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The Galerkin method for singular integral equations revisited

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Abstract

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In an earlier work, the author has obtained error bounds for the Galerkin method for solving Cauchy singular integral equations, discovering that the usually neglected constants contain the Riemann zeta function, when evaluated in the supremum norm. The aim of this investigation is twofold: to show that the occurrence of the Riemann zeta function in the error bound for the Chebyshev norm is sharp; and secondly to use this result to obtain a class of forcing functions for which the method does not yield an approximate solution differing from the analytical one by at most a prescribed error tolerance. These counterexamples indicate that in practical situations, for functions exhibiting a behavior similar to the one presented here, Galerkin's method might not lead to an acceptable solution.

Keywords: Galerkin method, singular integral equations.

1. Introduction

Singular integral equations arise in many problems of mathematical physics. They find applications in many important fields like fracture mechanics, aerodynamics, the theory of porous filtering, antenna problems in electromagnetic theory, and others. Their solutions can be obtained analytically, using the theory developed by Mushkelishvili [11]; but in practice, approximate methods are needed. The direct numerical methods are preferred, which attack the equation as it is written, without transforming it beforehand into a Fredholm equation. Among these, the oldest one is the Galerkin method, proposed originally by Erdogan [3]. The convergence proof for first-kind equations was first given by Linz [10], thereby justifying it theoretically. Afterwards, a number of papers for equations of the second kind appeared [1,5,6,8,9,15].

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The purpose of this study is to continue the investigation begun in the earlier work [14,15], where the author has obtained error bounds for the Galerkin method. A striking consequence was the discovery that the usually neglected constants contain the Riemann zeta function, if the error is evaluated in the supremum norm. However, this does not happen if the estimate is computed using the L_2-norm. The aim of this investigation is twofold. We want to show that the occurrence of the Riemann zeta function in the error bound for the supremum norm is sharp. Secondly we use this result to obtain a class of functions for which the method does not yield an approximate solution that differs from the analytical one by a given error tolerance. Thus convergence in practice does not occur, contrary to the common belief.

The counterexample, although somewhat artificial, indicates that in practical situations, within the limitations due to the memory of the system and the time requirements of the user, for functions exhibiting a behavior similar to the ones presented here, the method may not lead to an acceptable solution. Such situations do arise in engineering situations, as is outlined in [4].

The paper is organized as follows: in the next section we briefly review the basic formulae of the Galerkin method and recall the results derived earlier, for ease of the reader. In Section 3 we provide the counterexample for the dominant equation. The complete equation is examined in Section 4. A final discussion concludes the note.

2. Preliminaries

We consider here the dominant singular integral equation of the second kind

$$ag(x)+b\pi^{-1}\int_{-1}^{1} g(t)(t-x)^{-1}\,\mathrm{d}t=f(x), \quad -1<x<1. \tag{2.1}$$

The unknown function g is sought in the class of Hölder continuous functions over $(-1, 1)$. Its singular behavior at the endpoints is described by using the fundamental function

$$w(x)=(1-x)^{\alpha}(1+x)^{\beta},$$

with

$$\alpha=(2\pi\mathrm{i})^{-1}\log\left[\frac{a-\mathrm{i}b}{a+\mathrm{i}b}\right]+N, \qquad \beta=-(2\pi\mathrm{i})^{-1}\log\left[\frac{a-\mathrm{i}b}{a+\mathrm{i}b}\right]+M,$$

where M, N are integers chosen such that the index of the equation

$$\chi=-(\alpha+\beta)=-(M+N)$$

attains the values -1, 0, 1. Usually a new smooth unknown function $y(x)$ is defined by letting

$$g(x)=w(x)y(x).$$

The equation can be rewritten introducing the operator T as

$$T(y)\equiv aw(x)y(x)+b\pi^{-1}\int_{-1}^{1} w(t)y(t)(t-x)^{-1}\,\mathrm{d}t=f(x). \tag{2.2}$$

The formula [7, p.290]

$$\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}(t-x)^{-1}\,\mathrm{d}t$$
$$=\pi(1+x)^{\beta}(1-x)^{\alpha}\cot(\pi\alpha)-2^{-\chi}B(\alpha,\beta+1)\ {}_2F_1(\chi,1;1-\alpha;\tfrac{1}{2}(1-x)),$$

for $\chi \neq 1$, since χ is an integer, in our notation can be rewritten as

$$aw(x)+b\pi^{-1}\int_{-1}^{1} w(t)(t-x)^{-1}\,dt=\gamma P_{-\chi}^{(-\alpha,-\beta)}(x), \tag{2.3}$$

where we let

$$\gamma \equiv \frac{-2^{-\chi}b}{\sin(\pi\alpha)}.$$

However, (2.3) holds also for $\chi = 1$, by defining $P_{-1}(x) \equiv 0$ and remarking that $w(x)$ in such case is a solution of the homogeneous dominant singular integral equation. Note that if the equation is of the first kind, the Jacobi weight and polynomials reduce to the Chebyshev ones, of the first or second kind depending on χ being 1 or -1.

