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Density and O-Density of Beurling Generalized Integers

WEN-BIN ZHANG*

Department of Mathematics,
University of Illinois,
Urbana, Illinois 61801

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Let $\pi(x)$ and $N(x)$ be the respective counting functions of a set of generalized primes and a set of generalized integers in Beurling's sense. We consider weak conditions on $\pi(x)$ some of which yield $N(x) \sim cx$ for some positive c, and some of which yield nontrivial O-estimates for $N(x)/x$.

INTRODUCTION

Let $\mathscr{P} = \{p_i\}_{i=1}^{\infty}$ be a sequence of real numbers satisfying $1 < p_1 \leqslant p_2 \leqslant, ..., p_i \to \infty$. $\mathscr{P}$ generates a semi-group $\mathscr{N}$ under multiplication. We shall call $\mathscr{P}$ a system of Beurling generalized primes (briefly: g-primes) and $\mathscr{N}$ the associated g-integers (cf. [1, 2]). Arrange the elements of $\mathscr{N}$ in ascending order, so that

$$\mathscr{N} = \{n_i\}_{i=0}^{\infty}, \qquad n_0 = 1 < n_1 \leqslant n_2 \leqslant \cdots.$$

By analogy with classical prime number theory, we define

$$N(x) = \sum_{n_i \leqslant x} 1, \qquad \pi(x) = \sum_{p_i \leqslant x} 1,$$

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \cdots,$$

and

$$\Psi(x) = \int_1^x \log t \, d\Pi(t) = \sum_{p_i^{\alpha} \leqslant x} \log p_i \qquad (p_i \in \mathscr{P}, \alpha \in \mathbb{Z}^+).$$

Suppose that one of the sequences $\mathscr{N}$, $\mathscr{P}$ is distributed rather like the

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corresponding sequence of natural numbers or primes. We want to know whether the other sequence is also distributed like its classical counterpart. Beurling proved that $\pi(x) \sim x/\log x$, i.e., the prime number theorem holds for $\mathscr{P}$, if

$$N(x) = cx + O(x \log^{-\gamma} ex)$$

with constants $c > 0$ and $\gamma > \frac{3}{2}$. Diamond showed by example that the prime number theorem can fail for $\gamma = \frac{3}{2}$ (cf. [1, 2, 5, 6]).

In [7], Diamond considered weak conditions on $\pi(x)$ which enable one to deduce $N(x) \sim cx$ for some positive c, i.e., that $\mathscr{N}$ has a density. A closely related problem is the determination of when a multiplicative arithmetic function has a mean value (cf. [3, 8, 9, 13, 14]). Theorem 1 of Diamond [7] is optimal in some sense and a generalization of theorems of Delange [3] and Wirsing [13].

Diamond also asked the following question: What conditions on $\pi(x)$ will yield nontrivial O-estimates for $N(x)/x$? For convenience, we shall call this question the O-density problem.

In this paper, we shall give answers to the O-density problem in Theorem 4.1, 4.3, and 4.6. For this purpose, we shall prove Theorem 3.1 which is an O-type Hardy–Littlewood–Karamata Tauberian theorem and has interest in itself. Moreover, we shall relax the conditions of Theorem 1 of Diamond [7] a little in Theorem 1.1 and give an example to which the latter applies but the former does not. Our method of proof follows the general idea of Diamond.

In the proof of Theorems 1.1, 4.1, 4.3, and 4.6, we shall make frequent use of convolution techniques which have been described in detail in [4]. The notations and the relations that we shall use here have been summarized in [7].

1. Density of Beurling Generalized Integers

In this section, we shall prove the following

Theorem 1.1. *Suppose there exists a decomposition $\Pi = \Pi_1 + \Pi_2$ satisfying the following conditions*

$$\Pi_1(x) \uparrow, \tag{1.1}$$

$$\Pi_1(x) \sim x/\log x \qquad as \quad x \to \infty, \tag{1.2}$$

$$\lim_{s \to 1+} \left(\int_{1-}^{\infty} x^{-s} d\Pi_1(x) - \log \frac{s}{s-1} \right) = \log c, \tag{1.3}$$

in view of (14) and [18, p. 59]. Using [18, p. 59; 11, 5.8], we conclude that $\pi^{1-s}\Gamma(s-1)\, c_s(\omega)$ is invariant under $s \mapsto 3-s$. Thus another proof for the functional equation is complete.

In view of [18, p. 59], the only poles, which appear in the Fourier-expansion of $\tilde{\mathbb{E}}(w, s)$, are at $s=0, 1, \frac{3}{2}, 2, 3$ and they prove to be simple. Calculation of the residues implies that the poles at $s=\frac{3}{2}$ cancel. Since $\zeta(2s-2)$ possesses a pole at $s=\frac{3}{2}$ we conclude $\tilde{E}^*(w, \frac{3}{2})=0$. Using [18, p. 59] and (14) we calculate

$$\operatorname*{res}_{s=3} \tilde{E}^*(w, s) = \frac{\pi^2}{\Gamma(2)}\, \pi^{3/2}\, \frac{\Gamma(3/2)\cdot\zeta(3)}{\Gamma(3)\cdot\zeta(4)} = \frac{45}{2}\, \zeta(3).$$

