to minimize $|A \vee B|$. Note that theorem 2 does not extend to distributive lattices. Consider the lattice $a < b < \{c, d\} < e$, and let $\alpha = \beta = 2$. Here $\{b, c\} \wedge \{b, d\} = \{b\}$, which has cardinality 1. $\{a, b\}$ is the only ideal of size 2, and $\{a, b\} \wedge \{a, b\} = \{a, b\}$ still has size 2. However, $\{b, c\}$ and $\{b, d\}$ are compact sets, where a set S is compact if $x, y \in S$, $x \leqslant z \leqslant y$ implies $z \in S$. In other words, S is the intersection of a filter and an ideal. We conjecture that if $|A| = \alpha$ and $|B| = \beta$, then $|A \wedge B|$ is minimized by certain compact sets. If this does not hold for all lattices, it may still be true of distributive lattices.

For products of chains we now have three equivalent inequalities, (1), (2) and (3). The next theorem shows that in those inequalities we can replace cardinality by a more general weight function. First we must prove a lemma about sequences of real numbers.

LEMMA 1. *Suppose $u_0, u_1, \ldots, u_n$ and $v_0, v_1, \ldots, v_n$ are two sequences of non-negative real numbers. Let $\lambda_{rs} = max\{u_r v_i, u_i v_r : r \leqslant i \leqslant s\}$. Then*

(i) $\lambda_{rs} \leqslant \lambda_{rt}$ *for all* $r \leqslant s \leqslant t$

(ii) $u_s v_r + u_r v_s \leqslant u_s v_s + \max\{u_r v_r, u_s v_r, u_r v_s\}$

(iii) $\sum_{i=0}^{s-r} u_{s-i} v_{r+i} \leqslant \sum_{j=r}^{s} \lambda_{js}$.

Proof. (i) is obvious. For (ii) we have three cases.

(1) If $u_r \leqslant u_s$, then $u_r v_s \leqslant u_s v_s$ and $u_s v_r \leqslant \max\{u_r v_r, u_s v_r, u_r v_s\}$.

(2) If $v_r \leqslant v_s$, then $u_s v_r \leqslant u_s v_s$ and $u_r v_s \leqslant \max\{u_r v_r, u_s v_r, u_r v_s\}$.

(3) If $u_r > u_s$ and $v_r > v_s$, then

$$\begin{aligned} u_s v_r + u_r v_s &= u_s v_s + (v_r - v_s) u_s + (v_s - v_r) u_r + u_r v_r \\ &= u_s v_s + (u_r - u_s)(v_s - v_r) + u_r v_r \\ &\leqslant u_s v_s + u_r v_r . \end{aligned}$$

For (iii) we proceed by induction on $s - r$. If $r = s$, $u_s v_s = \lambda_{ss}$. If $s - r = 1$ we have (ii). Suppose $s - r > 1$ and the lemma is true for smaller values of $s - r$. Using (i) and (ii), we have

$$\begin{aligned} \sum_{i=0}^{s-r} u_{s-i} v_{r+i} &= u_s v_r + u_r v_s + \sum_{i=1}^{s-r-1} u_{s-i} v_{r+i} \\ &= u_s v_s + \max\{u_r v_r, u_s v_r, u_r v_s\} + \sum_{i=0}^{s-r-2} u_{s-1-i} v_{r+1+i} \\ &\leqslant \lambda_{ss} + \lambda_{rs} + \sum_{j=r+1}^{s-1} \lambda_{j,s-1} \leqslant \sum_{j=r}^{s} \lambda_{js} . \quad \blacksquare \end{aligned}$$

THEOREM 3. *Suppose L is a product of n chains. Choose non-negative real numbers $w_1, w_2, \ldots, w_n$ as "weights". For $a \in L$, let $w(a) = \prod_{i=1}^{n} w_i^{a_i}$. For $A \subset L$, let $w(A) = \sum_{a \in A} w(a)$. Then for all $A, B \subset L$,*

$$w(A)w(B) \leqslant w(L)w(A \wedge B).$$

Proof. We proceed by induction on n, the number of chains, by "factoring out" the part of the weight due to the nth chain. For $n = 0$ we may say $w(A) = 1$ for all $A \subset L$, so theorem holds trivially. Put $x = w_n$ and let k be the size of the nth chain. Suppose $n > 0$ and the theorem holds for smaller values of n.

For $C \subset L$ define $C_i^j = \{c \in C : c_i = j\}$. Define $f_i(C)$ as before, so $w(C_n^j) = x^j w(f_n^j(C_n^j))$.

In particular, if $\nu = w(L_n{}^0)$ is the weight of the sublattice obtained by dropping the last chain,

$$w(L) = \sum_{j=0}^{k-1} w(L_n{}^j) = \sum_{j=0}^{k-1} x^j w(f_n{}^j(L_n{}^j)) = \nu \sum_{j=0}^{k-1} x^j.$$

Now, let $D = A \wedge B$ and put

$$x^j u_j = w(A_n{}^j),\ x^j v_j = w(B_n{}^j),\ x^j z_j = w(D_n{}^j)$$

for $j = 0, 1, \ldots, k-1$, so that

$$w(A) = \sum_{j=0}^{k-1} u_j x^j \qquad w(B) = \sum_{j=0}^{k-1} v_j x^j \qquad w(A \wedge B) = \sum_{j=0}^{k-1} z_j x^j.$$

$D_n{}^j$ is composed of meets between elements of A and B, where the minimum of the two nth components is j. The nth component of the other element may also be decreased to j without changing the meet. That is,

$$\begin{aligned} D_n{}^j &= \bigcup_{i \geqslant j} (A_n{}^j \wedge B_n{}^i) \bigcup_{i \geqslant j} (A_n{}^i \wedge B_n{}^j) \\ &= \bigcup_{i \geqslant j} (A_n{}^j \wedge f_n^{i-j}(B_n{}^i)) \bigcup_{i \geqslant j} (f_n^{i-j}(A_n{}^i) \wedge B_n{}^j) \end{aligned}$$

Everything in the last expression has nth component j, so they are embedded in a sublattice on fewer chains. By the induction hypothesis, we have $u_j v_i \leqslant \nu z_j$ and $u_i v_j \leqslant \nu z_j$ for all $j \leqslant i < k$.

In fact, for all $0 \leqslant j \leqslant s < k$ we have

$$\lambda_{js} = \max\{u_r v_i\,,\ u_i v_r\ ;\ j \leqslant i \leqslant s\} \leqslant \nu z_j$$

Lemma 1 and the equations above allow us to bound $w(A)w(B)$.

$$\begin{aligned} w(A)\, w(B) &= \left(\sum_{j=0}^{k-1} u_j x^j\right)\left(\sum_{j=0}^{k-1} v_j x^j\right) \\ &= \sum_{j=0}^{k-1} x^j \sum_{i=0}^{j} u_{j-i} v_i + \sum_{j=k}^{2k-2} x^j \sum_{i=j-k+1}^{k-1} u_{j-i} v_i \\ &\leqslant \sum_{j=0}^{k-1} x^j \sum_{i=0}^{j} \lambda_{ij} + \sum_{j=k}^{2k-2} x^j \sum_{i=j-k+1}^{k-1} \lambda_{i,k-1} \\ &\leqslant \left(\sum_{j=0}^{2k-2} x^j\right)\left(\sum_{j=0}^{k-1} \lambda_{j,k-1} x^j\right) \leqslant \left(\sum_{j=0}^{k-1} x^j\right)\left(\sum_{j=0}^{k-1} \nu z_j x^j\right) \\ &= \left(\nu \sum_{j=0}^{k-1} x^j\right)\left(\sum_{j=0}^{k-1} w(D_n{}^j)\right) = w(L)\, w(A \wedge B). \quad \blacksquare \end{aligned}$$

3. Sets That Minimize $|A \wedge B|$

In this section we exhibit sets that minimize $|A \wedge B|$ on products of chains of equal length. Also, we analyze the sharpness of $|A||B|/|L|$ as a lower bound on that minimum. Intuitively, one would expect that restricting the weight of elements in the two sets as much as possible to different components will result in meets with very low rank, of which there are comparatively few. Theorem 4 formalizes this idea.