In the case $\chi = 1$, an extra normalization condition is added, usually of the form

$$\int_{-1}^{1} w(t)y(t)\,dt=1, \tag{2.4}$$

to obtain uniqueness of the solution. In case $\chi = 0$, no extra condition is needed, the solution being unique as stated by the theory, while for $\chi = -1$, the solution exists if and only if the following orthogonality condition is satisfied:

$$\int_{-1}^{1} \rho(t)f(t)\,dt \equiv \int_{-1}^{1} [w(t)]^{-1}f(t)\,dt=0. \tag{2.5}$$

To set up the Galerkin method, we need to define the weighted scalar product as follows:

$$\langle f, g\rangle_\omega \equiv \int_{-1}^{1} \omega(t)f(t)g(t)\,dt;$$

let $\|f\|_\omega$ be the corresponding norm. In our case $\omega(t)$ is the Jacobi weight $w(t)$ and the approximating functions are the corresponding orthogonal polynomials $P_n^{(\alpha,\beta)}(t)$.

The major results concerning convergence of the Galerkin method are summarized here for ease of reference. They can be found, together with the various constants L_r, $\hat{L}_r$, H_r, $\hat{H}_r$, in [15].

Theorem 1. *If* $f \in C^{(r+1)}[-1, 1]$, *then the error of the method satisfies*

$$\|e_N\|_w \leqslant L_r\|f^{(r+1)}\|_\infty N^{-r-1/2},$$

and if $r > \frac{3}{2}$,

$$\|e_N\|_\infty \leqslant \hat{L}_r\|f^{(r+1)}\|_\infty \zeta(r-\tfrac{1}{2})N^{-r+3/2}.$$

Together with (2.1) we will be concerned also with the complete equation, in which, in addition to the singular term, a Fredholm kernel is present; in extended form it is written

$$aw(x)y(x)+b\pi^{-1}\int_{-1}^{1} w(t)y(t)(t-x)^{-1}\,dt+\int_{-1}^{1} K(x,t)w(t)y(t)\,dt=f(x), \tag{2.6}$$

or, in operator notation,

$$T(y)+K(y)=f.$$

In this case the following results hold.

Theorem 2. *If $f \in C^{(r+1)}[-1, 1]$ and the Fredholm kernel $K \in C^{(r+1)}[-1, 1]$ with respect to both the variables, then the error of the method satisfies*

$$\| e_N \|_{w} \leqslant H_r \| f^{(r+1)} \|_\infty N^{-r-1/2},$$

and if $r > \frac{3}{2}$,

$$\| e_N \|_\infty \leqslant \hat{H}_r \| f^{(r+1)} \|_\infty \zeta(r - \tfrac{1}{2}) N^{-r+3/2}.$$

In the rest of the paper, some estimates will also be needed. We list them here, for ease of reference. From [13, p.168]:

$$\max_{-1<x<1} P_n^{(\alpha,\beta)}(x) = \binom{n+q}{n} \cong n^q, \tag{2.7}$$

where $q = \max(\alpha, \beta) \geqslant -\frac{1}{2}$. In view of the values attained by the index, this condition is always satisfied. Recall also that

$$\binom{n+q}{n} = \frac{(q+1)\cdots(q+n)}{n!}. \tag{2.8}$$

Finally, [13, pp. 58 and 63] yield respectively

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[P_n^{(\alpha,\beta)}(x) \right] = \tfrac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.10}$$

The L_2-norm of the Jacobi polynomials will also be of interest [13, p.68]:

$$\begin{aligned} \| P_n^{(\alpha,\beta)} \|_w^2 \equiv h_n^{(\alpha,\beta)} = {} & 2^{1-\chi} \Gamma(n+\alpha+1) \Gamma(n+\beta+1) \\ & \times \left[\Gamma(n+1) \Gamma(n-\chi+1)(2n+1-\chi) \right]^{-1}. \end{aligned} \tag{2.11}$$

