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Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta



Algebraic and combinatorial aspects of sandpile monoids on directed graphs

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ARTICLE INFO

Article history:

Received 29 April 2011

Available online 22 August 2012

Keywords:

Sandpile group

Sandpile monoid

Directed graph

Maximal subgroup

Distance-regular graph

ABSTRACT

The sandpile group of a graph is a well-studied object that combines ideas from algebraic graph theory, group theory, dynamical systems, and statistical physics. A graph's sandpile group is part of a larger algebraic structure on the graph, known as its sandpile monoid. Most of the work on sandpiles so far has focused on the sandpile group rather than the sandpile monoid of a graph, and has also assumed the underlying graph to be undirected. A notable exception is the recent work of Babai and Toumpakari, which builds up the theory of sandpile monoids on directed graphs from scratch and provides many connections between the combinatorics of a graph and the algebraic aspects of its sandpile monoid.

In this paper we primarily consider sandpile monoids on directed graphs, and we extend the existing theory in four main ways. First, we give a combinatorial classification of the maximal subgroups of a sandpile monoid on a directed graph in terms of the sandpile groups of certain easily-identifiable subgraphs. Second, we point out certain sandpile results for undirected graphs that are really results for sandpile monoids on directed graphs that contain exactly two idempotents. Third, we give a new algebraic constraint that sandpile monoids must satisfy and exhibit two infinite families of monoids that cannot be realized as sandpile monoids on any graph. Finally, we give an explicit combinatorial description of the sandpile group identity for every graph in a family of directed graphs which generalizes the family of (undirected) distance-regular graphs. This family includes many other graphs of interest, including iterated wheels, regular trees, and regular tournaments.

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<http://dx.doi.org/10.1016/j.jcta.2012.08.001>

1. Introduction

The sandpile model developed by Bak, Tang, and Wiesenfeld (BTW) is a mathematical model for dynamical systems that naturally evolve toward critical states, exemplifying the complex behavior known as self-organized criticality [3]. This model has been widely studied and used by various disciplines, including physics [3,12,14], computer science [1,7,8], and economics [6]. In 1990 Dhar generalized the BTW model into what is now known as the abelian sandpile model [12]. His results awakened the larger mathematical community to analyze the algebraic, geometric, and combinatorial structures arising from these dynamical systems. For example, one of Dhar's early contributions to the theory was his realization that the recurrent states in the sandpile model of a graph form a group, now called the *sandpile group* of the graph. The sandpile group of a graph has since been studied by a large number of authors—see, e.g., [2,16–18,21,22].

The sandpile group of a graph is a (usually strict) subset of a larger algebraic structure on the graph, called the *sandpile monoid* of the graph. Although Dhar introduced the sandpile monoid and the sandpile group of a graph at the same time, much of the subsequent work in the area has focused on the sandpile group. In addition, most work in the area so far has assumed the underlying graph to be undirected.

The recent paper of Babai and Toumpakari [2] forms the first published systematic study of the sandpile monoid of which we are aware. (Some of the work in [2] dates back to the 2005 PhD thesis of Toumpakari [21].) In [2] Babai and Toumpakari primarily consider directed graphs, and they build up the theory of sandpile monoids on directed graphs from scratch. In this paper we study the sandpile monoid of a finite directed graph, and we extend the existing theory in four main ways. We proceed as follows.

We begin in Section 2 by reviewing the basic definitions and observations for sandpile monoids and groups on directed graphs. We also recall the results of Babai and Toumpakari [2] that connect certain algebraic aspects of the sandpile monoid to combinatorial aspects of the underlying graph that we will need for our development.

In Section 3 we give our first main result, which is a combinatorial description of the maximal subgroups of a graph's sandpile monoid in terms of sandpile groups of easily-identified subgraphs. This is a nontrivial extension of the results of Babai and Toumpakari [2] on the idempotent structure of the sandpile monoid, and is accomplished in Theorem 3.5 (for the maximal subgroup corresponding to the 0 of the monoid) and Theorem 3.7 (for the maximal subgroups corresponding to the nonzero idempotents of the monoid).

In Section 4 we point out sandpile results for undirected graphs that are really results for sandpile monoids on directed graphs that contain exactly two idempotents (Theorems 4.2 and 4.3).

In Section 5 we give a new algebraic constraint that sandpile monoids must satisfy (Theorem 5.2) and we exhibit two infinite families of monoids that cannot be realized as sandpile monoids on any graph (Theorems 5.1 and 5.4).

Finally, in Section 6 we study a family of directed graphs that generalizes the family of undirected distance-regular graphs (Definition 6.4) and we give an explicit combinatorial description of the sandpile group identity for every graph in this family (Theorem 6.7). Our family includes distance-regular graphs as well as many other graphs of interest, including iterated wheels (Example 6.9), regular trees (Example 6.10), and regular tournaments (Example 6.11).

2. Basic algebraic theory for directed graphs

In this section we lay out the basic definitions and algebraic theory for sandpiles on directed graphs and we review Babai and Toumpakari's structural results on the sandpile monoid [2] that we will need for our development. Other good references include [13,16]. We use the notation established in this section throughout the paper.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Let \widehat{X} denote a finite weakly connected directed graph (weakly connected means that \widehat{X} is connected if we replace all directed edges with undirected edges) with vertex set \widehat{V} , directed edge set \widehat{E} , and a distinguished vertex (called the *sink*) which is reachable from all other vertices. We allow \widehat{X} to have loops as well as any number of edges in either direction between pairs

of vertices. X denotes the directed graph induced by removing the sink from \widehat{X} . Let V and E be the vertex and directed edge sets of X , respectively. We assume that V is nonempty (so that \widehat{X} has at least one non-sink vertex). For $v \in V$, let $\text{deg}^+(v)$ denote the out-degree of v . For $v, w \in \widehat{V}$, if there exists a path (trivial or not) from v to w in X , we write $v \rightarrow w$. In particular, $v \rightarrow v$ for all $v \in \widehat{V}$.

In the classical case, \widehat{X} is undirected and X is connected.

A sandpile configuration m on \widehat{X} (also called a configuration or a state) is a nonnegative integer number of “grains of sand” distributed, possibly unevenly, across the vertices of V . Given an ordering of the vertices of V , m can be described by a length- $|V|$ vector with entries in $\mathbb{N} \cup \{0\}$, where the k th entry of the vector gives the number of grains on the k th vertex of V .

Let m be a sandpile configuration on \widehat{X} . The support of m , denoted $\text{supp}(m)$, is the set of all $v \in V$ which hold at least one grain of sand. In the state m , a vertex $v \in V$ is said to be stable if the number of grains on v is less than $\text{deg}^+(v)$, and m itself is said to be stable if every vertex $v \in V$ is stable. If v is unstable, we may topple v , sending one grain of sand along each of its outward edges to its neighbors. This may cause previously stable vertices to become unstable. The sink swallows all grains of sand it receives and never topples, even if it has outward edges. Because there is a path from every vertex to the sink, if we continue toppling unstable vertices we will eventually reach a stable configuration. A sequence of topplings that terminates in a stable configuration is called an avalanche. Furthermore, the order of topplings in an avalanche is irrelevant—every sandpile configuration has a unique stable configuration [2,13].

2.1. The sandpile monoid and sandpile group of a directed graph

The sandpile monoid of (or on) \widehat{X} , denoted $M(\widehat{X})$, is the collection of all stable sandpile configurations on \widehat{X} . Given two configurations, we may add them pointwise. Denote this operation by $+$. Even if a and b are stable configurations, $a + b$ may not be. For any states a and b , denote the stabilization of $a + b$ by $a \oplus b$. The operation on $M(\widehat{X})$ is \oplus , which is well defined, associative, commutative, and has an identity [2,13]. The identity element is the empty configuration, which we denote by 0 . It is evident that

$$|M(\widehat{X})| = \prod_{v \in V} \text{deg}^+(v). \tag{1}$$

We follow Babai and Toumpakari [2] in our definitions of recurrence and the sandpile group of \widehat{X} . The development in [16] is also equivalent.

Definition 2.1. Let $a, b \in M(\widehat{X})$. We say that a is accessible from b (and that b can access a) if there exists $x \in M(\widehat{X})$ such that $b \oplus x = a$. For $a \in M(\widehat{X})$, we say that a is recurrent if a is accessible from every element $b \in M(\widehat{X})$. If a is not recurrent, a is said to be transient.

Definition 2.2. Let (S, \cdot) be a semigroup. If A and B are subsets of S , we define $A \cdot B = \{a \cdot b : a \in A \text{ and } b \in B\}$. $I \subseteq S$ is said to be an ideal of S if I is nonempty, $I \cdot S \subseteq I$, and $S \cdot I \subseteq I$. Let L be the intersection of all ideals of S . If L is nonempty, then L is said to be the minimal ideal of S , and if L is empty then S is said to have no minimal ideal.

If S is a finite semigroup then the minimal ideal of S exists, and if S is also commutative then its minimal ideal is a group [2].

Definition 2.3. The sandpile group of (or on) \widehat{X} , denoted $G(\widehat{X})$, is the minimal ideal of $M(\widehat{X})$.

By the above comment, $G(\widehat{X})$ is a group. As far as we know, Babai and Toumpakari were the first to give this definition of $G(\widehat{X})$. This is a useful definition (or characterization) of $G(\widehat{X})$ because it allows many of the basic properties of $G(\widehat{X})$ to be derived using straightforward semigroup theory. For several examples, see [2]. It is straightforward to show the following equivalent characterization of $G(\widehat{X})$ [2].

