Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices

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ABSTRACT

Let \( S = \{ S_i \} _{i \in I} \) be an arbitrary family of complex \( n \times n \) matrices, where \( 1 \leq n < \infty \). Let \( \hat{\rho}(S) \) denote the joint spectral radius of \( S \), defined as

\[
\hat{\rho}(S) = \limsup _{\ell \to +\infty} \left\{ \sup _{(i_1, \ldots , i_\ell) \in I^\ell} \| S_{i_1} \cdots S_{i_\ell} \| ^{1/\ell} \right\},
\]

which is independent of the norm \( \| \cdot \| \) used here. A semi-norm \( \| \cdot \| ^* \) on \( \mathbb{C}^n \) is called “extremal” of \( S \), fits satisfies

\[
\| x \| ^* \neq 0 \quad \text{and} \quad \| x \cdot S_i \| ^* \leq \hat{\rho}(S) \| x \| ^* \quad \forall x = (x_1, \ldots , x_n) \in \mathbb{C}^n \text{ and } i \in I.
\]

In this paper, using an elementary analytical approach, we show that if \( S \) is bounded in \( \mathbb{C}^{n \times n} \), then there always exists, for \( S \), an extremal semi-norm \( \| \cdot \| ^* \) on \( \mathbb{C}^n \); if additionally \( S \) is compact in \( (\mathbb{C}^{n \times n}, \| \cdot \|) \), this extremal semi-norm has the “Barabanov-type property”, i.e., to any \( x \in \mathbb{C}^n \), one can find an infinite sequence \( i_\ell : N \to I \) with \( \| x \cdot S_{i_1} \cdots S_{i_k} \| ^* = \hat{\rho}(S)^k \| x \| ^* \) for each \( k \geq 1 \). As a common starting point, this directly implies the fundamental results: Barabanov’s Norm Theorem, Berger–Wang’s Formula and Elsner’s Reduction Theorem.

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1. Introduction

Let \( I \) be a nonempty index set and \( S : I \to \mathbb{C}^{n \times n}; i \mapsto S_i \) a function from \( I \) into the space of all complex \( n \times n \) matrices, where \( 1 \leq n < \infty \). Naturally, \( S \) gives rise to a multiplicative semigroup, written as \( S^+ \), by the random finite-product with generator \( S \); that is to say,

\[
S^+ = \left\{ S_{i_1} \cdots S_{i_k} \mid (i_1, \ldots , i_k) \in I^k, \quad k = 1, 2, \ldots \right\} \quad \text{where } I^k = I \times \cdots \times I.
\]

Let \( \| \cdot \| \) be an arbitrary preassigned norm of \( \mathbb{C}^n \) that is thought of as the space of all \( n \)-dimensional complex row vectors \( x = (x_1, \ldots , x_n) \). Then for any \( A \in \mathbb{C}^{n \times n} \), by the same symbol \( \| A \| \) we denote the usual induced operator/matrix norm of \( A \), i.e., \( \| A \| = \max _{x \in \mathbb{C}^n; \| x \| = 1} \| x \cdot A \| \), associated to the preassigned vector norm \( \| \cdot \| \) on \( \mathbb{C}^n \); by \( \rho(A) \) the spectral radius of the matrix \( A \).

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We recall that the joint spectral radius of $S$, first introduced by G.-C. Rota and G. Strang in [22], is defined by

$$\hat{\rho}(S) = \limsup_{\ell \to +\infty} \left\{ \sup_{(i_1, \ldots, i_\ell) \in T^\ell} \| S_{i_1} \cdots S_{i_\ell} \|^{1/\ell} \right\} \quad \text{or equivalently} \quad \hat{\rho}(S) = \inf_{\ell \geq 1} \left\{ \sup_{(i_1, \ldots, i_\ell) \in T^\ell} \| S_{i_1} \cdots S_{i_\ell} \|^{1/\ell} \right\}.$$  

It is an extremely important quantity to capture the dynamical behaviors of the semigroup $S^+$. So, computing $\hat{\rho}(S)$ is one of the central tasks of matrix analysis. Although the quantity $\hat{\rho}(S)$ is independent of an explicit choice of the norm $\| \cdot \|$, yet its definition does depend on a choice of specific norm. This causes computing joint spectral radius to become a challenging problem, for example, in [22,9,12].

