The smallness problem for $C^*$-algebras

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A B S T R A C T

Akemann showed that any von Neumann algebra with a weak$^*$ separable dual space has a faithful normal representation on a separable Hilbert space. He posed the question: If a $C^*$-algebra has a weak$^*$ separable state space, must it have a faithful representation on a separable Hilbert space? Wright solved this question negatively and showed that a unital $C^*$-algebra has the weak$^*$ separable state space if and only if it has a unital completely positive map, into a type I factor on a separable Hilbert space, whose restriction to the self-adjoint part induces an order isomorphism. He called such a $C^*$-algebra almost separably representable. We say that a unital $C^*$-algebra is small if it has a unital complete isometry into a type I factor on a separable Hilbert space. In this paper we show that a unital $C^*$-algebra is small if and only if the state spaces of all $n$ by $n$ matrix algebras over the $C^*$-algebra are weak$^*$-separable. It is natural to ask whether almost separably representable algebras are small or not. We settle this question positively for simple $C^*$-algebras but the general question remains open.

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1. Introduction

Let us call a topological space separable if it has a countable dense subset. There are many examples of separable compact Hausdorff spaces which are not metrizable and which do not have a countable base. Many years ago, C.A. Akemann [1] showed that if the dual space of a von Neumann algebra is weak$^*$ separable then the algebra has a faithful normal representation on a separable Hilbert space. He posed the question: if a $C^*$-algebra has a separable state space must it have a faithful representation on a separable Hilbert space?

Wright [13] solved Akemann's problem negatively by exhibiting unital $C^*$-algebras which had separable state spaces but did not have faithful representations on a separable Hilbert space. Wright called such algebras almost separably representable. His results give:

Let $A$ be a unital $C^*$-algebra. Then $A$ is almost separably representable if and only if there exist a separable Hilbert space $\mathcal{H}$ and a completely positive unital map $\Gamma$ from $A$ into $L(\mathcal{H})$ such that $\Gamma$ induces an isometric order isomorphism of the self-adjoint part of $A$ into the self-adjoint part of $L(\mathcal{H})$.

By adjoining units we could talk about general $C^*$-algebras but, for convenience, we shall assume that all algebras we consider are unital.

Fairly recently, inspired by [6], we [10] used the notion of small $C^*$-algebras, which are defined as follows. A $C^*$-algebra, $A$ is said to be small if there exists a complete isometry $\gamma$ from $A$ into $L(\mathcal{H})$, where $\mathcal{H}$ is separable and $\gamma(1) = 1$. When such a $\gamma$ exists, it is completely positive. It turns out that all small $C^*$-algebras are almost separably representable. What about the converse? Is every almost separably representable $C^*$-algebra small?

This problem is open, although we shall settle the question positively for certain classes of algebras, for example, simple $C^*$-algebras (see Corollary 14 and Theorem 7).

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We shall see that every almost separably representable C*-algebra would be small if whenever \( A \) has a separable state space then so also does \( M_2(A) \). (See Lemma 8 and the remark after Lemma 11.)

We shall show that \( A \) has a separable state space if and only if the unit ball of the dual, \( A^* \) is separable in the (compact) weak* topology (see Proposition 5).

If \( A \) has a separable dual ball then, clearly, so does \( A \oplus A \oplus A \oplus A \). The latter is isomorphic, as a Banach space, to \( M_3(A) \), although these are very different as C*-algebras. Suppose we knew that when two Banach spaces are isomorphic and one of them has a separable dual ball, then so also does the other space. Then we could apply this to deduce that \( M_2(A) \) has a separable dual ball. But this approach does not work. Because, by using an ingenious construction of Richard Haydon, see Example 6, we shall show that having a separable dual ball is a property which is not preserved under Banach space isomorphisms. We are driven back from a Banach space approach to adopting C*-methods.

2. A characterization of small C*-algebras

We shall characterize small C*-algebras by making use of their state spaces. Assume that all C*-algebras, we consider, are unital.

For a C*-algebra \( B \), we denote the set of all states on \( B \) by \( \mathcal{S}_B \). Let us mind that \( \mathcal{S}_B \) is a weak* compact set that is the weak* closed convex hull of the set \( \mathcal{S}_0(B) \) of all pure states on \( B \).