3. Sharp error estimates

Our first task in this section is to derive the error equation. Let the solution of the equation be expressed as the infinite series of the Jacobi polynomials $P_n^{(\alpha,\beta)} \equiv \phi_n$ as follows:

$$y = \sum_{n=\chi}^{\infty} \beta_n \phi_n; \tag{3.1}$$

note that in case of negative index, β_{-1} is not present in the sum, since $\phi_{-1} \equiv 0$; also, if $\chi = 1$, β_0, which does not appear in the above sum, will be calculated from the normalization condition (2.4). We assume also that the right-hand side can be expressed as an infinite series in terms of the related Jacobi polinomials $P_{n-\chi}^{(-\alpha,-\beta)} \equiv \psi_{n-\chi}$, with coefficients $\alpha_{n-\chi}$ for the moment not yet specified:

$$f = \sum_{n=\chi}^{\infty} \alpha_{n-\chi} \psi_{n-\chi}.$$

Here it should be remarked that if $\chi = -1$, from the orthogonality condition it follows that $\alpha_0 = 0$, since (2.5) can be restated as

$$\langle f, \psi_0 \rangle_\rho = 0.$$

The approximate solution of the equation is obtained by truncating the series representation of y, denoting by c_n the approximations of the coefficients β_n,

$$y_N = \sum_{n=0}^{N-1} c_n \phi_n, \tag{3.2}$$

and imposing that the residual of the equation

$$r_N(x) \equiv f(x) - T(y_N) = \sum_{n=\chi}^{\infty} \alpha_{n-\chi} \psi_{n-\chi} - \sum_{n=\chi}^{N-1} c_n T(\phi_n)$$

$$= \sum_{n=\chi}^{\infty} \alpha_{n-\chi} \psi_{n-\chi} - \sum_{n=\chi}^{N-1} \gamma c_n \psi_{n-\chi}$$

be orthogonal to the first $N - \chi$ polynomials ψ_j, i.e.,

$$\langle r_N, \psi_j \rangle_\rho = 0, \quad j = \chi, \chi + 1, \dots, N-1.$$

This yields $N - \chi$ algebraic equations, the Galerkin equations, which are used to determine the values of the coefficients in the truncated expansion for the unknown function:

$$\sum_{n=\chi}^{\infty} \alpha_{n-\chi} \langle \psi_{n-\chi}, \psi_j \rangle_\rho = \sum_{n=\chi}^{N-1} \gamma c_n \langle \psi_{n-\chi}, \psi_j \rangle_\rho, \quad j = \chi, \chi+1, \dots, N-1, \tag{3.3}$$

that is,

$$\alpha_{j-\chi} = \gamma c_j, \quad j = \chi, \chi + 1, \dots, N-1.$$

Let us define the error

$$e_N(x) \equiv y(x) - y_N(x) = \sum_{n=\chi}^{\infty} \delta_n \phi_n(x).$$

Using (2.2) the error equation can be written as

$$T(e_N) \equiv T(y - y_N) = f - T(y_N) = \sum_{n=\chi}^{\infty} \alpha_{n-\chi} \psi_{n-\chi} - \sum_{n=\chi}^{N-1} \gamma c_n \psi_{n-\chi}$$

$$= \sum_{n=N}^{\infty} \alpha_{n-\chi} \psi_{n-\chi}.$$

We thus obtain

$$\sum_{n=\chi}^{\infty} \delta_n T(\phi_n) = \sum_{n=N}^{\infty} \alpha_{n-\chi} \psi_{n-\chi},$$

so that by equating coefficients of the same polynomials we obtain:

$$\delta_n = 0, \qquad n = \chi, \dots, N-1,$$
$$\delta_n = \frac{\alpha_{n-\chi}}{\gamma}, \quad n = N, N+1, \dots,$$

the former equation expressing the fact that $\beta_n = c_n$, $n = \chi, \dots, N-1$. This agrees with a fundamental lemma proven in [14,15], which shows that the first $N-\chi$ coefficients of the Fourier expansion of the solution must coincide with the ones obtained by solving the Galerkin equations. The expression for the error can then be written as:

$$e_N(x) = \gamma^{-1} \sum_{n=N}^{\infty} \alpha_{n-\chi} \phi_n(x). \tag{3.4}$$

We want to show here that the occurrence of the Riemann zeta function in the error bound obtained in [14,15] is sharp. Observe that using (2.9),

$$\| e_N \|_\infty \geqslant | e_N(1) | = \left| \gamma^{-1} \sum_{n=N}^{\infty} \alpha_{n-\chi} \phi_n(1) \right| \geqslant \left| \gamma^{-1} \sum_{n=N}^{\infty} \alpha_{n-\chi} \binom{n+\alpha}{n} \right|.$$

