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Best Constants for the Riesz Projection

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We prove the following inequality with a sharp constant,

$$\| P_+ f \|_{L^p(\mathbb{T})} \leqslant \csc \frac{\pi}{p} \, \| f \|_{L^p(\mathbb{T})}, \qquad f \in L^p(\mathbb{T}),$$

where $1 < p < \infty$, and $P_+ : L^p(\mathbb{T}) \to H^p(\mathbb{T})$ is the Riesz projection onto the Hardy

We also prove an analogous inequality in the nonperiodic case where $P_+ f = \mathscr{F}^{-1}(\chi_{\mathbb{R}_+} \mathscr{F} f)$ is the half-line Fourier multiplier on $\mathbb{R}$. Similar weighted inequalities with sharp constants for $L^p(\mathbb{R}, |x|^\alpha)$, $-1 < \alpha < p-1$, are obtained. In the multidimensional case, our results give the norm of the half-space Fourier multiplier on $\mathbb{R}^n$. © 2000 Academic Press

Key Words: Riesz projection; best constants; Fourier multipliers; plurisubharmonic functions; weighted L^p spaces.

1. INTRODUCTION

Let $f(\zeta) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \, \zeta^k$ be a complex-valued function on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$; here $\{\hat{f}(k)\}_{k \in \mathbb{Z}}$ are the Fourier coefficients of $f \in L^p(\mathbb{T})$, $1 < p < \infty$. The $H^p(\mathbb{T})$ space is a closed subspace of $L^p(\mathbb{T})$ which consists of functions f such that $\hat{f}(k) = 0$ for $k < 0$. The Riesz projection, $P_+ : L^p(\mathbb{T}) \to H^p(\mathbb{T})$, is the operator defined by

$$P_+ f(\zeta) = \sum_{k \geqslant 0} \hat{f}(k) \, \zeta^k, \qquad \zeta \in \mathbb{T}.$$

It follows from the classical M. Riesz theorem [Ri] that P_+ is a continuous operator on $L^p(\mathbb{T})$, $1 < p < \infty$. Let

$$\| P_+ \|_p = \sup \left\{ \frac{\| P_+ f \|_{L^p(\mathbb{T})}}{\| f \|_{L^p(\mathbb{T})}} : f \in L^p(\mathbb{T}), \, f \neq 0 \right\}$$

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be the operator norm of P_+. For $p=2$, P_+ is the orthogonal projection onto H^2, and hence $\|P_+\|_2=1$.

The main result of this paper is the following inequality,

$$\|P_+ f\|_{L^p(\mathbb{T})} \leqslant \frac{1}{\sin\frac{\pi}{p}} \|f\|_{L^p(\mathbb{T})}, \qquad 1<p<\infty, \tag{1.1}$$

for every $f \in L^p(\mathbb{T})$. A similar inequality holds for a nonperiodic analogue of P_+ on the real line $\mathbb{R}$ (the so-called half-line multiplier) defined below. This was first conjectured by Gohberg and Krupnik in [GKr1] where the lower estimate for the best constant in (1.1) was proved: $\|P_+\|_p \geqslant \frac{1}{\sin(\pi/p)}$ (see also [GKr2, GKr3]). The question about the exact value of $\|P_+\|_p$ was also stated by A. Pełczyński as Problem 3 in [Pe], and discussed in [KrV1, BMS, Pa].

As usual we identify $H^p(\mathbb{T})$ with the space of functions f analytic in the unit disc $\mathbb{D}$ so that

$$\|f\|_{H^p(\mathbb{T})} = \sup_{0<r<1} \left\{ \int_{\mathbb{T}} |f(r\zeta)|^p \, |d\zeta| \right\}^{1/p} < \infty.$$

Then $f^*(\zeta) = \lim_{r\to 1} f(r\zeta) \in L^p$, and $\|f^*\|_{L^p(\mathbb{T})} = \|f\|_{H^p(\mathbb{T})}$ (see [Gar, K, Z]). The Riesz projection P_+ may also be written as a Cauchy type integral,

$$P_+ f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\lambda)}{\lambda - z} \, d\lambda, \qquad z \in \mathbb{D},$$

with density $f \in L^p(\mathbb{T})$. We also define the co-analytic projection, $P_- = I - P_+$, and the involution operator $S = P_+ - P_-$, so that

$$P_- f(\zeta) = \sum_{k<0} \hat{f}(k) \, \zeta^k, \qquad Sf(\zeta) = \sum_{k\in\mathbb{Z}} \sigma(k) \, \hat{f}(k) \, \zeta^k, \quad \zeta \in \mathbb{T},$$

where $\sigma(k) = 1$ if $k \geqslant 0$ and $\sigma(k) = -1$ if $k<0$.

The analytic projection on the real line, $P_+ = P_+^{\mathbb{R}}$, can be defined as the half-line Fourier multiplier

$$P_+^{\mathbb{R}} f(x) = \mathscr{F}^{-1}(\chi_{\mathbb{R}_+} \mathscr{F} f)(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\mathscr{F} f$ and $\mathscr{F}^{-1} f$ are respectively the direct and inverse Fourier transforms of f. Then $P_-^{\mathbb{R}} = I - P_+^{\mathbb{R}}$ is the corresponding co-analytic projection, and $S^{\mathbb{R}} = P_+^{\mathbb{R}} - P_-^{\mathbb{R}}$. Analogously, one defines half-space multipliers for higher dimensions (see [St]): $P_+^j = \mathscr{F}^{-1} \chi_{\mathscr{S}_+^j} \mathscr{F}$ where $\mathscr{S}_+^j = \{x = (x_1, ..., x_n) \in \mathbb{R}^n \colon x_j > 0\}$, $j = 1, ..., n$, and $P_-^j = I - P_+^j$.

In this paper we will show that $\|P_{\pm}\|_p = \frac{1}{\sin(\pi/p)}$, $1 < p < \infty$, both on the circle (Section 2) and the line (Section 3), as well as on $\mathbb{R}^n$ (Corollary 3.2). In Section 2 we will actually prove a little stronger inequality for $1 < p \leqslant 2$ (Theorem 2.1):

$$\|\max(|P_+ f|, |P_- f|)\|_{L^p(\mathbb{T})} \leqslant \frac{1}{\sin \frac{\pi}{p}} \|f\|_{L^p(\mathbb{T})}, \qquad f \in L^p(\mathbb{T}). \tag{1.3}$$

We will also find in Section 4 the norm $\|P_{\pm}\|_{p,\alpha}$ for weighted spaces $L^p(\mathbb{R}, |x|^{\alpha})$ with power weights, together with a periodic analogue (Theorems 4.1 and 4.2):

$$\|P_{\pm}\|_{p,\alpha} = \frac{1}{\sin \frac{\pi}{r}}, \qquad r = \max\left(p, \frac{p}{p-1}, \frac{p}{1+\alpha}, \frac{p}{p-1-\alpha}\right). \tag{1.4}$$

This proves a conjecture stated in [KrV1]. (For $p = 2$, the proof of (1.4) was given in [Kr]; another proof based on the relation between $\|P_+\|_{2,\alpha}$ and the norm of the Hilbert matrix can be found in [KrV1].)