THEOREM 4. *Suppose L is a product of n chains of equal size k. The elements of L may be viewed as the set of n-digit k-ary numbers. Fix $|A| = \alpha$ and $|B| = \beta$. Then $\min|A \wedge B|$ is attained by letting B be the β smallest n-digit $k =$ ary numbers and A the α smallest written backwards.*

Proof. For any element $c \in L$ we define a forward and a backward value by $v(c) = \sum_{i=1}^{n} k^{n-i}c_i$ and $\bar{v}(c) = \sum_{i=1}^{n} k^{i-1}c_i$, respectively. v and $\bar{v}$ evaluate elements as k-ary numbers when the components are written in order or in reverse order. We also define an insertion operator g_i^j by $g_i^j(a) = (a_1, a_2, \ldots, a_{i-1}, j, a_i, \ldots, a_n)$. It sticks a j in the ith place and pushes every subsequent integer one component to the right.

Let $C_k(n, \alpha)$ and $D_k(n, \beta)$ be the prescribed sets of size α and β. Note that $C = \{c \in L: \bar{v}(c) < \alpha\}$ and $D = \{c \in L: v(c) < \beta\}$. C and D are clearly ideals, as required by theorem 2. For $n = 1$ and k arbitrary, C and D must minimize $|A \wedge B|$ since they are the only ideals of size α and β. We proceed by induction on n.

Suppose $|A \wedge B|$ is minimal over all $|A| = \alpha$ and $|B| = \beta$, but $B \neq D_k(n, \beta)$ or $A \neq C_k(n, \alpha)$. Then we can find $x, y \in L$ with $x \notin B$, $y \in B$, $v(x) < v(y)$ or $x \notin A$, $y \in A$, $\bar{v}(x) < \bar{v}(y)$. x and y may disagree in every component, or they may agree somewhere.

Suppose first that x and y agree in the ith component. Define A_i^j, B_i^j, and $E_i^j = (A \wedge B)_i^j$ as before. Since A and B are known to be ideals (by theorem 2), we have $B_i^0 \supset \delta_i B_i^1 \supset \cdots \supset \delta_i^{k-1} B_i^{k-1}$ and $A_i^0 \supset \delta_i A_i^1 \supset \cdots \supset \delta_i^{k-1} A_i^{k-1}$. This means $A_i^j \wedge B_i^j$ contains $A_i^j \wedge B_i^{j+r}$ and $A_i^{j+r} \wedge B_i^j$ for all $0 \leqslant r < k - j$, so $E_i^j = A_i^j \wedge B_i^j$.

Fixing i, the component where x and y agree, put $\alpha_j = |A_i^j|$ and $\beta_i = |B_i^j|$. Minimize $|A_i^j \wedge B_i^j|$ subject only to the values α_j and β_j. This will minimize $|E|$ subject to those values if when we put the results for all j together to form A and B we still have ideals. For a given value of j, minimizing $|A_i^j \wedge B_i^j|$ is a meet minimization problem on a sublattice with fewer chains. By induction, we replace A_i^j by $g_i^j(C_k(n-1, \alpha_j))$ and B_i^j by $g_i^j(D_k(n-1, \beta_j))$ for all j. Since $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1}$ and $\beta_0 < \beta_1 < \cdots < \beta_{k-1}$, the new A and B are also ideals. Recalling the pair (x, y), y now appears in the replacement set only if x does also.

This replacement procedure strictly decreases $\sum_{b\in B} v(b)$ or $\sum_{a\in A} \bar{v}(a)$, whichever generated the pair (x, y), without increasing $| A \wedge B |$. Since $A = C_k(n, \alpha)$ minimizes $\sum_{a\in A} \bar{v}(a)$ and $B = D_k(n, \beta)$ minimizes $\sum_{b\in B} v(b)$, it follows that repeating the procedure a finite number of times leads to one of the following situations. 1) A and B become $C_k(n, \alpha)$ and $D_k(n, \beta)$ without increasing $| A \wedge B |$. 2) Whenever $v(x) < v(y)$, $x \notin B$, $y \in B$ or $\bar{v}(x) < \bar{v}(y)$, $x \notin A$, $y \in A$, x and y differ in every component.

If the former occurs, we are finished. Suppose the latter occurs for B (the case for A follows by symmetry). The v-smallest $y \in B$ belonging to such a pair satisfies $v(y) = v(x) + 1$ for some such x. This occurs with disagreement in every component only when $x = (j - 1, k - 1, k - 1,..., k - 1)$ and $y = (j, 0, 0,..., 0)$ for some j, $x \notin B$, $y \in B$. We claim the only v-larger elements which can appear in B are $Y_r = \{j, 0, 0,..., 0, i)$: $0 \leqslant i \leqslant r\}$, where $0 \leqslant r < k - 1$. $(j, 0, 0,..., 0, k - 1)$ agrees with x in the last place, so it cannot be in B. If $v(j, 0, 0,..., 0, r + 1) \leqslant v(j, 0, k - 1,..., k - 1)$, z agrees with $(j, 0, 0,..., 0, r + 1)$ in the second place and must also avoid B. Now every digit has appeared in the third through $(n - 1)$st component of some element not in B, so no z with $v(z) > (j, 0, k - 1,..., k - 1)$ can lie in B. By similar reasoning, the only v-smaller elements which can *avoid* being in B are $X_s = \{(j - 1, k - 1, k - 1,..., k - 1, i)$: $s \leqslant i \leqslant k - 1\}$, where $r < s \leqslant k - 1$.

We are reduced to the case $\beta = jk^{n-1} + r + k - 1 - s$, $0 \leqslant r < s \leqslant k - 1$, $B = D_k(n, \beta + k - s) - X\rceil$. We now claim that $t = (j, 0,..., 0, r)$ can be replaced in B by $w = (j - 1, k - 1,..., k - 1, s)$ without increasing $| A \wedge B |$. For any $a \in A$, consider elements that might be added by the replacement, namely $z = a \wedge w$. If $z < w$, we have $z \in B$, so $z = a \wedge z$ is already present in $A \wedge B$. The remaining possibility is $a \wedge w = w$, in which case $a \geqslant w$. Since $r < s$, $\bar{v}(t) < \bar{v}(w) \leqslant \bar{v}(a)$. Applying the argument of the preceeding paragraph symmetrically to A and $\bar{v}$, we see that for this to happen A must have some element u with $u \geqslant t$. That guarantees $t \in A \wedge B$, since $t = u \wedge t$. Replacing t by w in B replaces t by w in $A \wedge B$, resulting in no net change. Repeating this procedure $k - s$ times transforms B into $D_k(n, \beta)$, which completes the proof. ∎

We note that theorem 4 can be proved by another argument in which elements of B are "pushed" towards elements of the same rank but smaller lexicographic value, while those of A are pushed toward larger lexicographic values. Always this is done without increasing $| A \wedge B |$. This efficacy of both proof by induction and proof by "pushing" toward the desired set occurs frequently in extremal set theory.