On the other hand, we also have the following upper bound:

$$\| e_N \|_\infty \leqslant | \gamma^{-1} | \sum_{n=N}^{\infty} | \alpha_{n-\chi} | \, \| \phi_n \|_\infty \leqslant | \gamma^{-1} | \sum_{n=N}^{\infty} | \alpha_{n-\chi} | \binom{n+q}{n},$$

where we have used (2.7). For the remainder of the paper, we make the assumption that

$$q = \alpha, \tag{3.5}$$

so that

$$-\beta \equiv \max(-\alpha, -\beta) = q^*. \tag{3.6}$$

Since we are about to provide a counterexample, this is not restrictive.

We can now specialize the constants $\alpha_{n-\chi}$; we take them to be

$$\alpha_{n-\chi} = \frac{n^{-p}}{\binom{n-\beta}{n}}, \qquad \alpha_0 \quad \text{for } \chi = 0, \qquad \alpha_0 = \alpha_1 = 0 \quad \text{for } \chi = -1, \tag{3.7}$$

so that the above estimates become, in view of (2.7) and of the fact that N is large,

$$\begin{aligned} \| e_N \|_\infty &\leqslant | \gamma^{-1} | \sum_{n=N}^{\infty} n^{-p} \binom{n+\alpha}{n} \Big/ \binom{n-\beta}{n} \cong | \gamma^{-1} | \sum_{n=N}^{\infty} n^{-p+\alpha+\beta} \\ &\leqslant | \gamma^{-1} | N^{1-p-\chi} \zeta(p+\chi) \end{aligned} \tag{3.8}$$

and denoting by L a suitable constant,

$$\| e_N \|_\infty \geqslant L | \gamma^{-1} | \sum_{n=N}^{\infty} n^{-p+\alpha+\beta} \geqslant L | \gamma^{-1} | N^{1-p-\chi} \left[\zeta(p+\chi) - 1 + \frac{1}{N} \right], \tag{3.9}$$

or, in summary

$$| \gamma^{-1} | N^{1-p-\chi} \zeta(p+\chi) \geqslant \| e_N \|_\infty \geqslant L | \gamma^{-1} | N^{1-p-\chi} \left[\zeta(p+\chi) - 1 + \frac{1}{N} \right]. \tag{3.10}$$

These bounds show that the actual growth of $\| e_N \|_\infty$ is like $N^{1-p-\chi}\zeta(p+\chi)$, i.e., the occurrence of the Riemann zeta function is sharp. Moreover, they yield theoretical convergence for the Galerkin method applied to the function f constructed in this way, provided that p is chosen so that $p > 1 - \chi$. Given this condition, the upper bound (3.10) does in fact approach 0 as N grows large.

We now examine the right-hand side, to understand the implications of the assumptions made so far. One extra difficulty with respect to the estimate obtained for the error is given here by the fact that the series for f starts from χ and not from N. We cannot use the asymptotic estimate (2.7) abruptly. However, an upper bound is immediately calculated:

$$\| f \|_\infty \leqslant \sum_{n=1}^{\infty} n^{-p} \| \psi_{n-\chi} \|_\infty \Big/ \binom{n-\beta}{n} = \zeta(p). \tag{3.11}$$

For $\chi = -1$, an upper bound on this norm is $\frac{1}{6}\pi^2$, for $\chi = 0$ the upper bound is finite, but the function f might have a large norm. In both such cases, being the uniform limit of continuous functions, f must be continuous. For $\chi = 1$ instead, the series on the right-hand side may diverge. In order to possibly understand more to what kind of forcing function the choice of the coefficients $\alpha_{n-\chi}$ leads, we investigate the lower bounds as well:

$$\| f \|_\infty \geqslant | f(1) | = \sum_{n=1}^{\infty} n^{-p} \binom{n-\alpha}{n} \Big/ \binom{n-\beta}{n} = \sum_{n=1}^{\infty} n^{-p} \prod_{k=1}^{n} \frac{k-\alpha}{k+\alpha+\chi}.$$

It is easily seen that

$$\prod_{k=1}^{n} \frac{k-\alpha}{k+\alpha+\chi} \geqslant \frac{1-\alpha}{\alpha+1+\chi} \prod_{k=2}^{n} \left[1 - \frac{2+\chi}{k+1+\chi} \right],$$

from which the estimate

$$\| f \|_\infty \geqslant G(\chi) \frac{1-\alpha}{\alpha+1+\chi} \zeta(p+2+\chi)$$

follows, with

$$G(\chi) = \chi(\chi+1) - (\chi^2 - 1) + \tfrac{1}{2}\chi(\chi-1), \quad \chi \in \{-1, 0, 1\}.$$