We obtain $\tilde{E}^*(w, 0) = -1$ from the formula for the residue and the functional equation. Again the functional equation yields the holomorphy of $\tilde{\mathbb{E}}(w, s)$ at $s=-1, -2, -3, \ldots$. In view of [18, p. 59], the function $\pi^{1-s}\Gamma(s-1)\, \zeta(2s-2)$ is holomorphic and non-zero at $s=-1, -2, -3, \ldots$. Then the poles of the gamma function imply $\tilde{E}^*(w, s)=0$ for $s=-1, -2, -3, \ldots$. ∎

Let γ denote the Euler constant and define

$$C := \gamma + \frac{1}{2} - \frac{4}{3}\log 2 + \frac{\zeta'}{\zeta}(2) + 2\,\frac{\zeta'}{\zeta}(3) - 2\,\frac{\zeta'}{\zeta}(4),$$

$$h(w) := \frac{1}{18}\, \pi^2 v^3 + \sum_{0 \neq \omega \in \operatorname{Im}\Lambda} c_3(\omega)\, \frac{1}{\zeta(3)} \times \frac{1 + 2\pi\, |\omega|\, v}{\pi^2\, |\omega|^{3/2}}\, e^{-2\pi|\omega| v + 2\pi i\langle u, \omega\rangle}.$$

Then the first Kronecker limt formula is described in

THEOREM 4. *One has*

$$\lim_{s \to 3} \left(\tilde{E}^*(w, s) - \frac{45}{2}\, \frac{\zeta(3)}{s-3} \right) = \frac{45}{2}\, \zeta(3)(C - \log v + h(w)).$$

Proof. Looking at the Fourier-expansion (14) we get

$$\begin{aligned}
&\lim_{s \to 3} \left(\tilde{E}^*(w, s) - \frac{45}{2}\, \frac{\zeta(3)}{s-3} \right) \\
&\quad = \frac{5}{4}\, \pi^2 \zeta(3)\, v^3 + \lim_{s \to 3} \left(f(s)\, \zeta(s-2) - \frac{f(3)}{s-3} \right) \\
&\qquad + \sum_{0 \neq \omega \in \operatorname{Im}\Lambda} c_3(\omega)\, \frac{90}{\pi}\, v^{3/2} K_{3/2}(2\pi\, |\omega|\, v)\, e^{2\pi i\langle u, \omega\rangle},
\end{aligned}$$

Proof. We have for $s > 1$

$$\zeta(s) = \frac{s}{s-1} \exp\left(s \int_1^\infty x^{-s-1}\{\Pi(x) - \tau(x)\}\, dx\right) \sim \frac{c}{s-1}.$$

The claimed estimate follows from the well-known Hardy–Littlewood–Karamata Tauberian Theorem [12, Chap. V, Sect. 4]. ∎

LEMMA 1.3 [11] (*Axer*). *Let dA, dB, and dA_1 be real-valued set functions on the Borel subsets of $[1, \infty)$ with the property that on any finite interval $[1, x]$ each is a finite measure. Assume that $dA_1 \geqslant 0$, $|A(x)| \leqslant A_1(x)$, and*

$$\int_1^\infty A_1(x)\, x^{-2}\, dx < \infty.$$

Also, assume that $B(x) = o(x)$ and $B_v(x) = O(x)$ where $B_v(x)$ is the total variation of $B(x)$. Then

$$\int_1^x dA * dB = o(x).$$

LEMMA 1.4. *Let $dA \geqslant 0$. If*

$$\left|\int_1^x (\exp d\Pi_2)(t)\right| \leqslant A(x) \tag{1.6}$$

and

$$\int_1^\infty A(x)\, x^{-2}\, dx < \infty \tag{1.7}$$

then $\int_1^\infty t^{-1}(\exp d\Pi_2)(t)$ is convergent.

Proof. We first note that $dA \geqslant 0$ implies $A(x)\uparrow$. This fact and (1.7) imply $A(x) = o(x)$. Let

$$E(u, x) = \int_u^x (\exp d\Pi_2)(t), \qquad u < x.$$

Then $|E(u, x)| \leqslant 2A(x)$. By integration by parts,

$$\int_u^v t^{-1}(\exp d\Pi_2)(t) = \frac{E(u, v)}{v} + \int_u^v \frac{E(u, t)}{t^2}\, dt.$$

It follows that

$$\left|\int_u^v t^{-1}(\exp d\Pi_2)(t)\right| \leqslant \frac{2A(v)}{v} + 2\int_u^v \frac{A(t)}{t^2}\,dt \to 0$$

as $u, v \to \infty$. ∎

Proof of Theorem 1.1. First, the condition $\Pi_1(x) \sim x/\log x$ implies that

$$\Psi_1(x) = \int_1^x \log t\,d\Pi_1(t) \sim x.$$

This fact and the analogue of Chebyshev's identity yield

$$T(x) := \int_1^x \log t\,dN_1(t) = \int_{1}^x \Psi_1\left(\frac{x}{t}\right) dN_1(t) \sim \int_1^x \frac{x}{t}\,dN_1(t),$$

where $dN_1 = \exp d\Pi_1$. The conditions (1.1) and (1.3) are equivalent to the hypotheses of Lemma 1.2. Hence $T(x) \sim cx \log x$. Now, for $x \geqslant 1$,

$$N_1(x) = 1 + \int_1^x \frac{1}{\log u}\,dT(u)$$

and integration by parts yields $N_1(x) \sim cx$.