Now, suppose we have an arbitrary product of chains with sizes $\mathbf{k} = (k_1,..., k_n)$. By convention, put $k_0 = k_{n+1} = 1$. If we let $v^{\mathbf{k}}(c) = \sum_{i=1}^{n}(\prod_{r=i+1}^{n+1} k_r)\, c_1$ and $\bar{v}^{\mathbf{k}}(c) = \sum_{i=1}^{n}(\prod_{r=0}^{i-1} k_r)\, c_i$, we can still define $C(\mathbf{k}, \alpha)$ and $D(\mathbf{k}, \beta)$ as the α

and β elements of the lattice with smallest values under $\bar{v}^{\mathbf{k}}$ and $v^{\mathbf{k}}$, respectively. When the k_i are all equal, these reduce to the definitions in theorem 4. Most of the argument of theorem 4 holds in this more general setting. Elements differeing by one under v still differ in every place only when $x = (j-1, k_2-1, k_2-1, ..., k_n-1)$ and $y = (j, 0, ..., 0)$. Creating sets Y_r and X_s as before, it is still possible to replace the largest element of Y_r by the smallest element of X_s without increasing $A \wedge B$. However, different orderings of the k_i may produce different values of $|C(\mathbf{k}, \alpha) \wedge D(\mathbf{k}, \beta)|$. This fouls up the induction step. Depending on α and β, the best ordering of the chains may vary. The simplest example has $n = 2$, $k_1 = 2$, $k_2 = 3$. Here $C(\mathbf{k}, 2) = \{00, 10\}$, $C(\mathbf{k}, 3) = \{00, 10, 01\}$, $D(\mathbf{k}, 3) = \{00, 01, 02\}$, and $D(\mathbf{k}, 4) = \{\{00, 01, 02, 10\}$. If on the other hand $k_1' = 3$ and $k_2' = 2$, then $C(\mathbf{k}', 2) = C(\mathbf{k}, 2)$, $C(\mathbf{k}', 3) = \{00, 10, 20\}$, $D(\mathbf{k}', 3) = C(\mathbf{k}, 3)$ and $D(\mathbf{k}', 4) = \{00, 01, 10, 11\}$. Now note that $C(\mathbf{k}, 2) \wedge D(\mathbf{k}, 3) = \{00\}$ and $C(\mathbf{k}, 3) \wedge D(\mathbf{k}, 4) = \{00, 10, 01\}$, but $C(\mathbf{k}', 2) \wedge D(\mathbf{k}', 3) = \{00, 10\} = C(\mathbf{k}', 3) \wedge D(\mathbf{k}', 4)$. $\mathbf{k}$ is preferable for $(\alpha, \beta) = (2, 3)$, but $\mathbf{k}'$ is preferable for $(\alpha, \beta) = (3, 4)$. In general, we conjecture that a permutation $\mathbf{k}$ can be determined as a function of α and β so that $C(\mathbf{k}, \alpha)$ and $D(\mathbf{k}, \beta)$ will minimize $|A \cap B|$. If β equals or is "slightly" less than a multiple of a product of certain chain sizes (where β divided by that product is as small as possible) let those chains be the right-most components, decreasing in size from the nth toward the left. If other chains can be chosen similarly for α, let *those* chains be the *left*-most components, decreasing in size from the left. The remaining chains, if any, occupy the middle in some convex distribution. Details and conflicts in the above procedure have yet to be resolved.

We now return to products of equal-length chains and analyze the accuracy of $|A||B|/|L|$ as a lower bound on $|A \wedge B|$.

THEOREM 5. *Suppose L is a product of n chains of size k, $0 \leqslant \alpha \leqslant k^n$, $0 \leqslant \beta \leqslant k^n$. Let $\mu_k(n, \alpha, \beta) = \min\{|A \wedge B| : |A| = \alpha, |B| = \beta\}$ and $\epsilon_k(n, \alpha, \beta) = \mu_k(n, \alpha, \beta) - \alpha\beta/k^n$. If $pk^{n-1} < \alpha \leqslant (p+1)k^{n-1}$ and $\beta \equiv r \bmod k$, then*

$$\mu_k(n, \alpha, \beta) = \mu_k\left(n-1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \begin{cases} 0; & p = 0 \\ \sum_{j=0}^{p-1} \left\lceil \frac{\beta - j}{k} \right\rceil; & p > 0 \end{cases} \tag{7}$$

$$\epsilon_k(n, \alpha, \beta) = \epsilon_k\left(n-1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \begin{cases} r\left[1 - \frac{\alpha}{k^n}\right]; & 0 \leqslant r \leqslant p \\ (k-r)\frac{\alpha}{k^n}; & p < r < k \end{cases} \tag{8}$$

Furthermore,

$$\epsilon_k(n, k^n - \alpha, k^n - \beta) = \epsilon_k(n, \alpha, \beta) \tag{9}$$

$$\mu_k(n, k^n - \alpha, k^n - \beta) = \mu_k(n, \alpha, \beta) + k^n - \alpha - \beta \tag{10}$$

and, finally,

$$0 \leqslant \epsilon_k(n, \alpha, \beta) \leqslant \frac{kn}{4} \tag{11}$$

Proof. Before proceeding, we note some additional relations and boundary values that are obvious. Namely,

$$\mu_k(n, \alpha, \beta) = \mu_k(n, \beta, \alpha); \; \epsilon_k(n, \beta, \alpha) = \epsilon_k(n, \alpha, \beta) \tag{12a}$$

$$\mu_k(n, 0, \beta) = 0; \; \mu_k(n, k^n, \beta) = \beta; \; \epsilon_k(n, 0, \beta) = \epsilon_k(n, k^n, \beta) = 0 \tag{12b}$$

(7) is the recursion on which all of this analysis rests. Fıom theorem 4, we have $\mu_k(n, \alpha, \beta) = | C_k(n, \alpha) \wedge D_k(n, \beta)|$. Put $E = C \wedge D$, and define E_r^j as usual, so that $\mu_k(n, \alpha, \beta) = \sum_{j=0}^{k-1} | E_n^j |$. We will establish the recursion by determining the sets in E_n^j.

Recall $E_n^j = C_n^j \wedge D_n^j$, since C and D are ideals. If $0 \leqslant j < p$, C_n^j is the entire sublattice on the first $n - 1$ components, with *nth* component j, so $E_n^j = C_n^j \wedge D_n^j = D_n^j$. $| D_n^j | = \lceil (\beta - j)/k \rceil$, since the first element of D_n^j is encountered when $\beta = j + 1$, and thereafter one is added for every increase of k in β. This yields the last term in (7). If $j > p$, $E_n^j = \varnothing$ since C_n^j is void. We need only show $| E_n^p | = \mu_k(n - 1, \alpha - pk^{n-1}, \lceil \beta - p/k \rceil)$. Again we are in a copy of the sublattice, $E_n^p = C_n^p \wedge D_n^p$, $| D_n^p | = \lceil \beta - p/k \rceil$, but now the only things in C_n^p are the "excess" after the first pk^{n-1} elements. They are the first $\alpha - pk^{n-1}$ elements in a copy of the sublattice. In fact $C_n^j = g_n^j(C_k(n - 1, \alpha - pk^{n-1}))$, $D_n^j = g_n^j(D_k(n - 1, \lceil (\beta - p)/k \rceil))$, which gives us the recursion.