The main theorem in this section is the following:

**Theorem 1.** Let \( A \) be a C*-algebra. Then, \( A \) is small if and only if \( \mathcal{S}_{M_n(A)} \) is weak*-separable for every \( n \in \mathbb{N} \). Here \( M_n(A) \) is the C*-algebra of all \( n \times n \) matrices over \( A \).

To prove the "only if part", suppose that \( A \) is small. We show that, for every \( n \in \mathbb{N} \), \( \mathcal{S}_{M_n(A)} \) is weak*-separable, however this is a direct consequence of the following lemma.

**Lemma 2.** Let \( B \) be any C*-algebra. Suppose that there exist a separable Hilbert space \( K \) and a unital isometry \( \Psi : B \rightarrow \mathcal{L}(K) \). Then \( \mathcal{S}_B \) is weak*-separable.

**Proof.** Since \( K \) is separable, there exists a countable norm dense subset \( \{ \xi_n \mid n = 1, 2, \ldots \} \) in the closed unit sphere \( \mathcal{S} \) of \( \mathcal{H} \).

Clearly, \( \{ \omega_{\xi_n} \mid n \in \mathbb{N} \} \) is norm dense in the set \( \mathcal{S}_{\mathcal{L}(K)} \), of all pure normal states, where \( \omega_{\xi} \) is the pure vector state on \( \mathcal{L}(K) \) defined by \( \omega_{\xi}(T) = \langle T\xi, \xi \rangle \in \mathbb{C} \). We show that the weak*-convex hull \( \mathcal{S}_B \) of \( \mathcal{S}_{\mathcal{L}(K)} \) is \( \mathcal{S}_{\mathcal{L}(K)} \). If there exists \( \varphi \in \mathcal{S}_{\mathcal{L}(K)} \setminus \mathcal{S}_B \), then by Hahn–Banach theorem, there exists \( T \in \mathcal{L}(K)_{sa} \) such that

\[
|\varphi(T)| > \sup \{|\psi(T)| \mid \psi \in \mathcal{S}_B\}.
\]

Since \( \omega_{\xi} \in \mathcal{S}_B \) for all \( \xi \in K \) with \( \|\xi\| = 1 \), it follows that \( \|T\| \geq \|\varphi(T)\| > \|T\| \). This is a contradiction. Hence \( \mathcal{S}_B = \mathcal{S}_{\mathcal{L}(K)} \). So, \( \mathcal{S}_{\mathcal{L}(K)} \) is weak*-separable. Note that the transposed map \( \Psi': \mathcal{S}_{\mathcal{L}(K)} \rightarrow \mathcal{S}_B \) is an affine weak*-continuous map, because \( \Psi \) is unital. Moreover, since the map \( \Psi \) is an isometry, the above map is onto. Indeed, since \( \mathcal{S}_{\mathcal{L}(K)} \) is weak*-compact, \( \Psi'(\mathcal{S}_{\mathcal{L}(K)}) \) is closed and convex in \( \mathcal{S}_B \) with respect to the weak*-topology. Suppose that there exists \( \psi \in \mathcal{S}_B \) such that \( \psi \neq \Psi'(\varphi) \). Again, by Hahn–Banach theorem, there exists \( s \in B_{sa} \) such that

\[
|\psi(s)| > \sup \{|\eta(\psi(s))| \mid \eta \in \mathcal{S}_{\mathcal{L}(K)}\}.
\]

Since \( \|s\| \geq |\psi(s)| > \|\psi(s)\| = \|s\| \), this is a contradiction. Hence it follows that \( \psi(\mathcal{S}_{\mathcal{L}(K)}) = \mathcal{S}_B \). So, \( \mathcal{S}_B \) is also weak*-separable. \( \square \)

A proof of Theorem 1 "the only if part": Since \( A \) is a small C*-algebra, there exist a separable Hilbert space \( \mathcal{H} \) and a complete isometry \( \Phi : A \rightarrow \mathcal{L}(\mathcal{H}) \). Take any \( n \in \mathbb{N} \). Since \( \Phi \) is a complete isometry,

\[
\Phi_n : M_n(A) \rightarrow M_n(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^n)
\]

is a unital isometry. Here \( \Phi_n \) is the map defined by

\[
\Phi_n((a_{ij})) = (\Phi(a_{ij})) \quad ((a_{ij}) \in M_n(A)).
\]

