JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **160**, 406–412 (1991)

An Existence Theorem for a Difference Inclusion in General Banach Spaces

SIMEON REICH

Center for Applied Mathematical Sciences,
Department of Mathematics,
University of Southern California,
Los Angeles, California 90089,
and Department of Mathematics,
The Technion-Israel Institute of Technology,
Haifa 32000, Israel

AND

ITAI SHAFRIR

Department of Mathematics,
The Technion-Israel Institute of Technology,
Haifa 32000, Israel,
and Analyse Numérique, Université Paris VI,
75252 Paris, France

Submitted by Ky Fan

Received March 9, 1990

The main purpose of this paper is to prove an existence theorem for the difference inclusion

$$\begin{aligned} &u_{i+1} - 2u_i + u_{i-1} \in c_i A u_i + f_i, \qquad i = 1, 2, ..., \\ &u_0 = x \\ &\sup\{|u_i| : i \geqslant 0\} < \infty, \end{aligned} \tag{1}$$

where A is a nonlinear (possibly discontinuous and set-valued) m-accretive operator in a Banach space $(X, |\cdot|)$, $\{c_i\}$ is a given sequence of positive numbers, and $\{f_i\}$ is a given sequence in X.

This problem is of interest because it is the discrete analog of the quasi-autonomous incomplete Cauchy problem

$$\begin{aligned} &u''(t) \in Au(t) + f(t), \qquad 0 < t < \infty \\ &u(0) = x \\ &\sup\{|u(t)| : t \geqslant 0\} < \infty, \end{aligned} \tag{2}$$

0022-247X/91 $3.00

the solutions of which have several remarkable properties [7]. It is also related to an optimization problem [6, p. 168]. For more motivation and information on the problems (1) and (2) see [1–9] and the references mentioned there.

Since our existence theorem is valid in all Banach spaces, it provides an affirmative answer to the question raised on p. 128 of [9], where X was assumed to have a strongly monotone duality map. We also include a new existence theorem for (a special case of) problem (2).

We begin by quoting an existence result [9, p. 123] for the boundary value problem

$$\begin{aligned} &u_{i+1} - 2u_i + u_{i-1} \in c_i A u_i + f_i, \qquad i = 1, 2, ..., n \\ &u_0 = x, \qquad u_{n+1} = y, \end{aligned} \tag{3}$$

where n is a positive integer, $\{c_i : 1 \leqslant i \leqslant n\}$ is given finite sequence of positive numbers, and $\{f_i : 1 \leqslant i \leqslant n\}$ is a given finite sequence of points in X.

PROPOSITION 1. *Let X be a Banach space and $A \subset X \times X$ an m-accretive operator. Then for each x and y in X and $\{f_i : 1 \leqslant i \leqslant n\} \subset X$, the problem* (3) *has a unique solution $\{u_i : 0 \leqslant i \leqslant n+1\} \subset X$.*

As already mentioned in [9], it is clear that in general the difference inclusion (1) has no solution even if $A = 0$ and $\{f_i\} \in l^1(X)$. It turns out, however, that if (1) has a solution for one point x in X, then it has a unique solution for all x in X.

THEOREM 2. *Let X be a Banach space and $A \subset X \times X$ an m-accretive operator. If problem* (1) *has a solution for some x in X, then it has a unique solution for all x in X.*

Proof. Let $w = \{w_i : i = 0, 1, 2, ...,\}$ be a solution to (1) with $w_0 = z$ and $\sup\{|w_i| : i \geqslant 0\} = K$, and let x be another point in X. For each $n \geqslant 1$, there exists, by Proposition 1, a unique solution u^n to (3) with $x = y$. Set $y_i = y_i^n = u_i^n - w_i$, and let J denote the duality map of X. Since A is accretive, there is a functional $j_i \in Jy_i$ such that

$$(y_{i+1} - 2y_i + y_{i-1}, j_i) \geqslant 0$$

for all $1 \leqslant i \leqslant n$. Hence $|y_i| \leqslant (1/2)(|y_{i-1}| + |y_{i+1}|)$, $|y_i| \leqslant \max\{|y_0|, |y_{n+1}|\}$, and