(8) follows from (7). Treating the summation in (7) as vacuously 0 if $p = 0$, we have

$$\begin{aligned}
\epsilon_k(n, \alpha, \beta) &= \mu_k(n, \alpha, \beta) - \frac{\alpha\beta}{k^n} \\
&= \mu_k\left(n - 1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \sum_{j=0}^{p-1} \left\lceil \frac{\beta - j}{k} \right\rceil - \frac{\alpha\beta}{k^n} \\
&= \epsilon_k\left(n - 1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \sum_{j=0}^{p-1} \left\lceil \frac{\beta - j}{k} \right\rceil \\
&\quad + \frac{(k - pk^{n-1})\lceil (\beta - p)/k \rceil}{k^{n-1}} - \frac{\alpha\beta}{k^n}.
\end{aligned}$$

We must evaluate the last three terms above. Suppose $\beta = r + mk$. We have two cases. If $0 \leqslant r \leqslant p$ then $\lceil \beta - p/k \rceil = m$ and $\sum_{j=0}^{p-1} \lceil \beta - j/k \rceil = r(m+1) + (p-r)m = pm + r$, so the last three terms sum to

$$pm + r + \frac{(\alpha - pk^{n-1})\, m}{k^{n-1}} - \frac{\alpha(r + mk)}{k^n} = r\left(1 - \frac{\alpha}{k^n}\right).$$

On the other hand, suppose $p < r < k$. Now $\lceil \beta - p/k \rceil = m + 1$ and $\sum_{j=0}^{p-1} \lceil (\beta - j)/k \rceil = p(m+1)$, so those terms sum to

$$p(m+1) + \frac{(\alpha - pk^{n-1})(m+1)}{k^{n-1}} - \frac{\alpha(r + mk)}{k^n} = (k - r)\frac{\alpha}{k^n}.$$

Combining the cases, we have (8).

(9) follows by induction from (8). For $n = 1$, $\mu_k(1, \alpha, \beta) = \min(\alpha, \beta)$. So, $\epsilon_k(1, k - \alpha, k - \beta) = \min(k - \alpha, k - \beta) - (k - \alpha)(k - \beta)/k = \min(k - \alpha, k - \beta) - k + \alpha + \beta - \alpha\beta/k = \min(\alpha, \beta) - \alpha\beta/k = \epsilon_k(1, \alpha, \beta)$. Suppose $n > 1$, and p as before. Then $(k - p)\, k^{n-1} > k^n - \alpha \geqslant (k - p - 1)\, k^{n-1}$. If $\beta \equiv r \bmod k$, then $k^n - \beta \equiv (k - r) \bmod k$. Suppose first that $\alpha \neq (p + 1)\, k^{n-1}$. We use $-\lfloor -x/k \rfloor = \lfloor (x + k - 1)/k \rfloor = \lceil x/k \rceil$, (8), and the induction hypothesis to show

$$\begin{aligned}
&\epsilon_k(n, k^n - \alpha, k^n - \beta) \\
&= \epsilon_k\left(n - 1, k^n - \alpha - (k - p - 1)\, k^{n-1}, \left\lceil \frac{k^n - \beta - (k - p - 1)}{k} \right\rceil\right) \\
&= \epsilon_k\left(n - 1, k^{n-1} - \alpha + pk^{n-1}, k^{n-1} + \left\lfloor \frac{-\beta + p}{k} \right\rfloor\right) \\
&\quad + \begin{cases} (k - r)\left(1 - \dfrac{k^n - \alpha}{k^n}\right); & 0 \leqslant k - r \leqslant k - p - 1 \\ [k - (k - r)]\left(\dfrac{k^n - \alpha}{k^n}\right); & k - p - 1 < k - r < k \end{cases} \\
&= \epsilon_k\left(n - 1, k^{n-1} - (\alpha - pk^{n-1}), k^{n-1} - \left\lceil \frac{\beta - p}{k} \right\rceil\right) \\
&\quad + \begin{cases} (k - r)\dfrac{\alpha}{k^n}; & p + 1 \leqslant r \leqslant k \\ r\left(1 - \dfrac{\alpha}{k^n}\right); & 0 < r < p + 1 \end{cases} \\
&= \epsilon_k\left(n - 1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \begin{cases} r\left(1 - \dfrac{\alpha}{k^n}\right); & 0 \leqslant r \leqslant p \\ (k - r)\dfrac{\alpha}{k^n}; & p < r < k \end{cases} \\
&= \epsilon_k(n, \alpha, \beta).
\end{aligned}$$

If, on the other hand, $\alpha = (p+1)\,k^{n-1}$ so that $k^n - \alpha = (k-p-1)\,k^{n-1}$, we use (12b) to find

$$
\begin{aligned}
\epsilon_k(n, k^n - \alpha, k^n - \beta)
&= \epsilon_k\left(n-1, k^{n-1}, \left\lfloor \frac{k^n - \beta + p + 1}{k} \right\rfloor\right) \\
&\quad + \begin{cases} (k-r)\left(1 - \dfrac{k^n - \alpha}{k^n}\right); & 0 \leqslant k - r \leqslant k - p - 1 \\ [k-(k-r)]\,\dfrac{k^n - \alpha}{k^n}; & k - p - 1 < k - r < k \end{cases} \\
&= 0 + \begin{cases} (k-r)\,\dfrac{p+1}{k}; & p + 2 \leqslant r \leqslant k \\ r\left(1 - \dfrac{p+1}{k}\right); & 0 < r < p + 2 \end{cases} \\
&= \epsilon_k\left(n-1, k^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) \\
&\quad + \begin{cases} r\left(1 - \dfrac{p+1}{k}\right); & 0 \leqslant r \leqslant p + 1 \\ (k-r)\,\dfrac{p+1}{k}; & p + 1 < r < k. \end{cases}
\end{aligned}
$$

For $r = p + 1$, $r(1 - (p+1)/k) = (k-r)(p+1)/k$, so the last line is again $\epsilon_k(n, \alpha, \beta)$.