The following lower estimates were obtained in [GKr1, GKr2] by means of the Fredholm theory of Toeplitz operators with piecewise continuous symbols (see also [GKr3]),

$$|||P_{\pm}|||_p \geqslant \frac{1}{\sin \frac{\pi}{p}}, \qquad |||S|||_p \geqslant \max\left(\tan \frac{\pi}{2p}, \cot \frac{\pi}{2p}\right), \tag{1.5}$$

for $p > 1$. Here $|||A|||_p$ is the essential norm of A: $L^p \to L^p$, i.e., the distance from A to the ideal of compact operators:

$$|||A|||_p = \inf\{\|A + T\|_p : \forall\, T \text{ compact on } L^p\}. \tag{1.6}$$

The Hilbert transform (conjugate function) on $\mathbb{T}$ is defined by

$$Hf(\zeta) = \tilde{f}(\zeta) = -i \sum_{k \in \mathbb{Z}} \operatorname{sign}(k)\, \hat{f}(k)\, \zeta^k, \qquad \zeta \in \mathbb{T}, \tag{1.7}$$

where $\operatorname{sign}(k) = 1$ if $k > 0$, $\operatorname{sign}(0) = 0$, and $\operatorname{sign}(k) = -1$ if $k < 0$. Clearly, $S = iH + K_0$ where $K_0 f = \hat{f}(0)$ is a rank one operator, and hence by (1.5)

$$|||H|||_p \geqslant \max\left(\tan \frac{\pi}{2p}, \cot \frac{\pi}{2p}\right). \tag{1.8}$$

Note that both Sf and Hf can be defined as singular integrals (see [GKr3, Z]):

$$Sf(\zeta)=\frac{1}{\pi i}\,\text{p.v.}\int_{\mathbb{T}}\frac{f(\lambda)}{\lambda-\zeta}\,d\lambda,\qquad \zeta\in\mathbb{T},$$

and

$$Hf(e^{ix})=\tilde{f}(e^{ix})=\frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi}\frac{f(e^{it})}{\tan\dfrac{x-t}{2}}\,dt,\qquad e^{ix}\in\mathbb{T}.$$

In the nonperiodic case, $H=H^{\mathbb{R}}$ is defined by

$$H^{\mathbb{R}}f(x)=-iS^{\mathbb{R}}f(x)=\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}}\frac{f(t)}{x-t}\,dt,\qquad x\in\mathbb{R}. \tag{1.9}$$

For $p=2^k$, $k\in\mathbb{Z}_+$, the exact values of $\|S\|_p$ on the circle and the line were found by Gohberg and Krupnik [GKr1]:

$$\|S\|_p=\max\left(\tan\frac{\pi}{2p},\ \cot\frac{\pi}{2p}\right). \tag{1.10}$$

The proof in [GKr1] is based on the important identity $f^2+(Sf)^2=2S(f\,Sf)$. For all other values of p, $1<p<\infty$, the fact that $\|H\|_p=\max\,(\tan\frac{\pi}{2p},\cot\frac{\pi}{2p})$ on the circle and the line was established in 1972 by S. Pichorides [P] and independently by B. Cole (unpublished; see [G]) for real L^p spaces. The subharmonic proof given in [P] is a refinement of A. Calderón's proof of the M. Riesz theorem (see [Z]). Other results on best constants for the Hilbert transform and singular integral operators, as well as different approaches, including probabilistic methods developed by D. L. Burkholder, can be found in [B, BMS, BW, Bur1, Bur2, D1, D2, GKr3, Pe].

The problem of the best constant for the Riesz projection P_+ turned out to be more difficult. Computing the norm of P_+ on the circle is essentially equivalent to finding the best constant in the inequality

$$\|\tfrac{1}{2}(f+i\tilde{f})\|_{L^p(\mathbb{T})}\leqslant C_p\,\|f\|_{L^p(\mathbb{T})},\qquad f\in L^p(\mathbb{T}), \tag{1.11}$$

since $P_+f=\frac{1}{2}(f+i\tilde{f})+\frac{1}{2}\hat{f}(0)$. For *real-valued* f, this constant was found in 1980 to be $C_p=\max\,(\frac{1}{2\cos(\pi/2p)},\frac{1}{2\sin(\pi/2p)})$ by Verbitsky [V], and later independently by M. Essén [E] (see also [BW] for a probabilistic proof). This gives the norm of the restriction of P_+ to real-valued f such that $\hat{f}(0)=0$. As is shown in [E, V], this result yields the sharp estimate for the Hilbert transform. However, P_+ is not a projection on a real L^p space;

e.g., the best constant in (1.11) for $p=2$ in this case is $C_2=1/\sqrt{2}<1$. For complex-valued f, the best constant becomes $C'_p=\frac{1}{\sin(\pi/p)}$ (Corollary 2.6).

In this paper, we give a "plurisubharmonic" proof of (1.3), (1.11), and some other related inequalities using functions of two complex variables. The main idea of the proof is that, in order to prove these inequalities it suffices to show that the function

$$\Phi(w, z)=\frac{1}{\sin^p \frac{\pi}{p}}\,|w+\bar{z}|^p-\max\left(|w|^p, |z|^p\right), \qquad (w, z)\in\mathbb{C}^2,$$

has a plurisubharmonic minorant $F(w, z)$ on $\mathbb{C}^2$ such that $F(0, 0)=0$. If this is the case, then

$$\max\left(|w|^p, |z|^p\right)\leqslant\frac{1}{\sin^p \frac{\pi}{p}}\,|w+\bar{z}|^p-F(w, z),$$

and (1.3) follows by letting $w=P_+ f$, $z=\overline{P_- f}$ in the preceding inequality, integrating both sides over $\mathbb{T}$, and using the sub-mean-value inequality for the composite subharmonic function $F(P_+ f, \overline{P_- f})$.

The details are given in the next section where we prove that $\Phi(w, z)$ does have a desired plurisubharmonic minorant. It can be found in the form $F(w, z)=b_p\,\mathrm{Re}[(w\,z)^{p/2}]$, where b_p is an appropriate positive constant. More precisely, we establish the following inequality which might be of independent interest,

$$\max\left(|w|^p, |z|^p\right)\leqslant a_p\,|w+\bar{z}|^p-b_p\,\mathrm{Re}[(w\,z)^{p/2}], \tag{1.12}$$

for every $(w, z)\in\mathbb{C}^2$ and $1<p\leqslant 2$, where the sharp constants a_p and b_p are given by

$$a_p=\frac{1}{\sin^p \frac{\pi}{p}}, \qquad b_p=\frac{2\left|\cos\frac{\pi}{p}\right|^{1-(p/2)}}{\sin\frac{\pi}{p}}, \tag{1.13}$$

where $b_2=2$ for $p=2$. The case $p>2$ is handled by duality. As we will show, the extremal points for (1.12) in $\mathbb{C}^2\backslash(0, 0)$ lie off the diagonal $w=z$, which explains why the best constants in the complex and real L^p inequalities are different. It is worth mentioning that before an analytic proof of (1.12) was found it was first verified using *Mathematica*.