$$|u_i^n| \leqslant |x| + 2K \tag{4}$$

for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$. Now let $n_0 < n_1 < n_2$, set $z_i = u_i^{n_1} - u_i^{n_2}$, $0 \leqslant i \leqslant n_1 + 1$, and for each such i, let the functional $j_i \in Jz_i$ satisfy

$$(z_{i+1} - 2z_i + z_{i-1}, j_i) \geqslant 0.$$

Since

$$(x - y, x^* - y^*) \geqslant (|x| - |y|)^2 \tag{5}$$

for all $x \in X$, $y \in X$, $x^* \in Jx$, and $y^* \in Jy$, we have

$$(z_{i+1} - z_i, j_i) - (z_i - z_{i-1}, j_{i-1}) \geqslant (|z_i| - |z_{i-1}|)^2$$

for all $1 \leqslant i \leqslant n_1$. Therefore

$$\sum_{i=1}^{k} (|z_i| - |z_{i-1}|)^2 \leqslant (z_{k+1} - z_k, j_k) \leqslant \tfrac{1}{2} (|z_{k+1}|^2 - |z_k|^2)$$

for all $n_0 \leqslant k \leqslant n_1$, and

$$\sum_{k=n_0}^{n_1} \sum_{i=1}^{k} (|z_i| - |z_{i-1}|)^2 \leqslant \tfrac{1}{2} |z_{n_1+1}|^2 \leqslant 2(|x| + 2K)^2.$$

It follows, in particular, that

$$(n_1 - n_0 + 1) \sum_{i=1}^{n_0} (|z_i| - |z_{i-1}|)^2 \leqslant 2(|x| + 2K)^2.$$

But

$$\begin{aligned}
(n_1 - n_0 + 1) &\sum_{i=1}^{n_0} (|z_i| - |z_{i-1}|)^2 \\
&= [(n_1 - n_0 + 1)/n_0] \, n_0 \sum_{i=1}^{n_0} (|z_i| - |z_{i-1}|)^2 \\
&\geqslant [(n_1 - n_0 + 1)/n_0] \left[\sum_{i=1}^{n_0} (|z_i| - |z_{i-1}|) \right]^2 \\
&= [(n_1 - n_0 + 1)/n_0] \, |z_{n_0}|^2,
\end{aligned}$$

so that

$$|z_{n_0}|^2 \leqslant 2n_0(|x| + 2K)^2/(n_1 - n_0 + 1).$$

Since n_0 is arbitrary, we see that $u_i = \lim_{n \to \infty} u_i^n$ exists for each $i = 1, 2, \ldots$. The sequence $u = \{u_i : i = 0, 1, 2, \ldots,\}$ is bounded by (4) and solves (1) because A is closed. If $v = \{v_i : i = 0, 1, 2, \ldots,\}$ is another solution of (1), then

$$|u_i - v_i| \leqslant \tfrac{1}{2} (|u_{i+1} - v_{i+1}| + |u_{i-1} - v_{i-1}|)$$

for all $i \geqslant 1$. Since the sequence $\{|u_i - v_i|\}$ is also bounded, it must be non-increasing. But $u_0 = v_0 = x$. Hence $u_i = v_i$ for all i and the proof is complete.

In view of Theorem 2, we can now improve upon Proposition 4 and Theorem 5 of [9].

PROPOSITION 3. *Let X be a Banach space and $A \subset X \times X$ an m-accretive operator with $0 \in R(A)$. If A is coercive, $\{c_i\}$ is bounded away from zero, and $\{f_i\}$ is bounded, then problem* (1) *has a unique solution for all x in X.*

THEOREM 4. *Let X be a Banach space and $A \subset X \times X$ an m-accretive operator. Assume that each bounded closed convex subset of X has the fixed point property for nonexpansive mappings. If problem* (1) *has a solution, and $\{f_i\}$ and $\{c_i\}$ are periodic of period N, then there is a solution of* (1) *which is also N-periodic.*

Turning our attention to the continuous problem (2), we first recall a known differentiation lemma and then present a new one.