(10) follows directly from (9):

$$
\begin{aligned}
\mu_k(n, k^n - \alpha, k^n - \beta) &= \epsilon_k(n, k^n - \alpha, k^n - \beta) + \frac{(k^n - \alpha)(k^n - \beta)}{k^n} \\
&= \epsilon_k(n, \alpha, \beta) + k^n - \alpha - \beta + \frac{\alpha\beta}{k^n} \\
&= \mu_k(n, \alpha, \beta) + k^n - \alpha - \beta.
\end{aligned}
$$

Finally, we come to (11). The lower bound on ϵ_k follows from theorem 1. For the upper bound, we examine the additive term in (8). If $0 \leqslant r \leqslant p$,

$$
r\left(1 - \frac{\alpha}{k^n}\right) \leqslant p\left(1 - \frac{pk^{n-1}}{k^n}\right) = \frac{p(k-p)}{k}
$$

If $p < r < k$,

$$
(k-r)\,\frac{\alpha}{k^n} \leqslant \frac{(k-p-1)(p+1)}{k}.
$$

These are maximized when p (resp. $p + 1$) equals $k/2$, and never exceed $k/4$. There are n such additive terms, since we may set $\epsilon_k(0, \alpha, \beta) = 0$, so

$\epsilon_k(n, \alpha, \beta) \leqslant kn/4$. When k is odd, integral p cannot equal $k/2$, and we have the slightly sharper $\epsilon_k(n, \alpha, \beta) \leqslant ((k - 1/k)n)/4$. ∎

If $\beta = r + mk$, the derivation of (8) shows we can rewrite (7) and (8) as

$$\mu_k(n, \alpha, \beta) = \begin{cases} \mu_k(n-1, \alpha - pk^{n-1}, m) + pm + r; & 0 \leqslant r \leqslant p \\ \mu_k(n-1, \alpha - pk^{n-1}, m+1) + pm + p; & p < r < k \end{cases} \tag{7a}$$

$$\epsilon_k(n, \alpha, \beta) = \begin{cases} \epsilon_k(n-1, \alpha - pk^{n-1}, m) + r\left(1 - \dfrac{\alpha}{k^n}\right); & 0 \leqslant r \leqslant p \\ \epsilon_k(n-1, \alpha - pk^{n-1}, m+1) + (k-r)\dfrac{\alpha}{k^n}; & p < r < k \end{cases} \tag{8a}$$

(9) and (10) are not surprising, because of the duality between meets of ideals and joins of filters. (11) is "asymptotically" best possible as a uniform bound for all (α. β), as seen in theorem 7. However, refinements to the basic argument for (11) yield better bounds for particular pairs (α, β).

For example, instead of $pk^{n-1} < \alpha \leqslant (p+1)\,k^{n-1}$, let us define p and q by $pk^q < \alpha \leqslant (p+1)\,k^q$, where q is the smallest possible. Writing α as a backwards k-ary number with n digits, the largest component with a nonzero entry is the $(q+1)st$, respresenting pk^q. Now we base the recursion on that component, rather than the last. The same argument used for (7) leads to

$$\mu_k(n, \alpha, \beta) = \mu_k\left(q, \alpha - pk^q, \left\lceil \frac{\beta - pk^{n-1-q}}{k^{n-q}} \right\rceil\right) + \sum_{j=0}^{p-1} \left\lceil \frac{\beta - jk^{n-1-q}}{k^{n-q}} \right\rceil \tag{7b}$$

Note that now $p = 0$ only if $q = 0$, $\alpha = 1$, in which case $\mu = 1$, $\epsilon = 1 - \beta/k^n$ so we may disregard that possibility. For an analogue to (8) let us suppose $\beta = r + mk^{n-q}$, $0 \leqslant r < k^{n-q}$. (7b) leads to

$$\epsilon_k(n, \alpha, \beta) = \epsilon_k\left(a, \alpha - pk^q, \left\lceil \frac{\beta - pk^{n-1-q}}{k^{n-q}} \right\rceil\right) + \sum_{j=0}^{p-1} \left(\frac{\beta - jk^{n-1-q}}{k^{n-q}}\right) + \frac{(\alpha - pk^q)\lceil(\beta - pk^{n-1-q})/k^{n-q}\rceil}{k^q} \frac{\alpha\beta}{k^n}$$

The two cases we must consider in evaluating this are $0 \leqslant r \leqslant pk^{n-1-q}$ and $pk^{n-1-q} < r < k^{n-q}$. In the former case let $r^* = \lceil r/k^{n-1-q} \rceil$. Then calculations like those used to prove (8) yield

$$\epsilon_k(n, \alpha, \beta) = \epsilon_k\left(q, \alpha - pk^q, \left\lceil \frac{\beta - pk^{n-1-q}}{k^{n-q}} \right\rceil\right) + \begin{cases} r^* - \dfrac{\alpha r}{k^n}; & 0 \leqslant r \leqslant pk^{n-1-q} \\ (k^{n-q} - r)\dfrac{\alpha}{k^n}; & pk^{n-1-q} < r < k^{n-q}. \end{cases} \tag{8b}$$

To obtain an analogue to (11), we bound the size of the additive terms above.

$$r^* - \frac{\alpha r}{k^n} < \frac{p(k-p+1)}{k}$$

$$(k^{n-q} - r)\frac{\alpha}{k^n} < \frac{(k-p)(p+1)}{k}$$

The former is maximized by $p = (k+1)/2$, the latter by $p = (k-1)/2$, but in either case the maximum is $(k+1)^2/4k$. We get a non-zero contribution to ϵ for each non-zero term in the k-ary expansion of α. Recalling (9) and (12a), let $\eta_{\alpha\beta}$ be the minimum number of non-zero components in the k-ary expansions of α, β, $k^n - \alpha$, and $k^n - \beta$. Then

$$\epsilon_k(n, \alpha, \beta) < \frac{(k + 2 + 1/k)\,\eta_{\alpha\beta}}{4} \tag{11b}$$

This bound is stronger than (11) when $\eta < nk/(k+2)$, and can be improved for even values of k by dropping $1/k$ from the numerator.

Another bound can be obtained by examining the additive term in (7a). It equals $pm + \min\{r, p\}$. Doing the recursion successively for all components, $\mu \leqslant \sum_{i=1}^{n}(p_i m_i + p_i)$, where p_i, m_i are p, m as determined when dropping the ith component. Now, $m_i k^{n+1-i} \leqslant \beta$ for all i, and $\sum_{i=1}^{n} p_i k^{i-1} = \alpha$. Let $\rho(\alpha) = \sum p_i$ be the sum of the components in the k-ary expansion of α. Then

$$\mu \leqslant \sum p_i m_i k^n/k^n + \sum p_i \leqslant \sum p_i k^{i-1}\beta/k^n + \rho(\alpha) = \alpha\beta/k^n + \rho(\alpha).$$

If $\rho_{\alpha\beta} = \min\{\rho(\alpha), \rho(\beta), \rho(k^n - \alpha), \rho(k^n - \beta)\}$, then we have

$$\epsilon_k(n, \alpha, \beta) \leqslant \rho_{\alpha\beta} \tag{11c}$$

In general, there is a lot of slack in this bound, since it yields a uniform bound of $(k-1)n/2$, which compares unfavorably with $kn/4$ unless $k = 2$. However, it yields good results for particular cases, and could be especially useful for applications to Boolean algebras ($k = 2$), where as a uniform bound it is best of the three bounds presented.

No examples have been found which achieve any of these upper bounds. Theorem 6 discusses when the lower bound $\epsilon = 0$ given by theorem 1 can be attained. Theorem 7 exhibits several classes of (n, α, β) with large ϵ_k, some of which appeoach $kn/4$.

THEOREM 6. *In the terminology of theorem 5, $\epsilon_k(n, \alpha, \beta) = 0$ if and only if* (i) *$\alpha\beta$ is a multiple of k^n and k divides both α and β, or* (ii) *trivially, α or β is k^n or 0.*

Proof. If α or β is k^n or 0, we apply (12). So assume $0 < k < k^n$, $0 < \beta < k^n$. $\epsilon_k(n, \alpha, \beta)$ will vanish if and only if the recursive term and the additive term in (8a) both vanish. The additive term vanishes iff $\beta = mk$ and $r = 0$.