Firstly, as a special case of the main result of Rota and Strang [22], the following holds

$$\hat{\rho}(S) = \inf_{\| \cdot \| \in \mathcal{N}} \left\{ \sup_{i \in I} \| S_i \| \right\},$$

where $\mathcal{N}$ denotes the set of all possible induced matrix norms for $\mathbb{C}^{n \times n}$; also see [10,23] for much shorter proofs for this special case. So, an important problem is whether or not the above $\inf_{\| \cdot \| \in \mathcal{N}}$ is actually attained by some induced matrix norm $\| \cdot \|$. For this, a norm $\| \cdot \|_*$ on $\mathbb{C}^n$ satisfying the condition

$$\hat{\rho}(S) = \sup_{i \in I} \| S_i \|_*, \quad \text{or equivalently} \quad \hat{\rho}(S) = \sup_{(i_1, \ldots, i_\ell) \in T^\ell} \| S_{i_1} \cdots S_{i_\ell} \|_*^{1/\ell} \quad \forall \ell \geq 1,$$

is called an “extremal norm” of $S$, for example, in [1,2,21,11,24,19,8]. In 1980s, in a series of papers, N. Barabanov published his celebrated result using linear dynamics approaches, which can be stated as follows.

**Barabanov’s Theorem.** (See [1].) If $S = \{S_i\}_{i \in I} \subset \mathbb{C}^{n \times n}$ is a compact family and irreducible, then there is an extremal norm $\| \cdot \|_*$ of $S$ on $\mathbb{C}^n$, such that for any $x \in \mathbb{C}^n$, there is an infinite sequence $i_i : \mathbb{N} \to I$ satisfying

$$\| \hat{x} \cdot S_{i_1} \cdots S_{i_k} \|_* = \hat{\rho}(S)^k \| \hat{x} \|_*$$

for all $k \geq 1$. Here $\| \cdot \|_* \in \{1, 2, \ldots \}$.  

Here “irreducible” means that there is no common, nontrivial, proper, and $S_i$-invariant linear subspaces of $\mathbb{C}^n$, for each $i \in I$. Such a norm given by Barabanov’s theorem is also called a “Barabanov norm”, for example, in [24,15,19,16]. A general extremal norm, even like the norm introduced by V. Protasov [20], need not be a Barabanov norm. V. Kozyakin in [15] proved compactness and uniform equivalence of all the Barabanov norms of an irreducible finite matrix set, corresponding to [26]. In [24], F. Wirth presented a new proof for Barabanov’s theorem using the technical tool “limit semigroup”. Both proofs presented in [1,24] are more or less intricate.

In [13], R. Jungers and V. Protasov, however, provided some examples that a bounded family of matrices having no extremal complex polytope norm; that is, an extremal norm $\| \cdot \|_*$ whose unit ball is a balanced complex polytope. And in [8], the authors provided an example that consists of two matrices not having the spectral finiteness property and any extremal norms.

For computing the joint spectral radius $\hat{\rho}(S)$ and other aims, not to involve any norms, I. Daubechies and J. Lagarias in [9] introduced the generalized spectral radius of $S$ by

$$\rho(S) = \limsup_{\ell \to +\infty} \left\{ \sup_{(i_1, \ldots, i_\ell) \in T^\ell} \rho(S_{i_1} \cdots S_{i_\ell})^{1/\ell} \right\},$$

and for studying the smoothness properties of compactly supported wavelets and solutions of two-scale dilation equations, they conjectured that $\rho(S) = \hat{\rho}(S)$ if $S$ is finite. This conjecture was proved, using advanced tools from ring theory, by M. Berger and Y. Wang in 1992 [2] even in a more general situation. Their celebrated work is known as the Generalized Gelfand Spectral-radius Formula, stated as follows:

**Berger–Wang’s Formula.** (See [2].) If $S = \{S_i\}_{i \in I}$ is bounded in $\mathbb{C}^{n \times n}$, then $\rho(S) = \hat{\rho}(S)$.