Theorem 2.4. *The elements of $G(\widehat{X})$ are precisely the recurrent elements of $M(\widehat{X})$.*

If the sink of \widehat{X} has any outward edges, denote the graph obtained by deleting them by \widehat{X}' . It is clear that the behavior of any sandpile configuration on \widehat{X} and \widehat{X}' is the same, and consequently that $M(\widehat{X}) \cong M(\widehat{X}')$ and $G(\widehat{X}) \cong G(\widehat{X}')$. So, why not just assume at the outset that the sink has no outward edges? It turns out that if \widehat{X} is classical (meaning that \widehat{X} is undirected and X is connected), then up to isomorphism $G(\widehat{X})$ does *not* depend on which vertex is chosen to be the sink. In fact, if \widehat{X} is Eulerian (meaning that there is a path from v to w in \widehat{X} for all $v, w \in \widehat{V}$, and $\text{indegree}(v) = \text{outdegree}(v)$ for all $v \in \widehat{V}$), then up to isomorphism $G(\widehat{X})$ does not depend on which vertex is the sink [16, Lemma 4.12]. Altering the location of the sink in these cases may, however, alter the structure of $M(\widehat{X})$.

2.2. Recurrence and the identity of $G(\widehat{X})$

Notation. The totally saturated configuration in $M(\widehat{X})$ (that is, the configuration on \widehat{X} in which every vertex holds as many grains of sand as possible without toppling) will play an important role in what follows. We denote it by MAX.

It is easy to see that MAX is accessible from any element $m \in M(\widehat{X})$ —just drop enough grains of sand on each vertex of V to totally saturate them. Hence $\text{MAX} \in G(\widehat{X})$.

The following theorem generalizes directly in its statement from the classical case [11] to the directed case.

Theorem 2.5. *Let $a \in M(\widehat{X})$.*

- (1) *a is recurrent if and only if a is accessible from MAX.*
- (2) *Let $e = [\text{MAX} - (\text{MAX} \oplus \text{MAX})] \oplus \text{MAX}$. Then e is the identity of $G(\widehat{X})$.*
- (3) *Let e denote the identity of $G(\widehat{X})$. Then a is recurrent if and only if $a \oplus e = a$.*

Proof. Part 1. If a is recurrent, then it is accessible from any configuration. On the other hand, if a is accessible from MAX, write $a = \text{MAX} \oplus m$ for some $m \in M(\widehat{X})$. Let $b \in M(\widehat{X})$. MAX is accessible from b , so we have $\text{MAX} = b \oplus m'$ for some $m' \in M(\widehat{X})$. Then $a = b \oplus m' \oplus m$, so a is accessible from b .

Part 2. e is accessible from MAX by definition, so $e \in G(\widehat{X})$. We show e is idempotent. Let δ denote MAX.

$$\begin{aligned} e \oplus e &= ([\delta - (\delta \oplus \delta)] \oplus \delta) \oplus (\delta \oplus [\delta - (\delta \oplus \delta)]) \\ &= ([\delta - (\delta \oplus \delta)] \oplus (\delta \oplus \delta)) \oplus [\delta - (\delta \oplus \delta)] \\ &= \delta \oplus [\delta - (\delta \oplus \delta)] = e, \end{aligned}$$

as $([\delta - (\delta \oplus \delta)] \oplus (\delta \oplus \delta)) = \delta$. Since any group contains exactly one idempotent—its identity— e is the identity of $G(\widehat{X})$.

Part 3. If a is recurrent, then $a \oplus e = a$ because $a \in G(\widehat{X})$ and e is the identity of $G(\widehat{X})$. On the other hand, if $a = a \oplus e$, then a is an element of $G(\widehat{X})$ because $G(\widehat{X})$ is an ideal. \square

Although part 2 of Theorem 2.5 gives us an easy way to compute the identity e of $G(\widehat{X})$ for any given digraph \widehat{X} , a description of e in terms of the combinatorics of \widehat{X} can be difficult to obtain. For example, many questions concerning the combinatorial structure of the identity element on the undirected $m \times n$ grid with an augmented sink remain open [16]. In Section 6 we give a combinatorial classification of e for every graph in a family of directed graphs that generalizes the family of undirected distance-regular graphs.

Also, note that parts 2 and 3 of Theorem 2.5 combine to provide a straightforward computational test for recurrence of a given state $a \in M(\widehat{X})$ —namely, if the identity e of $G(\widehat{X})$ is not already stored in memory, construct it, and then check whether $a \oplus e$ is a . There are tests which are generally

computationally faster (in terms of the number of topplings needed), however, which are known as the *burning algorithm* for undirected \widehat{X} [12] and the *script algorithm* for directed \widehat{X} [20].

We also have the following corollary of Theorem 2.5.

Corollary 2.6. *Let e denote the identity of $G(\widehat{X})$. Then $a \in M(\widehat{X})$ is recurrent if and only if a is accessible from e .*

Proof. If a is recurrent, then a is accessible from every element of $M(\widehat{X})$. On the other hand, if $a = e \oplus k$ for some $k \in M(\widehat{X})$, then $a \oplus e = e \oplus k \oplus e = e \oplus e \oplus k = e \oplus k = a$, and the result follows from part 3 of Theorem 2.5. \square

2.3. Cycles, idempotents, and structure

We now describe some of the results of Babai and Toumpakari on the relationship between the algebra of $M(\widehat{X})$ and the combinatorics of \widehat{X} [2,21].

In this paper we only consider directed cycles.

Definition 2.7. A cycle in a directed graph is a directed path of length 1 or greater from a vertex of the graph to itself.

Definition 2.8. A cycle in X is *accessible* from a vertex $v \in \widehat{V}$ if there is a path from v to a vertex in the cycle using only edges in E . In particular, no cycle in X is accessible from the sink.

We are now using the term *accessible* in two contexts: first, in terms of accessibility in the monoid $M(\widehat{X})$ (Definition 2.1), and second, in terms of accessibility in the underlying digraph \widehat{X} (Definition 2.8). The following theorem shows how these concepts are related.

Theorem 2.9. (See [2,21].) *Let $a, b \in M(\widehat{X})$.*

- (1) *0 is accessible from a if and only if a has no grains of sand on any vertex from which a cycle in X is accessible.*
- (2) *If there is a cycle of X on which a contains a grain of sand but on which b does not, then b is not accessible from a .*
- (3) *If $g \in G(\widehat{X})$, then g contains at least one grain of sand on every cycle of X .*
- (4) *The following are equivalent.*
 - *0 is recurrent.*
 - *$G(\widehat{X}) = M(\widehat{X})$.*
 - *$M(\widehat{X})$ contains exactly one idempotent.*
 - *X contains no cycles.*

Note that whenever a vertex on a cycle in X topples during an avalanche, it will leave at least one grain of sand on a vertex somewhere in that cycle, establishing the forward direction of part 1 and also part 2 of Theorem 2.9. For proofs of the remaining parts, see [2].

Notation. For $a \in M(\widehat{X})$, let e_a denote the (unique) idempotent of $M(\widehat{X})$ obtained by adding a to itself and stabilizing repeatedly.

Let us quickly establish that e_a is well defined.

Lemma 2.10. *Let (S, \cdot) be a finite semigroup, and let $a \in S$. Then there exists $k \in \mathbb{N}$ for which a^k is idempotent. Furthermore, there is only one idempotent in the list a, a^2, a^3, \dots , and it appears infinitely many times in this list.*

Proof. Consider $a, a^2, a^3, \dots, a^{|S|+1}$. These cannot all be distinct, so we have $a^m = a^n$ for some $1 \leq m < n \leq |S| + 1$. Thus we can write

$$a^m = a^m \cdot a^{n-m}. \tag{2}$$

Now consider $z = (a^m)^{n-m}$. Applying (2) m times to z expands z into $z \cdot z$, meaning that z is idempotent:

$$z = (a^m)^{n-m} = (a^m)^{n-m} \cdot (a^{n-m})^m = z \cdot z.$$

To finish the proof, suppose a^k and a^j are both idempotent for some $k, j \in \mathbb{N}$. Then $a^k = (a^k)^j = (a^j)^k = a^j$, and since a^k is idempotent, we have $a^k = a^{2k} = a^{3k} = \dots$. \square

Since outward edges from the sink in \widehat{X} do not affect the structure of $M(\widehat{X})$, we assume for the rest of the section that there are no outward edges from the sink in \widehat{X} , as it will make some of the following definitions and notation easier to state.

Definition 2.11. The closure of a subset $W \subseteq V$, denoted $\text{cl}(W)$, is given by

$$\text{cl}(W) = \{z \in V : \exists w \in W, w \rightarrow z\}.$$

In particular, no closure contains the sink (because V consists of the non-sink vertices of \widehat{X}), and for all $W \subseteq V$ we have $W \subseteq \text{cl}(W)$.

Notation. Given a set of non-sink vertices $S \subseteq V$, let $\iota(S)$ denote the subgraph of \widehat{X} induced by $\text{cl}(S) \cup \{\text{sink}\}$. That is, $\iota(S)$ is the graph whose vertex set is $\text{cl}(S) \cup \{\text{sink}\}$ and whose edge set consists of all edges in \widehat{X} which are directed outward from any vertex in $\iota(S)$ (including edges pointing to the sink).

Let $S \subseteq V$ be nonempty. Then $\iota(S)$ is a subgraph of \widehat{X} which is a legal graph for having a sandpile monoid, in the sense that $\iota(S)$ has at least one non-sink vertex and the sink of $\iota(S)$ is accessible from each of its non-sink vertices. Since toppling a state on \widehat{X} whose support lies in $\iota(S)$ is the same as toppling the same state on $\iota(S)$, we can view the sandpile monoid and sandpile group of $\iota(S)$ as a submonoid and subgroup, respectively, of $M(\widehat{X})$ by the natural inclusions.