Assume it does. For $n = 1$ this suffices to prove the claim. We proceed by induction on n. The induction hypothesis says the recursive term $\epsilon_k(n-1, \alpha - pk^{n-1}, m)$ vanishes iff k divides the last two arguments and k^{n-1} divides their product, or one of them is 0 or k^{n-1}. As p is defined, $\alpha - pk^{n-1}$ is never zero. It is k^{n-1} iff $k^{n-1} \mid \alpha$ and $k^n \mid \alpha\beta$, since $\beta = mk$. m is 0 or k^{n-1} iff β is 0 or k^n. Finally, $k \mid \alpha - pk^{n-1}$ iff $k \mid \alpha$, and $k^{n-1} \mid (\alpha - pk^{n-1})m$ iff $k^n \mid (\alpha - pk^{n-1})\, mk$ iff $k^n \mid \alpha\beta$. ∎

The result in [3] does not improve these results because of the way in which A and B can be chosen to minimize $| A \wedge B |$. If $\alpha\beta \geqslant k^n$ then $| A \vee B | = | L |$ and any derivable bounds are the same. On the other hand, if $\alpha\beta < k^n$ then $| A \wedge B | = 1$ and $| A | \, | B | / | A \vee B |$ is exact only when $| A |$ or $| B |$ is 1.

THEOREM 7. *Define sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ as follows:*

$$a_0 = 1; \qquad a_n = 2^n - a_{n-1}, \quad n \geqslant 1$$

$$c_n = \frac{k^n - 1}{2} \qquad b_n = \frac{k^n - 1}{k - 1} + 1 \qquad d_n = \frac{k}{2}\left(\frac{k^n - 1}{k - 1}\right).$$

Also let $r_k = (k-2)/(k-1)$ and $s_k = k/(k-1)$. Then

$$\epsilon_2(n, a_{2\lfloor n/2 \rfloor}, a_n) = \frac{n}{3} + \frac{1}{9} - \frac{(-1)^n}{9 \cdot 2^n} \tag{13}$$

$$\epsilon_k(n, b_n, b_n) = r_k n + r_k^2 - \frac{r_k^2}{k^n}; \qquad k > 2 \tag{14}$$

$$\epsilon_k(n, c_n, c_n) = \frac{(k-1)\, n}{4} + \frac{1}{4} - \frac{1}{3k^n}; \qquad k \text{ odd} \tag{15}$$

$$\epsilon_k(n, d_n, d_n) = r_k \frac{kn}{4} + \frac{s_k^2}{4} - \frac{s_k^2}{4k^n}; \qquad k \text{ even.} \tag{16}$$

Proof. (13)–(16) were derived by using the method of generating functions to solve the recursion relations that arise. Once known, however, they are much more easily verified by induction. When $n = 0$, the right sides above all vanish, so the theorem holds trivially. For the induction hypothesis, we assume it holds for smaller values of n.

Let us first consider (13). $\{a_n\}$ is a well-known sequence whose applications are referenced in [8]. We have four equivalent recursive definitions of a_n:

$$a_n = 2^n - a_{n-1} = 2^{n-1} + a_{n-2} = a_{n-1} + 2a_{n-2} = 2a_{n-1} + (-1)^n.$$

To prove the last from the third, for example, we have by induction $a_{n-1} = 2a_{n-2} + (-1)^{n-1}$. Substituting $2a_{n-2} = a_{n-1} - (-1)^{n-1}$ in $a_n = a_{n-1} + 2a_{n-2}$ yields the desired result. We note that $2^{n-1} < a_n < 2^n$ for $n > 1$ and every a_n is odd. Furthermore,

$$\begin{aligned} 9a_{n-1} - 3 \cdot 2^n + 2(-1)^n &= 6(a_{n-1} - 2^{n-1}) + 3a_{n-1} + 2(-1)^n \\ &= -6a_{n-2} + 3a_{n-1} + 2(-1)^n \\ &= 3(-1)^{n-1} + 2(-1)^n = -(-1)^n \end{aligned}$$

and

$$\left\lfloor \frac{a_{2m+1}}{2} \right\rfloor = \left\lfloor \frac{2a_{2m} + (-1)^{2m+1}}{2} \right\rfloor = a_{2m}$$

$$\left\lfloor \frac{a_{2m} - 1}{2} \right\rfloor = \left\lfloor \frac{2a_{2m-1} + (-1)^{2m} - 1}{2} \right\rfloor = a_{2m-1} \cdot$$

We claim $\epsilon_2(n, a_{2\lfloor n/2 \rfloor}, a_n) = \epsilon_2(n-1, a_{2\lfloor (n-1)/2 \rfloor}, a_{n-1}) + a_{n-1}/2^n$. This follows from (8) and the above whether n is even or odd.

$$\begin{aligned} &\epsilon_2(2m, a_{2m}, a_{2m}) \\ &= \epsilon_2 \left(2m - 1, a_{2m} - 2^{2m-1}, \left\lfloor \frac{a_{2m} - 1}{2} \right\rfloor \right) + \frac{2^{2m} - a_{2m}}{2^{2m}} \\ &= \epsilon_2(2m - 1, a_{2m-2}, a_{2m-1}) + \frac{a_{2m-1}}{2^{2m}} . \end{aligned}$$

$$\begin{aligned} &\epsilon_2(2m + 1, a_{2m}, a_{2m+1}) \\ &= \epsilon_2 \left(2m, a_{2m} - 0 \cdot 2^{2m-1}, \left\lfloor \frac{a_{2m+1}}{2} \right\rfloor \right) + \frac{a_{2m}}{2^{2m+1}} \\ &= \epsilon_2(2m, a_{2m}, a_{2m}) + \frac{a_{2m}}{2^{2m+1}} . \end{aligned}$$

Now we complete the induction.

$$\begin{aligned} \epsilon_2(n, a_{2\lfloor n/2 \rfloor}, a_n) &= \epsilon_2(n - 1, a_{2\lfloor (n-1)/2 \rfloor}, a_{n-1}) + \frac{a_{n-1}}{2^n} \\ &= \frac{n-1}{3} + \frac{1}{9} - \frac{(-1)^{n-1}}{9 \cdot 2^{n-1}} + \frac{a_{n-1}}{2^n} \\ &= \frac{n}{3} + \frac{1}{9} + \frac{-3 \cdot 2^n + 2(-1)^n + 9a_{n-1}}{9 \cdot 2^n} \\ &= \frac{n}{3} + \frac{1}{9} - \frac{(-1)^n}{9 \cdot 2^n} \end{aligned}$$

(14), (15), and (16) are verified in the same manner as (13), but with fewer preliminaries. In each case, we show that the recursive term arising from (8) is just the preceding term in the sequence, after which computation completes the induction.