Because of its importance, this Gelfand-type spectral-radius formula was reproved by using different approaches, for example, in [10,23,5,3], and extended in [25,18,6]. Particularly, in 1995 [10], as a tool of proving the Berger–Wang formula, L. Elsner proved, using analytic approach, an independently important reduction theorem.

**Elsner’s Theorem.** (See [10].) Let $S = \{S_i\}_{i \in I}$ be a bounded family in $\mathbb{C}^{n \times n}$ with $\hat{\rho}(S) = 1$. If the multiplicative semigroup $S^+$ is unbounded in $\mathbb{C}^{n \times n}$, then $S$ is reducible, i.e., each $S_i$ has a common, nontrivial, proper, and invariant linear subspace in $\mathbb{C}^n$.

Inspired by the papers mentioned above especially [10,1], we present, in this paper, the following more general “extremal semi-norm” and “Barabanov semi-norm” theorems neither imposing the irreducibility nor unboundedness of the semigroup $S^+$ generated by $S$, which thereof generalize the Barabanov theorem and the Elsner theorem.
Theorem A (Extremal semi-norm theorem). Let \( S = \{S_i\}_{i \in I} \subset \mathbb{C}^{n \times n} \) be an arbitrary bounded family. Then there always exists an extremal semi-norm, say \( \|\cdot\|_\ast \), on \( \mathbb{C}^n \); i.e.,

\[
\|\cdot\|_\ast \neq 0 \quad \text{and} \quad \sup_{i \in I} \|x \cdot S_i\|_\ast \leq \hat{\rho}(S)\|x\|_\ast \quad \forall x \in \mathbb{C}^n.
\]

We should notice here that the restriction \( \|\cdot\|_\ast \neq 0 \) is necessary; otherwise the trivial semi-norm \( \|x\|_\ast = 0 \) \( \forall x \in \mathbb{C}^n \) is extremal. As is shown by the well-known counterexample that \( S \) consists of the single matrix \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). \( S \) need not have an extremal norm. So, Theorem A is of interest.

Theorem B (Barabanov semi-norm theorem). Let \( S = \{S_i\}_{i \in I} \subset \mathbb{C}^{n \times n} \) be an arbitrary compact set. If there exists an extremal norm of \( S \), then one can find an extremal semi-norm \( \nu \) of \( S \) such that for any \( \hat{x} \in \mathbb{C}^n \), there is an infinite sequence \( i : \mathbb{N} \to I \) satisfying

\[
\nu(\hat{x} \cdot S_{i_1} \cdots S_{i_k}) = \hat{\rho}(S)^k \nu(\hat{x}) \quad \forall k \geq 1.
\]

Here \((i_k)_{k=1}^{+\infty}\) might depend upon the choice of the initial state \( \hat{x} \).

Both Theorems A and B above might be deduced from Barabanov’s theorem. However, we will prove them in Section 2, using an elementary, analytical, self-contained approach inspired by [10,1].

Barabanov’s norm theorem, Berger–Wang’s formula and Elsner’s reduction theorem all are powerful tools in many branches of pure and applied mathematics, such as for the spectral theory of random matrices, stability analysis of linear switched dynamics and so on. As a starting point in Section 3, we show how Theorems A and B directly imply Barabanov’s norm theorem, and then Berger–Wang’s formula and Elsner’s reduction theorem. These proofs presented here are much more accessible to readers than those available in literature.

We will end this paper with concluding remarks in Section 4.

2. The extremal and Barabanov semi-norms of matrix semigroups

This section is devoted to proving Theorems A and B stated in the introductory Section 1, using a simple approach.