Theorem 2.12. (See [2,21].) If $m \in M(\widehat{X})$ with $m \neq 0$, then the sequence

$$m, m \oplus m, m \oplus m \oplus m, \dots$$

eventually reaches a recurrent state in the sandpile monoid on $\iota(\text{supp}(m))$. Furthermore, if $e \in M(\widehat{X})$ is a nonzero idempotent, then e is recurrent in the sandpile monoid on $\iota(\text{supp}(e))$ —that is, e is the identity of the sandpile group on $\iota(\text{supp}(e))$.

Note that Lemma 2.10 and Theorem 2.12 together imply that if $m \in M(\widehat{X})$ with $m \neq 0$, then the sequence

$$m, m \oplus m, m \oplus m \oplus m, \dots$$

eventually reaches the identity of the sandpile group of $\iota(\text{supp}(m))$.

Definition. (See [21, Section 2.2].) Let $W \subseteq V$. W is said to be strongly connected if, for all $w, v \in W$, $w \rightarrow v$. W is a strongly connected component of \widehat{X} if W is strongly connected and is not contained in any other strongly connected subset of V . A strongly connected component of \widehat{X} is cyclic if it contains a cycle and acyclic otherwise. A vertex $v \in V$ is acyclic if it does not belong to a cycle.

Therefore, an acyclic strongly connected component consists of a single, loopless, non-sink vertex, and we consider the sink not to be an element of any strongly connected component. Note that a vertex $v \in V$ is acyclic if and only if $\{v\}$ is a strongly connected component of \widehat{X} and v is loopless. We will call the cyclic strongly connected components of \widehat{X} the *cyclic strong components* of \widehat{X} for short. Note that a cyclic strong component of \widehat{X} might contain just one vertex. Note that the adjectives *normal* and *abnormal* were used instead of *cyclic* and *acyclic* in [21].

The cyclic strong components of \widehat{X} form a partially ordered set, where $C_1 \leq C_2$ if and only if there is a path in X from some vertex (and hence every vertex) in C_1 to some vertex (and hence every vertex) in C_2 . In this case we say C_2 is accessible from C_1 , and we write $C_1 \rightarrow C_2$, so $C_1 \leq C_2 \iff C_1 \rightarrow C_2$.

Definition. Let P be a partially ordered set and let $F \subseteq P$. F is called a *filter* of P (or on P) if, for all $x, y \in P$, $x \in F$ and $y \geq x$ implies $y \in F$.

Remark. Our use of the word *filter* differs slightly from the normal order theoretic usage. In particular, we do not require our filters to have the property that for all $x, y \in F$, there exists $z \in F$ such that $z \leq x$ and $z \leq y$.

The following theorem, which is part of the main result of [2], tells us how to read the idempotent structure of $M(\widehat{X})$ from the cycle structure of \widehat{X} .

Theorem 2.13. *The idempotents of $M(\widehat{X})$ are in bijective correspondence with the filters on the set of cyclic strong components of \widehat{X} . Specifically, let E denote the set of idempotents of $M(\widehat{X})$ and let Q denote the set of filters on the cyclic strong components of \widehat{X} . Let $e \in E$. Define*

$$F : E \rightarrow Q$$

by $F(0) = \emptyset$ and, if $e \neq 0$, then $F(e)$ is the set of cyclic strong components of \widehat{X} contained in $\iota(\text{supp}(e))$. In other words, if C is a cyclic strong component of \widehat{X} , then $C \in F(e) \iff C \subseteq \iota(\text{supp}(e))$. Then F is a bijection.

Remark 2.14. Thus, given a nonempty filter $q = \{C_1, C_2, \dots, C_n\}$ on the set of cyclic strong components of \widehat{X} , there is exactly one nonzero idempotent e of $M(\widehat{X})$ whose support intersects each of the C_1, C_2, \dots, C_n and none of the other cyclic strong components of \widehat{X} . It is straightforward to show that if an acyclic vertex v is not accessible from at least one of the $C_i \in q$, then $v \notin \text{supp}(e)$ [2]. (However, we note that in general, acyclic vertices which are accessible from the C_i may be contained in $\text{supp}(e)$ —see, e.g., the idempotents e_2, e_3, e_4 , and e_5 in Example 2.15.) Thus by Theorem 2.12, we see that e can be found by placing a grain of sand on any vertex of each C_i and adding the resulting state to itself and stabilizing over and over until an idempotent is reached. However, by Theorem 2.12, e is the identity of the sandpile group of the graph $Y = \iota(\text{supp}(e)) = \text{cl}(\bigcup_{C \in q} C) \cup \{\text{sink}\}$, so a better way to find e is to use part 2 of Theorem 2.5. Specifically, if we let MAX_Y denote the totally saturated configuration on Y , then we have

$$e = [\text{MAX}_Y - (\text{MAX}_Y \oplus \text{MAX}_Y)] \oplus \text{MAX}_Y.$$

Example 2.15. Let \widehat{X} be the graph in Fig. 1.

The acyclic vertices are a_1, a_2, a_3, d_1, d_2 , and d_4 . The sets $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $D = \{d_3\}$ are the cyclic strong components. $P = \{B, C, D\}$ is the poset of cyclic strongly connected components, where $C < D$ and B is incomparable to C and D . The Hasse diagram of P is given in Fig. 2. Note that, since $\text{deg}^+(d_2) = 1$, d_2 holds no grains of sand in any stable configuration on \widehat{X} .

The filters on P are $F_0 = \{\}$, $F_1 = \{B\}$, $F_2 = \{D\}$, $F_3 = \{B, D\}$, $F_4 = \{C, D\}$, $F_5 = \{B, C, D\}$. Each filter corresponds to an idempotent in the following way:

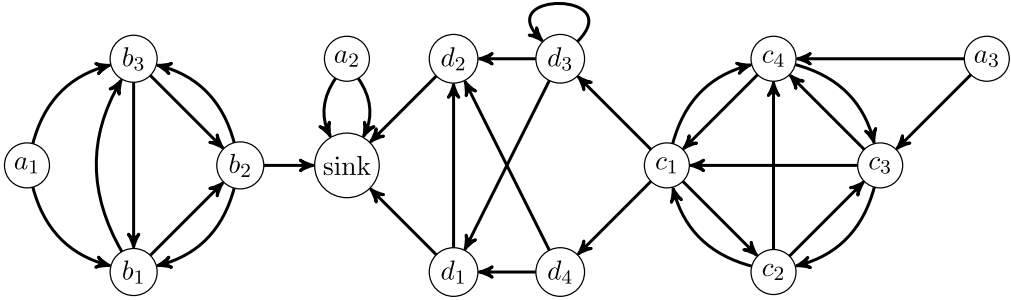


Fig. 1. \widehat{X} for Example 2.15.

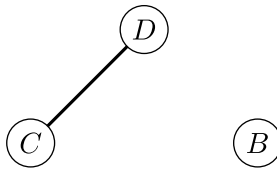


Fig. 2. Hasse diagram of the strongly connected components of \widehat{X} .

- F_0 corresponds to 0.
- F_1 corresponds to the idempotent e_1 where vertices b_1 and b_3 each have 1 grain, b_2 has 2 grains, and all other vertices have zero grains.
- F_2 corresponds to the idempotent e_2 where d_1 has 1 grain, d_3 has 2 grains, and all other vertices have zero grains.
- F_3 corresponds to the idempotent $e_3 = e_1 \oplus e_2$.
- F_4 corresponds to the idempotent e_4 where $c_1, c_3,$ and d_1 each have 1 grain, c_2 and d_3 each have 2 grains, and all other vertices have zero grains.
- F_5 corresponds to the idempotent $e_5 = e_1 \oplus e_4$, which is the identity of $G(\widehat{X})$.

3. The maximal subgroups of $M(\widehat{X})$

In this section we state and prove the first of our main results, which is a combinatorial classification of all the maximal subgroups of $M(\widehat{X})$. We prove in Theorems 3.5 and 3.7 that they are all sandpile groups on certain subgraphs of \widehat{X} which are easy to identify. In this section we assume the sink of \widehat{X} has no outward edges.

Definition. A *subgroup* of a semigroup is a nonempty subset of the semigroup which is a group under the semigroup operation. A subgroup of a semigroup is *maximal* if it is not properly contained in any other subgroup of the semigroup.

The following is a well-known result in semigroup theory [10, Section 1.7].

Theorem 3.1. Let (S, \cdot) be a semigroup. Each idempotent $e \in S$ is the identity for a unique maximal subgroup G_e of S , and G_e is the group of units of the monoid $\{e\} \cdot S \cdot \{e\}$, which has e as its identity. Every subgroup of S which contains e is contained in G_e .

In the language of sandpile monoids, we have:

Corollary 3.2. Each idempotent $e \in M(\widehat{X})$ is the identity for a unique maximal subgroup G_e of $M(\widehat{X})$, and G_e is the group of units of the monoid $\{e \oplus m : m \in M(\widehat{X})\}$, which has e as its identity. Every subgroup of $M(\widehat{X})$ which contains e is contained in G_e .

Given an idempotent $e \in M(\widehat{X})$, let us denote by G_e the corresponding maximal subgroup of $M(\widehat{X})$. $G(\widehat{X})$ is a subgroup of $M(\widehat{X})$. In fact it is a maximal subgroup:

Theorem 3.3. $G(\widehat{X})$ is a maximal subgroup of $M(\widehat{X})$.