Let \( \pi \) be the canonical *-isomorphism from \( M_n(\mathcal{L}(\mathcal{H})) \) onto \( \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^n) \). Since \( \mathcal{H} \otimes \mathbb{C}^n \) is separable and \( \pi \circ \Phi_n \) is a unital isometry, by Lemma 2, \( \mathcal{S}_{M_n(A)} \) is also weak*-separable. This completes the proof of the only if part.

The "if part": Take any C*-algebra \( A \) and suppose that for every \( n \in \mathbb{N} \), \( \mathcal{S}_{M_n(A)} \) is weak*-separable. We show that \( A \) is small. Take any \( n \in \mathbb{N} \). Note that \( M_n(H) \cong A \otimes M_n(\mathcal{C}) \) via the canonical isomorphism \( \pi_0 \). Let \( \Phi_n = \pi_0^{-1} \circ (\Phi \otimes 1_{M_n(\mathcal{C})}) \circ \pi_0 \). We may assume that \( M_n(A) = A \otimes M_n(\mathcal{C}) \). Here \( \pi_0 \) is given in the following manner. Let \( \{ e^{(n)}_{ij} \}_{1 \leq i, j \leq n} \) be the standard system of matrix units in \( M_n(\mathcal{C}) \) and \( \pi_0 \) is the map defined by

\[
\pi_0((a_{ij})) = \sum_{1 \leq i, j \leq n} a_{ij} \otimes e^{(n)}_{ij} \quad ((a_{ij}) \in M_n(A)).
\]
Since $\mathcal{A}_{\mathcal{M}_n(A)}$ is weak$^*$-separable, a weak$^*$-dense subset $\{\varphi(p, n) \mid p = 1, 2, \ldots\}$ in $\mathcal{A}_{\mathcal{M}_n(A)}$ does exist. Let us take any $p \in \mathbb{N}$ and let $\{\pi(p, n), \mathcal{H}(p, n), \xi(p, n)\}$ be the GNS-representation induced by $\varphi(p, n)$ of $\mathcal{M}_n(A)$. On noting that $\pi(p, n)(a \otimes 1_{C_n})(\pi(p, n)(1_a \otimes e_{ij}^{(n)})) = \pi(p, n)(1_a \otimes e_{ij}^{(n)})(\pi(p, n)(a \otimes 1_{C_n}))$ for every $a \in A$ and $1 \leq i, j \leq n$, it follows that $\phi(p, n)((a_{ij})) = (\pi(p, n)((a_{ij}))\xi(p, n), \xi(p, n))$ and

$$(\pi(p, n)((a_{ij}))\xi(p, n), \xi(p, n)) = \sum_{1 \leq i, j \leq n} (\pi(p, n)(a_{ij} \otimes e_{ij}^{(n)}))\xi(p, n), \xi(p, n))$$

$$= \sum_{1 \leq i, j \leq n} (\pi(p, n)(a_{ij} \otimes 1_{C_n}))\pi(p, n)(1_a \otimes e_{ij}^{(n)})\pi(p, n)(1_a \otimes e_{ij}^{(n)}))\xi(p, n), \xi(p, n))$$

$$= \sum_{1 \leq i, j \leq n} (\pi(p, n)(a_{ij} \otimes 1_{C_n}))\pi(p, n)(1_a \otimes e_{ij}^{(n)}))\xi(p, n), \xi(p, n))$$

for all $(a_{ij}) \in \mathcal{M}_n(A)$.

Let $\sigma(p, n)(a) = \pi(p, n)(a \otimes 1_{C_n})$ ($a \in A$).

Let $\{\pi, \mathcal{K}\}$ be the direct sum of $\{(\sigma(p, n), \mathcal{H}(p, n)) \mid n, p \in \mathbb{N}\}$. Then, $\pi$ is faithful, because $\{\varphi(p, n) \mid p \in \mathbb{N}\}$ is weak$^*$-dense in $\mathcal{A}_{\mathcal{A}}$.