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2. THE NORM OF THE RIESZ PROJECTION ON THE CIRCLE

THEOREM 2.1. *Let* $1 < p \leqslant 2$ *and* $f \in L^p(\mathbb{T})$ *be a complex-valued function such that* $f(\zeta) = \sum_{k \in \mathbb{Z}} \hat{f}(k)\, \zeta^k$ *where* $\zeta \in \mathbb{T}$. *Then*

$$\|\max(|P_+ f|, |P_- f|)\|_{L^p(\mathbb{T})} \leqslant \frac{1}{\sin \frac{\pi}{p}} \|f\|_{L^p(\mathbb{T})},$$

where $P_+ f(\zeta) = \sum_{k \geqslant 0} \hat{f}(k)\, \zeta^k$ *and* $P_- f(\zeta) = \sum_{k < 0} \hat{f}(k)\, \zeta^k$.

Proof. To proceed, we first need the following lemma.

LEMMA 2.2. *Let* $1 < p \leqslant 2$. *Then for every* $(w, z) \in \mathbb{C}^2$

$$\max(|w|^p, |z|^p) \leqslant a_p\, |w + \bar{z}|^p - b_p\, F(w, z), \tag{2.1}$$

where a_p *and* b_p *are defined by* (1.13), *and* $F(w, z)$ *is a PSH function defined by* $F(w, z) = \operatorname{Re}[(w\,z)^{p/2}]$. *More precisely,*

$$F(w, z) = r^{p/2} \rho^{p/2} \begin{cases} \cos\left[\dfrac{(t+\theta)\,p}{2}\right] & \text{if} \quad |t+\theta| \leqslant \pi, \\[2ex] \cos\left[\dfrac{(t+\theta)\,p}{2} - p\pi\right] & \text{if} \quad \pi < t+\theta \leqslant 2\pi, \\[2ex] \cos\left[\dfrac{(t+\theta)\,p}{2} + p\pi\right] & \text{if} \quad -2\pi \leqslant t+\theta < -\pi, \end{cases}$$

where $w = \rho e^{i\theta}$ *and* $z = re^{it}$ $(0 \leqslant \rho, r < \infty,\ -\pi \leqslant \theta, t \leqslant \pi)$.

Remark 2.3. Obviously $F(w, z)$ is a continuous function on $\mathbb{C}^2$. By Theorem 4.13 in [Ra], it is easily seen that $F(w, z)$ is a PSH function on $\mathbb{C}^2$ since $F(w, z) = \phi(w\,z)$ where $\phi(z)$ is a subharmonic function on $\mathbb{C}$ defined by

$$\phi(z) = \operatorname{Re}(z^{p/2}) = r^{p/2} \cos\left(\frac{tp}{2}\right), \qquad z = re^{it}, \qquad 0 \leqslant r < \infty, \quad |t| \leqslant \pi.$$

The fact that the function $\operatorname{Re}(z^\delta)$ is subharmonic for $0 < \delta = \frac{p}{2} \leqslant 1$ is easily verified: Note that $\phi(z)$ is harmonic if $z \in \mathbb{C} \setminus (-\infty, 0]$; it clearly satisfies the sub-mean-value inequality at $z = 0$, and $\phi(z) = \max[\phi_-(z), \phi_+(z)]$ if $\operatorname{Re} z < 0$, where ϕ_+ and ϕ_- are branches of $\operatorname{Re}(z^\delta)$ harmonic in the left half-plane.

Given the lemma, we can complete the proof of Theorem 2.1. Without loss of generality we may assume that f is a trigonometric polynomial $f(\zeta)=\sum_{k=-m}^{n}\hat{f}(k)\,\zeta^k$, $\zeta=e^{i\theta}\in\mathbb{T}$. Hence

$$f_+(\zeta)=P_+f(\zeta)=\sum_{k=0}^{n}\hat{f}(k)\,\zeta^k,\quad f_-(\zeta)=\overline{P_-f(\zeta)}=\sum_{k=-m}^{-1}\overline{\hat{f}(k)}\,\zeta^{-k}.$$

Note that f_+ and f_- are analytic trigonometric polynomials on $\mathbb{T}$; they can be extended to polynomials defined on $\mathbb{C}$. Putting in (2.1) the functions $f_+(\zeta)$ and $f_-(\zeta)$ in place of w and z, we obtain the inequality

$$\max\left(|f_+(\zeta)|^p, |f_-(\zeta)|^p\right)\leqslant\frac{|f(\zeta)|^p}{\sin^p\frac{\pi}{p}}-b_p\,F(f_+(\zeta), f_-(\zeta)),\tag{2.2}$$

where $\zeta\in\mathbb{T}$. Therefore, integrating both sides of (2.2) over $\mathbb{T}$ yields

$$\int_{\mathbb{T}}\max\left(|f_+(\zeta)|^p, |f_-(\zeta)|^p\right)|d\zeta| \leqslant\frac{1}{\sin^p\frac{\pi}{p}}\int_{\mathbb{T}}|f|^p\,|d\zeta|-b_p\int_{\mathbb{T}}F(f_+(\zeta), f_-(\zeta))\,|d\zeta|.$$

Since $f_-(z)$ and $f_+(z)$ are analytic functions, their composition with the plurisubharmonic function F yields a subharmonic function $F(f_+(z), f_-(z))$ on $\mathbb{C}$ ([Ra], Theorem 4.13). Also $F(f_+(0), f_-(0))=0$ because $f_-(0)=0$ and $F(w, 0)=0$ for every $w\in\mathbb{C}$ by the definition of F. Hence by the sub-mean-value inequality

$$\frac{1}{2\pi}\int_{\mathbb{T}}F(f_+(\zeta), f_-(\zeta))\,|d\zeta|\geqslant F(f_+(0), f_-(0))=0.$$

Noting that the constant b_p is positive for $1<p<\infty$ and applying the preceding inequality, we have

$$\int_{\mathbb{T}}\max\left(|f_+(\zeta)|^p, |f_-(\zeta)|^p\right)|d\zeta|\leqslant\frac{1}{\sin^p\frac{\pi}{p}}\int_{\mathbb{T}}|f|^p\,|d\zeta|.$$

Taking the p-th root of each side, we complete the proof of Theorem 2.1. ∎

It remains only to prove Lemma 2.2.

Proof. Note that by symmetry it suffices to show for $1 < p \leqslant 2$ that

$$|w|^p \leqslant a_p \, |w+\bar{z}|^p - b_p \, F(w, z), \tag{2.3}$$

for every $(w, z) \in \mathbb{C}^2$, where $F(w, z) = \text{Re}[(wz)^{p/2}]$. To prove (2.3) we start with the following observation: Inequality (2.3) is invariant under the transformation

$$(w, z) \to (\bar{\zeta} w, \zeta z), \qquad \zeta \in \mathbb{C}, \quad \zeta \neq 0.$$

Letting $\zeta = w/|w|^2$ so that $\bar{\zeta} w = 1$ and using the above transformation, we reduce (2.3), for any $w \in \mathbb{C}$, $w \neq 0$, to the inequality

$$1 \leqslant a_p \, |1+\zeta z|^p - b_p \, F(1, \zeta z). \tag{2.4}$$

Since (2.3) is obviously true for $w = 0$, it suffices to prove the following lemma.