Proof. Let e be the identity of $G(\widehat{X})$. $G(\widehat{X})$ is a subgroup of $M(\widehat{X})$ containing e , so $G(\widehat{X}) \subseteq G_e$. To explain why $G_e \subseteq G(\widehat{X})$, let $g \in G_e$. Then, since $e \in G(\widehat{X})$ and $G(\widehat{X})$ is an ideal, $g \oplus e = g$ is in $G(\widehat{X})$. \square

We already know from the results of Section 2.3 that, given a nonzero idempotent $e \in M(\widehat{X})$, e is the identity for the sandpile group on $\iota(\text{supp}(e))$, and this sandpile group is a subgroup of $M(\widehat{X})$ under the natural inclusion. However, this group is not necessarily the maximal subgroup in $M(\widehat{X})$ containing e . To give a combinatorial description of G_e we will use the following lemma.

Lemma 3.4. Let e be an idempotent of $M(\widehat{X})$ and let $a \in M(\widehat{X})$. Then $a \in G_e$ if and only if a and e are mutually accessible; that is, $a \in G_e$ if and only if $a = e \oplus m$ and $e = a \oplus k$ for some $k, m \in M(\widehat{X})$.

Proof. We have from Corollary 3.2 that G_e is the group of units of $\{e \oplus m : m \in M(\widehat{X})\}$, i.e.,

$$G_e = \{a \in M(\widehat{X}) : a = e \oplus m \text{ and } a \oplus (e \oplus k) = e \text{ for some } m, k \in M(\widehat{X})\}.$$

By definition, if $a \in G_e$ then a and e are mutually accessible. To show the reverse, suppose a and e are mutually accessible. Then $a = e \oplus m$ for some $m \in M(\widehat{X})$ and $e = a \oplus k$ for some $k \in M(\widehat{X})$. Adding e to each side of the last equation and using that e is idempotent, we obtain $a \oplus (e \oplus k) = e$. \square

We now describe the subgraph of \widehat{X} whose sandpile group will turn out to be G_e .

Notation. Let $e \in M(\widehat{X})$ be idempotent. Let $A(e)$ denote the set of acyclic vertices $v \in V$ for which $v \notin \text{cl}(\text{supp}(e))$ and for which the only cyclic strong components of \widehat{X} accessible from v are those which intersect $\text{supp}(e)$ (i.e., those which are contained in $\iota(\text{supp}(e))$). Let $I(e)$ denote the subgraph of \widehat{X} induced by

$$\text{cl}(\text{supp}(e)) \cup A(e) \cup \{\text{sink}\}.$$

Notice that, provided $I(e) \neq \{\text{sink}\}$ (in particular, when $e \neq 0$), $I(e)$ is a subgraph of \widehat{X} which is a legal graph for having a sandpile monoid, in the sense that $I(e)$ has at least one non-sink vertex and the sink of $I(e)$ is accessible from each of its non-sink vertices. On the other hand, if $I(e) = \{\text{sink}\}$, which is only possible if $e = 0$, then the only thing preventing $I(e)$ from being a legal graph for having a sandpile monoid is that it has no non-sink vertices. We therefore deal with G_0 first.

Theorem 3.5. If $I(0) = \{\text{sink}\}$, then $G_0 = \{0\}$. Otherwise, G_0 is equal to the sandpile group on $I(0)$, viewed as a subgroup of $M(\widehat{X})$ by the natural inclusion.

Proof. If $I(0) = \{\text{sink}\}$, then there is a path to a cycle of X from every vertex $v \in V$. By part 1 of Theorem 2.9, then, no nonzero state of $M(\widehat{X})$ can access 0, so $G_0 = \{0\}$.

On the other hand, if $I(0) \neq \{\text{sink}\}$, then $I(0) = A(0) \cup \{\text{sink}\}$. Let $a \in M(\widehat{X})$ be any state whose support is contained in $I(0)$. Certainly a is accessible from 0, and 0 is accessible from a by part 1 of Theorem 2.9. If $a \in M(\widehat{X})$ is any state with a grain outside of $I(0)$, then a cannot access 0 by part 1 of Theorem 2.9. Thus $G_0 = \{a \in M(\widehat{X}) : \text{supp}(a) \subseteq I(0)\}$, the sandpile monoid of $I(0)$. This is equal to the sandpile group on $I(0)$ by part 4 of Theorem 2.9. \square

We now deal with the nonzero idempotents. The following theorem allows us to build the G_e from the sandpile groups of the $\iota(\text{supp}(e))$.

Theorem 3.6. *Let $e \in M(\widehat{X})$ be idempotent with $e \neq 0$. Then*

$$G_e = \{g \oplus j : g, j \in M(\widehat{X}) \text{ where } g \text{ is recurrent on } \iota(\text{supp}(e)) \text{ and } \text{supp}(j) \subseteq A(e)\}.$$

Proof. We handle the reverse containment first. Consider $a = g \oplus j$, where g is recurrent on $\iota(\text{supp}(e))$ and $\text{supp}(j) \subseteq A(e)$. Then $a = g \oplus j = (e \oplus g) \oplus j$, so a is accessible from e . By part 3 of Theorem 2.9, g has a grain of sand on every cycle of $\iota(\text{supp}(e))$. The support of g and the support of j do not intersect, so $g \oplus j = g + j$. Thus the cyclic strong components of \widehat{X} accessible from the vertices of $\text{supp}(a)$ are precisely those contained in $\iota(\text{supp}(e))$. By Theorem 2.12 and Remark 2.14, then, $e_a = e$, i.e., e is accessible from a .

Now, suppose $a \in G_e$, i.e., a and e are mutually accessible. Thus $a = e \oplus m$ and $e = a \oplus k$ for some $m, k \in M(\widehat{X})$. Write $a = g \oplus j$ for the unique states $g, j \in M(\widehat{X})$ such that $\text{supp}(g) \subseteq \iota(\text{supp}(e))$ and $\text{supp}(j) \cap \iota(\text{supp}(e)) = \emptyset$. We claim that g is recurrent on $\iota(\text{supp}(e))$ and $\text{supp}(j) \subseteq A(e)$.

To see why, first, $a = e \oplus m = g \oplus j$, so $a \oplus e = e \oplus e \oplus m = g \oplus j \oplus e$, but $e \oplus e \oplus m = e \oplus m = a$, so $g \oplus j \oplus e = g \oplus j$. Since $\text{supp}(j) \cap \iota(\text{supp}(e)) = \emptyset$, $g \oplus e = g$, i.e., g is recurrent on $\iota(\text{supp}(e))$. Finally, if $\text{supp}(j) \not\subseteq A(e)$, then j has a grain of sand on a vertex from which some cyclic strong component $C \not\subseteq \iota(\text{supp}(e))$ of \widehat{X} is accessible via a path p which does not intersect $\iota(\text{supp}(e))$. Note that C does not intersect $\text{supp}(e)$. We have $a = g \oplus j$ and e is accessible from a , so $e = g \oplus j \oplus k$ for some $k \in M(\widehat{X})$. But then $\text{supp}(e) \neq \text{supp}(g \oplus j \oplus k)$ because $g \oplus j \oplus k$ must have a grain somewhere along p or on C . Hence $\text{supp}(j) \subseteq A(e)$. \square

Theorem 3.7. *Let $e \in M(\widehat{X})$ be idempotent with $e \neq 0$. Then G_e is equal to the sandpile group on $I(e)$, viewed as a subgroup of $M(\widehat{X})$ by the natural inclusion.*

Proof. We must show that G_e , as described in Theorem 3.6, is the set of recurrent elements of the sandpile monoid on $I(e)$. Let $a \in G_e$. Then $a = g \oplus j$ where g is recurrent on $\iota(\text{supp}(e))$ and $\text{supp}(j) \subseteq A(e)$. By definition, a is in the sandpile monoid of $I(e)$. Since e is the identity of the sandpile group on $\iota(\text{supp}(e))$, $a \oplus e = (g \oplus e) \oplus j = g \oplus j = a$, and since e is the identity of the sandpile group on $I(e)$, by part 3 of Theorem 2.5, a is recurrent in the sandpile monoid on $I(e)$. For the reverse inclusion, the sandpile group of $I(e)$ is a subgroup of $M(\widehat{X})$ which contains e , so it is contained in G_e by the maximality of G_e . \square

Example 3.8 (Example 2.15 continued). Consider the graph \widehat{X} in Fig. 1. Corresponding to the idempotents e_0 through e_5 of Example 2.15 we have the following maximal subgroups:

- $G_{e_0} = G_0$ is the sandpile group on $\{a_2, d_1, d_2, d_4, \text{sink}\}$.
- G_{e_1} is the sandpile group on $B \cup \{a_1, a_2, d_1, d_2, d_4, \text{sink}\}$.
- G_{e_2} is the sandpile group on $D \cup \{a_2, d_1, d_2, d_4, \text{sink}\}$. Note $A(e_2) = \{a_2, d_4\}$.
- G_{e_3} is the sandpile group on $B \cup D \cup \{a_1, a_2, d_1, d_2, d_4, \text{sink}\}$.
- G_{e_4} is the sandpile group on $C \cup D \cup \{a_2, a_3, d_1, d_2, d_4, \text{sink}\}$.
- G_{e_5} is the sandpile group of \widehat{X} . Note $A(e_5) = \{a_1, a_2, a_3\}$.

4. Classical and two-idempotent sandpile monoids

In this section we point out how two results for classical sandpile monoids generalize to directed sandpile monoids with two idempotents. If \widehat{X} is the graph for a classical sandpile monoid (i.e., \widehat{X} is undirected and X is connected), the results from Section 2.3 yield the well-known result that $M(\widehat{X})$ has one or two (usually two) idempotents. Let us quickly make this precise.

