Some $C^0$-continuous mixed formulations for general dipolar linear gradient elasticity boundary value problems and the associated energy theorems

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Abstract

The goal of this work is a systematic presentation of some classes of mixed weak formulations, for general multi-dimensional dipolar gradient elasticity (fourth order) boundary value problems. The displacement field main variable is accompanied by the double stress tensor and the Cauchy stress tensor (case 1 or $\mu - \tau - u$ formulation), the double stress tensor alone (case 2 or $\mu - u$ formulation), the double stress, the Cauchy stress, the displacement second gradient and the standard strain field (case 3 or $\mu - \tau - \kappa - e - u$ formulation) and the displacement first gradient, along with the equilibrium stress (case 4 or $u - \theta - \gamma$ formulation). In all formulations, the respective essential conditions are built in the structure of the solution spaces. For cases 1, 2 and 4, one-dimensional analogues are presented for the purpose of numerical comparison. Moreover, the standard Galerkin formulation is depicted. It is noted that the standard Galerkin weak form demands $C^1$-continuous conforming basis functions. On the other hand, up to first order derivatives appear in the bilinear forms of the current mixed formulations. Hence, standard $C^0$-continuous conforming basis functions may be employed in the finite element approximations. The main purpose of this work is to provide a reference base for future numerical applications of this type of mixed methods. In all cases, the associated quadratic energy functionals are formed for the purpose of completeness.

Keywords: Mixed formulations; Dipolar gradient elasticity; Mixed finite elements

1. Introduction

The scope of this work is a systematic presentation of some classes of mixed weak formulations, for general multi-dimensional dipolar gradient elasticity (fourth order) boundary value problems (Mindlin, 1964; Bleustein, 1967; Mindlin and Eshel, 1968; Georgiadis et al., 2004). Besides the displacement field, the main variables are the double stress tensor and the Cauchy stress tensor (case 1 or $\mu - \tau - u$ formulation), the double stress
tensor (case 2 or $\mu - u$ formulation), the double stress, the Cauchy stress, the displacement second gradient and the standard strain field (case 3 or $\mu - \tau - \kappa - \varepsilon - u$ formulation) and the displacement first gradient, along with the equilibrium stress (case 4 or $u - \theta - \gamma$ formulation). Moreover, for cases 1, 2 and 4, one-dimensional analogues are presented for the purpose of numerical comparison.

The main objective of this work is to provide a broad reference base for numerical applications of this type of mixed methods. The solution function spaces are appropriately defined and the final weak forms have the standard (symmetric) mixed structure. Furthermore, in all cases, the associated quadratic functionals (whose first variations result in the presented mixed formulations) are formed for the purpose of completeness.

The structure of the mixed methods of cases 1, 2 and 3 are based on generalizations of the Ciarlet–Raviart method (Ciarlet, 1978; Babuška et al., 1980), the Herrmann–Miyoshi technique (Herrmann, 1967; Miyoshi, 1973; Babuška et al., 1980) and the mixed method for biharmonic equations of Balasundaram and Bhattacharyya (1984), Bhattacharyya and Nataraj (2002). Theoretical analysis of the mixed method of case 1 (for both continuous and discrete formulations) can be found in Markolefas et al. (2007). Moreover, an alternative mixed formulation is developed in Markolefas et al. (2007), for the case of no coupling terms in the constitutive equations (with main variables the displacement field and the double stress tensor). In Section 5 of the current work, the latter formulation is extended to the more general case of non-zero coupling terms in the constitutive equations ($\mu - u$ formulation).

The mixed method of case 4 ($u - \theta - \gamma$) is a proper generalization of a mixed technique for Kirchhoff plate bending, see Braess (1997). It is noted that the mixed technique developed in Amanatidou and Aravas (2002), for general gradient elasticity problems (stated in Form I), is close in structure to the mixed method of case 4. Certain complications arise in the formulation $u - \theta - \gamma$, related to the satisfaction of the essential and natural conditions associated with the normal derivatives. The problem is resolved by proper decomposition of the trace of the displacement-gradient variable on the surface (with the introduction of a Lagrange multiplier on the surface).

The basic characteristic of the gradient elasticity theory (and the main difference from classical elasticity) is that the strain energy density is a positive-definite functional of the standard strain (as in the classical elasticity) and the second gradient of the displacement field (Form I) or the first gradient of strain (Form II). Other forms may be found in the literature, see Mindlin (1964), Mindlin and Eshel (1968). More general theories have been also developed, see for example Green and Rivlin (1964).

Energy theorems and weak mixed formulations of Hellinger–Reissner type are introduced in Georgiadis and Grentzelou (2006). Continuous/Discontinuous methods may be found in Engel et al. (2002). Boundary element techniques are developed in Polyzos et al. (2003), Karlis et al. (2007).

The new material constants, which relate generalized stress variables with generalized strains, contain certain characteristic lengths associated with the size and topology of the material micro-structure. In this way, size effects are incorporated in the stress analysis. Typical gradient elasticity models are concerned with materials having periodic microstructure like crystals (crystal lattices), polycrystal materials (crystallites), polymers (molecules) and granular materials (grains); the respective micro-media are shown in the parentheses (Mindlin, 1964).

The one-dimensional gradient elasticity model considered in the current work (see Section 2), is based on constitutive equations which take into account surface energy effects (Vardoulakis and Sulem, 1995). In terms of structure, the respective constitutive equations have the general form (with coupling terms), introducing two non-standard material constants, in addition to the standard Young’s modulus of Hooke’s law in one-dimension. For small characteristic lengths (small material constants), strong boundary layers appear in the stress fields at the boundary points. The numerical results of Section 8 demonstrate the effectiveness of the presented mixed formulations for both h- and p-extensions.

Resuming, the paper is organized into the following parts: The basic mathematical nomenclature and the dipolar gradient elasticity boundary value problems are introduced in Section 2. Section 3 contains the standard Galerkin formulations. Section 4 depicts the mixed formulation $\mu - \tau - u$ (case 1). The mixed formulation $\mu - u$ (case 2) is derived in Section 5, for general structure of constitutive equations. Section 6 introduces the mixed formulation $\mu - \tau - \kappa - \varepsilon - u$ (case 3). The mixed formulation $u - \theta - \gamma$ (case 4) is derived in Section 7. Numerical results, based on the one-dimensional analogues of cases 1, 2 and 4, are presented in Section 8. Finally, Section 9 contains closing remarks, conclusions and certain future research directions.
2. Nomenclature and definition of the strong forms of dipolar gradient elasticity problems

Let \( V \) be a bounded open subset of \( \mathbb{R}^n \) (\( n \) is the space dimension), with Lipschitz continuous boundary \( S = \partial V \). The standard notation for Sobolev spaces and norms is employed (Adams, 1975; Ciarlet, 1978; Kardestuncer and Norrie, 1987; Braess, 1997). We denote by \( \|w\|_r \), \( r \in \mathbb{R} \), the \( H^r \) norm of the (real valued) function \( w \), defined on \( V \). The Sobolev space \( H^r(V) \) is the space of functions \( w \) with finite norm \( \|w\|_r \). Note that \( r = 0 \) corresponds to \( L^2(V) := H^0(V) \) (space of square integrable functions on \( V \)).

The space \( H^1_0\times E(V) \) is the subspace of functions \( w \in H^1(V) \), such that \( w = 0 \) on \( S_E \subset \partial V \) (in the sense of trace). The space \( H^1_0(V) \) is the subspace of functions \( w \in H^1(V) \), such that \( w = 0 \) on \( \partial V \). Sobolev spaces are also defined on the boundary \( S = \partial V \) or on parts of the boundary. For example: \( H^1_0(S_E) = \{w|_{S_E} : w \in H^1(V)\} \) and \( H^1_0(S_E) = \{w|_{S_E} : w \in H^1(V) \cap V, w = 0 \text{ on } \partial V \setminus S_E\} \). The definitions of more general boundary function spaces may be found in Adams (1975), Kardestuncer and Norrie (1987).

The partial derivative of a given function \( y(x) \) with respect to \( x_i \) is denoted by \( \partial y/\partial x_i \) (Cartesian coordinates). The ordinary derivative \( du/dx \) is denoted by \( u' \). Standard indicial notation is used throughout the current work. The (underscored) symbol \( \underline{z} \) represents a vector (with components \( z_i \)) or a 2nd-rank tensor (with components \( z_{ij} \)) or a 3rd-rank tensor (with components \( z_{ijk} \)). Moreover, \( \underline{w} \) may denote an ordered tuple of (vector valued or tensor valued) functions.