LEMMA 2.4. *For all* $z \in \mathbb{C}$,

$$1 \leqslant a_p \, |1+z|^p - b_p \, \phi(z), \tag{2.5}$$

where $\phi(z) = F(1, z) = \text{Re}(z^{p/2})$ *and* $1 < p \leqslant 2$. *The inequality* (2.5) *is strict unless* $z = -\cos \frac{\pi}{p} \, e^{\pm \pi i/p}$.

Proof. If $p = 2$ then $a_2 = 1$, $b_2 = 2$, and (2.5) is obvious. To prove it for $1 < p < 2$, let $z = re^{it}$ and rewrite it in the equivalent form

$$1 \leqslant a_p \, (1 + 2r \cos t + r^2)^{p/2} - b_p \, r^{p/2} \cos \left(\frac{tp}{2} \right), \tag{2.6}$$

where $0 \leqslant r < \infty$, $|t| \leqslant \pi$. We set

$$H(t, r) = a_p \, (1 + 2r \cos t + r^2)^{p/2} - b_p \, r^{p/2} \cos \left(\frac{tp}{2} \right) - 1, \tag{2.7}$$

where (t, r) lies in the semi-infinite strip $S = [-\pi, \pi] \times [0, +\infty)$. We will show that

$$\min\{H(t, r) \colon (t, r) \in S\} = 0,$$

and the only points in S such that $H(t, r) = 0$ are $(\pm t^*, r^*)$ where $t^* = \frac{\pi}{p}$ and $r^* = |\cos \frac{\pi}{p}|$. To do this will require the following four steps.

Step 1. We show that the minimum of $H(t, r)$ is attained in the interior of S.

Clearly, for $r=0$ and $|t| \leqslant \pi$,

$$H(t, 0) = a_p - 1 = \frac{1}{\sin^p \frac{\pi}{p}} - 1 > 0.$$

If $r \to +\infty$, then for all $|t| \leqslant \pi$

$$H(t, r) \geqslant a_p\,(r-1)^p - b_p\, r^{p/2} - 1 \asymp \frac{r^p}{\sin^p \frac{\pi}{p}} > 0.$$

The first partial derivatives of H are given by

$$\frac{\partial H}{\partial t} = -a_p p(1+2r\cos t + r^2)^{(p/2)-1}\, r \sin t + b_p \frac{p}{2} r^{p/2} \sin\left(\frac{tp}{2}\right), \tag{2.8}$$

and

$$\frac{\partial H}{\partial r} = a_p p(1+2r\cos t + r^2)^{(p/2)-1}\,(r+\cos t) - b_p \frac{p}{2} r^{(p/2)-1} \cos\left(\frac{tp}{2}\right). \tag{2.9}$$

Note that $H(t, r)$ can be defined by (2.7) in a wider strip $[-2\pi, 2\pi] \times [0, +\infty)$, and hence (2.8) and (2.9) hold for $t = \pm\pi$ as well. For $t=\pi$, $0<r<\infty$, we get

$$\frac{\partial H}{\partial t} = b_p \frac{p}{2} r^{p/2} \sin\left(\frac{\pi p}{2}\right) > 0.$$

Similarly, $\frac{\partial H}{\partial t} < 0$ if $t = -\pi$, $0 < r < \infty$. Hence the minimum is attained in the interior of S.

Step 2. We now find the three critical points of $H(t, r)$ in the interior of S.

From (2.8) and (2.9) we see that the following equations hold at the critical points (t, r):

$$2a_p(1+2r\cos t + r^2)^{(p/2)-1} \sin t = b_p\, r^{(p/2)-1} \sin\left(\frac{tp}{2}\right), \tag{2.10}$$

and

$$2a_p(1+2r\cos t+r^2)^{(p/2)-1}\,(r+\cos t)=b_p\,r^{(p/2)-1}\cos\left(\frac{tp}{2}\right). \quad (2.11)$$

These equations are satisfied at $(\pm t^*, r^*)=(\pm\frac{\pi}{p}, |\cos\frac{\pi}{p}|)$ by our choice of b_p. However, (2.10) is also satisfied when $t=0$. Hence $(0, r_0)$ is a critical point if (2.11) holds for $r=r_0$:

$$2a_p(1+r_0)^{p-1}=b_p r_0^{(p/2)-1}. \quad (2.12)$$

The preceding equation can be rewritten in the form

$$(1+r_0)^{p-1}\,r_0^{1-(p/2)}=\left|\cos\frac{\pi}{p}\right|^{1-(p/2)}\sin^{p-1}\left(\frac{\pi}{p}\right). \quad (2.13)$$

The left-hand side of this equation is an increasing function of r_0, and hence (2.13) has a unique solution on $(0, \infty)$. We will show below in Step 4 that $H(t, r)$ has no other critical points in the interior of S.

Step 3. We next show that $(0, r_0)$ is a saddle point and hence is not a minimum point.

We evaluate the Hessian $\Phi(t, r)$ of H at $(0, r_0)$. The second partial derivatives are given by

$$\frac{\partial^2 H}{\partial r^2}=-a_p p(2-p)(1+2r\cos t+r^2)^{(p/2)-2}\,(r+\cos t)^2$$

$$+a_p p(1+2r\cos t+r^2)^{(p/2)-1}+b_p\,\frac{p\,(2-p)}{4}\,r^{(p/2)-2}\cos\left(\frac{tp}{2}\right),$$

$$\frac{\partial^2 H}{\partial r\partial t}=a_p p(2-p)\,(1+2r\cos t+r^2)^{(p/2)-2}\,(r+\cos t)\,r\sin t$$

$$-a_p p(1+2r\cos t+r^2)^{(p/2)-1}\sin t+b_p\,\frac{p^2}{4}\,r^{(p/2)-1}\sin\left(\frac{tp}{2}\right),$$

$$\frac{\partial^2 H}{\partial t^2}=-a_p p(2-p)\,(1+2r\cos t+r^2)^{(p/2)-2}\,r^2\sin^2 t$$

$$-a_p p(1+2r\cos t+r^2)^{(p/2)-1}\,r\cos t+b_p\,\frac{p^2}{4}\,r^{p/2}\cos\left(\frac{tp}{2}\right).$$

Hence

$$\frac{\partial^2 H}{\partial r^2}(0, r_0) = p(p-1)\, a_p(1+r_0)^{p-2} + b_p \frac{p\,(2-p)}{4} r_0^{(p/2)-2} > 0,$$

$$\frac{\partial^2 H}{\partial r \partial t}(0, r_0) = 0,$$

$$\frac{\partial^2 H}{\partial t^2}(0, r_0) = -pa_p(1+r_0)^{p-2}\, r_0 + b_p \frac{p^2}{4} r_0^{p/2}.$$