Theorem 4.1. *Let \widehat{X} be undirected and X connected. If X consists of a single loopless vertex, then $M(\widehat{X})$ has exactly one idempotent (namely, 0). Otherwise, $M(\widehat{X})$ has exactly two idempotents (0 and the identity of $G(\widehat{X})$).*

Proof. If X consists of a single loopless vertex, then X has no cycles so $M(\widehat{X})$ has only one idempotent by part 4 of Theorem 2.9. On the other hand, if X consists of anything other than a single loopless vertex, then X has a unique cyclic strong component, so by Theorem 2.13 $M(\widehat{X})$ has exactly two idempotents, 0 and the identity of $G(\widehat{X})$. \square

The case when X consists of a single loopless vertex is relatively uninteresting—in this case, $M(\widehat{X}) = G(\widehat{X})$ is just the cyclic group whose order is the outdegree of that vertex in \widehat{X} . So all classical sandpile monoids of interest have exactly two idempotents. Sometimes the correct setting for generalizations of theorems about classical sandpile monoids to those based on directed graphs is the setting when $M(\widehat{X})$ has exactly two idempotents. For example, the following theorem concerns an alternate definition of recurrence sometimes used for classical sandpile monoids.

Theorem 4.2. *Let $M(\widehat{X})$ contain exactly two idempotents and let $u \in M(\widehat{X})$. Then there exists $a \in M(\widehat{X})$ with $a \neq 0$ such that $u \oplus a = u$ if and only if u is recurrent.*

Remark. This theorem fails if we remove the hypothesis that $M(\widehat{X})$ has exactly two idempotents. If $M(\widehat{X})$ has one idempotent, $M(\widehat{X}) = G(\widehat{X})$, so 0 is the identity of $G(\widehat{X})$ and $u \oplus a = u$ for some $a, u \in M(\widehat{X})$ implies $a = 0$. If $M(\widehat{X})$ has more than 2 idempotents then there is an idempotent $a \in M(\widehat{X})$ separate from 0 and the identity of $G(\widehat{X})$, a being idempotent means $a \oplus a = a$, and a is not recurrent.

To prove Theorem 4.2 we will use the following theorem, whose proof we defer until Section 5.

Theorem (Theorem 5.2). *Let $u, a \in M(\widehat{X})$ such that $u \oplus a = u$. If $a \neq 0$, then a cannot access 0.*

Proof of Theorem 4.2. If $M(\widehat{X})$ contains exactly two idempotents then they are the identity of $G(\widehat{X})$, which we denote by e , and 0.

If $u \oplus a = u$ for some $a \in M(\widehat{X})$ with $a \neq 0$, then by Theorem 5.2, a cannot access 0. We also have

$$u = u \oplus a = u \oplus a \oplus a = \dots = u \oplus e_a.$$

Since a cannot access 0 and the only other idempotent in $M(\widehat{X})$ is e , we have $e_a = e$. Thus $u \oplus e = u$, so u is recurrent by part 3 of Theorem 2.5.

On the other hand, if u is recurrent then $u \oplus e = u$. \square

Second, for classical sandpile monoids it is well known that the process of adding a nonzero element to itself and stabilizing repeatedly eventually produces a recurrent configuration. For any directed graph \widehat{X} , if $M(\widehat{X})$ has only one idempotent, this is immediate (as all elements are recurrent), and if $M(\widehat{X})$ has two idempotents, we have the following generalization.

Theorem 4.3. *Let $M(\widehat{X})$ have exactly two idempotents and let $a \in M(\widehat{X})$. If a has a grain of sand on a vertex from which a cycle in X is accessible, then adding a to itself and stabilizing repeatedly will eventually reach a recurrent configuration. Conversely, if a has no sand on any vertex from which a cycle in X is accessible, then adding a to itself and stabilizing repeatedly will never reach a recurrent configuration.*

Proof. If a has a grain of sand on a vertex from which a cycle in X is accessible, by part 1 of Theorem 2.9 a cannot access 0. Since $M(\widehat{X})$ has only two idempotents (0 and the identity of $G(\widehat{X})$), e_a must be the identity of $G(\widehat{X})$.

For the converse, since $M(\widehat{X})$ has two idempotents X must contain at least one cycle. If a has no sand on any vertex from which a cycle in X is accessible, then the process of adding a to itself and stabilizing repeatedly will never transfer any sand to a cycle. The converse thus follows from part 3 of Theorem 2.9. \square

In fact, this theorem is easy to generalize beyond two-idempotent sandpile monoids. Recall from Section 2.3 that the number of idempotents of $M(\widehat{X})$ is equal to the number of filters on the set of cyclic strong components of \widehat{X} .

Theorem 4.4. *Let $a \in M(\widehat{X})$. If $\iota(\text{supp}(a))$ contains every cycle of X then adding a to itself and stabilizing repeatedly will eventually reach a recurrent configuration. Conversely, if there is a cycle of X not contained in $\iota(\text{supp}(a))$ then adding a to itself and stabilizing repeatedly will never reach a recurrent configuration.*

Proof. Let e denote the identity of $G(\widehat{X})$. By part 3 of Theorem 2.9, e has a grain of sand on every cycle of X , so $\text{supp}(e)$ intersects every cyclic strong component of \widehat{X} . By Theorem 2.13, e is the only idempotent of $M(\widehat{X})$ whose support intersects every cyclic strong component of \widehat{X} . If $\iota(\text{supp}(a))$ contains every cycle of X , then e_a also has this property, so $e_a = e$. On the other hand, if there is a cycle of X not contained in $\iota(\text{supp}(a))$ then adding a to itself and stabilizing repeatedly will never transfer sand to this cycle. The converse follows from part 3 of Theorem 2.9. \square

5. Monoids that are not sandpile monoids

In this section we give a new algebraic structural constraint for sandpile monoids and we exhibit two infinite families of finite commutative monoids which cannot be realized as sandpile monoids on any graph.

Every finite commutative group is isomorphic to the sandpile group of some graph. To see this, let \widehat{X}_k denote the graph which consists of two vertices (one ordinary vertex and the sink) and k edges from the vertex to the sink. Then $G(\widehat{X}_k) = M(\widehat{X}_k) \cong \mathbb{Z}_k$. If we create the graph \widehat{X} by identifying the sinks of $\widehat{X}_{k_1}, \widehat{X}_{k_2}, \dots, \widehat{X}_{k_n}$, then

$$M(\widehat{X}) = G(\widehat{X}) \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \dots \times \mathbb{Z}_{k_n}.$$

Thus, we also see that every finite commutative group is isomorphic to the sandpile monoid of some graph.

Remark. Another way to prove that every finite commutative group is isomorphic to the sandpile group of some graph is by identifying undirected cycles at their sinks, as the sandpile group of the undirected cycle C_n is \mathbb{Z}_n [11].

A natural question, then, is whether every finite commutative monoid is the sandpile monoid of some graph. The answer is no, and in this section we give two infinite families of monoids that cannot be realized as sandpile monoids on any graph. Here is our first family.

Theorem 5.1. *Let p be a prime number with $p > 2$, and let M_p denote a linearly ordered set of p elements under the meet operation. Then M_p is an idempotent commutative monoid which is not a sandpile monoid.*

Proof. It is immediate that M_p is an idempotent commutative monoid, and $|M_p| = p$ by definition.

Suppose M_p is isomorphic to the sandpile monoid of some graph \widehat{X} . Then $|M(\widehat{X})| = p$ and thus, by (1), X has exactly one vertex v with $\text{deg}^+(v) = p$ and the rest of the vertices in X have out-degree 1. Furthermore, $M(\widehat{X})$ is idempotent.

Let 1_v denote the configuration on \widehat{X} consisting of one grain of sand on v and none elsewhere. Then, since $M(\widehat{X})$ is idempotent, $1_v \oplus 1_v = 1_v$, meaning that $1_v + 1_v$ is an unstable configuration, which implies that $\text{deg}^+(v) \leq 2$, a contradiction. \square

Next we prove Theorem 5.2, which provides an algebraic constraint that every sandpile monoid must satisfy.

Theorem 5.2. *Let $u, a \in M(\widehat{X})$ such that $u \oplus a = u$. If $a \neq 0$, then a cannot access 0.*

Proof. Suppose that $u, a \in M(\widehat{X})$, $u \oplus a = u$, and $a \neq 0$ (i.e., $\text{supp}(a)$ is nonempty). Suppose further that a can access 0. Then for all $w \in \text{supp}(a)$, by part 1 of Theorem 2.9, there is no path from w to any cycle in X .

Let w denote any member of $\text{supp}(a)$ for which there is no path of length 1 or greater in X from any member of $\text{supp}(a)$ to w . To see that such a w exists, suppose not. Then we have $w_1 \in \text{supp}(a)$, to which there is a path from $w_2 \in \text{supp}(a)$, to which there is a path from $w_3 \in \text{supp}(a)$, etc. Since $\text{supp}(a)$ is finite, we eventually get a cycle in X that contains a member of $\text{supp}(a)$, a contradiction.

Let a_w, u_w , and $(u \oplus a)_w$ indicate the number of grains of sand the configurations a, u , and $u \oplus a$, respectively, have on the vertex w . We have $u_w = (u \oplus a)_w$ and since $w \in \text{supp}(a)$ and both a and u are stable we also have

$$\begin{aligned} 1 &\leq a_w < \text{deg}^+(w), \\ 0 &\leq u_w < \text{deg}^+(w). \end{aligned} \tag{3}$$

Now, note that the only vertices that may have grains of sand transferred to them by topplings in the process of stabilizing $u + a$ are vertices to which there is a path in X from some member of $\text{supp}(a)$. Thus, during the stabilization of $u + a$, w will not have any grains of sand transferred to it through toppling. Therefore, no grains may be transferred away from w in the stabilization of $u + a$ either, for if some were then we would have $(u \oplus a)_w < u_w$ by (3) (in particular, $a_w, u_w < \text{deg}^+(w)$), which would imply $u \oplus a \neq u$.