The strong form of the model problem to be considered is (see Fig. 1):

2.1. Strong formulation (general multi-dimensional dipolar)

Gradient elasticity formulation, Forms I, II (Mindlin, 1964; Bleustein, 1967; Mindlin and Eshel, 1968; Markolefas et al., 2007). Find \( u_i \in H^r_{S_E}(V) \), \( i = 1, 2, \ldots, n, r \geq 4 \), such that,

\[
\partial_j (\tau_{jk} - \partial_i \mu_{ijk}) + f_k - \partial_j \Phi_{jk} = 0 \text{ in } V \tag{2.1}
\]

(equilibrium equations):

\[
\text{traction boundary conditions):}
\]

\[
n_j (\tau_{jk} - \partial_i \mu_{ijk}) - D_j (n_i \mu_{ijk}) + (D_m n_i) n_j n_i \mu_{ijk} = t_k + n_j \Phi_{jk} + (D_m n_i) n_j T_{jk} - D_j T_{jk} \quad \text{on } S^k_{N,J} \tag{2.2a}
\]

(moment boundary conditions):

\[
n_n n_j \mu_{ijk} = n_j T_{jk} \quad \text{on } S^k_{N,m} \tag{2.2b}
\]

Fig. 1. Domain nomenclature for the general gradient elasticity problem.
\( (\text{jump conditions}) \):
\[
[m, n, j] \mu_{j k} = [m, T_{j k}] \quad \text{on } C
\] (2.2c)

\( (\text{general constitutive equations}) \):
\[
ev_{ij} = C_{ijkl} \tau_{kl} + F_{klm} \mu_{klm}
\] (2.3a)
\[
\kappa_{ijkl} = A_{ijklm} \mu_{lmn} + F_{ijklm} \tau_{lm}
\] (2.3b)

\( (\text{strain–displacement, gradient–displacement relations}) \):
\[
ev_{ij} := \frac{1}{2} (\partial_i u_j + \partial_j u_i) := u_{(ij)}
\] (2.4a)
\[
\kappa_{ij} := \partial_i \partial_j u_k \quad (\text{Form I statement})
\] (2.4b)
or
\[
\kappa_{ij} := \partial_i \tilde{e}_{jk} \quad (\text{Form II statement})
\] (2.4c)

The variables and parameters that appear in the above statements are defined in the following:

- \( u_i \) components of the displacement field
- \( \tau_{ij} \) components of the (symmetric) Cauchy stress tensor
- \( \mu_{ijkl} \) components of the double stress tensor (recall the symmetry conditions \( \mu_{ijk} = \mu_{jik} \) for Form I and \( \mu_{ijkl} = \mu_{ilkj} \) for Form II)
- \( f_k \) body force per unit volume
- \( t_k \) components of surface (true) traction (force per unit area)
- \( T_{j k} \) components of surface (true) double force per unit area (double traction)
- \( \Phi_{j k} \) body double force per unit volume
- \( H^j_S(V) \) contains functions of \( H(V) \), satisfying the essential boundary conditions of the strong formulation (2.1), on \( S^E \subset \partial V, i = 1, 2, \ldots , n \) (for each component \( u_i \))
- \( S^E_i \) part of the boundary where \( u_i \) is prescribed or \( Du_i \) (normal derivative) is prescribed or both are prescribed. In many places of the following work, \( S^E_{i,t} \) denotes the part where \( u_i \) is prescribed and \( S^E_{i, m} \) denotes the part where \( Du_i \) is specified
- \( S^E_{k, j} \) part of the boundary where the \( k \) component of the right hand side of the traction condition Eq. (2.2a) is specified \( (S^E_{k, j} \cup S^E_{k, l} = S, S^E_{k, j} \cap S^E_{k, l} = \emptyset) \)
- \( S^E_{k, m} \) part of the boundary where the \( k \) component of the right hand side of the moment condition Eq. (2.2b) is specified \( (S^E_{k, j} \cup S^E_{k, m} = S, S^E_{k, j} \cap S^E_{k, m} = \emptyset) \)
- \( n_j \) components of the outer unit vector normal to the surface
- \( D_j(\cdot) := (\partial_j - n_j \delta_j(\cdot)) \quad \text{Surface gradient operator} \)
- \( \delta_j(\cdot) \) components of Kronecker delta operator
- \( Du_k := n_k \delta_j(\cdot) \mu_k \) Normal gradient of \( u_k \)
- \( C \) the boundary curve(s) where the normal unit vector exhibits jumps (i.e., corners, edges etc.) and (or) the boundary curve(s) on which the external load \( T_{j k} \) exhibits jumps
- \( [y] \) the difference of the values of quantity \( y \) between both sides of curve \( C \)
- \( m_j \) \( e_{l k j} \mu_{l k} \), where \( s_j \) denotes the components of the tangential vector of curve \( C \) and \( e_{l k j} \) is the well known alternating tensor
- \( C_{ijkl}, F_{klm}, A_{ijklm} \) components of material constant tensors (general anisotropic material behavior)

Upon substituting Eqs. (2.3) and (2.4) into Eq. (2.1) and (2.2) a fourth order boundary value problem with respect to \( u_i \) is formulated. This type of problem is usually referred to as dipolar gradient elasticity boundary value problem, or strain gradient elasticity problem. The equations are written in the so-called Form I or Form II. It is noteworthy that both forms possess exactly the same structure (Mindlin, 1964; Mindlin and Eshel, 1968).

Typical example of dipolar gradient elasticity formulation is the so-called Toupin’s generalization of couple stress theory or elastic theory of micro-homogeneous material (Mindlin, 1964; Bleustein, 1967). An important
point is the nature of the true loads one must impose in order to have a well defined problem, see (2.1) and the right hand sides of (2.2). The true loads should be meaningful, primarily from an engineering point of view. For more details, see Bleustein (1967), Markolefas et al. (2007).

For the purpose of numerical verification, the final form of a general one-dimensional gradient elasticity analogue, is given in the following (Tsepoura et al., 2002).

2.2. Strong formulation (one-dimensional problem)

Gradient elastic bar in tension: Find \( u(x) \in H^4_{S^g}(V) \) such that

\[
g^2u''' - u'' = \frac{f}{AE} = f \quad \text{in } V = (0, 1)
\]

(2.5)

where \( u(x) \) is the axial displacement distribution, \( f(x) \) is the distributed axial load per unit length, \( AE \) is the axial stiffness of the bar and \( g \) represents material length related to volumetric elastic strain energy \((AE\text{ and } g^2 \text{ are constants})\). The strong formulation (2.5) is accompanied by essential and Neumann conditions. For example:

\[
u(0) = 0 \quad \text{(essential condition)}
\]

(2.6a)

\[
u'(1) = \epsilon_1 \quad \text{(essential condition)}
\]

(2.6b)

\[
AE(u'(1) - g^2u'''(1)) := P(1) = P_1 \quad \text{(Neumann condition)}
\]

(2.6c)

\[
AE(u(0) + g^2u''(0)) := R(0) = 0 \quad \text{(Neumann condition)}
\]

(2.6d)

where \( P(x) \) is the axial force, \( R(x) \) is the double force, \( l \) represents a (constant) material length related to surface elastic strain energy and \( P_1, \epsilon_1 \) are given constants.

The above one-dimensional model is based on the following set of constitutive equations,

\[
\tau = Eu' + lEu''
\]

(2.7a)

\[
\mu = lEu' + g^2Eu''
\]

(2.7b)

where \( \tau(x) \) is the Cauchy stress and \( \mu(x) \) is the double stress.

Note that (2.7) represent an one-dimensional analogue of the (inverse of the) general case Eq. (2.3). In fact, Eq. (2.7) is an 1-D analogue of (3.2), see Section 3. In order to get a positive definite strain energy functional, certain restrictions on the values of \( g \) and \( l \) must be imposed (i.e., \( l < g^2 \) or \( l < g \), Georgiadis et al. (2004); Markolefas et al. (2007)).

Applications of the above constitutive model for bending and buckling of gradient elastic beams can be found in Papargyri-Beskou et al. (2003), Giannakopoulos and Stamoulis (2007), Tsamasphyros et al. (2007).

One of the basic characteristics of gradient elasticity models, which is captured by the above one-dimensional analogue, is the appearance of high gradients (boundary layers) in the stresses, for the case of small non-standard material constants \( l,g \) (more generally, for small characteristic lengths). Regarding the given one-dimensional model problem, the boundary layers appear near the boundaries, mainly in the vicinity of the point load, see Section 8.

3. Standard Galerkin formulations

The standard Galerkin formulation can be deduced from the strong formulation 2.1, using an appropriate weighted residual technique. Multiplying (2.1) by a weighting function \( s_j \), integrating over the whole problem domain and employing properly the Gauss and Stokes theorems, the following weak form is derived (Bleustein, 1967):

3.1. Standard Galerkin formulation

Find displacement field \( u_j \in H^2_{S^g}(V) \), such that,
\[
\int_V \tau_{ij} s_{(ij)} dV + \int_V \mu_{ijk} \partial_i \partial_j s_k dV = \int_V f_j s_j dV + \int_V \Phi_j \partial_j (s_k) dV + \int_{S_{h,t}} t_s s dS + \int_{S_{h,m}} T_{jk} D_j (s_k) dS
\]

for every \( s_j \in H^2_{0,S_{h}^e} (V) \), where,

- \( H^2_{S_{h}^e} (V) \): Space of functions belonging to \( H^2 (V) \), satisfying explicitly the essential conditions of the strong formulation (2.1) (for the component \( u_j \)).
- \( H^2_{0,S_{h}^e} (V) \): Space of functions belonging to \( H^2 (V) \), satisfying explicitly homogeneous (zero) essential conditions (for the component \( s_j \)).

It is noteworthy that formulation 3.1, which is the expression of the virtual work principle for the dipolar gradient elasticity theory, is in fact the traditional starting point for the derivation of the strong form (2.1) (Mindlin, 1964; Bleustein, 1967).