Consider (14). $b_n = (k^n - 1)/(k - 1) + 1 = 1 + \sum_{i=1}^{n-1} k^i$. We note $k^{n-1} < b_n < 2k^{n-1}$ so $p = 1$, also $b_n - k^{n-1} = b_{n-1}$, $b_n \equiv 2 \bmod k$, and $\lceil (b_n - 1)/k \rceil = b_{n-1} (k > 2)$. We shall also need

$$(k - 2)\, b_n - (k - 1)\, r_k^2 = r_k(k^n - 1 + k - 1 - k + 2) = r_k k^n.$$

Now to complete the induction,

$$\begin{aligned}
\epsilon_k(n, b_n, b_n) &= \epsilon_k(n - 1, b_{n-1}, b_{n-1}) + (k - 2)\frac{b_n}{k^n} \\
&= r_k(n - 1) + r_k^2 - \frac{r_k^2}{k^{n-1}} + (k - 2)\frac{b_n}{k^n} \\
&= r_k n + r_k^2 - \frac{r_k^2}{k^n} - \frac{(k - 1)\, r_k^2}{k^n} + \frac{(k - 2)\, b_n}{k^n} - r_k \\
&= r_k n + r_k^2 - \frac{r_k^2}{k^n}.
\end{aligned}$$

Consider (15). $((k - 1)/2)\, k^{n-1} < (k^n - 1)/2 < ((k + 1)/2/\, k^{n-1}$, so $p = (k - 1)/2$. $(k^n - 1)/2 = (k - 1)/2 \sum_{i=0}^{n-1} k^i$, so $c_n - ((k - 1)/2)\, k^{n-1} = c_{n-1}$ and $c_n \equiv (k - 1)/2 \bmod k$. Also $\lceil (c_n - (k - 1)/2)/k \rceil = c_{n-1}$, so we can complete the induction with

$$\begin{aligned}
\epsilon_k(n, c_n, c_n) &= \epsilon_k(n - 1, c_{n-1}, c_{n-1}) + \left(\frac{k - 1}{2}\right)\left(1 - \frac{c_n}{k^n}\right) \\
&= \frac{(k - 1)(n - 1)}{4} + \frac{1}{4} - \frac{1}{4k^{n-1}} + \left(\frac{k - 1}{2}\right)\left(\frac{k^n + 1}{2k^n}\right) \\
&= \frac{(k - 1)\, n}{4} + \frac{1}{4} - \frac{1}{4k^n} - \frac{(k - 1)}{4k^n} - \frac{(k - 1)}{4} \\
&\quad + \frac{k - 1}{4}\left(1 + \frac{1}{k^n}\right) \\
&= \frac{(k - 1)\, n}{4} + \frac{1}{4} - \frac{1}{4k^n}.
\end{aligned}$$

Similarly for (16). $d_n = k/2((k^n - 1)/(k - 1)) = \frac{1}{2}\sum_{i=0}^{n-1} k^i$. $(k/2)\, k^{n-1} < d_n < (k/2 + 1)\, k^{n-1}$, so $p = k/2$. Also, we have $d_n - (k/2)\, k^{n-1} = d_{n-1}$, $d_n \equiv (k/2) \bmod k$, and $\lceil (d_n - k/2)/k \rceil = d_{n-1}$. We shall need the following computation.

$$\frac{k}{2}\left(1-\frac{d_n}{k^n}\right)=\frac{k}{2}-\frac{k^2}{4k^n}\frac{k^n-1}{k-1}$$
$$=\frac{k^2}{4k^n(k-1)}-\frac{k^2}{4(k-1)}+\frac{2k(k-1)}{4(k-1)}$$
$$=\frac{k^2}{4k^n(k-1)}+\frac{k(k-2)}{4(k-1)}=\frac{s_k^2(k-1)}{4k^n}+\frac{kr_k}{4}$$

Finally, to complete the induction

$$\epsilon_k(n, d_n, d_n)=\epsilon_k(n-1, d_{n-1}, d_{n-1})+\frac{k}{2}\left(1-\frac{d_n}{k^n}\right)$$
$$=\frac{k(n-1)}{4}r_k+\frac{s_k^2}{4}-\frac{s_k^2}{4k^{n-1}}+\frac{k}{2}\left(1-\frac{d_n}{k^n}\right)$$
$$=\frac{kn}{4}r_k+\frac{s_k^2}{4}-\frac{s_k^2}{4k^n}-\frac{s_k^2(k-1)}{4k^n}-\frac{kr_k}{4}+\frac{k}{2}\left(1-\frac{d_n}{k^n}\right)$$
$$=\frac{kn}{4}r_k+\frac{s_k^2}{4}-\frac{s_k^2}{4k^n}. \quad \blacksquare$$

Note that for $k=3$, $b_n=3^n$, so by (9) $\epsilon_3(n, b_n, b_n)=\epsilon_3(n, c_n, c_n)$. Similarly, for $k=4$, $b_n+d_n=4^n$ and $\epsilon_4(n, b_n, b_n)=\epsilon_4(n, d_n, d_n)$. $\{c_n\}$ and $\{d_n\}$ are attempts to realize the worst case behavior described in the proof of (11). As k and n become large they achieve it asymptotically. When $k=2$ and we have the lattice of subsets of a set, there seems to be some essential difference. It would be interesting to know if, in fact, the values of α and β for which ϵ_k is maximized are $\{a_n\}$ ($k=2$), $\{c_n\}$ (k odd), and $\{d_n\}$ (k even, $k>2$).

4. Related Results

In this section we obtain two related results. They follow from the inequality (1) of the introduction and can be subjected to analysis like that of the preceding section.

Hilton [6] proved that if A and B are subsets of a Boolean algebra each containing no element and its complement, and if no element of A is related to any element of B, then $|A \cup B| \leqslant \frac{1}{2}|L|$. We generalize this, giving a proof different from his. We shall need a lemma about intersections of filters and ideals, for which we introduce additional notation.

For a subset C of a lattice, let uC and lC denote the filter and the ideal generated by C. That is, $uC=\{c \in L: \exists a \leqslant c,\ a \in C\}$, $lC=\{c \in L: \exists a \geqslant c,\ a \in C\}$. Clearly, $C \subset uC \cap lC$ and in fact $uC \cap lC$ is the smallest compact set containing C. By a *polarity* σ of a lattice we mean an order-reversing bijection whose square is the identity. That is, $a \leqslant b$ implies $\sigma b \leqslant \sigma a$ and $\sigma(\sigma a) = a$. Complementation, for example, is a polarity. We may call σa the "polar image" of a.

LEMMA 2. *Suppose L satisfies* (1) *and has a polarity σ. Then for any $C \subset L$,*

$$|C| \leqslant |uC \cap u(\sigma C)| \quad |C| \leqslant |lC \cap l(\sigma C)|$$

Proof. We first note $|u(\sigma C)| = |lC|$, since $u(\sigma C) = \{a: a \geqslant \sigma c, c \in C\} = \{a: \sigma a \leqslant c, c \in C\} = \{a: \sigma a \in LC\}$. Using (1) and (2), we then have

$$\begin{aligned} |L|\,|C| &\leqslant |L|\,|uC \cap lC| \leqslant |uC|\,|lC| \\ &= |uC|\,|u(\sigma C)| \leqslant |L|\,|uC \cap u(\sigma C)| \end{aligned}$$

and similarly for $|lC \cap l(\sigma C)|$. ∎

This leads to

THEOREM 8. *Suppose L is a finite lattice with a polarity σ, in which* (1) *holds. Let π be the number of nontrivial orbits of σ (i.e., unordered pairs $\{e, \sigma e\}$ where $e \neq \sigma e$). If A and B are subsets of L each containing no element and its polar image, and if no element of A is related to any element of B, then*

$$|A \cup B| \leqslant \pi \leqslant \tfrac{1}{2}|L|.$$

Proof. Let π_i be the set of non-trivial orbits under σ which have i elements in $A \cup B$. Clearly,

$$\pi = |\pi_0| + |\pi_1| + |\pi_2|$$

$$|A \cup B| = |\pi_1| + 2|\pi_2|$$

so to prove the theorem it suffices to show $|\pi_2| \leqslant |\pi_0|$. Let C be the subset of A whose polar images lie in B, and let D be the subset of B whose polar images lie in A. Clearly $D = \sigma C$. By lemma 2, it suffices to show that $|C| = |\pi_2|$ and $|uC \cap uD| \leqslant |\pi_0|$. Each element of C is contained in exactly one pair in π_2, and every element (pair) of π_2 contains exactly one element of C, so $|\pi_2| = |C| = |D|$.