Now, since no grains of sand are transferred to or from w in the stabilization of $u + a$, we have $(u \oplus a)_w = u_w + a_w$. Since $(u \oplus a)_w = u_w$, we have $a_w = 0$, a contradiction. \square

Remark 5.3. Theorem 5.2, said another way, says that if $(M, +)$ is a finite commutative monoid with identity 0 and elements u, a, k satisfying $u + a = u$, $a \neq 0$, and $a + k = 0$, then M cannot be realized as a sandpile monoid on any graph. This theorem gives a straightforward way to show that a monoid cannot be a sandpile monoid, although there are monoids—for example, those in Theorem 5.1—which cannot be sandpile monoids that are not detected by this test.

Now, if $M(\widehat{X})$ is a classical sandpile monoid, then $M(\widehat{X})$ contains exactly one or two idempotents, so another question is whether every finite commutative monoid containing exactly one or two idempotents is isomorphic to the sandpile monoid of some graph. First, if M is a finite commutative monoid with exactly one idempotent, then M is actually a group, so the answer in this case is yes. However, as the next theorem shows, there are infinitely many finite commutative monoids with exactly two idempotents that are not the sandpile monoids of any (directed or undirected) graph.

Theorem 5.4. Let $n > 2$ and let $G = \{0, 1, \dots, n - 2\}$ be the cyclic group of order $n - 1$ (with operation denoted by $+$). Let $(M, +)$ be $G \cup \{\infty\}$ under the same operation as G , with

$$\infty + \infty = \infty \quad \text{and} \quad g + \infty = \infty + g = \infty \quad \text{for all } g \in G.$$

Then M is a commutative monoid of order n with exactly two idempotents, and M is not a sandpile monoid.

Proof. The idempotents of M are 0 (the identity) and ∞ . Let $u = \infty$, $a = 1$, $k = n - 2$. Then $u + a = u$, $a \neq 0$, and $a + k = 0$, so by Remark 5.3, M is not the sandpile monoid of any graph. \square

Remark. In [2] it is shown that the lattice of idempotents of a sandpile monoid is distributive. Therefore, any finite commutative monoid whose lattice of idempotents is not distributive is not a sandpile monoid. In particular, any finite non-distributive lattice is a finite (idempotent) commutative monoid that is not a sandpile monoid. The two families of monoids in this section are both families of monoids whose lattices of idempotents are distributive, and yet the monoids themselves are still not sandpile monoids.

6. Sink-distance-regular directed graphs

In [5, Section 5] Biggs gives an explicit combinatorial description of the sandpile group identity for any (undirected) distance-regular graph. In this section we study a family of directed graphs that generalizes the family of distance-regular graphs and we give an explicit combinatorial description (Theorem 6.7) of the sandpile group identity for every member of this family. For distance-regular graphs, our description recovers that of Biggs. Our family also includes many other families of interest, including iterated wheels (Example 6.9), regular trees (Example 6.10), and regular tournaments (Example 6.11).

First we recall what it means for a graph to be distance-regular.

Definition 6.1. Let Y be a finite undirected graph. For vertices x, y of Y , the *distance* $\delta(x, y)$ from x to y is the length of the shortest path from x to y . If Y is connected, the *diameter* of Y , denoted $\text{diam}(Y)$, is given by $\text{diam}(Y) = \max\{\delta(x, y) : x, y \in Y\}$.

Let Y be a finite, connected, undirected graph with no loops and with at most one edge between each pair of vertices. Given two vertices x, y of Y of distance $\delta(x, y) = i$, define $c_i(x, y)$, $a_i(x, y)$, and $b_i(x, y)$ to be the number of vertices z adjacent to y such that $\delta(x, z)$ is $i - 1$, i , or $i + 1$, respectively. (The integer $c_i(x, y)$ counts the vertices z that are adjacent to y but “closer” to x , and $b_i(x, y)$ counts those z adjacent to y that are “beyond” y .) Note that, by definition, for a fixed pair of vertices x, y with $\delta(x, y) = i$, the sum $c_i(x, y) + a_i(x, y) + b_i(x, y)$ is the degree of y , and in general the degree of a vertex x is $b_0(x, x)$.

Definition 6.2. Let Y be a finite, connected, undirected graph with no loops and with at most one edge between each pair of vertices. Y is said to be *distance-regular* if the integers $c_i(x, y)$, $a_i(x, y)$, and $b_i(x, y)$ are constants, independent of the choice of vertices x, y of distance i . That is, a graph is distance-regular if, for each integer i , for all vertices x_1, y_1, x_2, y_2 such that $\delta(x_1, y_1) = \delta(x_2, y_2) = i$, we have $a_i(x_1, y_1) = a_i(x_2, y_2)$, $b_i(x_1, y_1) = b_i(x_2, y_2)$, and $c_i(x_1, y_1) = c_i(x_2, y_2)$.

Thus, if a graph is distance-regular we can simply refer to the numbers a_i, b_i , and c_i . A distance-regular graph is certainly regular, as the degree of a vertex x is $b_0(x, x)$, and $b_0(x, x) = b_0(y, y)$ for all vertices x, y . Basic material on distance-regular graphs may be found in Chapters 20 and 21 of [4], the text [9], Sections 1.6 and 4.5 of [15], and Chapter 21 of [19]. It is often fruitful in the study of a distance-regular graph Y to begin by fixing a vertex s and partitioning the vertices of Y based on their distance from s , defining $\Gamma_i = \{y \in Y : \delta(y, s) = i\}$, for $0 \leq i \leq \text{diam}(Y)$. In such a partition, vertices of Γ_i can only be adjacent to vertices in Γ_{i-1}, Γ_i , or Γ_{i+1} .

We now consider the following generalization to directed graphs. We remind the reader that the digraph \widehat{X} may have loops and multiple edges.

Definition 6.3. For vertices $x, y \in \widehat{X}$, the *distance* $\delta(x, y)$ from x to y is the length of the shortest directed path (of length 0 or greater) from x to y in \widehat{X} , if such a path exists, and is ∞ otherwise. If i is a nonnegative integer, the *i th neighborhood* of y , denoted $\Gamma_i(y)$, is

$$\Gamma_i(y) = \{x \in \widehat{X} : \delta(x, y) = i\}.$$

Denote the sink of \widehat{X} by s , and denote $\Gamma_i(s)$ by Γ_i . Since there is a path from every vertex to the sink and \widehat{X} is finite, $d = \max_{v \in \widehat{X}} \{\delta(v, s)\}$ is an integer, and $\{\Gamma_0, \Gamma_1, \dots, \Gamma_d\}$ forms a partition of the vertices of \widehat{X} , with $\Gamma_0 = \{s\}$. If (v, w) is a directed edge of \widehat{X} from v to w , we call v and w the *tail* and the *head*, respectively, of the edge (v, w) .

Definition 6.4. We call \widehat{X} *sink-distance-regular* if the sink of \widehat{X} has outdegree 0 and the neighborhoods $\{\Gamma_0, \Gamma_1, \dots, \Gamma_d\}$ of the sink satisfy the following properties:

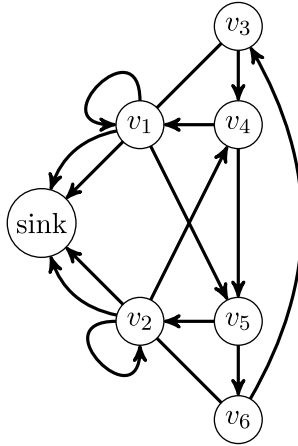


Fig. 3. A small sink-distance-regular graph \widehat{X} .

- (1) if $v \in \Gamma_i$ and (v, w) is a directed edge of \widehat{X} , then $w \in \Gamma_{i-1}$, $w \in \Gamma_i$, or $w \in \Gamma_{i+1}$ (where Γ_{d+1} is interpreted as $\{\}$), and
- (2) there exist collections of nonnegative integers $\{a_i\}_{i=1}^d$ and positive integers $\{b_i\}_{i=1}^{d-1}$ and $\{c_i\}_{i=1}^d$, such that
 - (a) if $v \in \Gamma_1$, then v is the tail of precisely c_1 edges whose heads are the sink,
 - (b) if $v \in \Gamma_i$ where $i > 1$, then v is the tail of precisely c_i edges whose heads are in Γ_{i-1} and v is the head of precisely c_i edges whose tails are in Γ_{i-1} , and
 - (c) if $v \in \Gamma_i$ where $i \geq 1$, then v is the tail of precisely a_i edges whose heads are in Γ_i , v is the head of precisely a_i edges whose tails are in Γ_i , v is the tail of precisely b_i edges whose heads are in Γ_{i+1} , and v is the head of precisely b_i edges whose tails are in Γ_{i+1} (where b_d is interpreted as 0).

Remark 6.5. When \widehat{X} is sink-distance-regular, the neighborhoods of the sink form a directed version of the concept of “equitable partition” appearing in pp. 195–198 of [15]. If Y is a distance-regular graph, then after choosing a vertex of Y to be the sink and removing the outward edges from the sink we obtain a graph that is sink-distance-regular. However, the converse is not true even for undirected graphs—every distance-regular graph is regular, whereas there exist sink-distance-regular graphs (where every edge not including the sink is undirected) which are not regular—see, for example, the graphs in Examples 6.9 and 6.10.

Sink-distance-regular graphs are far more common than distance-regular graphs, and it is simple to construct examples of them one neighborhood at a time, beginning with Γ_0 . For a small example of a sink-distance-regular graph that consists mostly of directed edges, see Fig. 3 (where each undirected edge is to be interpreted as a pair of directed edges, one in each direction). This graph is sink-distance-regular, with $d = 2$, $\Gamma_1 = \{v_1, v_2\}$, $\Gamma_2 = \{v_3, v_4, v_5, v_6\}$, $c_1 = 2$, $a_1 = 1$, $b_1 = 2$, $c_2 = 1$, and $a_2 = 1$.