The right hand side of (3.1) belongs to the dual space of the solution space \( H^2_{0,S_{h}^e} (V) \), provided that the data \( f_j, \Phi_j, t_j \) and \( T_{jk} \) are sufficiently regular. Furthermore, the variables \( \tau_{ij} \) and \( \mu_{ijk} \) are written explicitly in terms of \( u_j \), via the constitutive relations Eqs. (2.3a) and (2.3b). Note that (2.3) must be written in the usual form, i.e., expressing stresses in terms of strains (Mindlin, 1964),

\[
\tau_{ij} (u) = \epsilon_{ijk} u_{(k,l)} + f_{klmij} \partial_k \partial_l u_{(m)} \quad \text{and} \quad \mu_{ijk} (u) = a_{ijklmn} \partial_k \partial_l u_{(m)} + f_{ijklm} u_{(m)}
\]

(Form I)

A basic feature of the Galerkin formulation 3.1 is that its conforming finite element approximations demand \( C^1 \)-continuity for the basis functions. This is a result of the appearance of second order derivatives of \( u_j \) in the definition of the bilinear form, at the left hand side of (3.1). Obviously, (3.1) may be written in the well-known general form:

Find \( u \) with components \( u_j \in H^2_{S_{h}^e} (V) \), such that

\[
a(u, z) = F(z)
\]

for every \( z \) whose components \( z_j \) belong to \( H^2_{0,S_{h}^e} (V) \).

It is not difficult to verify that the positive definite functional \( \frac{1}{2} a(u, u) \) represents the elastic strain energy stored in the structure. Moreover \( a(u, z) \) is a symmetric bilinear form. Hence the solution of (3.1) minimizes in \( H^2_{S_{h}^e} (V) \) the functional,

\[
I (z) = \frac{1}{2} a(z, z) - F(z)
\]

or equivalently, the energy functional,

\[
I (z) = \frac{1}{2} \int_V \tau_{ij} (z) z_{(ij)} dV + \frac{1}{2} \int_V \mu_{ijk} (z) \partial_i \partial_j z_k dV - \int_V f_j z_j dV - \int_V \Phi_j \partial_j (z_k) dV - \int_{S_{h,t}} t_s z_d S + \int_{S_{h,m}} T_{jk} D_j (z_k) dS
\]

Assuming proper essential conditions, the bilinear form \( a(u, z) \) becomes coercive on \( H^2_{0,S_{h}^e} (V) \) (or \( H^2_{0,S_{h}^e} (V) \)-elliptic). Then, the well-posedness of the above weak formulation may be deduced directly from the well-known Lax–Milgram theorem (Ciarlet, 1978; Braess, 1997).

For the purpose of completeness, the standard Galerkin formulation of the 1-D model problem (2.5) is also depicted. For derivation details, see Tsepoura et al. (2002).

3.2. Standard Galerkin formulation (1-D model problem)

Find \( u(x) \in H^2_{S_{h}} (V) \), such that,
\[
\int_0^1 AE(g''u''s'' + u's' + lu''s' + lu's'')dx = \int_0^1 f(s)dx + P(1)s(1) - R(0)s'(0)
\]
for every \( s(x) \in H^2_{0,S_k}(V) \).

Based on the given essential boundary conditions (2.6a) and (2.6b),
\[
H^2_{0,S_k}(V) := \{ s \in H^2(V) : s(0) = 0, s'(1) = 0 \}
\]
\[
H^2_{S_k}(V) := \{ u \in H^2(V) : u(0) = 0, u'(1) = \varepsilon_1 \}
\]

It is not difficult to verify the symmetry and positive-definiteness (assuming \( l < g \)) of the bilinear functional on the left hand side of (3.6), (Tsepoura et al., 2002; Markolefas et al., 2007). Applying (3.4), the corresponding quadratic energy functional is
\[
I(z) = \frac{1}{2} \int_0^1 AE(g''(z'')^2 + (z')^2 + 2lz''z')dx - \int_0^1 f(z)dx - P(1)z(1) + R(0)z'(0)
\]
The exact solution of formulation 3.2 minimizes \( I(z) \) over the solution space \( H^2_{S_k}(Q) \).

4. Case 1: Mixed formulation with main variables the double stress, the Cauchy stress and the displacement field \((\mu - \tau - \nu \text{ formulation})\)

The details of the development of mixed formulation \( \mu - \tau - \nu \), as well as, further mathematical analysis, may be found in Markolefas et al. (2007). The final results, as well as, the associated nomenclature are summarized.

Let \( \mu_{ijk} \in H^1(V), \tau_{ij} \in L^2(V), u_k \in H^1_{S_k}(V) \subset H^1(V) \), where \( H^1_{S_k}(V) \) is the subset of \( H^1(V) \), containing functions with a priori specified trace \( (u_k = U_k) \) on the part of the boundary \( S_{E,l} \subset S \). The functions \( \mu_{ijk} \) are constrained to satisfy the condition (2.2b), \( n_j\mu_{ijk} = n_jT_{jk} \), on the respective part \( S_{N,m}^k \) of the boundary. The functions \( \tau_{ij} \) are not constrained. Let \( r_{ijk} \in H^1(V), \phi_{ij} \in L^2(V), s_k \in H^1_{0,S_k}(V) \subset H^1(V) \), where \( H^1_{0,S_k}(V) \) is the subset of \( H^1(V) \) containing functions with zero trace on the part of the boundary \( S_{E,l}^h \subset S \). The functions \( r_{ijk} \in H^1(V) \) are constrained to satisfy the condition \( n_i n_j r_{ijk} = 0 \) on the part \( S_{N,m}^k \) of the boundary.

The mixed formulation reads as follows.

4.1. Mixed formulation \((\mu - \tau - \nu)\)

(Form I, with coupling terms in the constitutive equations): Find \( \mu_{ijk} \in H^1(V) \) (with the above mentioned constraints), \( \tau_{ij} \in L^2(V) \) and \( u_k \in H^1_{S_k}(V) \) such that, for every \( r_{ijk} \in H^1(V) \) (with the above mentioned constraints), \( \phi_{ij} \in L^2(V) \) and \( s_k \in H^1_{0,S_k}(V) \) the following weak statements hold.

\[
\int_V r_{ijk} F_{ijkl}^m \tau_{im}dV + \int_V r_{ijk} A_{ijkl}^m \mu_{jm}dV + \int_V \partial_j u_k \partial_j r_{ijk} dV - \int_S (D_{jk} u_k) n_j r_{ijk} dS = \int_{S_{E,m}^h} (D u_k) n_j r_{ijk} dS \tag{4.1a}
\]

\[
\int_V \tau_{kl} C_{ijkl} \phi_{ij} dV + \int_V \mu_{klm} F_{klm}^j \phi_{ij} dV - \int_V \phi_{ij} u_{(i,j)} dV = 0 \tag{4.1b}
\]

\[
\int_S \partial_i \mu_{ijk} \partial_i s_k dS - \int_S (D_j s_k) n_j \mu_{ijk} dS - \int_V \tau_{jk} s_{(k,j)} dV = - \int_{S_{N,m}^h} s_k (f_k - \partial_i \Phi_{ij}) dV - \int_{S_{N,m}^h} (t_k + n_j \Phi_{ij} - D_j T_{jk}) s_k dS - \int_{S_{N,m}^h} (D_{mn}) n_j T_{jk} s_k dS - \int_C [m_j T_{jk}] s_k dC \tag{4.1c}
\]

Without loss of generality, in the following part of the current section, it is assumed that \( H^1_{0,S_k}(V) = H^1_{S_k}(V) \), i.e., \( u_k = 0 \) on \( S_{E,l}^h \). Moreover, the trial space of double stresses \( \mu_{ijk} \) is the same as the test space of the respective weighting functions \( r_{ijk} \) \( (n_j r_{ijk} = 0 \) on \( S_{N,m}^h \), i.e., homogeneous essential conditions for the mixed formulation are considered).

The mixed formulation 4.1 may be written in the standard mixed form as follows (Brezzi and Bathe, 1990; Braess, 1997; Markolefas et al., 2007).
Find \((w, u) \in (U \times Q)\), such that
\[
A(w, z) + B(z, u) = y(z), \quad \forall z \in U
\]
\[
B(w, s) = g(s), \quad \forall s \in Q
\]
where \(U, Q\) are infinite dimensional Hilbert spaces, endowed with inner products \((\cdot, \cdot)_U\), \((\cdot, \cdot)_Q\) and associated norms \(\|\cdot\|_U\), \(\|\cdot\|_Q\) respectively, while \(A(\cdot, \cdot) : U \times U \to \mathbb{R}\) and \(B(\cdot, \cdot) : U \times Q \to \mathbb{R}\) are continuous (bounded) bilinear forms.