2.1. Extremal semi-norm theorem

Let \( S = \{S_i\}_{i \in I} \) be an arbitrary bounded family in \( \mathbb{C}^{n \times n} \) where \( 1 \leq n < \infty \). By \( \|\cdot\|_2 \), we denote the standard Euclidean vector norm on \( \mathbb{C}^n \) and its induced matrix norm on \( \mathbb{C}^{n \times n} \). A norm \( \|\cdot\| \) on \( \mathbb{C}^n \) is called “normalized”, provided that

\[
\max_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|x\|}{\|x\|_2} = 1.
\]

Recall that a real-valued function \( p \), defined on the linear space \( \mathbb{C}^n \), is called a “semi-norm”, if the following conditions are satisfied:

\[
p(x + y) \leq p(x) + p(y) \quad \text{and} \quad p(\alpha x) = |\alpha| p(x) \quad \forall x, y \in \mathbb{C}^n \quad \text{and} \quad \alpha \in \mathbb{C}.
\]

In addition, if \( p(x) = 0 \iff x = 0 \), then \( p \) is just a “norm” on \( \mathbb{C}^n \).

For proving our extremal semi-norm theorem, our starting point is the following lemma that is essentially due to Rota and Strang [22].

**Lemma 2.1.** For \( S \), there holds

\[
\hat{\rho}(S) = \inf_{\|\cdot\| \in \mathcal{N}} \left( \sup_{i \in I} \|S_i\| \right),
\]

where \( \mathcal{N} \) denotes the set of matrix norms on \( \mathbb{C}^{n \times n} \) induced by normalized vector norms on \( \mathbb{C}^n \).

**Proof.** For any vector norm \( \|\cdot\| \) on \( \mathbb{C}^n \), let \( \beta = \max_{\|x\|_2 = 1} \|x\| \) then \( \|\cdot\|' = \beta^{-1} \|\cdot\| \) is the normalization of \( \|\cdot\| \) having \( \|A\| = \|A\|' \) for every \( A \in \mathbb{C}^{n \times n} \). So, the statement comes from Rota and Strang. \( \square \)

Note that since \( S \) is bounded, we have \( \hat{\rho}(S) < \infty \) and \( \|S\| := \sup_{i \in I} \|S_i\| < \infty \) for any induced matrix norm \( \|\cdot\| \) on \( \mathbb{C}^{n \times n} \).

Now, we can simply prove our extremal semi-norm theorem.

**Proof of Theorem A.** By Lemma 2.1, one can choose a sequence of normalized norms \( \|\cdot\|_{(k)} \) \( k \geq 1 \) on \( \mathbb{C}^n \) such that

\[
\|S\|_{(k)} \searrow \hat{\rho}(S) \quad \text{as} \ k \to +\infty.
\]
Since \( \|x\|_{(k)} \leq \|x\|_2 \) and \( \|x - y\|_{(k)} \leq \|x - y\|_2 \) for any pair \( x, y \in \mathbb{C}^n \) and \( k \geq 1 \), \( \|\cdot\|_{(k)} \) is pointwise bounded and equi-continuous on the compact set \( S = \{x \in \mathbb{C}^n : \|x\|_2 \leq 1\} \). Then, the Arzelà-Ascoli theorem ensures that there exists a subsequence, say \( \{k_j\} \supseteq \mathbb{N} \), such that
\[
\|x\|_{(k_j)} \rightarrow \|x\|_* \quad \text{as} \quad j \rightarrow +\infty, \quad \forall x \in \mathbb{C}^n.
\]
Here the limit function
\[
\|\cdot\|_* : \mathbb{C}^n \rightarrow [0, \infty)
\]
is obviously a semi-norm on \( \mathbb{C}^n \). By \( \|x \cdot S_i\|_{(k)} \leq \|S_i\|_{(k)} \|x\|_{(k)} \) and letting \( j \rightarrow +\infty \), we have
\[
\|x \cdot S_i\|_* \leq \hat{\rho}(S) \|x\|_* \quad \forall x \in \mathbb{C}^n \text{ and } i \in \mathcal{I}.
\]
Next, we will assert that \( \|\cdot\|_* \neq 0 \) on \( \mathbb{C}^n \). In fact, by the normality of each \( \|\cdot\|_{(k)} \) there is a sequence \( x_j \in \mathbb{C}^n \) such that
\[
\|x_j\|_{(k)} = \|x_j\|_2 = 1.
\]
By the compactness of \( S \), we may assume, without loss of generality, that \( x_j \rightarrow \hat{x} \) for some \( \hat{x} \in \mathbb{C}^n \) as \( j \rightarrow +\infty \). Then, it is easy to see that \( \|\hat{x}\|_* = \|\hat{x}\|_2 = 1 \), as desired.