Now, let \widehat{X} be a sink-distance-regular graph with $d \geq 1$. For $i \geq 1$, let γ_i denote the configuration on \widehat{X} , stable or not, with one grain of sand on each vertex of Γ_i and none elsewhere. Note that if $v \in \Gamma_i$ and $i \geq 1$, then $\text{deg}^+(v) = a_i + b_i + c_i$ —thus every vertex of Γ_i has the same outdegree. We write $\text{deg}^+(\Gamma_i)$ for this common outdegree. Also, if m is a sandpile configuration on \widehat{X} and j is a nonnegative integer, we denote the state (stable or not)

$$\underbrace{m + m + \dots + m}_j$$

by jm .

If x is an unstable configuration on \widehat{X} in which each vertex of Γ_i holds at least $\text{deg}^+(\Gamma_i)$ grains of sand, let $T_i(x)$ denote the configuration (stable or not) created by toppling each vertex of Γ_i exactly once. Then it follows from the definitions of the parameters $a_i, b_i,$ and c_i that if $d = 1,$ then

$$T_1(x) = T_d(x) = x - c_1\gamma_1,$$

and if $d > 1,$ then

$$\begin{aligned} T_1(x) &= x - (c_1 + b_1)\gamma_1 + c_2\gamma_2, \\ T_d(x) &= x - c_d\gamma_d + b_{d-1}\gamma_{d-1}, \quad \text{and} \\ T_i(x) &= x - (c_i + b_i)\gamma_i + b_{i-1}\gamma_{i-1} + c_{i+1}\gamma_{i+1} \end{aligned}$$

for $1 < i < d.$

Let U_i denote the following composition of these topplings:

$$U_i = T_d T_{d-1} \cdots T_{i+1} T_i,$$

where $U_d = T_d.$ We will use the following lemma regarding the U_i to prove that our description of the sandpile group identity of \widehat{X} in Theorem 6.7 is correct.

Lemma 6.6. *Let $1 \leq i \leq d.$ Suppose x is an unstable sandpile configuration in which each vertex of Γ_i is ready to topple (that is, each vertex of Γ_i holds at least $\text{deg}^+(\Gamma_i)$ grains of sand) and, for all j with $i < j \leq d,$ each vertex of Γ_j holds at least $\text{deg}^+(\Gamma_j) - c_j$ grains of sand. Then U_i is a legal sequence of topplings to apply to x and, for $1 < i \leq d$ we have*

$$U_i(x) = x + b_{i-1}\gamma_{i-1} - c_i\gamma_i,$$

and for $i = 1$ we have

$$U_1(x) = x - c_1\gamma_1.$$

Proof. Certainly T_i may be applied to $x.$ If $i < d,$ this moves c_{i+1} grains of sand onto each vertex of $\Gamma_{i+1},$ so we may apply T_{i+1} to $T_i(x).$ In general, if $i < j < d,$ none of $T_i, T_{i+1}, \dots, T_{j-1}$ affect the number of grains of sand on the vertices of $\Gamma_{j+1},$ and since T_j moves c_{j+1} grains of sand onto each vertex of $\Gamma_{j+1},$ T_{j+1} may be applied to $T_j T_{j-1} \cdots T_i(x).$ This justifies the legalities of the topplings used in $U_i(x).$

Next we verify the formulas for the $U_i(x).$ The result is immediate for $d = 1$ and a simple calculation verifies $d = 2,$ so for the remainder of the proof assume $d > 2.$

First, $T_d = U_d$ and the formula for $T_d(x)$ agrees with the one we claimed for $U_d(x).$ Next, for $i = d - 1$ we have

$$U_{d-1}(x) = T_d(T_{d-1}(x)) = x + b_{d-2}\gamma_{d-2} - c_{d-1}\gamma_{d-1},$$

as claimed. Next, if $1 < i < d - 1,$ then a straightforward induction shows that

$$T_j T_{j-1} \cdots T_{i+1} T_i(x) = x + b_{i-1}\gamma_{i-1} - c_i\gamma_i - b_j\gamma_j + c_{j+1}\gamma_{j+1}$$

for all $i < j < d.$ In particular,

$$T_{d-1} T_{d-2} \cdots T_{i+1} T_i(x) = x + b_{i-1}\gamma_{i-1} - c_i\gamma_i - b_{d-1}\gamma_{d-1} + c_d\gamma_d.$$

Applying T_d to this we obtain

$$\begin{aligned} U_i(x) &= x + b_{i-1}\gamma_{i-1} - c_i\gamma_i - b_{d-1}\gamma_{d-1} + c_d\gamma_d + b_{d-1}\gamma_{d-1} - c_d\gamma_d \\ &= x + b_{i-1}\gamma_{i-1} - c_i\gamma_i, \end{aligned}$$

as claimed. Finally, for $i = 1$ we have

$$\begin{aligned}
 U_1(x) &= U_2(T_1(x)) = U_2(x - (c_1 + b_1)\gamma_1 + c_2\gamma_2) \\
 &= x - (c_1 + b_1)\gamma_1 + c_2\gamma_2 + b_1\gamma_1 - c_2\gamma_2 \\
 &= x - c_1\gamma_1. \quad \square
 \end{aligned}$$

Recall that $c_i > 0$ for all $1 \leq i \leq d$. Now, we use the floor function to define the parameter

$$n_d = \left\lfloor \frac{\text{deg}^+(G_d) - 1}{c_d} \right\rfloor.$$

Note that $\text{deg}^+(G_d) \geq 1$, so $n_d \geq 0$. Thus $0 \leq n_d c_d \leq \text{deg}^+(G_d) - 1$ but $n_d c_d + c_d > \text{deg}^+(G_d) - 1$. In other words, the configuration $(n_d c_d)\gamma_d$ is stable but adding another c_d grains of sand to each vertex of G_d makes it unstable.

Define, furthermore, for $1 \leq i < d$,

$$n_i = \left\lfloor \frac{\text{deg}^+(G_i) - 1 + n_{i+1} b_i}{c_i} \right\rfloor.$$

Thus $n_i \geq 0$, $n_i c_i - n_{i+1} b_i \leq \text{deg}^+(G_i) - 1$, and $n_i c_i - n_{i+1} b_i + c_i > \text{deg}^+(G_i) - 1$. Also let $n_{d+1} = 0$.

Theorem 6.7. *Let \widehat{X} be a sink-distance-regular directed graph, with d , $\{a_i\}_{i=1}^d$, $\{b_i\}_{i=1}^d$, $\{c_i\}_{i=1}^d$ as in Definition 6.4, and $\{n_i\}_{i=1}^{d+1}$ as above. Then the identity element of $G(\widehat{X})$ is the element $e \in M(\widehat{X})$ given by*

$$e = \sum_{i=1}^d (n_i c_i - n_{i+1} b_i) \gamma_i.$$

Proof. First we show that e really is an element of $M(\widehat{X})$. For all $1 \leq i \leq d$, we have

$$n_i c_i - n_{i+1} b_i + c_i > \text{deg}^+(G_i) - 1,$$

so

$$n_i c_i - n_{i+1} b_i \geq \text{deg}^+(G_i) - c_i,$$

and since $c_i \leq \text{deg}^+(G_i)$,

$$n_i c_i - n_{i+1} b_i \geq 0,$$

so e has a nonnegative number of grains of sand on each vertex. Also for all $1 \leq i \leq d$,

$$n_i c_i - n_{i+1} b_i \leq \text{deg}^+(G_i) - 1,$$

so e is stable. Thus $e \in M(\widehat{X})$.

Now, to show that e is the identity of $G(\widehat{X})$, by Remark 2.14 it suffices to show that e is idempotent and that e has a grain of sand on every cycle of \widehat{X} . First, we show e is idempotent.

By the definition of the parameters n_i , e is only barely stable, in the sense that if, for any $1 \leq i \leq d$, c_i additional grains of sand are added to each vertex of G_i , the resulting configuration is unstable and we may apply U_i . Lemma 6.6 then gives

$$U_i(e + m + c_i \gamma_i) = e + m + b_{i-1} \gamma_{i-1}$$

for any $m \in M(\widehat{X})$ and any i with $1 < i \leq d$. For $i = 1$ we have

$$U_1(e + m + c_1 \gamma_1) = e + m$$

for any $m \in M(\widehat{X})$. If k is a nonnegative integer, we denote the composition of the toppling operator U_i with itself k times by U_i^k . We compute $e \oplus e$ by toppling the configuration $e + e$ as follows.

If $d = 1$, then $e = n_1c_1\gamma_1$, so $e + e = 2n_1c_1\gamma_1$, so U_1 may be applied n_1 times to $e + e$. Thus $U_1^{n_1}(e + e) = U_1^{n_1}(e + n_1c_1\gamma_1) = e$, so e is idempotent.

If $d = 2$, a similar argument shows that U_2 may be applied to $e + e$ n_2 times, and U_1 may then be applied n_1 times to the resulting state, and that $U_1^{n_1}(U_2^{n_2}(e + e)) = e$, so e is idempotent.

Let $d > 2$. Then a straightforward induction shows that, if $1 < j \leq d$, then $U_j^{n_j} \dots U_d^{n_d}$ is a legal sequence of topplings to apply to $e + e$, and

$$U_j^{n_j} \dots U_d^{n_d}(e + e) = e + n_{j-1}c_{j-1}\gamma_{j-1} + \sum_{i=1}^{j-2} (n_i c_i - n_{i+1} b_i) \gamma_i.$$

In particular, for $j = 2$, we have

$$U_2^{n_2} \dots U_d^{n_d}(e + e) = e + n_1c_1\gamma_1,$$

so $U_1^{n_1} U_2^{n_2} \dots U_d^{n_d}(e + e) = U_1^{n_1}(e + n_1c_1\gamma_1) = e$. Thus e is idempotent.