Then, consider

$$\begin{aligned} dN &= dN_1 * \exp d\Pi_2 \\ &= (c\,dt + c\delta) * \exp d\Pi_2 + (dN_1 - c\,dt - c\delta) * \exp d\Pi_2. \end{aligned}$$

Condition (1.4) implies, by Lemma 1.4, that $\int_{1-}^{\infty} t^{-1}(\exp d\Pi_2)(t)$ converges. Thus we have

$$\begin{aligned} \int_{1-}^x (c\,dt + c\delta) * (\exp d\Pi_2)(t) &= \int_{1-}^x c\frac{x}{t}(\exp d\Pi_2)(t) \\ &= cx\left(\int_{1-}^{\infty} t^{-1}(\exp d\Pi_2)(t) + o(1)\right). \end{aligned} \tag{1.8}$$

Note that

$$\int_1^x (dN_1 - c\,dt - c\delta) = N_1(x) - cx = o(x)$$

and

$$\left(\int_1^x (dN_1 - c\,dt - c\delta)\right)_v = N_1(x) + cx = O(x).$$

These two facts and the condition (1.4) imply, by Lemma 1.3, that

$$\int_1^x (dN_1 - c\,dt - c\delta) * (\exp d\Pi_2)(t) = o(x). \tag{1.9}$$

Combining estimates, we arrive at

$$\begin{aligned} N(x) &= \int_{1-}^x (c\,dt + c\delta) * (\exp d\Pi_2)(t) \\ &\quad + \int_{1-}^x (dN_1 - c\,dt - c\delta) * (\exp d\Pi_2)(t) \\ &= \left(c \int_{1-}^{\infty} t^{-1}(\exp d\Pi_2)(t) + o(1)\right) x. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\int_{1-}^{\infty} t^{-1}(\exp d\Pi_2)(t) \\ &\quad = \lim_{s \to 1+} \int_{1-}^{\infty} t^{-s}(\exp d\Pi_2)(t) \\ &\quad = \lim_{s \to 1+} \exp\left(\int_{1-}^{\infty} t^{-s}\, d\Pi_2(t)\right) = \exp\left(\lim_{s \to 1+} \int_{1-}^{\infty} t^{-s}\, d\Pi_2(t)\right) > 0 \end{aligned}$$

provided that $\lim_{s \to 1+} \int_{1-}^{\infty} t^{-s} d\Pi_2(t)$ exists. ∎

COROLLARY 1.5. *If we replace* (1.4) *by*

$$\int_{1-}^{\infty} x^{-1}\, |\exp d\Pi_2|(x) < \infty \tag*{$(1.4)_1$}$$

then the theorem is still true.

Proof. Take

$$A(x) = \int_1^x |\exp d\Pi_2|(t)$$

Then we have

$$\left|\int_1^x (\exp d\Pi_2)(t)\right| \leqslant A(x)$$

and, by integration by parts,

$$\int_1^x A(t)\, t^{-2}\, dt = -\frac{A(x)}{x} + \int_1^x t^{-1}\, |\exp d\Pi_2|(t) \leqslant \int_1^\infty x^{-1}\, |\exp d\Pi_2|(x) < \infty.$$

Therefore the hypotheses of Theorem 1.1 are satisfied. ▮

COROLLARY 1.6 (*Diamond*). *If we replace* (1.4) *by*

$$\int_{1-}^\infty x^{-1}\, |d\Pi_2|(x) < \infty \tag{1.4$_2$}$$

then there exists a positive constant c_1 *such that* $N(x) \sim c_1 x$ *holds as* $x \to \infty$.

Proof. We have

$$\int_1^\infty x^{-1}\, |\exp d\Pi_2|(x) \leqslant \int_1^\infty x^{-1}(\exp |d\Pi_2|)(x) = \exp\left\{\int_1^\infty x^{-1}\, |d\Pi_2|(x)\right\}$$

which is finite. Moreover, $(1.4)_2$ implies the existence of $\lim_{s\to 1+} \int_{1-}^\infty x^{-s} d\Pi_2(x)$. Therefore, we have $N(x) \sim c_1 x$ with positive constant

$$c_1 = c \int_1^\infty x^{-1}(\exp d\Pi_2)(x) > 0. \quad ▮$$

2. AN EXAMPLE

In this section, we give an analysis of an example of Diamond [5].

The analysis shows that Theorem 1.1 applies to the example but Corollary 1.6 does not.

Following [5], we define $\tau(x)$ by setting

$$\tau(x) = \int_1^x \{1 - \cos(\log t)\}(\log t)^{-1}\, dt$$

for $x \geqslant 1$ and $\tau(x) = 0$ for $x < 1$. Let p_r, the rth g-prime, be defined by $p_r = \tau^{-1}(r)$. Then, as Diamond showed,

$$N(x) = c_1 x + O\{x(\log x)^{-3/2}\},$$

where $c_1 = \exp\{\int_1^\infty t^{-1}(d\Pi - d\tau)(t)\} > 0$.