The linear forcing functionals \(y(\cdot)\) and \(g(\cdot)\) belong to the dual spaces \(U' \times \mathbb{K}LQ'\), respectively. The respective main variables, functionals and norms are formally defined as follows:

\(w := (\mu, \tau)\): where \(\mu\) denotes the double stress tensor and \(\tau\) denotes the Cauchy stress tensor

\(z := (r, \phi)\): test function for \(w\) (ordered pair of the respective weighting tensor functions)

\(u\): displacement field

\(s\): test function for \(u\)

\[
A(w, z) := \int_V r_{ijk} F_{ijklmn} \tau_{lm} dV + \int_V r_{ijk} A_{ijklmn} \mu_{lmm} dV + \int_V \tau_{kl} C_{ijkl} \phi_{ij} dV + \int_V \mu_{klmn} F_{klmj} \phi_{ij} dV
\]

\[
B(w, s) := \int_V \partial_1 \mu_{ijk} \partial_1 s_k dV - \int_S (D_j s_k) n_j \mu_{ijk} dS - \int_V \tau_{jkl} s_{kl} dV
\]

\[
y(z) := \int_{S_{h,1}} D u_k n_k r_{ijk} dS
\]

\[
g(s) := - \int_V s_k (f_k - \partial_1 \phi_{ijk}) dV - \int_{S_{h,1}} (t_k + n_j \phi_{jk} - D_j T_{jk}) s_k dS - \int_{S_{h,1}} (D_j m_j) n_j T_{jk} s_k dS - \int_C |m_j T_{jk}| s_k dC
\]

Moreover,

\[
\|w\|_U := \sqrt{\|\mu\|_1^2 + \|\tau\|_0^2}
\]

\[
\|u\|_Q := \|s\|_1^2 := \int_V (\partial_1 u_i \partial_1 u_i + u_i u_i) dV
\]

\(U := (U_1, U_2)\)

\(Q := (V^1, V^2, V^3)\) (for \(3 - D\)) or \(Q := (V^1, V^2)\) (for \(2 - D\))

The space \(U_1\) is a subspace of a product space, whose components are equal to \(H^1(V)\), while \(U_2\) is a product space, whose components are equal to the space \(L^2(V)\) (the above imply an arbitrary ordering of the components of the tensors involved). The number of components of typical elements of \(U_1\) and \(U_2\) equals the number of components of \(w\) and \(s\), respectively. Both trial and test tensor functions must satisfy \(a priori\) the symmetry conditions \(\mu_{ijk} = \mu_{kji}\) and \(\tau_{ijk} = \tau_{kji}\). Note that, in the general case, \(U_1\) contains ordered tuples of components \(\mu_{ijk}\) (or \(r_{ijk}\)) that satisfy the moment condition \(n_j \mu_{ijk} = 0\) on \(S_{h,m}^k\).

Setting \(z = w\) in (4.3a) we get,

\[
A(w, w) := 2 \int_V \mu_{ijk} F_{ijklmn} \tau_{lm} dV + \int_V \mu_{ijk} A_{ijklmn} \mu_{lmm} dV + \int_V \tau_{kl} C_{ijkl} \tau_{ij} dV
\]
It can be easily seen that $A(w, w)$ is twice the elastic energy stored in the structure, given the field $w := (u, z)$. Therefore, $A(w, w)$ is a positive definite functional (Mindlin, 1964; Tsepoura et al., 2002; Georgiadis and Grentzelou, 2006). Furthermore, $A(w, z)$ is a symmetric bilinear form (assuming certain symmetries for the material tensors, Mindlin, 1964). Hence, the exact solution of mixed formulation 4.1 is a stationary point of the following quadratic functional (Braess, 1997).

$$I(z, \phi) = \frac{1}{2} A(\phi, \phi) + B(\phi, \phi) - \dot{g}(\phi)$$

then, the following quadratic functional (Braess, 1997).

$$A(\phi, \phi) = \frac{1}{2} \int_V \left( \int_V \left( \int_V \phi_{ij} \phi_{ij} \right) + \int_V \partial_j \phi_{ij} \partial_j \phi_{ij} \right) dV$$

Furthermore, $A(w, z)$ is a constrained to satisfy the condition (2.2b), on the respective part $S_{N,m}$. The functions $\mu_{ijk}$ are constrained to satisfy condition (2.2b), on the respective part $S_{N,m}$ of the boundary. Let $r_{ijk} \in H^1(V)$, $s_k \in H^1(\gamma^E)$. The functions $r_{ijk}$ and $s_k$ are constrained to satisfy the condition $n_\mu r_{ijk} = 0$ on the part $S_{N,m}$ of the boundary.

5. Case 2: Mixed formulation with main variables the Double stress and the Displacement field ($\mu - u$ formulation)

The starting point is the constitutive Eq. (3.2b). Solving for the second gradient of displacements, we get, 

$$\partial_j \partial_i u_k = a_{ijk}^{rst} u_{rst} - a_{ijk}^{rst} f_{rst} u_{t,m}$$

where $a^{-1}$ denotes the inverse of tensor $a$.

Multiplying (5.1) by a proper weighting function $r_{ijk}$ and integrating by parts, it follows

$$\int_V r_{ijk} a_{ijk}^{rst} u_{rst} dV - \int_V r_{ijk} a_{ijk}^{rst} f_{rst} u_{t,m} dV + \int_V \partial_i u_k \partial_j r_{ijk} dV - \int_S (D_{ij} u_k) n_i r_{ijk} dS$$

Substitution of Eq. (5.1) in (3.2a) gives the Cauchy stress in terms of double stress tensor and the displacement vector,

$$\tau_{ij} = (C_{ijkl} - f_{kl} a_{kl}^{rst} f_{rst} u_{t,r}) u_{t,m} + f_{kl} a_{kl}^{rst} u_{t,r}$$

Substitution of Eqs. (5.3) into (4.1c) gives

$$\int_V \partial_i u_k \partial_j s_k dV - \int_S (D_{ij} s_k) n_i u_k dS - \int_S (C_{ijkl} - f_{kl} a_{kl}^{rst} f_{rst} u_{t,r}) u_{t,m} dV - \int_V s_{t,m} f_{kl} a_{kl}^{rst} u_{t,r} dV$$

$$= - \int_v s_k (f_k - \partial_i \phi_{ik}) dV - \int_S (t_k + n_i \phi_{ik} - D_{ij} T_{jk}) s_k dS - \int_S s_k (D_{ij} u_k) n_i T_{jk} s_k dS - \int_C [m_j T_{jk}] s_k dC$$

Let $\mu_{ijk} \in H^1(V)$, $u_k \in H^{1, \gamma^E}(V)$. The functions $\mu_{ijk}$ are constrained to satisfy condition (2.2b), on the respective part $S_{N,m}$. Let $r_{ijk} \in H^1(V)$, $s_k \in H^{1, \gamma^E}(V)$. The functions $r_{ijk} \in H^1(V)$ are constrained to satisfy the condition $n_\mu r_{ijk} = 0$ on the part $S_{N,m}$ of the boundary.

5.1. Mixed formulation ($\mu - u$)

(Form I, with coupling terms in the constitutive equations): Find $\mu_{ijk} \in H^1(V)$ (with the above mentioned constraints) and $u_k \in H^{1, \gamma^E}(V)$ such that, for every $r_{ijk} \in H^1(V)$ (with the above mentioned constraints) and $s_k \in H^{1, \gamma^E}(V)$, the weak statements (5.2) and (5.4) hold.
Assuming homogeneous essential conditions for the main variables, as we did in formulation 4.1, weak statements Eqs. (5.2) and (5.4) can be cast in the following general mixed structure:

\[
\begin{align*}
\tilde{A}(\mu, r) + \tilde{B}(r, u) &= \tilde{y}(r), \quad \forall r \in U \\
\tilde{B}(\mu, s) - C(u, s) &= \tilde{g}(s), \quad \forall s \in Q
\end{align*}
\]

(5.5a) (5.5b)

where \( C(u, s) \) and \( \tilde{A}(\mu, r) \) are positive definite symmetric bilinear forms.

The latter can be shown by using the standard symmetries for the material tensors, i.e., \( a_{ijklmn} = a_{klmijn} \), \( c_{ijkl} = c_{klij} \), as well as, physical considerations. Moreover, \( U \) coincides with \( U_1 \) of formulation 4.1 (\( \mu - \tau - u \)). The bilinear forms and functionals of the above mixed method are defined as follows,

\[
\begin{align*}
\tilde{A}(\mu, r) &= \int_V r_{ijk} a_{ijkrel} \mu_{rst} dV \\
\tilde{B}(\mu, s) &= \int_V \partial_i \mu_{ijk} \partial_j s_k dV - \int_S (D_i s_k) n_i \mu_{ijk} dS - \int_V s_{(i,j)} f_{klmjp} a_{klmjp} \mu_{rst} dV \\
C(u, s) &= \int_V s_{(i,j)} (c_{ijzp} - f_{klmjp} a_{klmjp} f_{rstzp}) u(z,p) dV \\
\tilde{y}(r) &= \int_{S_{h,i}} D u_k n_i r_{ijk} dS \\
\tilde{g}(s) &= -\int_V s_k (f_k - \partial_j \Phi_{jk}) dV - \int_{S_{h,i}} (t_k + n_j \Phi_{jk} - D_j T_{jk}) s_k dS - \int_{S_{h,i}} (D_l n_j) n_j T_{jk} s_k dS - \int_C [m_j T_{jk}] s_k dC
\end{align*}
\]

(5.6a) (5.6b) (5.6c) (5.6d) (5.6e)

It is not difficult to verify that, in the absence of coupling terms in the constitutive equations, i.e, \( f_{ijkln} = 0 = F_{ijklm} \), then \( A_{ijklmn} = a_{ijklmn} \), see (2.3) and (3.2). In this case the above \( \mu - u \) formulation is the same as the respective formulation developed in Markolefas et al. (2007). It is also mentioned that the above general \( \mu - u \) formulation (with coupling terms), may be derived directly from the formulation 4.1 (\( \mu - \tau - u \)) by eliminating the Cauchy stress from (4.1a). However, the algebra involved will be somehow more cumbersome, compared to the above derivation.