To prove $|uC \cap uD| \leqslant |\pi_0|$ we will exhibit a distinct pair in π_0 for each element of $uC \cap uD$. If $d \in uC$ then there is an $a \in C$ with $a \leqslant d$, and then $\sigma d \leqslant \sigma a \in D \subset B$. Since no elements of A and B are related, $d \notin B$ and $\sigma d \notin A$. For the same reason we cannot have $a \leqslant d = \sigma d \leqslant \sigma a$, so $d \neq \sigma d$. Similarly, if $d \in uD$, then $d \notin A$, $\sigma d \notin B$, $d \neq \sigma d$. Thus if d is in both uC and uD, then $\{d, \sigma d\}$ is in π_0. Now suppose $d, d' \in uC \cap uD$ both give rise to the same pair in π_0. That is, $d \neq d'$ but $\sigma d = d'$. $d, \sigma d \in uC$ implies there are

$a_1, a_2 \in C$ with $a_1 \leqslant d$ and $a_2 \leqslant \sigma d$. This leads to $a_2 \leqslant \sigma d \leqslant \sigma a_1 \in D$, which contradicts the hypothesis about relations between elements of A and B. This proves $|uC \cap uD| \leqslant |\pi_0|$ and establishes the theorem. ∎

We may form a set A by taking one element from every non-trivial orbit under σ, so that $|A| = \pi$. Taking this A and $B = \varnothing$ shows that the theorem is best possible. When L is a Boolean algebra and σ is complementation, we have Hilton's result. He exhibits other sets which achieve the bound. If A consists of all subsets of cardinality at most h containing the element j and B has the subsets of cardinality at least h not containing j, then A and B satisfy the conditions of the theorem and $|A| + |B| = \sum_{i=0}^{h-1}\binom{n-1}{i} + \sum_{i=h}^{n-1}\binom{n-1}{i} = 2^{n-1}$. As in the preceding section, we can ask several questions: for fixed $|A| = \alpha$ how big can B be satisfying the conditions of the theorem, what A and B achieve this maximum, for which α is their union of size π, and for which α is their union farthest from π?

For our final result we return to products of chains. The restriction of this theorem to a Boolean algebra was proved by Hilton as corollary 2 of his result in [6]. We define a binary operation $+$ on elements of such a lattice to be their sum as vectors, $a + b = (a_1 + b_1, \ldots, a_n + b_n)$. Let k_i be the length of the ith chain. Note that $a + b \in L$ if $a_i + b_i \leqslant k_i$ for all i. If on the other hand, $a_1 + b_1 \geqslant k_i$ for *all i*, we say that the pair $\{a, b\}$ *escapes L*. We have the following

THEOREM 9. *Suppose A is a subset of a product of chains L and the sum of any two elements of A neither lies in L nor escapes L. Then $|A| \leqslant \frac{1}{4}|L|$.*

Proof. Viewing the lattice as a multiset with bounded multiplicities k_i define the complement $\bar{a}$ of an element $a = (a_1, \ldots, a_n)$ by $\bar{a} = (k_1 - a_1, \ldots, k_n - a_n)$. We claim neither uA nor lA can contain both a and $\bar{a}$ for any a. If $a, \bar{a} \in uA$, then for any $a_1, a_2 \in A$ with $a_1 \leqslant a$ and $a_2 \leqslant \bar{a}$, $(a_1 + a_2)$ lies in L. Similarly, if $a, \bar{a} \in lA$, then for any $a_1, a_2 \in A$ with $a_1 \geqslant a$ and $a_2 \geqslant a$, $(a_1 + a_2)$ escapes L. Complementation is a polarity, so $|uA| \leqslant \frac{1}{2}|L|$, $|lA| \leqslant \frac{1}{2}|L|$. Applying (1) we have

$$|A|\,|L| \leqslant |uA \cap lA|\,|L| \leqslant |uA|\,|lA| \leqslant \tfrac{1}{4}|L|^2$$

as required. ∎

A product of chains of *lengths* $k_1, \ldots, k_n$ is isomorphic to the lattice of divisors of an integer $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$, where the p_i are distinct primes. $a + b$ in the lattice of multisets corresponds to $a \cdot b$ in the lattice of divisors. $a + b \in L$ means $ab \mid N$ and $a + b$ escapes L means $N^2/ab \mid N$. So, the theorem states that a collection A of divisors of N for which $ab \nmid N$, $N^2/ab \nmid N$ whenever $a, b \in A$ contains at most one-quarter of the divisors of N.

If there is only one chain, A can have no elements. If there are two or more chains and at least two have odd length, say i and j, then theorem 9 is best possible, i.e., the bound is achievable. A may include, for example, all elements a with $a_i < k_i/2$ and $a_j > k_j/2$. If there is exactly one chain of odd length, the theorem is almost best possible. Order the chains so that k_1 is odd, $k_2 \leqslant k_3 \leqslant \cdots \leqslant k_n$ are even. For all i with $2 \leqslant i \leqslant n$, let $A_i = \{a \in L: a_1 < k_1/2,\ a_i > k_i/2$, and $a_j = k_j/2$ for all $2 \leqslant j < i\}$, and let $A = \bigcup_{i=2}^n A_i$. Then A satisfies the conditions and

$$| A | = \sum | A_i | = \sum_{i=2}^{n} \frac{k_1 + 1}{2} \cdot \frac{k_i}{2} \prod_{j=i-1}^{n} (k_j + 1).$$

Computation shows this to be close to $\frac{1}{4} | L | = \frac{1}{4} \prod_{i=1}^n (k_i + 1)$. Another possibility would be a maximum-sized subset such that for all $a \in A$, (i) $a_1 < k_1/2$, (ii) $\sum_{i=2}^n a_i \geqslant \frac{1}{2} \sum_{i=2}^n k_i$, and (iii) $(a_1, k_2 - a_2, \ldots, k_n - a_n) \notin A$. When N is a square we do not know the best bound for $| A |$. Perhaps it can be generated as in one of the examples above, with one set of components providing high values and the remainder providing low values.

The case of theorem 9 in which N is square free has been proved in different ways by Lovász, by Seymour, and by Schönheim (see [5]). Also, that case has been generalized in different directions by Anderson [1] and Frankel [3]. Finally, if the lattice is of subsets of a set, $a + b \notin L$ means $a \cap b \neq \varnothing$, while $a + b$ not escaping L is the same as $a \cup b \neq \{1, \ldots, n\}$. For a product of arbitrary chains, let us say a *intersects* b if $a + b \notin L$, and a *self-intersecting family* is a subset A where $a + b \notin L$ whenever $a, b \in A$. It would be interesting to generalize [7] to products of arbitrary chains and determine in that setting the maximal size of the union of k self-intersecting families.

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