Recalling the conditions for the convergence of the error, if follows that these lower bounds even in case of positive index do not imply that the norm of the forcing function is necessarily large. For $\chi = 1$ more insight is obtained by looking at the estimate, using (2.9), (2.7), and where C denotes a suitable constant,

$$\begin{aligned} \| f \|_\infty &\geqslant \sum_{n=1}^{N-1} n^{-p} \left[\binom{n-\alpha}{n} \Big/ \binom{n-\beta}{n} - C n^{-\alpha+\beta} \right] + C \sum_{n=1}^{\infty} n^{-p-\alpha+\beta} \\ &= \text{constant} + C\zeta(p+1+2\alpha). \end{aligned} \tag{3.12}$$

For $p \cong 0$ and $\alpha \cong 0$, the series on the right-hand side converges, but its sum is large, implying necessarily a large norm for the forcing function.

Notice that the convergence of the method must be shown since the function f does not necessarily satisfy the sufficient conditions yielding convergence in the supremum norm, given in Theorem 1, as it is clear from the above considerations.

In the choice made above for the coefficients $\alpha_{n-\chi}$ nothing has been said about the parameter p, which for the moment is still free, with the only restriction $p > 1-\chi$ in order to attain convergence.

Now suppose we want the solution of the equation with a maximum tolerance, ϵ, i.e., $\| e_N \|_\infty < \epsilon$. Recall that N is the size of the Galerkin system (3.3) that has to be solved. It is thus limited by the amount of storage available on the system in use. Equivalently one could talk about execution time, which is related to the size of the system. In fact in the program implementing the Galerkin method there should be an exit test on the maximum iterations, or the maximum execution time, the algorithm should perform. In any case it is enough to remark that N is system dependent, and therefore it is fixed a priori; then choose p so that

$$\gamma^{-1} N^{1-p-\chi}\left[\zeta(p+\chi)-1+\frac{1}{N}\right] \geqslant 1. \tag{3.13}$$

This is certainly possible, since $\zeta(z)$ has a pole for $z=1$; it is then sufficient to choose p larger but close enough to $1-\chi$, to obtain the above inequality. The right-hand side constructed in this way, whose coefficients are determined by the formulae (3.7) and (3.13) leads to a singular integral equation which cannot be solved within the prescribed error tolerance by the Galerkin method: indeed its approximate solution will differ from the analytical one by more than the tolerance at least at some point.

It is also interesting to note that the convergence condition for the lower bound of (3.12) is $p > -2\alpha$, in case $\chi = 1$. If $\alpha \cong -\frac{1}{2}$, then $p > 2$. In such situation from the lower bound for the error norm we obtain that the condition $N^{-p}/|\gamma|[\zeta(p+1)-1+1/N] \geqslant 1$ does not hold any more, since $\zeta(p+1) \leqslant \zeta(2) = \frac{1}{6}\pi^2$. The bounds for the error become in this case

$$\frac{\frac{1}{6}\pi^2 N^{-1-\epsilon}}{|\gamma|} \geqslant \| e_N \|_\infty \geqslant \frac{\left[\frac{1}{6}\pi^2 - 1 + 1/N\right]}{|\gamma|} N^{-1-\epsilon}, \quad \epsilon > 0.$$

In other words, the maximum size N of the system which can be stored in the computer memory determines the maximum accuracy $1/N$, attainable in solving the Galerkin equations. This is another instance for which the first-kind singular integral equations are better behaved than second-kind ones.

An application

In order to substantiate the above analysis, we briefly discuss an engineering application in fracture mechanics. In [4, p.557] it is shown how to obtain a second kind singular integral equation from a system of singular integral equations for two bonded half planes containing a series of interface cracks. For a single crack, assumed to be the interval $(-1, 1)$, the system is:

$$\pi^{-1}\int_{-1}^{1} f_1(t)(t-x)^{-1}\,\mathrm{d}t + \gamma f_2(x) = (2\mu_0)^{-1} p_2(x),$$

$$\pi^{-1}\int_{-1}^{1} f_2(t)(t-x)^{-1}\,\mathrm{d}t - \gamma f_1(x) = (2\mu_0)^{-1} p_1(x),$$

where μ_0 and γ represent material constants, p_1 and p_2 represent the external loads, f_1 and f_2 are auxiliary unknown functions related to the displacements. Introducing the functions