The exact solution of formulation 5.1 is a stationary point of the following functional,

\[
I(r, s) := \frac{1}{2} \tilde{A}(r, r) - \frac{1}{2} C(s, s) + \tilde{B}(r, s) - \tilde{y}(r) - \tilde{g}(s)
\]

\[
= \frac{1}{2} \int_V r_{ijk} a_{ijkrel} r_{rst} dV - \frac{1}{2} \int_V s_{(i,j)} (c_{ijzp} - f_{klmjp} a_{klmjp} f_{rstzp}) s_{(z,p)} dV + \int_V \partial_i r_{ijk} \partial_j s_k dV
\]

\[
- \int_S (D_i s_k) n_i r_{ijk} dS - \int_S s_{(i,j)} f_{klmjp} a_{klmjp} r_{rst} dV - \int_{S_{h,i}} D u_k n_i r_{ijk} dS + \int_V s_k (f_k - \partial_j \Phi_{jk}) dV
\]

\[
+ \int_{S_{h,i}} (t_k + n_j \Phi_{jk} - D_j T_{jk}) s_k dS + \int_{S_{h,i}} (D_l n_j) n_j T_{jk} s_k dS + \int_C [m_j T_{jk}] s_k dC
\]

(5.7)

The development of the one-dimensional analogue of the above formulation starts from constitutive (2.7b)

\[
\frac{\mu}{g^2 E} = \frac{lu}{g^2} + u''
\]

(5.8)

Multiplying (5.8) by a weight function \( r \in H^1(V) \), integrating over the problem domain and integrating by parts the last term, gives,

\[
\int_0^1 r \mu \frac{d}{dx} \frac{1}{g^2 E} dx - \int_0^1 \frac{1}{g^2} r u \frac{d}{dx} dx + \int_0^1 r' u' dx = u'(1)r(1) - u'(0)r(0)
\]

(5.9)

From (2.7a) and (5.8) we get,
\[ \tau = E\ddot{u} + 1E \left( \frac{\mu}{g^2} - \frac{l\ddot{u}}{g^2} \right) \]  \hspace{1cm} (5.10)

Now, (5.10) is used to eliminate the Cauchy stress from the second equation of the 1D \( \mu - \tau - u \) formulation developed in Section 7 of Markolefas et al. (2007). After some algebra, the final result is,

\[ \int_0^1 \mu'\sigma'dx - \int_0^1 \frac{l}{g^2} \mu\sigma' dx - \int_0^1 \left( 1 - \frac{l^2}{g^2} \right) Eu'\sigma' dx = - \int_0^1 s\frac{j}{g} dx - \left( \frac{P}{A} s \right)^1. \]  \hspace{1cm} (5.11)

Assuming the specific boundary conditions (2.6), the 1-D \( \mu-u \) mixed formulation is stated as follows.

5.2. Mixed formulation (\( \mu-u \)) (axial tension problem of a gradient elastic bar)

Find \( \mu \in U := H^1_{0,S_{N,m}}(V), u \in Q := H^1_{0,S_{E}}(V) \) such that for every \( r \in H^1_{0,S_{N,m}}(V), s \in H^1_{0,S_{E}}(V) \), the following holds

\[ \int_0^1 \frac{r\mu}{g^2}E dx - \int_0^1 \frac{l}{g^2} ru' dx + \int_0^1 r'u' dx = \varepsilon_1 r(1) \]  \hspace{1cm} (5.12a)

\[ \int_0^1 \mu'\sigma' dx - \int_0^1 \frac{l}{g^2} \mu\sigma' dx - \int_0^1 \left( 1 - \frac{l^2}{g^2} \right) Eu'\sigma' dx = - \int_0^1 s\frac{j}{g} dx - \frac{P_1}{A} s(1) \]  \hspace{1cm} (5.12b)

where \( H^1_{0,S_{E}}(V) = H^1_{0,S_{N,m}}(V) = \{ r \in H^1(V) : r(0) = 0 \} \).

Formulation 5.2 has the general mixed structure of (5.5). Its exact solution is a stationary point of the following quadratic functional,

\[ I(\varepsilon, s) := \frac{1}{2} \overline{A}(\varepsilon, \varepsilon) - \frac{1}{2} \overline{C}(s, s) + \overline{B}(\varepsilon, s) - \overline{h}(\varepsilon) - \overline{g}(s) \]

\[ = \frac{1}{2} \int_0^1 \frac{r^2}{g^2} dx - \frac{1}{2} \int_0^1 \left( 1 - \frac{l^2}{g^2} \right) E(s)^2 dx + \int_0^1 r'\sigma' dx - \int_0^1 \frac{l}{g^2} r\sigma' dx - \varepsilon_1 r(1) + \int_0^1 s\frac{j}{g} dx - \frac{P_1}{A} s(1) \]

\hspace{1cm} (5.13)

6. Case 3: Mixed formulation with main variables the Double stress, the Cauchy stress, the Displacement second gradient, the standard strain and the Displacement field (\( \mu - \tau - \kappa - \varepsilon - u \) formulation)

Multiplying (2.4b) by a test function \( r_{ij} \), corresponding to the double stress variable and integrating over the problem domain, we get,

\[ \int_V \kappa_{ij} r_{ij} dV = \int_V r_{ij} \partial_i \sigma_{jk} dV \]  \hspace{1cm} (6.1)

Using the identity \( \partial_i \partial_j \mu_k r_{ijk} = \partial_i (\partial_j \mu_k r_{ijk}) - \partial_j \mu_k \partial_i r_{ijk} \) and applying the Gauss theorem, as well as the decomposition \( \partial_j u_k = D_{jk} + n_j Du_k \), (6.1) finally gives,

\[ \int_V \kappa_{ij} r_{ij} dV + \int_V \partial_j u_k \partial_i r_{ijk} dV - \int_S (D_j u_k) n_i r_{ijk} dS = \int_S (Du_k) n_i n_j r_{ijk} dS \]  \hspace{1cm} (6.2)

The test functions \( r_{ij} \in H^1(V) \) satisfy the condition \( n_i n_j r_{ijk} = 0 \) on the part \( S^e_{N,m} \) of the boundary. Hence, from (6.2)

\[ \int_V \kappa_{ij} r_{ij} dV + \int_V \partial_j u_k \partial_i r_{ijk} dV - \int_S (D_j u_k) n_i r_{ijk} dS = \int_S^e (Du_k) n_i n_j r_{ijk} dS \]  \hspace{1cm} (6.3)

Multiplying (2.4a) by a test function \( \phi_{ij} \) corresponding to the Cauchy stress variable and integrating over the problem domain, we get,
Next, the constitutive Eqs. (3.2) are multiplied by proper weighting functions. More specifically, (3.2a) is multiplied by the test function, \( z_{ij} \), associated to the strain variable and (3.2b) by the test function, \( \theta_{ijk} \), associated with the second gradient of displacement variable.

\[
\int_{V} z_{ij} \tau_{ij} dV = \int_{V} z_{ij} (c_{ijkl} \varepsilon_{kl} + f_{klmij} \kappa_{klm}) dV \\
\int_{V} \theta_{ijk} \mu_{ijk} dV = \int_{V} \theta_{ijk} (a_{ijklmn} \kappa_{lmn} + f_{ijklm} \ell_{lm}) dV
\]

The last weak equations is (4.1c), i.e., the virtual work principle. Combining Eqs. (6.3)–(6.5) with (4.1c), we arrive at the following mixed formulation, expressed in the standard mixed structure (4.2),

\section{Mixed formulation \((\mu - \tau - \kappa - \varepsilon - u)\)}

(Form I, with coupling terms in the constitutive equations): Find fields \( ((\mu, \tau, \kappa, \varepsilon, u), \nu) \) such that,

\[
\int_{V} z_{ij} \tau_{ij} dV - \int_{V} z_{ij} (c_{ijkl} \varepsilon_{kl} + f_{klmij} \kappa_{klm}) dV + \int_{V} \theta_{ijk} \mu_{ijk} dV - \int_{V} \int_{V} \theta_{ijk} (a_{ijklmn} \kappa_{lmn} + f_{ijklm} \ell_{lm}) dV \\
+ \int_{V} \kappa_{ijk} r_{ijk} dV + \int_{V} \phi_{ij} e_{ij} dV + \int_{V} \phi_{ij} e_{ij} dV - \int_{S} (D_{ijkl}) n_{ijkl} dS - \int_{V} \phi_{ij} u_{ijkl} dV \\
= \int_{V} D_{ijkl} n_{ijkl} dS
\]

(6.6a)

for every \( (r, \phi, \theta, z) \)

\[
\int_{V} \partial_{ij} \mu_{ijk} \partial_{sk} dV - \int_{V} (D_{ijkl}) n_{ijkl} dS - \int_{V} \tau_{ijk} s_{(k,j)} dV \\
= - \int_{V} s_{k} (f_{k} - \partial_{j} \Phi_{jk}) dV - \int_{S_{(k,j)}} (t_{k} + m_{ij} \Phi_{jk} - D_{ijkl} s_{(k,j)} s_{k} dS - \int_{S_{(k,j)}} (D_{ijkl}) n_{ijkl} dS - \int_{C} [m_{ijkl}] s_{k} dC
\]

(6.6b)

for every \( s \)

The proper function spaces for \( \mu_{ijkl}, \partial_{j} \) and \( u_{k} \) are exactly the same as the respective spaces of mixed formulation 4.1. Moreover, the proper space for both \( \kappa_{ijkl} \) and \( \varepsilon_{ij} \) is \( L^2(V) \), since no derivatives of those variables appear in (6.6).