This proves the theorem. \( \square \)

2.2. Barabanov semi-norm theorem

In the situation of Theorem B, by re-indexing \( S \) if necessary, there is no loss of generality in assuming that \( \mathcal{I} \) is a compact metric space with a metric \( d(\cdot, \cdot) \) and that \( i \mapsto S_i \) is continuous. Since the statement holds trivially in the case \( \hat{\rho}(S) = 0 \), we assume \( \hat{\rho}(S) = 1 \) without loss of generality.

Proof of Theorem B. Let \( \|\cdot\|_* \) be an extremal norm of \( S \). Then \( \|x \cdot S_i\|_* \leq \|x\|_* \) for all \( x \in \mathbb{C}^n \), since \( \hat{\rho}(S) = 1 \). Put
\[
v(x) = \limsup_{k \rightarrow +\infty} \left\{ \max_{(1, \ldots, k)} \|x \cdot S_{i_1} \cdots S_{i_k}\|_* \right\} \quad \forall x \in \mathbb{C}^n.
\]
It is easy to see that \( v \) is a semi-norm on \( \mathbb{C}^n \) having the properties \( v \neq 0 \) and \( v(x \cdot S_i) \leq v(x) \) for all \( x \in \mathbb{C}^n \) and \( i \in \mathcal{I} \). Next, we claim that \( v \) satisfies the requirements of Theorem B.

Let \( \hat{x} \in \mathbb{C}^n \) be arbitrarily given. As \( S_i \) is continuous in \( i \in \mathcal{I} \), it follows from the compactness of \( \mathcal{I} \) that there is a sequence of switching signals \( i(\ell) : \mathbb{N} \rightarrow \mathcal{I} \) such that
\[
\|\hat{x} \cdot S_{i(\ell)} \cdots S_{i(\ell)}\|_* = \max_{(1, \ldots, k)} \|\hat{x} \cdot S_{i_1} \cdots S_{i_k}\|_* \quad \forall \ell \geq 1.
\]
By the compactness of \( \mathcal{I} \) and \( \mathcal{I}^N = \{i : \mathbb{N} \rightarrow \mathcal{I}\} \), we could choose a sequence \( \{i(\ell)_k\} \) satisfying
\[
v(\hat{x}) = \lim_{k \rightarrow +\infty} \|\hat{x} \cdot S_{i(\ell)_1} \cdots S_{i(\ell)_k}\|_* \quad \text{and} \quad i(\ell)_k \rightarrow i^{(\infty)}_k = \left(i^{(\infty)}_j\right)_{j=1}^{+\infty}
\]
Then for any \( j \geq 1 \), we have
\[
v(\hat{x} \cdot S_{i(\infty)_1} \cdots S_{i(\infty)_j}) \geq \limsup_{k \rightarrow +\infty} \|\hat{x} \cdot S_{i(\ell)_1} \cdots S_{i(\ell)_k}\|_* 
\]
\[
\geq \limsup_{k \rightarrow +\infty} \|\hat{x} \cdot S_{i(\ell)_1} \cdots S_{i(\ell)_k}\|_* \cdot \|S_{i(\ell)_1} \cdots S_{i(\ell)_j}\|_*
\]
\[
= v(\hat{x}),
\]
noting that \( \|S_{i(\ell)_1} \cdots S_{i(\ell)_k}\|_* \leq 1 \). So, \( v(\hat{x} \cdot S_{i(\infty)_1} \cdots S_{i(\infty)_j}) = v(\hat{x}) \) for each \( j \geq 1 \).

This completes the proof of Theorem B. \( \square \)

3. Applications of the extremal and Barabanov semi-norms

Using our extremal semi-norm theorem as the starting point, we can simply prove the Barabanov theorem, the Berger-Wang formula and the Elsner theorem in this section. In what follows, let \( S = \{S_i\}_{i \in \mathcal{I}} \) be an arbitrary bounded family in \( \mathbb{C}^{n \times n} \), where \( 2 \leq n < \infty \). We note that the case of \( n = 1 \) is trivial.