We first show that Corollary 1.6 does not apply to this example. Suppose that we have a decomposition $\Pi = \Pi_1 + \Pi_2$ with Π_1 satisfying the condition (1.2). Then it can be shown that Π_2 does not satisfy $(1.4)_2$. Let

$$\Pi_{2v}(x) = \int_{1-}^{x} |d\Pi_2|(t).$$

Then

$$\Pi_{2v}(x) \geqslant |\Pi_2(x)| - c_0.$$

We know that [5]

$$\begin{aligned}\Pi(x) &= \tau(x) + O(x^{1/2})\\ &= li(x) - \frac{x}{\sqrt{2}\log x}\sin\left(\log x + \frac{\pi}{4}\right) + O(x\log^{-2} x).\end{aligned}$$

Therefore we have

$$\begin{aligned}\Pi_2(x) &= \Pi(x) - \Pi_1(x)\\ &= -\frac{1}{\sqrt{2}}\frac{x}{\log x}\sin\left(\log x + \frac{\pi}{4}\right) + o\left(\frac{x}{\log x}\right)\end{aligned}$$

and hence

$$|\Pi_2(x)| = \frac{1}{\sqrt{2}}\frac{x}{\log x}\left|\sin\left(\log x + \frac{\pi}{4}\right)\right| + o\left(\frac{x}{\log x}\right).$$

It follows that

$$\begin{aligned}\int_e^x \frac{\Pi_{2v}(t)}{t^2}\,dt &\geqslant \int_e^x \frac{|\Pi_2(t)|}{t^2}\,dt - c_0\left(\frac{1}{e} - \frac{1}{x}\right)\\ &= \int_e^x \frac{1}{\sqrt{2}}\frac{1}{t\log t}\left|\sin\left(\log t + \frac{\pi}{4}\right)\right|\,dt + o(\log\log x)\\ &= \frac{\sqrt{2}}{\pi}\log\log x + o(\log\log x)\end{aligned}$$

since

$$\int_e^x \frac{1}{t\log t}\left|\sin\left(\log t + \frac{\pi}{4}\right)\right|\,dt = \frac{2}{\pi}\log\log x + O(1).$$

Therefore

$$\int_{1-}^{x} t^{-1}\,|d\Pi_2|(t)=\frac{\Pi_{2v}(x)}{x}+\int_{1}^{x}\frac{\Pi_{2v}(t)}{t^2}\,dt$$
$$\geqslant\frac{\sqrt{2}}{\pi}\log\log x+o(\log\log x)\to\infty$$

as $x\to\infty$ and $(1.4)_2$ cannot be satisfied.

We then show that Theorem 1.1 applies to the example. Set

$$\Pi_1(x)=\int_1^x\frac{1-t^{-1}}{\log t}\,dt$$

for $x\geqslant 1$ and $\Pi_1(x)=0$ for $x<1$ and

$$\Pi_2(x)=\Pi(x)-\Pi_1(x).$$

Then (1.1) and (1.2) are satisfied. Moreover, since

$$\int_1^\infty x^{-s}d\Pi_1(x)=\int_1^\infty x^{-s}\frac{1-x^{-1}}{\log x}\,dx=\log\frac{s}{s-1},\qquad s>1,$$

the condition (1.3) with $c=1$ holds.

It remains to show the existence of the function $A(x)$ in (1.4). Actually, we have the estimate that

$$\int_{1-}^{x}(\exp d\Pi_2)(t)\ll x\log^{-3/2}x. \tag{2.1}$$

To show (2.1), we define $dT=\exp d\tau$, $C=\log c_1$, and

$$dv=d\Pi-d\tau-C\delta$$

as in [5]. Then we have

$$\exp(d\Pi_2)=\exp\{d\tau-d\Pi_1\}*\exp\{d\Pi-d\tau\}$$
$$=c_1(dT-T(t)\,t^{-1}\,dt)*\exp\{dv\}.$$

By a technique of Dirichlet, we have

$$\int_{1-}^{x}(\exp d\Pi_2)(t)=c_1\int_{1-}^{\sqrt{x}+}\left(T\left(\frac{x}{u}\right)-\int_{1-}^{x/u^+}T(t)\,t^{-1}\,dt\right)(\exp\{dv\})(u)$$
$$+c_1\int_{1-}^{\sqrt{x}+}\left(\int_{\sqrt{x}+}^{x/u^+}\exp\{dv\}\right)(dT(u)-T(u)\,u^{-1}\,du). \tag{2.2}$$

We shall show that both terms on the right-hand side are $O(x\log^{-3/2}x)$.

Let

$$f(y)=\int_{1-}^{y+} u^{-1}(\exp\{dv\}-\delta).$$

Then by Lemma 4 in [5], for each (fixed) positive number k, $f(y)=O(\log^{-k} y)$ as $y\to\infty$. We have the formula

$$\int_{1-}^{y+} \exp\{dv\}=\int_{1-}^{y+} \{\delta+udf(u)\}$$

and integration by parts yields

$$\int_{1-}^{y+} udf(u)=O(y\log^{-3} y).$$

Thus

$$\int_{1-}^{\sqrt{x}+}\left(\int_{\sqrt{x}+}^{x/u+} \exp\{dv\}\right)(dT(u)-T(u)\,u^{-1}\,du)$$

$$\ll\int_{1-}^{\sqrt{x}+} \frac{x}{u}\left(\log\frac{x}{u}\right)^{-3} dT(u)+\int_{1-}^{\sqrt{x}+} \frac{x}{u}\left(\log\frac{x}{u}\right)^{-3} T(u)\,u^{-1}\,du$$

$$=O(x\log^{-2} x)$$

since $dT\geqslant 0$, $T(u)=O(u)$, and

$$\int_{1-}^{\sqrt{x}} u^{-1}\,dT(u)=O(\log x).$$

From Lemma 1 in [5],

$$T(x)=x+2\mathscr{R}e\sum_{j=1}^{2} d_j x^{1+i}(\log x)^{-j-1/2}+O(x\log^{-7/2} x).$$