From this it follows that

$$\det \Phi(0, r_0) = \frac{\partial^2 H}{\partial r^2}(0, r_0)\, \frac{\partial^2 H}{\partial t^2}(0, r_0) < 0$$

if and only if $\frac{\partial^2 H}{\partial t^2}(0, r_0) < 0$, which is equivalent to the estimate

$$a_p(1+r_0)^{p-2} > \frac{p}{4} b_p r_0^{(p/2)-1}. \tag{2.14}$$

Using (2.12) we rewrite (2.14) in the equivalent form

$$r_0 < \frac{2}{p} - 1. \tag{2.15}$$

From this we see that $\det \Phi(0, r_0) < 0$ if and only if (2.15) holds. Let $s = \frac{2}{p} - 1$, $0 < s < 1$; then (2.15) takes the form $r_0 < s$, or equivalently,

$$(1+r_0)^{1-s}\, r_0^s < (1+s)^{1-s}\, s^s. \tag{2.16}$$

By (2.13) the left-hand side of the preceding inequality equals

$$(1+r_0)^{1-s}\, r_0^s = \cos^{1-s}\left(\frac{s\pi}{2}\right) \sin^s\left(\frac{s\pi}{2}\right). \tag{2.17}$$

Hence the Hessian determinant at $(0, r_0)$ is negative for every $1 < p < 2$ provided the inequality

$$\cos^{1-s}\left(\frac{s\pi}{2}\right) \sin^s\left(\frac{s\pi}{2}\right) < (1+s)^{1-s}\, s^s \tag{2.18}$$

holds for $0 < s < 1$. To prove (2.18), let

$$x = \frac{\cos\left(\frac{s\pi}{2}\right)}{1+s}, \qquad y = \frac{\sin\left(\frac{s\pi}{2}\right)}{s},$$

so that it boils down to $x^{1-s}\, y^s < 1$. Applying the inequality

$$x^{1-s}\, y^s \leqslant (1-s)\, x + sy$$

we get

$$x^{1-s} y^s \leqslant \frac{1-s}{1+s} \cos\left(\frac{s\pi}{2}\right) + \sin\frac{s\pi}{2}.$$

Thus the estimate $x^{1-s} y^s < 1$ holds if the right-hand side of the preceding inequality is less than 1, i.e., the following inequality is true:

$$\sin\left(\frac{s\pi}{2}\right) < 1 - \frac{1-s}{1+s} \cos\left(\frac{s\pi}{2}\right) = 1 - \cos\left(\frac{s\pi}{2}\right) + \frac{2s}{1+s} \cos\left(\frac{s\pi}{2}\right). \quad (2.19)$$

Rewrite both sides of (2.19) using the half-angle formula:

$$2 \sin\left(\frac{s\pi}{4}\right)\left(\cos\frac{s\pi}{4}\right) < 2 \sin^2\left(\frac{s\pi}{4}\right) + \frac{2s}{1+s}\left[\cos^2\left(\frac{s\pi}{4}\right) - \sin^2\left(\frac{s\pi}{4}\right)\right].$$

Dividing both sides of the preceding inequality by $2\cos^2\left(\frac{s\pi}{4}\right)$ we have

$$\tan\left(\frac{s\pi}{4}\right) < \tan^2\left(\frac{s\pi}{4}\right) + \frac{s}{1+s}\left[1 - \tan^2\left(\frac{s\pi}{4}\right)\right],$$

or equivalently

$$\frac{\tan\left(\frac{s\pi}{4}\right)}{1 + \tan\left(\frac{s\pi}{4}\right)} < \frac{s}{1+s}.$$

This is obviously equivalent to the inequality $\tan\left(\frac{s\pi}{4}\right) < s$ which follows from the convexity of $\tan\theta$ on $(0, \frac{\pi}{4})$. Thus (2.18) holds which proves that $(0, r_0)$ is a saddle point for H.

Step 4. We now show that besides $(0, r_0)$ and $(\pm t^*, r^*)$, there are no other critical points in the interior of S.

Since $H(t, r) = H(-t, r)$, and $(0, r_0)$ is the only critical point on the r-axis, it suffices to consider the case $0 < t < \pi$. Let (t, r) be a critical point. Then dividing the left-hand side and right-hand side of (2.11) by the corresponding parts of (2.10) respectively we get

$$r + \cos t = \sin t \cot\left(\frac{tp}{2}\right). \tag{2.20}$$

Hence

$$r = \sin t \cot\left(\frac{tp}{2}\right) - \cos t = \sin t\left[\cot\left(\frac{tp}{2}\right) - \cot t\right] = \frac{\sin\left(t\,\frac{2-p}{2}\right)}{\sin\left(\frac{tp}{2}\right)}.$$

Thus, if (t, r) is a critical point for $t \in (0, \pi)$, then

$$r = \frac{\sin\left(t\,\frac{2-p}{2}\right)}{\sin\left(\frac{tp}{2}\right)}. \tag{2.21}$$

Taking squares of both sides in (2.20) we obtain

$$r^2 + 2r\cos t + 1 = \frac{\sin^2 t}{\sin^2\left(\frac{tp}{2}\right)}. \tag{2.22}$$

Substituting (2.22) into (2.10) we get the following equation for $t \in (0, \pi)$:

$$\frac{\sin^{p-1}(t)\left[\sin\left(t\,\frac{2-p}{2}\right)\right]^{(2-p)/2}}{\sin^{p/2}\left(\frac{tp}{2}\right)} = \frac{b_p}{2a_p} = \left|\cos\frac{\pi}{p}\right|^{1-(p/2)} \sin^{p-1}\left(\frac{\pi}{p}\right). \tag{2.23}$$

Since r is determined from (2.21), it remains to show that $t = \frac{\pi}{p}$ is the only solution of (2.23) on $(0, \pi)$. To this end we verify that the left-hand side of (2.23) is a decreasing function of $t \in (0, \pi)$. Taking the logarithm of the left-hand side of (2.23) we set

$$h(t) = (p-1)\ln(\sin t) + \frac{2-p}{2}\ln\left[\sin\left(t\,\frac{2-p}{2}\right)\right] - \frac{p}{2}\ln\left[\sin\left(\frac{tp}{2}\right)\right]. \tag{2.24}$$

We show

$$h'(t) = (p-1)\cot(t) + \frac{(2-p)^2}{4}\cot\left(t\,\frac{2-p}{2}\right) - \frac{p^2}{4}\cot\left(\frac{tp}{2}\right) < 0.$$

Since

$$\frac{(2-p)^2}{p^2} = 1 - \frac{4(p-1)}{p^2},$$

we have

$$h'(t) = \frac{p^2}{4}\left[\frac{4(p-1)}{p^2}\left(\cot t - \cot\left(t\,\frac{2-p}{2}\right)\right) + \left(\cot\left(t\,\frac{2-p}{2}\right) - \cot\left(\frac{tp}{2}\right)\right)\right]$$

$$= \frac{p^2}{4}\,\frac{1}{\sin\left(t\,\dfrac{2-p}{2}\right)}\left[-\frac{4(p-1)}{p^2}\,\frac{\sin\left(\dfrac{tp}{2}\right)}{\sin t} + \frac{\sin(t(p-1))}{\sin\left(\dfrac{tp}{2}\right)}\right].$$

Hence $h'(t) < 0$ if and only if

$$\frac{\sin t\,\sin(t(p-1))}{\sin^2\left(\dfrac{tp}{2}\right)} < \frac{4(p-1)}{p^2}. \tag{2.25}$$