$$\varphi(t) = f_1(t) + \mathrm{i} f_2(t), \qquad P(x) = (2\mu_0)^{-1}[p_1(x) - \mathrm{i} p_2(x)],$$

the above system reduces to

$$(\pi i)^{-1}\int_{-1}^{1}\varphi(t)(t-x)^{-1}\,dt-\gamma\varphi(x)=P(x).$$

If the external load is expressed as a series of Jacobi polynomials $\psi_{n-\chi}(x)$, with coefficients $\alpha_{n-\chi}$ expressed by (3.7), where p is small, we fall exactly into the case discussed above. For this problem, the algorithm will not give a solution within the prescribed error tolerance, and the minimum obtainable error will be given by an estimate of the type (3.13).

In practical engineering situations, loads acting as delta functions are not uncommon. Since they have a mass concentrated in a neighborhood of a point and are nearly zero everywhere else, their supremum norm is large. Even though they can be approximated by smooth functions, the latter will also possess large norms. Thus we have a potentially dangerous situation, of the type described above. In such a case, care should be taken in trusting results obtained with the Galerkin method.

4. The complete equation

What has been presented in the previous section can be extended to the complete equation. The algebraic system to be solved is modified, but again the choice for the Fredholm kernel, as well as the one for the right-hand side, is delayed as done in the previous section. We suppose however that the kernel can be expressed in terms of the Fourier series

$$K(x,t)=\sum_{i=\chi}^{\infty}\sum_{j=0}^{\infty}K_{i-\chi,j}\psi_{i-\chi}(x)\phi_j(t),\tag{4.1}$$

and approximate it with the truncated series

$$K_N(x,t)=\sum_{i=\chi}^{N-1}\sum_{j=0}^{N-1}K_{i-\chi,j}\psi_{i-\chi}(x)\phi_j(t).\tag{4.2}$$

If we apply the operator (4.1) to the representation (3.1) for the solution, we obtain that (2.6) yields

$$\sum_{n=\chi}^{\infty}\psi_{n-\chi}\left[\gamma\beta_n+\sum_{j=0}^{\infty}\beta_jK_{n-\chi,j}\|\phi_j\|_w^2-\alpha_{n-\chi}\right]=0\tag{4.3}$$

or

$$\gamma\beta_n+\sum_{j=0}^{\infty}\beta_jK_{n-\chi,j}\|\phi_j\|_w^2=\alpha_{n-\chi},\quad n=\chi,\chi+1,\ldots.\tag{4.4}$$

Substitution of the representation (4.2) and of the approximation (3.2) into (2.6), and use of the orthogonality relation of the residual against the first polynomials ψ_j, yields the Galerkin equations

$$\sum_{n=\chi}^{N-1}\left[\gamma c_n+\sum_{j=0}^{N-1}K_{n-\chi,j}c_j\|\phi_j\|_w^2\right]\langle\psi_{n-\chi},\psi_m\rangle_\rho=\sum_{n=\chi}^{\infty}\alpha_{n-\chi}\langle\psi_{n-\chi},\psi_m\rangle_\rho,$$
$$m=\chi,\ldots,N-1,\tag{4.5}$$

where again the c_n's are used as approximations of the coefficients β_n of (3.1). The approximate equations are then the following:

$$\gamma c_{n+\chi} + \sum_{j=0}^{N-1} K_{n,j} \|\phi_j\|_w^2 c_j = \alpha_n, \quad n = \chi, \ldots, N-1. \tag{4.6}$$

Unlike in the situation for the dominant equation, these equations do not coincide with the first $N+\chi$ equations (4.4). Before proceeding any further, we need to specialize the choice for the kernel $K(x, t)$. The example provided here is simple, but along the same lines infinitely many others can be constructed. Suppose the kernel can be written in separated form as follows:

$$K(x, t) = k(x)\phi_1(t) = \sum_{n=\chi}^{\infty} K_{n-\chi,1}\psi_{n-\chi}(x)\phi_1(t). \tag{4.7}$$

In other words, $K_{ij} = 0$ for $i = \chi, \chi+1, \ldots$, and $j = 2, 3, \ldots$. Equations (4.4) reduce then to the following ones:

$$\gamma\beta_n + K_{n-\chi,1}\|\phi_1\|_w^2\beta_1 = \alpha_{n-\chi}, \quad n = \chi, \chi+1, \ldots, \tag{4.8}$$

while the approximate equations (4.6) become

$$\gamma c_n + K_{n-\chi,1}\|\phi_1\|_w^2 c_1 = \alpha_{n-\chi}, \quad n = \chi, \ldots, N-1. \tag{4.9}$$