Hence

$$\int_{1-}^{x+} T(t)\,t^{-1}\,dt=x+2\mathscr{R}e\sum_{j=1}^{2} d_j\int_{e}^{x} t^{i}(\log t)^{-j-1/2}\,dt+O(x\log^{-7/2} x).$$

Therefore, the first term on the right-hand side of (2.2) equals

$$2c_1 \mathscr{Re} \sum_{j=1}^{2} d_j \left\{ \int_{1-}^{\sqrt{x}+} \left(\frac{x}{u}\right)^{1+i} \left(\log \frac{x}{u}\right)^{-j-1/2} (\exp\{dv\})(u) \right.$$
$$\left. - \int_{1-}^{\sqrt{x}+} \left(\int_{e}^{x/u+} t^i (\log t)^{-j-1/2}\, dt\right) (\exp\{dv\})(u) \right\}$$
$$+ \int_{1-}^{\sqrt{x}+} O\left(\frac{x}{u}\left(\log \frac{x}{u}\right)^{-7/2}\right) (\exp\{dv\})(u).$$

We treat separately the three resulting integrals $I - II + III$. The integral I is a linear combination of the following integrals

$$J_I = \int_{1-}^{\sqrt{x}+} \left(\frac{x}{u}\right)^{1\pm i} \left(\log \frac{x}{u}\right)^{-j-1/2} (\exp\{dv\})(u).$$

By the definition of f,

$$J_I = x^{1\pm i} \int_{1-}^{\sqrt{x}+} u^{\pm i} \left(\log \frac{x}{u}\right)^{-j-1/2} df(u) + x^{1\pm i} (\log x)^{-j-1/2}.$$

Integrating by parts and noting that $\int_1^\infty u^{-1}\, |f(u)|\, du < \infty$, we see that

$$\int_{1-}^{\sqrt{x}} u^{\pm i} \left(\log \frac{x}{u}\right)^{-j-1/2} df(u) = O((\log x)^{-j-1/2})$$

and thus $I = O\{x(\log x)^{-3/2}\}$. The integral II is a linear combination of the following integrals:

$$J_{II} = \int_{1-}^{\sqrt{x}+} \left(\int_{e}^{x/u+} t^i (\log t)^{-j-1/2}\, dt\right) (\exp\{dv\})(u)$$
$$= \int_{1-}^{\sqrt{x}+} \left(\int_{e}^{x/u+} t^i (\log t)^{-j-1/2}\, dt\right) u\, df(u) + \int_{e}^{x} t^i (\log t)^{-j-1/2}\, dt.$$

Integrating by parts and noting that

$$\int_{e}^{x/u+} t^i (\log t)^{-j-1/2}\, dt = O\left(\frac{x}{u}\left(\log \frac{x}{u}\right)^{-j-1/2}\right),$$

we see that

$$\int_{1-}^{\sqrt{x}+} \left(\int_{e}^{x/u+} t^i (\log t)^{-j-1/2}\, dt\right) u\, df(u) = O(x(\log x)^{-j-1/2})$$

and thus $II = O\{x(\log x)^{-3/2}\}$. Finally,

$$|III| \ll \int_{1-}^{\sqrt{x}+} \frac{x}{u}\left(\log\frac{x}{u}\right)^{-7/2} |\exp\{d\Pi - d\tau - C\delta\}|$$

$$\ll x \log^{-7/2} x \left(\int_{1-}^{\sqrt{x}+} u^{-1} \exp d\Pi\right)\left(\int_{1-}^{\sqrt{x}+} u^{-1} \exp d\tau\right)$$

$$\ll x \log^{-3/2} x$$

since

$$\int_{1-}^{\sqrt{x}+} u^{-1}(\exp d\Pi)(u) = \int_{1}^{\sqrt{x}+} u^{-1} dN(u) = O(\log x),$$

$$\int_{1-}^{\sqrt{x}+} u^{-1}(\exp d\tau)(u) = \int_{1-}^{\sqrt{x}+} u^{-1} dT(u) = O(\log x).$$

This completes the proof of (2.1).

Finally, it is easy to see that

$$\int_{1-}^{\infty} x^{-s} d\Pi_2(x) = \int_{1-}^{\infty} x^{-s} \frac{x^{-1} - \cos(\log x)}{\log x}\, dx + \int_{-1}^{\infty} x^{-s}(d\Pi - d\tau)(x)$$

for $s > 1$ and

$$\lim_{s \to 1+} \int_{1-}^{\infty} x^{-s} d\Pi_2(x)$$

exist.

3. An O-Type Hardy–Littlewood–Karamata Tauberian Theorem

The purpose of this section is to set up the following O-type Hardy–Littlewood–Karamata Tauberian theorem which we shall apply to the O-density problem. This theorem is of independent interest.

Theorem 3.1. *Let $F(x)$ be a nondecreasing function with support $\subset$ $[1, \infty)$. Suppose that the Mellin transform $\int_{1-}^{\infty} x^{-s} dF(x)$ converges for all $s > 1$. If there exist constants $K > 0$ and $\delta > 0$ such that*

$$(s-1)\int_{1-}^{\infty} x^{-s} dF(x) \leqslant K \tag{3.1}$$

holds for $1 < s < 1 + \delta$ *then there exists a constant* $k > 0$ *for which*

$$\int_{1-}^{x} t^{-1} dF(t) \leqslant k \log x \tag{3.2}$$

holds for $x \geqslant x_0$. *Furthermore, if there also exists a constant* $K' > 0$ *such that*

$$K' \leqslant (s-1) \int_{1-}^{\infty} x^{-s} \, dF(x) \leqslant K \tag{3.3}$$

holds for $1 < s < 1 + \delta$ *then there exists a constant* $k' > 0$ *for which*

$$\int_{1-}^{x} t^{-1} dF(t) \geqslant k' \log x \tag{3.4}$$

holds for $x \geqslant x_0$.