It is easy to verify that the respective functional \( A((\mu, \tau, \kappa, \varepsilon, u), (r, \phi, \theta, z)) \) in (6.6a) is symmetric. Hence, the exact solution of mixed formulation 6.1 is a stationary point of the following functional,

\[
I((r, \phi, \theta, z), \nu) := \frac{1}{2} \left( 2 \int_{V} z_{ij} \phi_{ij} dV - \int_{V} z_{ij} (c_{ijkl} \varepsilon_{kl} + f_{klmij} \kappa_{klm}) dV + 2 \int_{V} \theta_{ijk} r_{ijk} dV \right) - \int_{V} \theta_{ijk} (a_{ijklmn} \kappa_{lmn} + f_{ijklm} \ell_{lm}) dV + \int_{V} \partial_{ij} s_{k} \partial_{j} \Phi_{k} dV - \int_{S} (D_{ijkl}) n_{ijkl} s_{k} dS \\
- \int_{V} \phi_{ij} s_{(i,j)} dV - \int_{S_{k,m}} D_{ijkl} n_{ijkl} s_{k} dS + \int_{V} s_{k} (f_{k} - \partial_{j} \Phi_{jk}) dV \\
+ \int_{S_{k,m}} (t_{k} + m_{ij} \Phi_{jk} - D_{ijkl} s_{(k,j)} s_{k} dS + \int_{S_{k,m}} (D_{ijkl}) n_{ijkl} s_{k} dS + \int_{C} [m_{ijkl}] s_{k} dC
\]

(6.7)
7. Case 4: Mixed formulation with main variables the Displacement vector, the Displacement gradient field and the equilibrium stress (\(u - \theta - \gamma\) formulation)

The starting point is the definition of the equilibrium stress in dipolar gradient elasticity theory (Mindlin, 1964; Mindlin and Eshel, 1968),

\[
\gamma_{jk} := \tau_{jk} - \partial_i \mu_{ijk}
\]  

(7.1)

Let \(\psi_{jk}\) be a weighting function which corresponds to the displacement gradient field, see Eq. (7.3). From (7.1) it follows,

\[
\begin{align*}
\int_V \gamma_{jk} \psi_{jk} \, dV &= \int_V \tau_{jk} \psi_{jk} \, dV - \int_V \partial_i \mu_{ijk} \psi_{jk} \, dV \\
\int_V \gamma_{jk} \psi_{jk} \, dV &= \int_V \tau_{jk} \psi_{jk} \, dV - \int_S n_i \mu_{ijk} \psi_{jk} \, d\Sigma + \int_V \mu_{ijk} \partial_i \psi_{jk} \, dV \\
\int_V \tau_{jk} \psi_{jk} \, dV + \int_V \mu_{ijk} \partial_i \psi_{jk} \, dV - \int_V \gamma_{jk} \psi_{jk} \, dV &= \int_S n_i \mu_{ijk} \psi_{jk} \, d\Sigma
\end{align*}
\]  

(7.2)

The second weak equation is based on the introduction of the displacement first gradient as independent main variable:

\[
\theta_{ij} := \partial_i u_j
\]  

(7.3)

Let \(n_{ij}\) be a weighting function which corresponds to the equilibrium stress field \(\zeta\). From (7.3) there follows,

\[
\int_V \partial_i u_j n_{ij} \, dV - \int_V \theta_{ij} n_{ij} \, dV = 0
\]  

(7.4)

The equilibrium equation (2.1) is expressed in terms of \(\zeta\),

\[
\partial_j \gamma_{jk} + f_k - \partial_i \Phi_{ik} = 0
\]  

(7.5)

Let \(s_k\) be a weighting function which corresponds to the displacement field. From (7.5) we have,

\[
\begin{align*}
\int_V \partial_i \gamma_{jk} \, dV + \int_V s_k f_k \, dV - \int_V \partial_i \Phi_{ik} \, dV &= 0 \\
\int_S S_k \gamma_{jk} \, d\Sigma - \int_S \partial_j S_k \gamma_{jk} \, d\Sigma + \int_V s_k (f_k - \partial_i \Phi_{ik}) \, dV &= 0 \\
\int_S \partial_j S_k \gamma_{jk} \, d\Sigma &= \int_S n_i \mu_{ijk} \, d\Sigma + \int_V s_k (f_k - \partial_i \Phi_{ik}) \, dV
\end{align*}
\]  

(7.6)

The traction boundary condition (2.2a), on the part \(S_{N,t}^k\) of the boundary, is written as follows,

\[
n_i \gamma_{jk} = D_j (n_i \mu_{ijk}) + (D_j n_i) n_i n_j \mu_{ijk} = t_k + n_i \Phi_{ik} + (D_j n_i) n_j T_{jk} - D_j T_{jk}
\]  

(7.7)

Substitution of Eq. (7.7) into (7.6) gives,

\[
\begin{align*}
\int_V \partial_j S_k \gamma_{jk} \, dV - \int_S s_k D_j (n_i \mu_{ijk}) \, d\Sigma + \int_S s_k (D_j n_i) n_i n_j \mu_{ijk} \, d\Sigma \\
= \int_V s_k (f_k - \partial_i \Phi_{ik}) \, dV + \int_S s_k (t_k + n_i \Phi_{ik} + (D_j n_i) n_j T_{jk} - D_j T_{jk}) \, d\Sigma
\end{align*}
\]  

(7.8)

Application of the Stokes theorem (Mindlin, 1964) on the first boundary term of the left hand side gives,

\[
\begin{align*}
\int_S s_k D_j (n_i \mu_{ijk}) \, d\Sigma &= \int_S D_j (s_k n_i \mu_{ijk}) \, d\Sigma - \int_S D_j (s_k) n_i \mu_{ijk} \, d\Sigma \\
= \int_S s_k (D_j n_i) n_i n_j \mu_{ijk} \, d\Sigma + \oint_{C} [m_i n_j \mu_{ijk}] s_k \, dC - \int_S D_j (s_k) n_i \mu_{ijk} \, d\Sigma
\end{align*}
\]  

(7.9)
Using Eq. (7.9), from (7.8) it follows,
\[
\int_V (\partial_j s_k) \gamma_{jk} \, dV + \int_S D_j (s_k) n_i \mu_{jk} \, dS = \int_\Omega s_k (f_k - \partial_j \Phi_{jk}) \, dV \\
+ \int_S s_k (t_k + n_j \Phi_{jk} + (D_j n_l) n_l T_{jk} - D_j T_{jk}) \, dS + \oint_C \mu_{jk} s_k \, dC
\]
(7.10)
where the jump condition (2.2c) has been used.

The essential condition on the displacement field \(u_k\), on \(S_{E,m}^k\), can be easily incorporated in the structure of the solution space, as previously. The difficulty lies in the incorporation of the condition \(D u_k = V_k\) on \(S_{E,m}^k\), which refers to normal derivative of the component \(u_k\). This refers only to a part of the associated main variable \(\theta_{jk} := \partial_j u_k\), since
\[
\partial_j u_k = D_j u_k + n_j D u_k
\]
(7.11)
Moreover, the introduction of the moment boundary condition (2.2b), on the right hand side of Eq. (7.2) is not straightforward. It must be noted that (2.2b) is a natural boundary condition for the current formulation.

To solve both problems simultaneously, we assume a specific decomposition of the traces on the surface of both trial and test functions \(\theta_{jk}\) and \(\psi_{jk}\),
\[
\theta_{jk}|_S = \tilde{\theta}_{jk} + n_j d_k
\]
(7.12a)
\[
\psi_{jk}|_S = y_{jk} + n_j z_k
\]
(7.12b)
where
\[
\tilde{\theta}_{jk} := \theta_{jk} - n_j \theta_{jk}
\]
(7.12c)
\[
d_k := n_j \theta_{jk}
\]
(7.12d)
\[
y_{jk} := \psi_{jk} - n_j \psi_{jk}
\]
(7.12e)
\[
z_k := n_j \psi_{jk}
\]
(7.12f)
Based on the given boundary conditions,
\[
d_k|_{S_{E,m}^k} := V_k \quad \text{(essential boundary condition on } D u_k)\]
(7.13a)
\[
\tilde{\theta}_{jk}|_{S_{E,m}^k} := D_j u_k = D_j U_k \quad \text{(since } u_k \text{ is prescribed on } S_{E,m}^k)\]
(7.13b)
\[
z_k|_{S_{N,m}^k} = 0
\]
(7.13c)
\[
y_{jk}|_{S_{N,m}^k} = 0
\]
(7.13d)
It is also necessary to introduce a Lagrange multiplier on \(S_{N,m}^k\), in order to retain the symmetry of the formulation. This is also mentioned in Amanatidou and Aravas (2002). Define \(p_{jk} := n_j \mu_{jk}|_{S_{N,m}^k}\). Then, we introduce the following weak constraint equation,
\[
\int_{S_{N,m}^k} (D_j u_k - \tilde{\theta}_{jk}) \tilde{\lambda}_{jk} \, dS = 0
\]
(7.13e)
where \(\tilde{\lambda}_{jk}\) is the weight function corresponding to \(p_{jk}\).