Evidently, now these are the first $N-\chi$ of the equations (4.8). Thus the unknowns c_n and β_n satisfy the same finite system, and therefore coincide,

$$c_n = \beta_n, \quad n = \chi, \ldots, N-1.$$

We can solve the system (4.9) to obtain

$$\beta_1 = \alpha_{1-\chi}\left[\gamma + K_{1-\chi,1}\|\phi_1\|_w^2\right]^{-1}, \tag{4.10}$$

$$\beta_n = \frac{\alpha_{n-\chi} - K_{n-\chi,1}\|\phi_1\|_w^2\beta_1}{\gamma}, \quad n = 2, 3, \ldots . \tag{4.11}$$

Notice that the very first equation for $\chi = -1$ is identically satisfied by the convention $\psi_{-1} \equiv 0$. Also for $\chi = 0, -1$, β_0 is still evaluated with formula (4.11).

Consider now the error: in view of the previous remarks,

$$e_N = y - y_N = \sum_{n=N}^{\infty} \beta_n\phi_n(t).$$

An upper bound for the supremum norm then can be calculated by again letting $q = \max(\alpha, \beta) \equiv \alpha \geqslant -\frac{1}{2}$:

$$\begin{aligned}\|e_N\|_\infty &\leqslant \sum_{n=N}^{\infty} |\beta_n| \, \|\phi_n\|_\infty \leqslant \sum_{n=N}^{\infty} |\beta_n| \binom{n+\alpha}{n} \\ &\leqslant |\gamma^{-1}| \sum_{n=N}^{\infty} \left\{ |\alpha_{n-\chi}| + \left|\beta_1 K_{n-\chi,1}\|\phi_1\|_w^2\right| \right\} \binom{n+\alpha}{n}.\end{aligned} \tag{4.12}$$

Now let us specialize the constants, as done in the previous section:

$$\alpha_{n-\chi} = K_{n-\chi,1} = \frac{n^{-p}}{\binom{n-\beta}{n}}. \tag{4.13}$$

From the previous estimate, using the fact that N large allows us to use the asymptotic estimate (2.7), we obtain

$$\| e_N \|_\infty \leqslant |\gamma^{-1}| \sum_{n=N}^{\infty} \frac{n^{-p}}{\binom{n-\beta}{n}} \{1 + |\beta_1| \, \|\phi_1\|_w^2\} \binom{n+\alpha}{n}$$
$$\cong J \sum_{n=N}^{\infty} n^{-p-\chi} = JN^{1-p-\chi}\zeta(p+\chi), \tag{4.14}$$

where J is a constant. Convergence is ensured if $p + \chi > 1$, and again we would like to obtain a lower bound containing the zeta function. We can compute it as follows:

$$\| e_N \|_\infty \geqslant \left| \sum_{n=N}^{\infty} \beta_n \phi_n(1) \right| \geqslant |\gamma|^{-1} \left| \sum_{n=N}^{\infty} \left[\alpha_{n-\chi} - \beta_1 K_{n-\chi,1} \|\phi_1\|_w^2\right] \binom{n+\alpha}{n} \right|. \tag{4.15}$$

We finally choose the two remaining constants, in the formula for β_1; we want $\alpha_{1-\chi} < 0$ and $K_{1-\chi,1} > -\gamma/\|\phi_1\|_w^2$, so that, with these choices, $\beta_1 < 0$. Then asymptotically,

$$\| e_N \|_\infty \geqslant |\gamma|^{-1} \left| \sum_{n=N}^{\infty} \frac{n^{-p}}{\binom{n-\beta}{n}} \left[1 - \beta_1 \|\phi_1\|_w^2\right] \binom{n+\alpha}{n} \right|$$
$$\cong |\gamma|^{-1} \sum_{n=N}^{\infty} n^{-p+\beta+\alpha} \left[1 + |\beta_1| \, \|\phi_1\|_w^2\right]$$
$$\geqslant HN^{1-p-\chi}\left[\zeta(p+\chi) - 1 + \frac{1}{N}\right], \tag{4.16}$$

where again H represents a constant. We can now repeat the argument at the end of Section 3 to show that in this case as well, *in practice*, by a suitable choice of the parameter p, the approximate solution obtained with the Galerkin method will differ from the analytical solution more than the required tolerance. Also, similar computations to the ones already performed at the end of Section 3 on $\| f \|_\infty$, repeated here for $\| f \|_\infty$ and for $\| K \|_\infty$, show that in some cases these functions may well be badly behaved. Finally, we observe that not both these functions need to have a bad behavior; indeed an alternative choice to (4.13) is to take the constants $K_{n-\chi,1} \geqslant 0$ and in such a way that their series converges:

$$\sum_{n=\chi}^{\infty} K_{n-\chi,1} \binom{n+\alpha}{n} = S < \infty. \tag{4.17}$$