Observe that

$$1 - \frac{\sin t\,\sin\,(t(p-1))}{\sin^2\left(\dfrac{tp}{2}\right)} = \frac{\sin^2\left(\dfrac{tp}{2}\right) - \dfrac{1}{2}\,[\cos(t(2-p)) - \cos(tp)]}{\sin^2\left(\dfrac{tp}{2}\right)}$$

$$= \frac{2\sin^2\left(\dfrac{tp}{2}\right) + \cos(tp) - \cos(t(2-p))}{2\sin^2\left(\dfrac{tp}{2}\right)}$$

$$= \frac{1-\cos(t(2-p))}{2\sin^2\left(\dfrac{tp}{2}\right)} = \left[\frac{\sin\left(t\,\dfrac{2-p}{2}\right)}{\sin\left(\dfrac{tp}{2}\right)}\right]^2.$$

Hence (2.25) is equivalent to the inequality

$$1-\frac{4(p-1)}{p^2}=\frac{(2-p)^2}{p^2}<\left[\frac{\sin\left(t\,\frac{2-p}{2}\right)}{\sin\left(\frac{tp}{2}\right)}\right]^2. \tag{2.26}$$

Clearly the preceding inequality holds if and only if

$$\frac{\sin\left(t\,\frac{2-p}{2}\right)}{2-p}>\frac{\sin\left(\frac{tp}{2}\right)}{p}. \tag{2.27}$$

To prove (2.27) we set

$$g(t)=\frac{\sin\left(t\,\frac{2-p}{2}\right)}{2-p}-\frac{\sin\left(\frac{tp}{2}\right)}{p}.$$

Then $g(0)=0$ and

$$g'(t)=\frac{1}{2}\left[\cos\left(t\,\frac{2-p}{2}\right)-\cos\left(\frac{tp}{2}\right)\right]=\sin\left(\frac{t}{2}\right)\sin\left(t\,\frac{p-1}{2}\right)>0.$$

Thus g is increasing and (2.27) holds. From this it follows that the function $h(t)$ defined by (2.24) is decreasing, and hence (2.23) has a unique solution $t^*=\frac{\pi}{p}$ on $(0,\pi)$, which is the only critical point if $0<t<\pi$. This completes the proof of Step 4.

It is not difficult to verify that $H(t,r)$ has a local minimum at $(t^*,r^*)=(\frac{\pi}{p},|\cos\frac{\pi}{p}|)$. Indeed, the Hessian of H at (t^*,r^*) is given by

$$\Phi(t^*,r^*)=\begin{pmatrix} \dfrac{p}{\sin^2\dfrac{\pi}{p}} & \dfrac{-p(2-p)}{2\sin\dfrac{\pi}{p}} \\ \dfrac{-p(2-p)}{2\sin\dfrac{\pi}{p}} & \dfrac{p(p-1)\cos^2\dfrac{\pi}{p}}{\sin^2\dfrac{\pi}{p}} \end{pmatrix},$$

and hence

$$\det\Phi(t^*,r^*)=\frac{p^4}{4\sin^4\dfrac{\pi}{p}}\left[\cos^2\frac{\pi}{p}-\frac{(2-p)^2}{p^2}\right].$$

Letting $s=\frac{2}{p}-1$ $(0<s<1)$ again we get

$$\det \Phi(t^*, r^*) = \frac{4}{(1+s)^4 \cos^4\left(\dfrac{s\pi}{2}\right)} \left[\sin^2\left(\frac{s\pi}{2}\right) - s^2\right] > 0,$$

and so $\Phi(t^*, r^*)$ is positive definite. From the discussion above we have

$$\min\{H(r, t) : (r, t) \in S\} = H(\pm t^*, r^*) = 0,$$

which completes the proofs of Lemma 2.4 and Theorem 2.1. ∎

COROLLARY 2.5. *Let $1<p<\infty$. Then*

$$\| P_+ \|_p = \| P_- \|_p = \frac{1}{\sin \dfrac{\pi}{p}}. \tag{2.28}$$

Because it has already been shown in [GKr1] that the estimates hold from below, Corollary 2.5 is immediate from Theorem 2.1.

COROLLARY 2.6. *Let $1<p<\infty$. Then*

$$\left\| \frac{f+i\tilde{f}}{2} \right\|_{L^p(\mathbb{T})} \leqslant \frac{1}{\sin \dfrac{\pi}{p}} \|f\|_{L^p(\mathbb{T})}, \tag{2.29}$$

for every complex-valued $f \in L^p(\mathbb{T})$. The constant in this inequality is sharp.

Proof. Let $Af = \frac{f+i\tilde{f}}{2}$. Then $A-P_+$ is a rank one operator, and the fact that the constant in (2.29) is sharp follows from (1.5). Applying Lemma 2.2 with $\frac{f+i\tilde{f}}{2}$ in place of w and $\frac{f-i\tilde{f}}{2}$ in place of $\bar{z}$, and proceeding as in the proof of Theorem 2.1 for $1<p\leqslant 2$, we get

$$\left\| \frac{f+i\tilde{f}}{2} \right\|^p_{L^p(\mathbb{T})} + b_p \frac{|\hat{f}(0)|^p}{2^p} \leqslant \frac{1}{\sin^p \dfrac{\pi}{p}} \|f\|^p_{L^p(\mathbb{T})}. \tag{2.30}$$

The case $p>2$ follows by duality. ∎

Remark 2.7. We can prove that the constant in (2.29) is sharp by considering a function of the form $f_\gamma = \alpha \operatorname{Re} g_\gamma + i\beta \operatorname{Im} g_\gamma$, where $\alpha, \beta \in \mathbb{R}$ and $g_\gamma(z) = (\frac{1+z}{1-z})^{2\gamma/\pi}$. Note that g_γ is the function used in [P] such that

$|\text{Im } g_\gamma| = (\tan\gamma) \text{ Re } g_\gamma$ on $\mathbb{T}$. (A similar function was used in [GKr1] on $\mathbb{R}$.) To show the constant is sharp, we need only compute

$$\frac{\|f_\gamma + i\tilde{f}_\gamma\|_{L^p(\mathbb{T})}}{\|f_\gamma\|_{L^p(\mathbb{T})}} = \frac{\left(\int_{-\pi}^{\pi} |\alpha+\beta|^p \, |\text{Re } g_\gamma(e^{i\theta}) + i \text{ Im } g_\gamma(e^{i\theta})|^p \, d\theta\right)^{1/p}}{\left(\int_{-\pi}^{\pi} |\alpha \text{ Re } g_\gamma(e^{i\theta}) + i\beta \text{ Im } g_\gamma(e^{i\theta})|^p \, d\theta\right)^{1/p}}$$

$$= \frac{2^{1/p} \, |\alpha+\beta|}{(|\alpha\cos\gamma + i\beta\sin\gamma|^p + |\alpha\cos\gamma - i\beta\sin\gamma|^p)^{1/p}}.$$

Substituting $\alpha = \sin^2\gamma$ and $\beta = \cos^2\gamma$ into the above, we have

$$\frac{\|f_\gamma + i\tilde{f}_\gamma\|_{L^p(\mathbb{T})}}{\|f_\gamma\|_{L^p(\mathbb{T})}} = \frac{2}{\sin 2\gamma}.$$

Letting $\gamma \uparrow \frac{\pi}{2p}$ yields the desired result.