Proof. If we take $s = 1 + (\log x)^{-1}$ then, by (3.1), we have

$$\frac{1}{e} \int_{1-}^{x} t^{-1} \, dF(t) \leqslant \int_{1-}^{x} t^{-s} \, dF(t) \leqslant \int_{1-}^{\infty} t^{-s} \, dF(t) \leqslant K \log x.$$

This proves (3.2) with $k = eK$.

We then assume (3.3). Choose $\alpha > 0$ satisfying

$$\frac{K'}{\alpha + 1} - \frac{K}{e^{\alpha}} > 0.$$

Then we have

$$u^{-\alpha} - \frac{1}{e^{\alpha}} \leqslant \begin{cases} (1 - 1/e^{\alpha}) u, & \text{if } \ 1 \leqslant u \leqslant e, \\ 0, & \text{if } \ e < u. \end{cases}$$

If we take $s = 1 + (\log x)^{-1}$ then $1 \leqslant t^{s-1} \leqslant e$ for $1 \leqslant t \leqslant x$ and $t^{s-1} > e$ for $t > x$. Therefore, by (3.3), we have

$$\begin{aligned} &\int_{1-}^{x} t^{-1} \, dF(t) \\ &\quad = \int_{1-}^{x} t^{-s} t^{s-1} \, dF(t) \\ &\quad \geqslant (1 - e^{-\alpha})^{-1} \int_{1-}^{x} t^{-s} (t^{-\alpha(s-1)} - e^{-\alpha}) \, dF(t) \\ &\quad \geqslant (1 - e^{-\alpha})^{-1} \left\{ \int_{1-}^{\infty} t^{-(s-1)(1+\alpha)-1} \, dF(t) - e^{-\alpha} \int_{1-}^{\infty} t^{-s} \, dF(t) \right\} \\ &\quad \geqslant (1 - e^{-\alpha})^{-1} \left(\frac{K'}{\alpha + 1} - \frac{K}{e^{\alpha}} \right) \log x. \end{aligned}$$

This proves (3.4) with

$$k' = (1-e^{-\alpha})^{-1}\left(\frac{K'}{\alpha+1} - \frac{K}{e^{\alpha}}\right). \quad \blacksquare$$

Remark. We note that if (3.2) holds for $x \geqslant x_0$ sufficiently large then (3.1) follows for $1 < s < 1+\delta$. Also, if (3.4) holds for $x \geqslant x_0$ then the lower bound in (3.3) follows for $1 < s < 1+\delta$. Therefore, the upper bound hypothesis in (3.3) might seem unnatural. The following example shows, however, that (3.4) may fail to be true without the upper bound in (3.3).

EXAMPLE. Define recursively two sequences a_m, b_m by setting

$$a_1 = e, \qquad b_m = e a_m, \qquad a_{m+1} = e^{m b_m}.$$

Then $a_{m+1} > b_m$ and $a_m \geqslant e^m$ hold. Let

$$f(n) = \begin{cases} n^{1/m}, & \text{if } \ a_m < n \leqslant b_m, \qquad \text{for } \ m \in \mathbb{N}, \\ 0, & \text{else.} \end{cases}$$

The function $F(x) = \sum_{n \leqslant x} f(n)$ is nondecreasing. Let $2 > s > 1$. For $m > 2/(s-1)$, we have

$$\sum_{a_m < n \leqslant b_m} n^{-s+1/m} \leqslant a_m^{1/m-(s-1)} \sum_{a_m < n \leqslant b_m} \frac{1}{n}$$

$$\leqslant a_m^{1/m-(s-1)} \left\{\frac{1}{a_m} + 1\right\} < 2a_m^{-1/m} = 2e^{-(1-m^{-1})b_{m-1}}.$$

Therefore, $\sum_{m > 2/(s-1)} \sum_{a_m < n \leqslant b_m} n^{-s+1/m}$ converges and hence

$$\int_{1-}^{\infty} x^{-s}\, dF(x) = \sum_{m=1}^{\infty} \sum_{a_m < n \leqslant b_m} n^{-s+1/m}$$

converges too for all $s > 1$. Furthermore, for m satisfying $1/2(m+1) < s-1 \leqslant 1/2m$, we have

$$\sum_{a_m < n \leqslant b_m} n^{-s+1/m} > \int_{a_m}^{b_m} x^{-s+1/m}\, dx - 1$$

$$= \frac{1}{1-s+1/m} (b_m^{1-s+1/m} - a_m^{1-s+1/m}) - 1$$

$$\geqslant 2m a_m^{1/2m}(e^{1/2m} - 1) - 1 > a_m^{1/2m} - 1.$$

Hence, for $m \geqslant 4$,

$$(s-1)\int_{1-}^{\infty} x^{-s}\,dF(x) \geqslant (s-1)(a_m^{1/2m}-1)$$

$$\geqslant \frac{1}{2(m+1)}(a_m^{1/2m}-1)$$

$$\geqslant \frac{1}{2(m+1)}(e^{a_{m-1}}-1) \to \infty.$$

Therefore, we have $\lim_{s\to 1+}(s-1)\int_{1-}^{\infty} x^{-s}\,dF(x)=\infty$ and the upper bound in (3.3) does not hold.