Then we would have

$$\| e_N \|_\infty \leqslant |\gamma^{-1}| \sum_{n=N}^{\infty} n^{-p} \binom{n+\alpha}{n} \Big/ \binom{n-\beta}{n} + |\gamma^{-1}| \sum_{n=N}^{\infty} |\beta_1| K_{n-\chi,1} \|\phi_1\|_w^2 \binom{n+\alpha}{n}$$
$$\cong N^{1-p-\chi}\zeta(p+\chi) + (S - S_N)\beta_1 \|\phi_1\|_w^2 |\gamma|^{-1},$$

S_N denoting the partial sum of the series (4.17); the right-hand side thus converges and for the lower bound a similar analysis yields again the occurrence of the Riemann zeta function, so that the same conclusion holds. Indeed note that the sequence of the partial sums of the series (4.17) is monotonically increasing, so that

$$\|e_N\|_\infty \geq |\gamma|^{-1} \sum_{n=N}^{\infty} n^{-p}\binom{n+\alpha}{n}\Big/\binom{n-\beta}{n} + |\gamma^{-1}| \sum_{n=N}^{\infty} |\beta_1| K_{n-\chi,1}\|\phi_1\|_w^2\binom{n+\alpha}{n}$$
$$\cong N^{1-p-\chi}\left[\zeta(p+\chi) - 1 + \frac{1}{N}\right] + (S - S_N)\beta_1\|\phi_1\|_w^2 |\gamma|^{-1}$$
$$\geq N^{1-p-\chi}\left[\zeta(p+\chi) - 1 + \frac{1}{N}\right].$$

Alternatively, the roles of $\alpha_{n-\chi}$ and $K_{n-\chi,1}$ in (4.13) and (4.17) could be interchanged and still the same analysis carries through. In either way one set of coefficients can be chosen so as to make either the right-hand side or the kernel a function as smooth as we please. The other function instead might possibly exhibit a bad behavior, as the analysis at the end of Section 3 shows. Thus it is likely that a bad behavior of either the kernel or the forcing function is reflected in the corresponding behavior of the error bounds and thus in poor performance of the algorithm.

An application

We refer again to [4, p.568]. A torsion problem for an infinitely long elastic shaft bonded to an elastic disk of finite width and different elastic constant is expressed as a singular integral equation with a generalized Cauchy kernel, of the form

$$\pi^{-1}\int_{-1}^{1}\varphi(t)\left[(t-x)^{-1} - (t+x-2)^{-1} - (t+x+2)^{-1}\right] dt = f(x).$$

Notice that the terms apart from the singular value are bounded for every t, $x \in (-1, 1)$, but become unbounded as both x and t simultaneously approach the endpoints. In the same paper also, other types of kernels are discussed, containing weak or logarithmic singularities in addition to the principal value integral. Even though it is possible to represent within machine accuracy such functions by means of smooth functions, these will exhibit large norms, and thus fail into the class of badly behaved functions discussed above. It thus would be unrealistic to claim convergence for the Galerkin method and trust blindly the computer results.

5. Conclusions

In this study we have examined how the error bounds affect the Galerkin method, when applied to the solution of singular integral equations. We have seen that the occurrence of the Riemann zeta function in the error bound for the method, discovered in [14,15], is sharp. This in turn shows that in particularly nasty cases, the method may very well fail to give a reliable approximate solution *in practice*, because the size of the linear algebraic system needed to achieve a given tolerance is too large to fit in the memory of the system or, equivalently, that the time required to solve it is too large. The occurrence of such situations in practical problems has been examined, by discussing some instances where engineering problems lead to forcing functions or kernels with large norms. One should also mention that these problems do not arise if the norm used is the L_2-norm, as illustrated in [14,15]. Also, it is interesting to compare this analysis with the one done for direct quadrature-collocation methods, which discretize the singular integral via an appropriate integration formula and then obtain a linear algebraic system by collocating the functional equation at a discrete set of points. It is well known that there is a relationship between the Galerkin method and the quadrature-collocation method. As it has been shown in [12], the condition number relative to the L_2-norm of the linear algebraic system obtained via quadrature-collocation grows like $N^{3/2}$, where N is its size. Evidently, for N large, this entails practical problems of the type addressed here.

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