3. THE NORM OF THE HALF-LINE MULTIPLIER

THEOREM 3.1. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R})$ be a complex-valued function. Then*

$$\|P_\pm f\|_{L^p(\mathbb{R})} \leqslant \frac{1}{\sin\dfrac{\pi}{p}} \|f\|_{L^p(\mathbb{R})}, \tag{3.1}$$

where $\frac{1}{\sin \pi/p}$ is the best constant.

The lower estimate $\|P_\pm\|_p \geqslant \frac{1}{\sin\pi/p}$ is known (see [GKr1, GKr3]). We prove Theorem 3.1 by reducing it to the periodic case and applying Corollary 2.6. This idea is due to A. Zygmund ([Z], Chapter XVI, Theorem 3.8). Other proofs can be given by modifying the argument used in the proof of Theorem 2.1 to the nonperiodic situation (cf. [E, Gr]).

Proof. Noting that $P_+^{\mathbb{R}} f = \frac{1}{2}(f + iH^{\mathbb{R}} f)$ we will need to show

$$\|f + iH^{\mathbb{R}} f\|_{L^p(\mathbb{R})} \leqslant c_p \, \|f\|_{L^p(\mathbb{R})}, \qquad 1 < p < \infty, \tag{3.2}$$

where $c_p = \frac{2}{\sin \pi/p}$. Letting $f = u + iv$, where u and v are the real and imaginary parts of f, respectively, we define the functions

$$g_n(x) = \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} u(t) \cot \frac{x-t}{2n} \, dt \qquad \text{and}$$

$$h_n(x) = \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} v(t) \cot \frac{x-t}{2n} \, dt.$$

As Zygmund proves [Z], $g_n \to H^{\mathbb{R}} u$ and $h_n \to H^{\mathbb{R}} v$ a.e. A substitution shows that when $|y| < \pi$, $g_n(ny) = H^{\mathbb{T}}(u(ny))$ and $h_n(ny) = H^{\mathbb{T}}(v(ny))$ are the conjugate functions (periodic Hilbert transforms) of $u(ny)$ and $v(ny)$, respectively. Therefore, by letting $x = ny$ we have

$$\left(\int_{-\pi n}^{\pi n} |f(x) + i(g_n(x) + ih_n(x))|^p \, dx \right)^{1/p}$$
$$= \left(n \int_{-\pi}^{\pi} |f(ny) + iH^{\mathbb{T}}(f(ny))|^p \, dy \right)^{1/p}.$$

We now use Corollary 2.6 to obtain

$$\left(\int_{-\pi}^{\pi} |f(ny) + iH^{\mathbb{T}}(f(ny))|^p \, dy \right)^{1/p} \leqslant c_p \left(\int_{-\pi}^{\pi} |f(ny)|^p \, dy \right)^{1/p}.$$

Combining the above results yields

$$\left(\int_{-\pi n}^{\pi n} |f(x) + i(g_n(x) + ih_n(x))|^p \, dx \right)^{1/p} \leqslant c_p \left(n \int_{-\pi}^{\pi} |f(ny)|^p \, dy \right)^{1/p}$$
$$= c_p \left(\int_{-\pi n}^{\pi n} |f(x)|^p \, dx \right)^{1/p} \leqslant c_p \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}.$$

We can now apply Fatou's Lemma to obtain

$$\left(\int_{-a}^{a} |f + iH^{\mathbb{R}} f|^p \, dx \right)^{1/p} \leqslant c_p \left(\int_{\mathbb{R}} |f|^p \, dx \right)^{1/p}$$

for every $a > 0$. Letting $a \to +\infty$ yields (3.2). A similar argument proves the result for P_-f. Thus, Theorem 3.1 is proved. ∎

Let $\mathcal{S}_{\pm}^{j} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_j \lessgtr 0\}$, $j = 1, ..., n$. Define by $P_{\pm}^{j} f = \mathcal{F}^{-1}(\chi_{\mathcal{S}_{\pm}^{j}} \mathcal{F} f)$ the corresponding half-space multipliers. The following corollary is derived from Theorem 3.1 by applying the one-dimensional Riesz projection to the j-variable as is shown in [St], Chapter 4, Sec. 4.2.3. (The lower estimate $\|P_{\pm}^{j}\|_{L^p(\mathbb{R}^n)} \geqslant \frac{1}{\sin \pi/p}$ follows as in the one-dimensional case.)

COROLLARY 3.2. *Let* $1 < p < \infty$. *Then*

$$\| P_{\pm}^{j} f \|_{L^p(\mathbb{R}^n)} \leqslant \frac{1}{\sin \frac{\pi}{p}} \| f \|_{L^p(\mathbb{R}^n)}, \qquad j = 1, ..., n,$$

for every complex-valued $f \in L^p(\mathbb{R}^n)$, *where the constants in the preceding inequalities are sharp.*

4. BEST CONSTANTS FOR WEIGHTED L^p SPACES

Let $L^p(\mathbb{R}, |x|^\alpha)$ $(1 < p < \infty, \alpha > -1)$ be a weighted L^p space on the line with norm

$$\| f \|_{L^p(\mathbb{R}, |x|^\alpha)} = \left\{ \int_{\mathbb{R}} |f(x)|^p \, |x|^\alpha \, dx \right\}^{1/p}.$$

We may restrict ourselves to the case $-1 < \alpha < p - 1$ when the weight $|x|^\alpha$ is in the Muckenhoupt class A_p (see e.g. [Gar]) and hence the operators $P_{\pm} = P_{\pm}^{\mathbb{R}}$ and $H = H^{\mathbb{R}}$ defined in the Introduction are continuous on $L^p(\mathbb{R}, |x|^\alpha)$.

We denote by $\| A \|_{p, \alpha}$ and $|\!|\!| A |\!|\!|_{p, \alpha}$ respectively the norm and essential norm of the operator A on the weighted space $L^p(\mathbb{R}, |x|^\alpha)$. For the Hilbert transform H, they were computed in [KrV2] (see also [GKr3], Theorem 3.1, p. 208),

$$\| H \|_{p, \alpha} = |\!|\!| H |\!|\!|_{p, \alpha} = \cot \frac{\pi}{2r}, \qquad r = \max \left(p, \frac{p}{p-1}, \frac{p}{1+\alpha}, \frac{p}{p-1-\alpha} \right), \tag{4.1}$$

both on the line and circle.