The closed subspace of $\mathcal{K}$, generated by $\{\pi(p, n)(1_a \otimes e_{ij}^{(n)}))\xi(p, n) \mid i, j = 1, \ldots, n; n \in \mathbb{N} \text{ and } p \in \mathbb{N}\}$ shall be denoted by $\mathcal{H}$. Then clearly, $\mathcal{H}$ is a separable Hilbert space. Let $Q$ be the orthogonal projection $P_{\mathcal{H}}$ on $\mathcal{H}$ and let us define the map $\Gamma$ by $\Gamma(a) = Q \pi(a) Q |_{\mathcal{H}} (a \in A)$. Then, clearly, $\Gamma$ is a C*-map from $A$ into $\mathcal{L}(\mathcal{H})$ such that $\Gamma(1_a) = 1_{\mathcal{H}}$.

Take any $n \in \mathbb{N}$. Suppose that $(a_{ij}) \in \mathcal{M}_n(A)$ satisfies $\Gamma_n((a_{ij})) \geq 0$, that is, $(\Gamma((a_{ij})) \geq 0$. We show that $(a_{ij}) \geq 0$ in $\mathcal{M}_n(A)$. To show this, take any $x_1, \ldots, x_n \subseteq \mathcal{H}$ and $\xi \in \mathcal{H}$. Since $(\Gamma((a_{ij})) \geq 0$, it follows that

$$\left( \sum_{1 \leq i, j \leq n} x_i^* \Gamma(a_{ij}) x_j, \xi, \xi \right) \geq 0.$$ 

That is,

$$\left( \sum_{1 \leq i, j \leq n} x_i^* Q \pi(a_{ij}) Q x_j, \xi, \xi \right) \geq 0. \quad (1)$$

Note that $\pi(p, n)(1_a \otimes e_{ij}^{(n)})\mathcal{H}(n, p) \subset \mathcal{H}$ for all $i$ and $j$ with $1 \leq i, j \leq n$. Put $\xi = \xi(p, n)$ and $x_j = \pi(p, n)(1_a \otimes e_{ij}^{(n)})|_{\mathcal{H}} (j = 1, \ldots, n)$. Since $\pi|_{\mathcal{H}(n, p)} = \sigma(p, n)$, it follows that

$$Q \pi(p, n)(1_a \otimes e_{ij}^{(n)})\xi(p, n) = \pi(p, n)(1_a \otimes e_{ij}^{(n)})\xi(p, n) \quad (j = 1, \ldots, n).$$

Hence by (1), we have

$$\sum_{1 \leq i, j \leq n} (\sigma(p, n)(a_{ij}))\pi(p, n)(1_a \otimes e_{ij}^{(n)})\xi(p, n), \pi(p, n)(1_a \otimes e_{ij}^{(n)})\xi(p, n)) \geq 0.$$

So it follows that

$$\sum_{1 \leq i, j \leq n} (\pi(p, n)(a_{ij} \otimes e_{ij}^{(n)}))\xi(p, n), \xi(p, n)) \geq 0,$$

and hence, $\varphi(p, n)((a_{ij})) \geq 0$ for all $p \in \mathbb{N}$. Since $\{\varphi(p, n) \mid p \in \mathbb{N}\}$ is weak$^*$-dense in $\mathcal{A}_{\mathcal{M}_n(A)}$, we have that $\varphi((a_{ij})) \geq 0$ ($\varphi \in \mathcal{A}_{\mathcal{M}_n(A)}$). This implies that $(a_{ij}) \in \mathcal{M}_n(A)$ and so $(a_{ij}) \geq 0$ follows. Thus, for any $(a_{ij}) \in \mathcal{M}_n(A)$, $\Gamma_n((a_{ij})) \geq 0$ if and only if $(a_{ij}) \geq 0$.

To show that $\Gamma$ is a complete isometry, we need the following lemma:

**Lemma 3** (Choi and Effros). (See [4, Proposition 1.3.2].) Let $n \in \mathbb{N}$ be any given and fix it. Let $\mathcal{K}$ be a Hilbert space and let $B \in \mathcal{M}_n(\mathcal{L}(\mathcal{K}))$. Then, $\|B\| \leq 1$ if and only if $\|1_n B 1_n\| \geq 0$.