We now see that, for $x=a_{m_0+1}$, we have

$$\int_{1-}^{x} t^{-1}\,dF(t)=\sum_{m\leqslant m_0}\ \sum_{a_m<n\leqslant b_m} n^{-1+1/m}\leqslant b_{m_0}=\frac{1}{m_0}\log a_{m_0+1}=\frac{1}{m_0}\log x.$$

Hence, (3.4) fails to be true.

4. O-Density of Beurling g-Integers

In this section, we shall give answers to the O-density question.

THEOREM 4.1. *Suppose that $\Pi(x)\uparrow$. If $\Pi(x)$ satisfies the conditions*

$$\Pi(x) \ll x/\log x \tag{4.1}$$

$$\lim_{s\to 1+}\sup\left(\int_1^{\infty} x^{-s}\,d\Pi(x)-\log\frac{1}{s-1}\right)<\infty. \tag{4.2}$$

then $N(x)\ll x$ holds. Moreover, if $\Pi(x)$ satisfies (4.2) *and the conditions*

$$\Pi(x) \gg x/\log x \tag{4.3}$$

$$\lim_{s\to 1+}\inf\left(\int_1^{\infty} x^{-s}\,d\Pi(x)-\log\frac{1}{s-1}\right)>-\infty, \tag{4.4}$$

then $N(x)\gg x$ holds.

To prove Theorem 4.1, we need the following

LEMMA 4.2. *Assume* (4.2). *Then we have*

$$\int_1^{x} t^{-1}\,dN(t)\ll \log x. \tag{4.5}$$

Moreover, assume (4.2) *and* (4.4). *Then*

$$\int_1^x t^{-1}\, dN(t) \gg \log x. \tag{4.6}$$

Proof. Assume (4.2), then we have

$$\zeta(s) = \int_1^\infty x^{-s}\, dN(x) = \frac{s}{s-1} \exp\left(\int_1^\infty x^{-s}\, d\Pi(x) - \log \frac{s}{s-1}\right) \leqslant \frac{s}{s-1} e^c$$

for $s \to 1+$. Therefore, by Theorem 3.1, (4.5) holds.

In the same way, we can prove (4.6). ∎

Proof of Theorem 4.1. We first assume (4.1) and (4.2). Then

$$\Psi(x) = \int_1^x L\, d\Pi \ll x.$$

Chebyshev's identity and Lemma 4.2 imply

$$T(x) := \int_1^x L\, dN = \int_{1-}^x \Psi\left(\frac{x}{t}\right) dN(t) \ll x \log x.$$

By integration by parts, we have

$$N(x) = 1 + \int_1^x \log^{-1} u\, dT(u) \ll x.$$

We then assume (4.2), (4.3), and (4.4). In the same way, we can prove $N(x) \gg x$. ∎

THEOREM 4.3. *Suppose there exists a decomposition* $\Pi = \Pi_1 + \Pi_2$ *where* $\Pi_1(x) \uparrow$ *and* Π_1 *satisfies the conditions* (4.1) *and* (4.2). *Moreover, suppose there exists a nondecreasing function* $A(x)$ *such that*

$$\left| \int_{1-}^x (\exp d\Pi_2)(t) \right| \leqslant A(x)$$

and

$$\int_1^\infty A(x)\, x^{-2}\, dx < \infty.$$

Then $N(x) \ll x$ *holds.*

Proof. The hypotheses (4.1) and (4.2) imply $N_1(x) = \int_1^x \exp d\Pi_1 \ll x$. It follows that

$$\begin{aligned} N(x) &= \int_1^x dN_1 * \exp d\Pi_2 \leqslant \int_{1-}^x A\left(\frac{x}{t}\right) dN_1(t) \\ &= \int_1^x A\left(\frac{x}{t}\right)(K\delta + K\,dt) + \int_{1-}^x A\left(\frac{x}{t}\right)(dN_1(t) - K\delta - K\,dt) \\ &\leqslant 2KA(x) + K\int_{1-}^x A\left(\frac{x}{t}\right) dt \end{aligned}$$

since $A(x/t)\downarrow$ in t. If we change the variable in the last integral then we have

$$N(x) \leqslant 2KA(x) + Kx\int_1^x A(u)\,u^{-2}\,du = O(x)$$

since $A(x) = o(x)$. ∎

By the same way that we deduced Corollaries 1.5 and 1.6, the following two corollaries are established.

COROLLARY 4.4. *Suppose that* $\Pi_1(x)\uparrow$ *and that* Π_1 *satisfies the conditions* (4.1) *and* (4.2). *Moreover, suppose that*

$$\int_1^\infty x^{-1}\,|\exp d\Pi_2|(x) < \infty$$

holds. Then $N(x) \ll x$.

COROLLARY 4.5. *Suppose that* $\Pi_1(x)$ *satisfies the conditions of Corollary* 4.4. *Moreover, suppose that*

$$\int_1^\infty x^{-1}\,|d\Pi_2|(x) < \infty$$

holds. Then $N(x) \ll x$.

The conditions which guarantee a lower bound for $N(x)/x$ are more complicated as we show in the following.