Finally, we show that e has a grain of sand on every cycle of X . First, if $1 \leq i < d$, then $n_i c_i - n_{i+1} b_i + c_i \geq \text{deg}^+(\Gamma_i)$, so

$$\begin{aligned} n_i c_i - n_{i+1} b_i &\geq \text{deg}^+(\Gamma_i) - c_i \\ &= (a_i + b_i + c_i) - c_i \\ &= a_i + b_i \\ &\geq 1 \end{aligned}$$

because $b_i \geq 1$ and $a_i \geq 0$. Thus every vertex of Γ_i holds at least one grain of sand in e , for $1 \leq i < d$.

Suppose now for the sake of contradiction that there exists a cycle C of X such that no vertex of C holds a grain of sand in e . Then every vertex of C must lie in Γ_d , so Γ_d contains a cycle, and so $a_d \geq 1$. But we also have $n_d c_d + c_d \geq \text{deg}^+(\Gamma_d)$, so

$$\begin{aligned} n_d c_d &\geq \text{deg}^+(\Gamma_d) - c_d \\ &= (a_d + c_d) - c_d \\ &= a_d \\ &\geq 1. \end{aligned}$$

Thus every vertex of Γ_d holds at least one grain of sand on in e , contradicting our assumption.

Thus e has a grain of sand on every cycle of X . \square

Example 6.8. Let Y be a distance-regular graph of diameter d . Then Y is regular, say of degree k . Form the graph \widehat{X} by choosing any vertex of Y to be the sink and deleting all outward edges from the sink. Then \widehat{X} is sink-distance-regular, where the constants $\{a_i\}_{i=1}^d, \{b_i\}_{i=1}^d, \{c_i\}_{i=1}^d$ agree with those that arise from the distance-regularity of Y . The identity e of $G(\widehat{X})$ is then

$$e = \sum_{i=1}^d (n_i c_i - n_{i+1} b_i) \gamma_i,$$

where $n_{d+1} = 0$,

$$n_d = \left\lfloor \frac{k-1}{c_d} \right\rfloor,$$

and, for $1 \leq i < d$,

$$n_i = \left\lfloor \frac{k-1 + n_{i+1} b_i}{c_i} \right\rfloor.$$

This agrees with Biggs' description of the identity of $G(\widehat{X})$ [5].

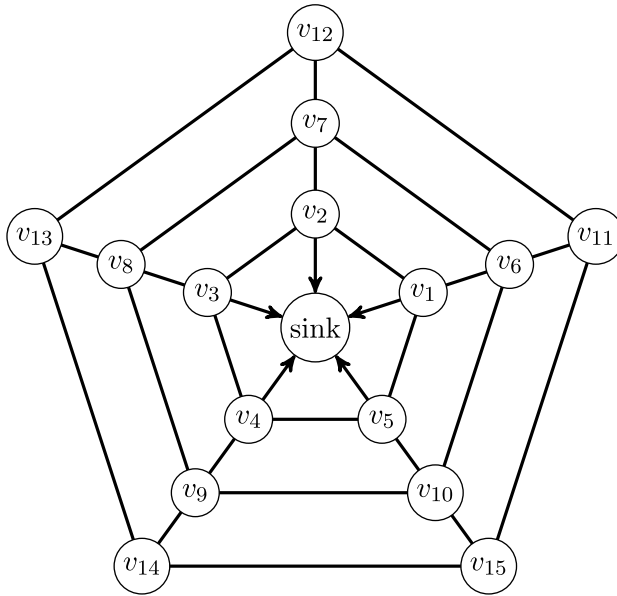


Fig. 4. The 3-iterated 5-wheel \widehat{X} ; $\Gamma_1 = \{v_1, \dots, v_5\}$, $\Gamma_2 = \{v_6, \dots, v_{10}\}$, $\Gamma_3 = \{v_{11}, \dots, v_{15}\}$.

We now give some examples to show how Theorem 6.7 generalizes beyond distance-regular graphs.

Example 6.9. Let n and d be integers with $n \geq 2$ and $d \geq 1$. Let Y be the undirected prism graph $Y_{n,d} = C_n \times P_d$ (where C_n is the undirected n -cycle and P_d is the path graph on d vertices), with the C_n 's of Y arranged in the plane in a concentric fashion. Form the graph \widehat{X} by adding a central vertex (the sink) and adding an edge from each of the innermost vertices of Y to the sink. We call \widehat{X} the d -iterated n -wheel; when $d = 1$ \widehat{X} is the n -wheel, whose sandpile group was considered as an example in [11]. Fig. 4 depicts the 3-iterated 5-wheel (where each undirected edge is to be interpreted as a pair of directed edges, one in each direction).

\widehat{X} is sink-distance-regular. For $1 \leq i \leq d$, Γ_i consists of the vertices of the i th cycle of \widehat{X} (counted starting from the sink and moving outward), $c_i = 1$, and $a_i = 2$. We also have $b_i = 1$ for all $1 \leq i < d$. A straightforward induction shows that $n_i = 2 + 3(d - i)$ for $1 \leq i \leq d$, so $n_d c_d = 2$ and $n_i c_i - n_{i+1} b_i = 3$ for $1 \leq i < d$. The identity of $G(\widehat{X})$ is therefore

$$\begin{aligned} e &= \sum_{i=1}^d (n_i c_i - n_{i+1} b_i) \gamma_i \\ &= 2\gamma_d + \sum_{i=1}^{d-1} 3\gamma_i \\ &= \text{MAX}. \end{aligned}$$

Example 6.10. Let n and d be integers with $n \geq 3$ and $d \geq 2$. Let Y be the n -regular (undirected) tree of depth d —that is, let Y be a finite rooted tree where each non-leaf vertex has degree n (so the root has n children and each non-root, non-leaf vertex has $n - 1$ children) and where the path from any leaf to the root has $d - 1$ edges. Form the graph \widehat{X} by adding a new vertex (the sink) and adding $n - 1$ directed edges from each leaf of Y to the sink. The sandpile group of \widehat{X} was studied in [22]. A variant was studied in [17].

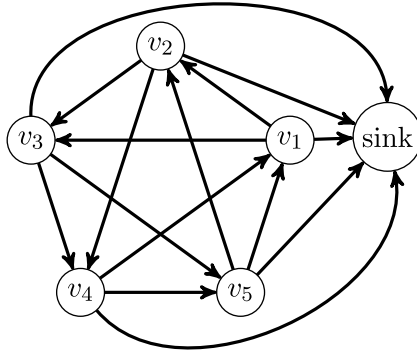


Fig. 5. The graph \widehat{X} obtained from a regular tournament on 5 vertices ($r = 1$).

\widehat{X} is sink-distance-regular. For $1 \leq i \leq d$, Γ_i consists of the vertices of distance $d - i$ from the root and $a_i = 0$. We also have $c_i = n - 1$ and $b_i = 1$ for $1 \leq i < d$, and $c_d = n$. A straightforward induction shows that $n_d = 0$ and $n_i = 1$ for $1 \leq i < d$, so identity of $G(\widehat{X})$ is

$$\begin{aligned}
 e &= \sum_{i=1}^d (n_i c_i - n_{i+1} b_i) \gamma_i \\
 &= 0 \gamma_d + (n - 1) \gamma_1 + \sum_{i=1}^{d-2} (n - 2) \gamma_i,
 \end{aligned}$$

the configuration of $M(\widehat{X})$ that holds no grains on the root, $n - 1$ grains on each child of the root, and $n - 2$ grains on every other non-sink vertex of \widehat{X} .

Example 6.11. Let k be a positive integer and let Y be a regular tournament on $2k + 1$ vertices—that is, let Y be a directed graph with $2k + 1$ vertices where there is exactly one directed edge between every pair of vertices and where each vertex has in-degree k and out-degree k . Let r be a positive integer and form the graph \widehat{X} by adding a vertex (the sink) and adding r directed edges from each vertex of Y to the sink. The graph \widehat{X} obtained from a regular tournament on 5 vertices (with $r = 1$) is depicted in Fig. 5.

\widehat{X} is sink-distance regular, with $d = 1$. Γ_1 consists of all non-sink vertices of \widehat{X} , $a_1 = k$, and $c_1 = r$, so the identity of $M(\widehat{X})$ is

$$e = n_1 c_1 \gamma_1 = \left(\left\lfloor \frac{k+r-1}{r} \right\rfloor r \right) \gamma_1.$$

When $r = 1$ this is MAX.

Example 6.12. Let \widehat{X} be the graph in Fig. 3, where each undirected edge is to be interpreted as a pair of directed edges, one in each direction. \widehat{X} is sink-distance-regular, with $d = 2$, $\Gamma_1 = \{v_1, v_2\}$, $\Gamma_2 = \{v_3, v_4, v_5, v_6\}$, $c_1 = 2$, $a_1 = 1$, $b_1 = 2$, $c_2 = 1$, and $a_2 = 1$. We have

$$n_3 = 0, \quad n_2 = 1, \quad n_1 = 3,$$

so the identity of $G(\widehat{X})$ is

$$e = (3 \cdot 2 - 1 \cdot 2) \gamma_1 + (1 \cdot 1) \gamma_2 = 4 \gamma_1 + 1 \gamma_2 = \text{MAX}.$$

Acknowledgments

We thank László Babai and the anonymous referees for their comments and suggestions, which have helped us substantially improve our results in Section 6 and our presentation of the material throughout this paper.

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