The boundary functions \(\tilde{\theta}_{jk}\) and \(d_k\) are unknown on the parts of the boundary where the respective test functions \(y_{jk}\) and \(z_k\) do not vanish identically (and vice-versa). Moreover, \(\tilde{\theta}_{jk}|_{S_{N,m}^k}\) is weakly related to \(D u_k\) on \(S_{N,m}^k\) via (7.13e).

Without loss of generality we assume homogeneous essential conditions in the rest of the current section. Hence \(d_k|_{S_{N,m}^k} = V_k = 0\), \(u_k|_{S_{N,m}^k} = U_k = 0\) and \(\tilde{\theta}_{jk}|_{S_{N,m}^k} = D_j u_k = 0\). Weak Eq. (7.2) can now be written,
the solution (and test) space for the last variable is plates with similar structure in the constraint Eq. (7.15b), the proper space should be larger. More precisely, obviously, variable $\gamma_{jk}$ admits no essential boundary conditions. Moreover, for 1-D problems, by definition

$$H^{-1}(\text{div}, V) := \left\{ n : n \in H^{-1}(V), \frac{dn}{dx} := n' \in H^{-1}(V) \right\} = L^2(V)$$

The proper function space for the surface trace $\xi_{jk}$ is the space $H^1_{00}(S_{N,j}^k)$, see (7.12a), (7.13b). The proper space for the Lagrange multiplier $p_{jk}$ however, is not the respective dual space $H^{-1}(S_{N,j}^k)$, due to the term $D_j u_k|_{S_{N,j}^k}$, for every $(s, \psi)$ such that,

$$\int_V \tau_{jk} \psi_{jk} \, dV + \int_V \mu_{ijk} \partial_i \psi_{jk} \, dV - \int_V \gamma_{jk} \psi_{jk} \, dV = \int_S n_i \mu_{ijk} (y_{jk} + n_j z_k) \, dS \Rightarrow$$

$$(7.14)$$

Eqs. (7.4), (7.10), (7.13e) and (7.14) can be cast in the standard mixed form.

7.1. Mixed formulation $(u - \theta - \gamma)$

Find fields $(u, \theta, \psi, \gamma)$ such that,

$$\int_V \tau_{jk}(\theta) \psi_{jk} \, dV + \int_V \mu_{ijk}(\theta) \partial_i \psi_{jk} \, dV + \int_{S_{N,j}^k} D_j(s_k) p_{jk} \, dS - \int_{S_{N,j}^k} p_{jk} \gamma_{jk} \, dS - \int_V \gamma_{jk} \psi_{jk} \, dV$$

$$+ \int_V (\partial_j s_k) \gamma_{jk} \, dV$$

$$= \int_{S_{N,j}^k} T_jk n_j z_k \, dS + \int_V s_k (f_k - \partial_j \Phi_{jk}) \, dV + \int_{S_{N,j}^k} s_k (t_k + n_j \Phi_{jk} + (D_j n_i) n_j T_{jk} - D_j T_{jk}) \, dS$$

$$+ \int_C [n_j T_{jk}] s_k \, dC$$

(7.15a)

for every $(s, \psi)$ and

$$\int_{S_{N,j}^k} D_j u_k \lambda_{jk} \, dS - \int_{S_{N,j}^k} \xi_{jk} \lambda_{jk} \, dS - \int_V \theta_{jk} n_j \, dV + \int_V (\partial_j n_k) n_j \, dV = 0$$

(7.15b)

for every $(\lambda, n)$

(7.16)

endowed with the respective graph norm,

$$\|n\|_{H^{-1}(\text{div}, V)} := \sqrt{\|n\|_{-1}^2 + \sum_k \|\partial_j n_k\|_{-1}^2}$$

Obviously, variable $\gamma_{jk}$ admits no essential boundary conditions. Moreover, for 1-D problems, by definition

$$H^{-1}(\text{div}, V) := \left\{ n : n \in H^{-1}(V), \frac{dn}{dx} := n' \in H^{-1}(V) \right\} = L^2(V)$$

The proper function space for the surface trace $\xi_{jk}$ is the space $H^1_{00}(S_{N,j}^k)$, see (7.12a), (7.13b). The proper space for the Lagrange multiplier $p_{jk}$ however, is not the respective dual space $H^{-1}(S_{N,j}^k)$, due to the term $D_j u_k|_{S_{N,j}^k}$,
which belongs to this dual space. Another selection, which secures at least continuity of the bilinear functionals, is simply \( p_{jk} \in H^1(S_{N,k}^k) \).

Formulation 7.1 \((u - \theta - \gamma)\) has the standard mixed structure of (4.2). Hence, its exact solution is a stationary point of the following quadratic functional,

\[
I((\tilde{s}, \tilde{\psi}, \tilde{y}, (\tilde{z}, \tilde{u}))) = \frac{1}{2} \left( \int_V \tau_{jk}(\tilde{\psi})\tilde{\psi}_{jk}dV + \int_V \mu_{jk}(\tilde{\psi})\partial_{\tilde{\psi}}\tilde{\psi}_{jk}dV \right) + \int_{S_{N,k}^k} D_j(s_k)\tilde{\lambda}_{jk}dS \\
- \int_{S_{N,k}^k} \tilde{\lambda}_{jk}y_{jk}dS - \int_V \tilde{\psi}_{jk}n_{jk}dV + \int_V (\partial_j s_k)n_{jk}dV \\
- \int_{S_{N,m}^m} T_{jk}n_{jk}dS - \int_V s_k(f_k - \partial_j\Phi_{jk})dV - \int_C [n_i T_{jk}]s_k dC \\
- \int_{S_{N,k}^k} s_k(t_k + n_j\Phi_{jk} + (D_m l)n_{Tjk} - D_j T_{jk})dS
\]  

(7.19)

The respective mixed formulation for the one-dimensional model problem is derived in the following. We start with the definition of the equilibrium stress \( (Tsepoura et al., 2002) \),

\[
c := \frac{s}{\mu} \cdot \lambda_0 
\]

(7.20)

where \( \tau \) is the Cauchy stress and \( \mu \) is the double stress.

Note that \( \gamma \) is the total or true stress, since there are no other terms in the definition of the true traction for one-dimensional problems. Let \( \psi \) be a weighting function which corresponds to the displacement gradient field. From Eq. (7.20) it follows,

\[
\int_0^1 \gamma \psi dx = \int_0^1 \tau \psi dx + \int_0^1 \mu \psi' dx - (\mu \psi) \bigg|_0^1 \\
\int_0^1 \tau \psi dx + \int_0^1 \mu \psi' dx - \int_0^1 \gamma \psi dx = (\mu \psi) \bigg|_0^1
\]

(7.21)

The value of \( \psi \) equals to zero at the portion of the boundary where the double stress is not known (i.e., where the displacement gradient \( u' \) is known). The second weak equation is derived from the displacement gradient independent variable,

\[
\theta := u'
\]

(7.22)

Let \( n \) be a test function which corresponds to the total stress field. From (7.22) there follows,

\[
\int_0^1 n u' dx - \int_0^1 n \theta dx = 0
\]

(7.23)

The equilibrium equation, \((Tsepoura et al., 2002)\), is expressed in terms of \( \gamma \),

\[
(\tau - \mu')' + \frac{\tilde{f}}{A} = \gamma' + \frac{\tilde{f}}{A} = 0
\]

(7.24)

The so-called true traction (axial force) is defined as follows,

\[
P(x) := (\tau - \mu')A = \gamma A \Rightarrow \gamma = \frac{P}{A}
\]

(7.25)

Let \( s \) be a weighting function which corresponds to the displacement field. From (7.24) we get,

\[
\int_0^1 \gamma s' dx = \int_0^1 s \frac{\tilde{f}}{A} dx + \left( s \frac{P}{A} \right) \bigg|_0^1
\]

(7.26)

Using the definition of \( \theta \) in the constitutive relations (2.7), we get
\[ \tau = E\theta + lE\theta' = \tau(0) \]  
\[ \mu = lE\theta + g^2E\theta' = \mu(0) \]  
(7.27a)  
(7.27b)

Eqs. (7.21), (7.23) and (7.26) may be cast in the standard mixed form.

### 7.2. Mixed formulation \((u - \theta - \gamma)\)

Find \((u, \theta); \gamma\) such that,

\[
\int_0^1 \tau(\theta)\psi d\xi + \int_0^1 \mu(\theta)\psi' d\xi + \int_0^1 \gamma s' d\xi - \int_0^1 \gamma \psi d\xi = (\mu \psi)|_0^1 + \int_0^1 s \bar{f} \psi d\xi + \left(\frac{P_1}{A}\right)|_0^1
\]

for every \((s, \psi)\)

\[
\int_0^1 m u' d\xi - \int_0^1 n \theta d\xi = 0 \quad \text{for every } n
\]

(7.28a)  
(7.28b)

The above formulation is exactly analogous to the previous formulation 7.1. For the specific boundary value problem defined by Eqs. (2.5), Eq. (2.6), the forcing term in (7.28a) becomes,

\[
F((s, \psi)) := \int_0^1 s \bar{f} \psi d\xi + s(1) \frac{P_1}{A}
\]

since \(R(0) = \mu(0) = 0, u(0) = s(0) = 0 \) and \(\Psi(1) = 0 \) \(\theta(1) = u'(1) = \epsilon_1\).