LEMMA 5. *Let $J : X \to X^*$ be the duality map of a smooth Banach space X, $u : [0, \infty) \to X$, and define $p : [0, \infty) \to [0, \infty)$ by $p(t) = (1/2)|u(t)|^2$. If u is differentiable, then so is p and $p'(t) = (u'(t), Ju(t))$ for all t.*

LEMMA 6. *Let $J : X \to X^*$ be the duality map of a smooth Banach space X. Suppose $u : [0, T] \to X$ is continuously differentiable, u' is absolutely continuous, and $u'' \in L^1(0, T; X)$. If $p(t) = (1/2)|u(t)|^2$ and $q : [0, T] \to R$ is defined by $q(t) = (u'(t), Ju(t))$, then*

(a) *q is differentiable almost everywhere;*

(b) *$\int_s^t q'(r)\, dr \leqslant q(t) - q(s)$ for all $0 \leqslant s \leqslant t \leqslant T$;*

(c) *$2p(t)\, q'(t) \geqslant 2p(t)(u''(t), Ju(t)) + (q(t))^2$ for almost all t.*

Proof. For parts (a) and (b), see the proof of [8, Lemma 2.5]. To establish (c), we first note that by (5),

$$
\begin{aligned}
|x+y|^2 - |x|^2 - 2(y, Jx) &= 2 \int_0^1 (y, J(x+ry) - Jx)\, dr \\
&\geqslant 2 \int_0^1 r[(|x+ry| - |x|)/r]^2\, dr \\
&\geqslant 2 \int_0^1 r[(y, Jx)/|x|]^2\, dr \\
&= [(y, Jx)/|x|]^2
\end{aligned}
$$

for all x ($\neq 0$) and y in X. Therefore

$$4p(t)[p(t+h)-p(t)-(u(t+h)-u(t), Ju(t))] \geqslant (u(t+h)-u(t), Ju(t))^2$$

and

$$4p(t)[p(t-h)-p(t)-(u(t-h)-u(t), Ju(t))] \geqslant (u(t-h)-u(t), Ju(t))^2.$$

Hence

$$\begin{aligned}4p(t)[&p(t+h)-2p(t)+p(t-h)\\ &-(u(t+h)-2u(t)+u(t-h), Ju(t))]\\ &\geqslant (u(t+h)-u(t), Ju(t))^2+(u(t-h)-u(t), Ju(t))^2\end{aligned}$$

and (c) follows.

In general, the incomplete Cauchy problem (2) has no solution even if $A=0$ and $f \in W^{1,2}(0, \infty; X)$. We now present a positive result for the problem

$$\begin{aligned}&u''(t)=A_r u(t)+f(t), \qquad 0<t<\infty\\ &u_0=x\\ &\sup\{|u(t)| : t \geqslant 0\}<\infty,\end{aligned} \tag{6}$$

which is obtained from (2) by replacing A with its Yosida approximation A_r, $r>0$. In contrast with previous results in this direction [7, 8], we no longer assume that the Banach space X has a strongly monotone duality map. Therefore Theorem 8 partially answers the question raised on p. 528 of [8]. To prove it, we need an existence result [7, p. 387] for the boundary value problem

$$\begin{aligned}&u''(t)=A_r u(t)+f(t), \qquad 0<t<\infty\\ &u_0=x, \qquad u(T)=y.\end{aligned} \tag{7}$$

PROPOSITION 7. *Let X be a Banach space and $A \subset X \times X$ an m-accretive operator. Then for each x and y in X and f in $L^2(0, T; X)$ the problem* (7) *has a unique solution in $W^{2,2}(0, T; X)$.*

THEOREM 8. *Let X be a smooth Banach space, $A \subset X \times X$ an m-accretive operator, and $f \in L^2_{\mathrm{loc}}(0, \infty; X)$. If problem* (6) *has a solution in $C([0, \infty); X) \cap W^{2,2}_{\mathrm{loc}}(0, \infty; X)$ for some x in X, then it has a unique solution for all x in X.*