3.1. Proof of Barabanov’s theorem

Proof. Let \( \|\cdot\|_* \) be an arbitrary extremal semi-norm on \( \mathbb{C}^n \) of \( S \), given by the extremal semi-norm theorem (Theorem A).

Let \( \mathcal{H} = \{x \in \mathbb{C}^n : \|x\|_* = 0\} \).
Clearly, \( \mathcal{K} \) is a proper linear subspace of \( \mathbb{C}^n \). The extremality of \( \| \cdot \|_* \) implies that \( \mathcal{K} \) is invariant by each \( S_i \), i.e., \( x \cdot S_i \in \mathcal{K} \) for all \( x \in \mathcal{K} \). Because \( S \) is irreducible, we have \( \mathcal{K} = \{ \mathbf{0} \} \). So, \( \| \cdot \|_* \) is exactly a norm on \( \mathbb{C}^n \). This implies that \( \sup_{i \in I} \| S_i \|_* \leq \hat{\rho}(S) \) and hence \( \hat{\rho}(S) = \| S \|_* \). If \( \{ S_i \}_{i \in I} \) is compact in \( \mathbb{C}^{n \times n} \), then the supremum can be attained. Further, the statement of Barabanov’s theorem follows from Theorem B at once. This completes the proof of Barabanov’s theorem. \( \square \)

As a result of Barabanov’s theorem, we can obtain a concise proof of the following classical theorem of J. Levitzki at once.

**Corollary 3.1.** (See Levitzki [17].) If \( S \) is irreducible, then \( \hat{\rho}(S) > 0 \).

This implies that we can always normalize an irreducible bounded set of matrices \( S \) to \( S/\hat{\rho}(S) \) which has the joint spectral radius 1.

### 3.2. Proof of the Elsner theorem

**Proof.** If \( S \) is irreducible with \( \hat{\rho}(S) = 1 \), then from the Barabanov theorem, it follows that \( \| S_i \|_* \leq 1 \) for all \( i \in I \), for some extremal norm \( \| \cdot \|_* \) of \( S \). This contradicts the unboundedness of the multiplicative semigroup \( S^+ \) generated by \( S \). This proves the Elsner reduction theorem. \( \square \)

We notice here that if one wants to prove Barabanov’s theorem from Elsner’s reduction theorem, then there needs the following basic result besides Theorem B and Corollary 3.1.

**Lemma 3.2.** (See [4, Theorem 1], also [14, Theorem 3].) \( S^+ \) is bounded in \( \mathbb{C}^{n \times n} \) under a norm on \( \mathbb{C}^n \) if and only if one can define a norm \( \| \cdot \| \) on \( \mathbb{C}^n \) such that \( \| S_i \| \leq 1 \) for all \( i \in I \).

**Proof.** Let \( S^+ \) be bounded and nontrivial. Define \( \| x \| = \sup\| x \cdot A \|_2 : A \in S^+ \), where \( \| \cdot \|_2 \) denotes the standard Euclidean vector norm on \( \mathbb{C}^n \). Then, \( \| S_i \| \leq 1 \) for all \( i \in I \). The sufficiency holds trivially. \( \square \)

Here we can present another proof of the Barabanov theorem using Elsner’s reduction theorem.

**Another proof of Barabanov’s theorem.** In fact, letting \( \tilde{S} = S/\hat{\rho}(S) \) by Corollary 3.1, it follows from Elsner’s reduction theorem that \( \tilde{S}^+ \) is bounded. So, by Lemma 3.2, \( \| \tilde{S} \|_* \leq 1 \) and then \( \| S \|_* \leq \hat{\rho}(S) \) for some norm \( \| \cdot \|_* \) on \( \mathbb{C}^n \). Clearly, this norm is just extremal of \( S \). Then, the statement comes from Theorem B. \( \square \)

### 3.3. Proof of the Berger–Wang formula

From the basic inequality \( \rho(A) \leq \| A \| \) for any \( A \in \mathbb{C}^{n \times n} \) and any norm \( \| \cdot \| \) on \( \mathbb{C}^n \), we see that \( \rho(S) \leq \hat{\rho}(S) \). So, if \( \hat{\rho}(S) = 0 \) then \( \rho(S) = 0 \). Next, we will assume \( \hat{\rho}(S) > 0 \). Moreover, replacing \( S \) by \( S/\hat{\rho}(S) \) if necessary, we might assume \( \hat{\rho}(S) = 1 \) without loss of generality.