Analogously to the multi-dimensional formulation, the proper function spaces for the above variables are defined as follows:

\[ u(x) \text{ and } s(x) \text{ belong to } \{ s \in H^1(V), s(0) = 0 \} \]

(7.30a)

\[ \theta(x) \text{ belongs to } \{ \theta \in H^1(V), \theta(1) = \epsilon_1 \} \]

(7.30b)

\[ \psi(x) \text{ belongs to } \{ \psi \in H^1(V), \psi(1) = 0 \} \]

(7.30c)

\[ \gamma(x) \text{ and } n(x) \text{ belong to } H^{-1}(\text{div}, V) = L^2(V) \]

(7.30d)

Without loss of generality, assume \(\epsilon_1 = 0\). Then, the exact solution of mixed formulation 7.2 is a stationary point of the quadratic functional,

\[
I((s, \psi), n) := \frac{1}{2} \left( \int_0^1 \tau(\psi)\psi' d\xi + \int_0^1 \mu(\psi)\psi' d\xi \right) + \int_0^1 ns' d\xi - \int_0^1 m \psi d\xi - \int_0^1 s \bar{f} d\xi - s(1) \frac{P_1}{A}
\]

(7.31)

### 8. Numerical results

The current section presents numerical results based on the one-dimensional mixed formulations (case 1: \(\mu - \tau - u\), case 2: \(\mu - u\) and (case 4: \(u - \theta - \gamma\)). The adopted properties for the material, bar and external load, are as follows: \(g = 0.02, l = 0.0001, \epsilon_1 = 0, AE = 2, P_1 = 1, f(x) = 0\). Note that \(C^0\)-continuous basis functions are used in all cases, with equal interpolation order for all main variables (Markolefas et al., 2007).

Figs. 2, 4 and 3, 5 depict the exact versus finite element solutions for the double stress \(\mu(x)\) and Cauchy stress \(\tau(x)\), respectively. A 16 element uniform mesh is employed, with polynomial interpolation orders \(p = 2 \& p = 4\) \((p\text{-extension})\). Figs. 6, 8 and 7, 9; also, depict the exact versus finite element solutions for the double stress and Cauchy stress, respectively, but for polynomial interpolation order \(p = 1\) and uniform meshes of 32 & 64 elements \((h\text{-extension})\). Note that for better presentation of the results, in some cases, zooming is performed near the boundary layers.

As far as the 1st derivative of the double stress \((\mu'(x))\) and the Cauchy stress \((\tau'(x))\) are concerned, the numerical results are depicted in Figs. 10–17 More specifically, Figs. 10, 12 and 11, 13 show the aforementioned derivatives for a uniform mesh of 16 elements and interpolation orders of \(p = 2 \& p = 4\) \((p\text{-extension})\). Figs.
Fig. 2. Exact vs FE solutions for the double stress. Using 16 elements & $p = 2$.

Fig. 3. Exact vs FE solutions for the Cauchy stress. Using 16 elements & $p = 2$. 
14, 16 and 15, 17 depict the results for the same derivatives but for a polynomial interpolation order of $p = 1$ and uniform meshes of 32 & 64 elements ($h$-extension).

It is mentioned that the $l$, $s$ (secondary) variables of the $u/C_0$ formulation (case 4) as well as the $s$ (secondary) variable of the $l$-u formulation (case 2), are computed based on post-processing of the respective main variables. It is also observed that, for both stresses, strong boundary layers appear at the boundaries, mainly at $x = 1$. The boundary layers are steeper for the derivatives of the variables.

A general conclusion from the numerical results is that all formulations converge for all main and secondary variables, especially for high polynomial orders and/or fine meshes. The $u/C_0$ formulation exhibits pollution error, in the form of oscillations, for the (secondary) variables $l$, $s$. On the other hand, the (secondary) variable $s$ of the $l$-u formulation converges much faster.

Extensive numerical results regarding the convergence rates and relative errors in the solution spaces norms for all variables of the formulations has been also gathered and will be presented in a future publication. Briefly one can say that the observed convergence rates for all formulations are quasi-optimal (e.g., see Markolefas et al., 2007 for the theoretical explanation regarding the $l$-u formulation).

9. Closing discussion and future research directions

A systematic presentation of some classes of mixed weak formulations, for general multi-dimensional dipolar gradient elasticity (fourth order) boundary value problems has been attempted in this work. The important feature of the mixed formulations is that $C^0$-continuity conforming basis functions can be employed in the respective finite element approximations (or even $C^{-1}$ basis functions for some of the main variables). Moreover, despite their complexity, the formulations can be stated according to the well-known (symmetric) mixed structure of the standard Brezzi theory. In case 2 ($\mu - u$) an extra positive definite symmetric term appears in the constraint equation, see (5.5b).
Fig. 5. Exact vs FE solutions for the Cauchy stress. Using 16 elements & $p = 4$.

Fig. 6. Exact vs FE solutions for the double stress. Using 32 elements & $p = 1$. 
The main goal of this work is to form a broad reference base for future applications of this type of mixed formulations (e.g., numerical applications). One-dimensional analogues have been developed for the purpose of numerical comparison. The 1-D model problem is based on general constitutive relations, which include

Fig. 7. Exact vs FE solutions for the Cauchy stress. Using 32 elements & $p = 1$.

Fig. 8. Exact vs FE solutions for the double stress. Using 64 elements & $p = 1$.
Fig. 9. Exact vs FE solutions for the Cauchy stress. Using 64 elements & $p = 1$.

Fig. 10. Exact vs FE solutions for the 1st derivative of double stress. Using 16 elements & $p = 2$. 
Fig. 11. Exact vs FE solutions for the 1st derivative of Cauchy stress. Using 16 elements & $p = 2$.

Fig. 12. Exact vs FE solutions for the 1st derivative of double stress. Using 16 elements & $p = 4$. 
Fig. 13. Exact vs FE solutions for the 1st derivative of Cauchy stress. Using 16 elements & $p = 4$.

Fig. 14. Exact vs FE solutions for the 1st derivative of double stress. Using 32 elements & $p = 1$. 
coupling terms. The numerical results of Section 8 verify the effectiveness and performance of the presented mixed methods.

In terms of practical applications, case 2 ($\mu - u$ formulation) demands the smallest number of unknowns. Case 1 ($\mu - \tau - u$ formulation) is an alternative, when it is necessary to compute all stresses directly from the solution process. Case 3 ($\mu - \tau - \kappa - \varepsilon - u$ formulation) is an alternative for cases 1 and 2, with theoretical, rather than practical interest.
Case 4 ($u - \theta - \gamma$ formulation) has a different structure. Part of the complexity here lies in the fact that the, essential for this formulation, boundary condition on $Du_0$ corresponds to a part of the trace of $\theta$, see Eqs. (7.3), (7.11).

Besides $C^0$-continuity, all the presented mixed formulations correspond to what is called in the literature canonical forms (Braess, 1997). This roughly means that the displacement field is a $C^0$-continuous main variable, with essential boundary conditions imposed on it. Furthermore, the weak equations implicitly contain the virtual work principle.

A mathematical analysis of case 1 may be found in Markolefas et al. (2007). This analysis can be easily extended to cases 2 and 3. As far as the $u - \theta - \gamma$ formulation (case 4) is concerned, the associated mathematical analysis can be based on the respective results of Braess (1997).

Extensive numerical experimentation of the above mixed formulations is the first step for practical applications of the presented mixed methods; see for example Amanatidou and Aravas (2002), Tsamasphyros et al. (2005, 2007). The results of Section 8 employ uniform $h$- and $p$-extensions. The effectiveness of the approximate solutions will be improved dramatically with proper mesh refinement near the boundary layers. Therefore, the development of a posteriori error estimators and adaptive techniques (Babuška and Suri, 1990; Szabo and Babuška, 1991) are necessary to effectively capture the errors at high-gradient regions with relatively low computational cost.

It is noteworthy that one could develop $J - e - u$ formulations or $J - u$ formulations (where $e$ is the standard strain and $J$ is the displacement second gradient or the first strain gradient, see Section 2). Moreover, instead of $\theta$ (or $\gamma$) in case 4 one could select other combinations of partial derivatives of $u$, up to third order.

Finally, it is noteworthy that all the current mixed formulations can be augmented so that the respective essential conditions are weakly satisfied. This is useful for the cases of complex non-uniform essential conditions and can be achieved with the introduction of proper Lagrange multipliers on the boundary. The details of the derivations, as well as, the deductions of the respective mixed structures, may constitute the subject matter of a future work.

Fig. 17. Exact vs FE solutions for the 1st derivative of Cauchy Stress. Using 64 elements & $p = 1$. 

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**1st Derivative of Main variables**

**h- Extension**

Comparison of Exact Vs FEM for 1st Derivative of Cauchy Stress

- $\mu - u$
- $\mu - \tau - u$
- $\kappa - u - \gamma$
- Exact Solution

![Graph showing comparison of Exact Vs FEM for 1st Derivative of Cauchy Stress](image-url)
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