Proof. Let w be a solution to (6) with $w(0)=z$ and $\sup\{|w(t)| : t \geqslant 0\} = K$, and let x be another point in X. For each $n \geqslant 1$, there exists, by Proposition 7, a unique solution $u_n \in W^{2,2}(0, n; X)$ of problem (7) with $T=n$ and $x=y$. Let J denote the duality map of X and set $v(t)=v_n(t)=u_n(t)-w(t)$, $0 \leqslant t \leqslant n$. Since the Yosida approximation A_r is accretive, we can apply Lemma 6 to $v(t)$ and conclude that $|v(t)|^2$ is convex. Hence $|v(t)| \leqslant \max\{|v(0)|, |v(n)|\}$ and

$$|u_n(t)| \leqslant |x| + 2K \tag{8}$$

for all $n \geqslant 1$ and $0 \leqslant t \leqslant n$. Now let $T<m<n$ and set $p(t) = (1/2)|u_n(t) - u_m(t)|^2$, $0 \leqslant t \leqslant m$. Using Lemmata 5 and 6, we see that

$$\begin{aligned}(p(t))^2 &= \left(\int_0^t p'(s)\, ds\right)^2 \\ &\leqslant t \int_0^t (p'(s))^2\, ds \leqslant 2t \int_0^t p(s)\, p''(s)\, ds \\ &\leqslant 2tp(t)\, p'(t).\end{aligned}$$

Hence $p(t) \leqslant 2tp'(t)$ and $\int_T^m (p(t)/t)\, dt \leqslant 2p(m)$. Since p is convex and $p(0)=0$, it is nondecreasing on $[0, m]$. Therefore

$$p(T) \log(m/T) \leqslant 2p(m)$$

and, by (8),

$$|u_n(t) - u_m(t)|^2 \leqslant 8(|x| + K)^2/\log(m/T)$$

for all $0 \leqslant t \leqslant T$. Thus $u(t) = \lim_{n \to \infty} u_n(t)$ exists uniformly in every bounded interval $[0, T]$. The function $u: [0, \infty) \to X$ is bounded by (8) and solves (6) because A_r is continuous. If v is another solution, then the function $|u(t) - v(t)|^2$ is convex by Lemma 6. Since it is also bounded, it must be non-increasing. Hence $u = v$ and the proof is complete.

The case $f=0$ of Theorem 8 shows that if $0 \in R(A)$, then (in the notation of [8, Sect. 3]) the semigroup $(S_r)_{1/2}$ can be defined in all smooth Banach spaces.

Finally, we mention the following partial improvement of [7, Theorem 10]. It is also a consequence of Theorem 8.

THEOREM 9. *Let X be a smooth Banach space and $A \subset X \times X$ an m-accretive operator. Assume that each bounded closed convex subset of X has the fixed point property for nonexpansive mappings. If problem* (6) *has a solution and f is periodic of period T, then there is a solution of* (6) *which is also T-periodic.*

References

1. V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff, Leyden, 1976.
2. R. E. Bruck, Periodic forcing of solutions of a boundary value problem for a second order differential equation in Hilbert space, *J. Math. Anal. Appl.* **76** (1980), 159–173.
3. E. Mitidieri, Some remarks on the asymptotic behavior of the solutions of second order evolution equations, *J. Math. Anal. Appl.* **107** (1985), 211–221.
4. E. Mitidieri and G. Morosanu, Asymptotic behavior of the solutions of second order difference equations associated to monotone operators, *Numer. Funct. Anal. Optim.* **8** (1986), 419–434.
5. G. Morosanu, Second order difference equations of monotone type, *Numer. Funct. Anal. Optim.* **1** (1979), 441–450.
6. G. Morosanu, "Nonlinear Evolution Equations and Applications," Reidel, Dordrecht, 1988.
7. E. I. Poffald and S. Reich, A quasi-autonomous second-order differential inclusion, *in* "Nonlinear Analysis," pp. 387–392, North-Holland, Amsterdam, 1985.
8. E. I. Poffald and S. Reich, An incomplete Cauchy problem, *J. Math. Anal. Appl.* **113** (1986), 514–543.
9. E. I. Poffald and S. Reich, A difference inclusion, *in* "Nonlinear Semigroups, Partial Differential Equations and Attractors," pp. 122–130, Lecture Notes in Mathematics, Vol. 1394, Springer, Berlin, 1989.