The following result holds trivially by induction on \( n \), which is a standard result in the theory of linear algebras.

**Lemma 3.3.** (See [1].) There exists a nonsingular matrix \( P \in \mathbb{C}^{n \times n} \) and \( r \) positive integers \( n_1, \ldots, n_r \) with \( n_1 + \cdots + n_r = n \) such that for each \( i \in I \),

\[
PS_iP^{-1} = \begin{bmatrix}
S_i^{(1,1)} & 0_{n_1 \times n_2} & \cdots & 0_{n_1 \times n_r} \\
0_{n_1 \times n_2} & S_i^{(2,2)} & \cdots & 0_{n_2 \times n_r} \\
\vdots & \vdots & \ddots & \vdots \\
0_{n_1 \times n_2} & 0_{n_2 \times n_r} & \cdots & S_i^{(r,r)}
\end{bmatrix},
\]

where \( S_i^{(k,k)} := \{ S_i^{(k,k)} \}_{k \in I} \subseteq \mathbb{C}^{n_k \times n_k} \) is irreducible for each \( 1 \leq k \leq r \).

When \( S \) is itself irreducible, \( r = 1 \) in Lemma 3.3. The following is well known and obvious by the principle of diagonal majority.

**Lemma 3.4.** Under the block-triangular decomposition of Lemma 3.3, \( \hat{\rho}(S) = \max_{1 \leq k \leq r} \hat{\rho}(S_i^{(k,k)}) \).

One can find an ergodic version of the above statement from [7]. In addition, this statement still holds if the sub-blocks systems \( S_i^{(k,k)} \), given by Lemma 3.3, are not necessarily irreducible, for \( 1 \leq k \leq r \).
On the other hand, by \( \rho(A)^\ell = \rho(A^\ell) \) for any \( \ell \geq 1 \) and any \( A \in \mathbb{C}^{n \times n} \), we can easily obtain

\[
\rho(S) = \sup_{\ell \geq 1} \left\{ \sup_{(i_1, \ldots, i_{\ell}) \in \mathcal{I}^\ell} \rho(S_{i_1} \cdots S_{i_{\ell}})^{1/\ell} \right\}.
\]

To prove the Berger–Wang formula, based on Lemmas 3.3 and 3.4, we then need to prove only the following simple case.

**Theorem 3.5.** Let \( S = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{n \times n} \) be a bounded, irreducible family with \( \rho(S) = 1 \). Then, it holds \( \rho(S) = 1 \).

**Proof.** Let \( \text{Cl}(S) \) be the closure of \( \{S_i\}_{i \in \mathcal{I}} \) in \( \mathbb{C}^{n \times n} \). Since \( \rho(S) = \rho(\text{Cl}(S)) \) and \( \hat{\rho}(S) = \hat{\rho}(\text{Cl}(S)) \) by definitions, there is no loss of generality in assuming that \( S \) is a compact set in \( \mathbb{C}^{n \times n} \). Let \( \| \cdot \|_a \) be a norm on \( \mathbb{C}^n \) given by Barabanov’s theorem for \( S \). Then, \( \max_{(i_1, \ldots, i_{\ell}) \in \mathcal{I}^\ell} \|S_{i_1} \cdots S_{i_{\ell}}\|_a = 1 \) for any \( \ell \geq 1 \) and, moreover, one could find, by the Barabanov-type property, some sequence \( i_\ell : \mathbb{N} \to \mathcal{I} \) such that \( \|S_{i_1} \cdots S_{i_k}\|_a = 1 \) for all \( k \geq 1 \). Using this boundedness, one could pick out a positive integer sequence, say \( \{k_\ell\}_{\ell=1}^{+\infty} \), with \( k_{\ell+1} - k_\ell \geq 1 \) for \( \ell \geq 1 \), such that

\[
C_\ell := S_{i_1} \cdots S_{i_{k_\ell}} \to C \neq 0_{n \times n} \quad \text{as} \quad \ell \to +\infty.
\]