For $P_{\pm}$, the following conjecture was stated in [KrV1]:

$$\| P_{\pm} \|_{p, \alpha} = \frac{1}{\sin \frac{\pi}{r}} = \max \left(\frac{1}{\sin \frac{\pi}{p}}, \frac{1}{\sin \frac{\pi(1+\alpha)}{p}} \right). \tag{4.2}$$

In the case $p = 2$, it is not difficult to see that (4.2) is equivalent to (4.1); independent proofs can be found in [Kr], [KrV1] (see also an interpolation proof below). The corresponding lower estimate is known ([GKr3], Theorem 9.1, p. 101):

$$\| P_\pm \|_{p,\alpha} \geqslant ||| P_\pm |||_{p,\alpha} \geqslant \max\left(\frac{1}{\sin\frac{\pi}{p}}, \frac{1}{\sin\frac{\pi(1+\alpha)}{p}}\right). \tag{4.3}$$

It remains to prove the following inequalities:

$$\| P_\pm \|_{p,\alpha} \leqslant \frac{1}{\sin\frac{\pi}{p}} \qquad \text{if} \quad \alpha \in [\min(0, p-2), \max(0, p-2)], \tag{4.4}$$

and

$$\| P_\pm \|_{p,\alpha} \leqslant \frac{1}{\sin\frac{\pi(1+\alpha)}{p}} \qquad \text{if} \quad \alpha \notin [\min(0, p-2), \max(0, p-2)]. \tag{4.5}$$

THEOREM 4.1. *Let* $1<p<\infty$, $-1<\alpha<p-1$. *Then* (4.4) *and* (4.5) *hold*; *i.e., for any complex-valued function* $f \in L^p(\mathbb{R}, |x|^\alpha)$,

$$\| P_\pm f \|_{L^p(\mathbb{R}, |x|^\alpha)} \leqslant \max\left(\frac{1}{\sin\frac{\pi}{p}}, \frac{1}{\sin\frac{\pi(1+\alpha)}{p}}\right) \| f \|_{L^p(\mathbb{R}, |x|^\alpha)}. \tag{4.6}$$

The constant in the preceding inequality is sharp so that (4.2) holds. The corresponding lower estimate $\| P_\pm \|_{p,\alpha} \geqslant \max\left(\frac{1}{\sin(\pi/p)}, \frac{1}{\sin \pi(1+\alpha)/p}\right)$ follows from (4.3).

Proof. To prove (4.4), define the operator B: $L^p(\mathbb{R}, |x|^{p-2}) \to L^p(\mathbb{R})$ by $Bf(x) = \frac{1}{x} f(\frac{1}{x})$. Then it is easily seen that $BP_\pm f = P_\mp Bf$, and (cf. [GKr3], p. 207)

$$\| Bf \|_{L^p(\mathbb{R})} = \| f \|_{L^p(\mathbb{R}, |x|^{p-2})}. \tag{4.7}$$

Hence, by (4.7) and Theorem 3.1,

$$\begin{aligned} \| P_\pm f \|_{L^p(\mathbb{R}, |x|^{p-2})} &= \| BP_\pm f \|_{L^p(\mathbb{R})} = \| P_\mp Bf \|_{L^p(\mathbb{R})} \\ &\leqslant \frac{1}{\sin\frac{\pi}{p}} \| Bf \|_{L^p(\mathbb{R})} = \frac{1}{\sin\frac{\pi}{p}} \| f \|_{L^p(\mathbb{R}, |x|^{p-2})}, \end{aligned}$$

which proves (4.4) for $\alpha = p-2$. Then using interpolation between $L^p(\mathbb{R})$ and $L^p(\mathbb{R}, |x|^{p-2})$ we obtain (4.4) for $\min(0, p-2) \leqslant \alpha \leqslant \max(0, p-2)$.

The proof of (4.5), which uses the same idea as in [KrV2], is based on complex interpolation between the spaces $L^s(\mathbb{R})$ and $L^r(\mathbb{R}, |x|^{r-2})$, where $s<p<r$. It suffices to prove (4.5) for $p \geqslant 2$, $\alpha \notin [0, p-2]$, since the case $p<2$ then follows by duality. We first consider the case where $p \geqslant 2$ and $p-2<\alpha<p-1$. Set $r=\frac{p}{p-1-\alpha}$ and $s=\frac{p}{1+\alpha}$ so that $s<p<r$. Then $\frac{1}{p}=\frac{1-t}{r}+\frac{t}{s}$, $\alpha=\frac{p}{r}(1-t)(r-2)$, where $t=\frac{2-p+\alpha}{2-p+2\alpha} \in (0, 1)$. Observe that

$$\|P_\pm\|_s = \frac{1}{\sin\frac{\pi}{s}} = \frac{1}{\sin\frac{\pi(1+\alpha)}{p}} \quad \text{and} \quad \|P_\pm\|_{r,r-2} = \frac{1}{\sin\frac{\pi}{r}} = \frac{1}{\sin\frac{\pi(1+\alpha)}{p}}.$$

Then by complex interpolation with a change of weights,

$$\|P_\pm\|_{p,\alpha} \leqslant \|P_\pm\|_{r,r-2}^{1-t} \, \|P_\pm\|_s^t = \frac{1}{\sin\frac{\pi(1+\alpha)}{p}}.$$

In particular, for $p=2$ and $-1<\alpha<1$ this, together with (4.3), gives another proof of the statement

$$\|P_\pm\|_{2,\alpha} = \frac{1}{\sin\frac{\pi(1+\alpha)}{2}} = \frac{1}{\sin\frac{\pi(1-|\alpha|)}{2}}, \tag{4.8}$$

since the case $-1<\alpha<0$ follows by duality.

We now consider the case $p>2$, $-1<\alpha<0$. Let $r=\frac{p}{1+\alpha}$, $s=2$, and $\beta=\frac{2+2\alpha-p}{p}$. Then $2<p<r$, $-1<\beta<0$, and again using interpolation and (4.8), we get:

$$\|P_\pm\|_{p,\alpha} \leqslant \|P_\pm\|_{2,\beta}^{1-t} \, \|P_\pm\|_r^t = \frac{1}{\sin\frac{\pi(1+\alpha)}{p}},$$

where $\frac{1}{p}=\frac{1-t}{2}+\frac{t}{r}$, $\alpha=\frac{p}{2}(1-t)\,\beta$. ∎

In the periodic case, denote by $L^p(\mathbb{T}, |\zeta-\zeta_0|^\alpha)$ $(\zeta_0 \in \mathbb{T})$ the weighted space with norm

$$\|f\|_{L^p(\mathbb{T}, |\zeta-\zeta_0|^\alpha)} = \left\{ \int_{\mathbb{T}} |f(\zeta)|^p \, |\zeta-\zeta_0|^\alpha \, |d\zeta| \right\}^{1/p}.$$

Then the following theorem holds (the proof is analogous to that of Theorem 4.1).

THEOREM 4.2. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$, *and* $f \in L^p(\mathbb{T}, |\zeta - \zeta_0|^\alpha)$, $\zeta_0 \in \mathbb{T}$. *Then*

$$\|P_\pm^{\mathbb{T}} f\|_{L^p(\mathbb{T}, |\zeta-\zeta_0|^\alpha)} \leqslant \max\left(\frac{1}{\sin\frac{\pi}{p}}, \frac{1}{\sin\frac{\pi(1+\alpha)}{p}}\right) \|f\|_{L^p(\mathbb{T}, |\zeta-\zeta_0|^\alpha)}.$$

As in the nonperiodic case, the constant in the preceding inequality is sharp, so that $\|P_\pm^{\mathbb{T}}\|_{p,\alpha} = \max\left(\frac{1}{\sin(\pi/p)}, \frac{1}{\sin\pi(1+\alpha)/p}\right)$.

The same formula is valid for the norm of the half-space Fourier multiplier (see Section 3) on the weighted space $L^p(\mathbb{R}^n, |x|^\alpha)$, for $1 < p < \infty$, and $-1 < \alpha < p-1$.

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