THEOREM 4.6. *Suppose there exists a decomposition* $\Pi = \Pi_1 + \Pi_2$ *where* $\Pi_1(x)\uparrow$ *and* Π_1 *satisfies the conditions*

$$x/\log x \ll \Pi_1(x) \ll x/\log x \tag{4.7}$$

$$\int_1^\infty x^{-s}\, d\Pi_1(x) - \log \frac{s}{s-1} = O(1) \qquad \text{as} \quad s \to 1+. \tag{4.8}$$

(*Accordingly, by Theorem* 4.1, *we have*

$$\limsup_{x\to\infty} \frac{N_1(x)}{x} = \limsup_{x\to\infty} \frac{\int_1^x (\exp d\Pi_1)(t)}{x} = K < \infty,$$

$$\liminf_{x\to\infty} \frac{N_1(x)}{x} = \liminf_{x\to\infty} \frac{\int_1^x (\exp d\Pi_1)(t)}{x} = K' > 0.)$$

Moreover, suppose that Π_2 *satisfies the conditions*

$$\int_1^\infty x^{-1}\, |d\Pi_2|(x) < \infty \tag{4.9}$$

$$K' > \frac{K}{2} \int_1^\infty x^{-1}(|d\Pi_2| - d\Pi_2)(x). \tag{4.10}$$

Then we have $x \ll N(x) (\ll x)$.

Proof. Let $d\Pi_3 = \frac{1}{2}(|d\Pi_2| - d\Pi_2)$ be the lower variation of the Jordan decomposition of $d\Pi_2$ [10]. Then $d\Pi_3 \geqslant 0$. We note that condition (4.9) implies the convergence of $\int_1^\infty x^{-1}(\exp d\Pi_3)(x)$. Consider

$$\begin{aligned}
&\int_1^x N_1\left(\frac{x}{t}\right) \exp(-d\Pi_3)(t)\\
&\quad = \int_1^x N_1\left(\frac{x}{t}\right)\left(\delta + \frac{d\Pi_3^2}{2!} + \frac{d\Pi_3^4}{4!} + \cdots\right)(t)\\
&\qquad - \int_1^x N_1\left(\frac{x}{t}\right) d\Pi_3 * \left(\delta + \frac{d\Pi_3^2}{3!} + \frac{d\Pi_3^4}{5!} + \cdots\right)(t)\\
&\quad \geqslant \int_1^x N_1\left(\frac{x}{t}\right)(\delta - d\Pi_3) * \left(\delta + \frac{d\Pi_3^2}{2!} + \frac{d\Pi_3^4}{4!} + \cdots\right)(t).
\end{aligned}$$

Given $\varepsilon > 0$ sufficiently small, there exists $B > 1$ such that for $x \geqslant B$ we have

$$(K' - \varepsilon)\, x \leqslant N_1(x) \leqslant (K + \varepsilon)\, x.$$

Therefore

$$T_1(x) := \int_1^x N_1\left(\frac{x}{t}\right)(\delta - d\Pi_3)(t) = N_1(x) - \int_1^x N_1\left(\frac{x}{t}\right) d\Pi_3(t)$$

$$\geqslant (K' - \varepsilon)\, x - (K + \varepsilon)\, x \int_1^{x/B} t^{-1} d\Pi_3(t) - Mx \int_{x/B}^x t^{-1} d\Pi_3(t)$$

$$\geqslant x \left\{ K' - \varepsilon - (K + \varepsilon) \int_1^{\infty} t^{-1} d\Pi_3(t) - M \int_{x/B}^x t^{-1} d\Pi_3(t) \right\}$$

$$\geqslant cx,$$

where $c > 0$, for $x \geqslant B_1$ sufficiently large. Moreover, we have

$$|T_1(x)| \leqslant N_1(x) + \int_1^x N_1\left(\frac{x}{t}\right) d\Pi_3(t) \ll x + x \int_1^x t^{-1} d\Pi_3(t) \ll x.$$

It follows that

$$\int_1^x N_1\left(\frac{x}{t}\right) \exp(-d\Pi_3)(t) \geqslant \int_1^x T_1\left(\frac{x}{t}\right)\left(\delta + \frac{d\Pi_3^2}{2!} + \frac{d\Pi_3^4}{4!} + \cdots\right)(t)$$

$$\geqslant \int_1^{x/B_1} T_1\left(\frac{x}{t}\right)\left(\delta + \frac{d\Pi_3^2}{2!} + \frac{d\Pi_3^4}{4!} + \cdots\right)(t)$$

$$- \int_{x/B_1}^x \left|T_1\left(\frac{x}{t}\right)\right|\left(\delta + \frac{d\Pi_3^2}{2!} + \frac{d\Pi_3^4}{4!} + \cdots\right)(t)$$

$$\geqslant cx - Mx \int_{x/B_1}^x t^{-1}(\exp d\Pi_3)(t) \geqslant \frac{c}{2}\, x$$

for $x \geqslant B_2$ sufficiently large.

Finally, we have

$$N(x) = 1 + \int_1^x N_1\left(\frac{x}{t}\right)(\exp\ d\Pi_2)(t)$$

$$= 1 + \int_1^x N_1\left(\frac{x}{t}\right) \exp(-d\Pi_3) * \exp\left(\frac{|d\Pi_2| + d\Pi_2}{2}\right)(t) \gg x. \quad \blacksquare$$

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