Now, define \( B_\ell := S_{i_{k_\ell+1}} \cdots S_{i_{k_{\ell+1}}} \) and so \( C_{\ell+1} = C \ell B_\ell \). Using the boundedness again, we could pick out a positive integer sequence \( \{\ell_j\}_{j=1}^{+\infty} \) satisfying

\[
B_{\ell_j} \to B \in \mathbb{C}^{n \times n} \quad \text{as} \quad j \to +\infty.
\]

Then, \( C = CB \), \( C \neq 0_{n \times n} \), and \( \rho(B) = \lim_{j \to +\infty} \rho(B_{\ell_j}) \). But \( \text{Im}(C) \cdot B = \text{Im}(C) \neq \{0\} \), so \( B_{\text{Im}(C)} \) is the identity. Thus, \( \rho(B) \geq 1 \). So, \( \rho(S) = 1 \) from \( 1 \geq \rho(S) \geq \rho(S)^{k_{\ell+1}+k_\ell} \geq \rho(B_{\ell_j}) \) for all \( j \geq 1 \).

This therefore proves the statement of Theorem 3.5. \( \square \)

Therefore, the proof of the Berger–Wang formula is completed.

4. Concluding remarks

For an arbitrary bounded family \( S = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{n \times n} \) with \( \hat{\rho}(S) > 0 \), there always exists an extremal semi-norm \( \| \cdot \|_a \) of \( S \) on \( \mathbb{C}^n \). Let \( \mathcal{X}_{\| \cdot \|_a} \), be the kernel of the semi-norm \( \| \cdot \|_a \), i.e.,

\[
\mathcal{X}_{\| \cdot \|_a} = \{ x \in \mathbb{C}^n : \|x\|_a = 0 \}.
\]

Then, we have \( 0 \leq \dim \mathcal{X}_{\| \cdot \|_a} \leq n - 1 \) and it is a common invariant subspace for each \( S_i \) from our extremal semi-norm theorem. Therefore, if \( \dim \mathcal{X}_{\| \cdot \|_a} = 0 \) then \( \| \cdot \|_a \) is just an extremal norm of \( S \); if \( 1 \leq \dim \mathcal{X}_{\| \cdot \|_a} \leq n - 1 \) then \( S \) is reducible.

If there is an extremal norm \( \| \cdot \|_a \) of \( S \) and \( S \) is compact, then the semi-norm \( v \), defined by

\[
v(x) = \limsup_{k \to +\infty} \left\{ \max_{(i_1, \ldots, i_k) \in \mathcal{I}^k} \hat{\rho}(S)^{-k}\|x \cdot S_{i_1} \cdots S_{i_k}\|_a \right\} \quad \forall x \in \mathbb{C}^n.
\]

is an extremal semi-norm of \( S \), which has the Barabanov-type property, i.e., to any \( x \in \mathbb{C}^n \), there is an infinite sequence \( i_\ell : \mathbb{N} \to \mathcal{I} \) such that

\[
v(x \cdot S_{i_1} \cdots S_{i_k}) = \hat{\rho}(S)^k v(x) \quad \forall k \geq 1.
\]

If \( S \) is irreducible, then \( v \) is exactly a norm. The Barabanov-type property is itself an important dynamical property of the linear switched dynamics induced by \( S \).

We notice here that our defining of the Barabanov norm is different from Barabanov’s means presented in [1] for irreducible \( S \) by

\[
v(x) = \sup_{i_\ell \in \mathbb{N}} \left\{ \limsup_{k \to +\infty} \hat{\rho}(S)^{-k}\|x \cdot S_{i_1} \cdots S_{i_k}\|_a \right\} \quad \forall x \in \mathbb{C}^n.
\]

Theorems A and B presented in this note generalize Barabanov’s theorem and Elsner’s theorem.

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References