We show that $\Gamma$ is a complete isometry. To prove this, take any $n \in \mathbb{N}$ and fix it. Since $\Gamma_n$ is a unital completely positive map (CP-map), it follows that $\|\Gamma_n(x)\| \leq \|x\|$ for all $x \in \mathcal{M}_n(A)$. Hence we only have to check that $\|\Gamma_n(x)\| = \|x\|$ for all $x \in \mathcal{M}_n(A)$. To do this, take any $x \in \mathcal{M}_n(A)$ and suppose that $\|\Gamma_n(x)\| \leq 1$. Then, as was shown in Lemma 3, on noting that $\Gamma_n$ is self-adjoint, $\Gamma_n(x^*)^* \Gamma_n(x^*) = 0 \in \mathcal{M}_{2n}(\mathcal{L}(\mathcal{H}))$. So, $0 \leq (\Gamma_n(x^*), \Gamma_n(x^*)^*) = \Gamma_n((x^* x)_{1,1})$, since for any $y \in \mathcal{M}_{2n}(A), \Gamma_{2n}(y) \geq 0$ if and only if $y \geq 0$, it follows that $\|1_n x^* 1_n, n \| \geq 0$ in $\mathcal{M}_{2n}(A)$. Hence by Lemma 3, we have that $\|x\| \leq 1$.
Now take any \( x \in M_n(A) \). For every positive real number \( \varepsilon \), consider \( y = \frac{x}{\| x \|_0 + \varepsilon} \in M_n(A) \). Clearly \( \| \Gamma(y) \| \leq 1 \). Hence by the argument in the above paragraph, it follows that \( \| y \| \leq 1 \), that is, \( \| x \| \leq \| \Gamma_n(x) \| + \varepsilon \) for all \( \varepsilon > 0 \). Hence we have \( \| x \| \leq \| \Gamma_n(x) \| \) for all \( x \in M_n(A) \). Thus \( \Gamma \) is a complete isometry from \( A \) into \( L(H) \). This implies that \( A \) is small. This completes the proof of Theorem 1.

3. When is an almost separably representable algebra small?

Wright [13] introduced the concept of almost separably representable \( C^* \)-algebras which can be characterized as \( C^* \)-algebras with the weak\(^*\)-separable state spaces.

The following question naturally arises.

Are almost separably representable \( C^* \)-algebras small?

That is, for any \( C^* \)-algebra \( A \), if \( \mathfrak{S}_A \) is weak\(^*\)-separable, then are all \( \mathfrak{S}_{M_n(A)} \) weak\(^*\)-separable for \( n \in \mathbb{N} \)?

The next proposition, which is proved by Takesaki, implies that to show that any almost separably representable \( C^* \)-algebra is small, we only have to check that the closed unit ball of the dual \( M_2(A) \) of \( M_2(A) \) is separable in the weak\(^*\)-topology if the closed unit ball of the dual \( A^* \) of a \( C^* \)-algebra \( A \) is separable in the weak\(^*\)-topology.

**Lemma 4.** Let \( \phi \) be a pure state of a unital \( C^* \)-algebra \( A \). Then \( \phi \) is an extreme point of \( S_{R_1} \). Here \( S_{R_1} \) is the unit ball of the dual \( A_{sa}^* \) of the real Banach space \( A_{sa} \).

**Proof.** Assume that this is false. Then \( \phi = t f + (1 - t) g \) where \( 0 < t < 1 \) and \( f, g \) are in \( S_{R_1} \), with \( f \neq g \). So

\[
1 = \phi(1) = t f(1) + (1 - t) g(1) \leq t \| f \| + (1 - t) \| g \| \leq t + (1 - t) = 1.
\]

From this it follows that \( f(1) = \| f \| = 1 \) and \( g(1) = \| g \| = 1 \). Hence \( f \) and \( g \) are states. But this implies that \( \phi \) is not an extreme point of the state space \( \mathfrak{S} \) of \( A \). This is a contradiction. \( \square \)

**Proposition 5.** If \( A \) is a unital \( C^* \)-algebra such that the closed unit ball \( S \) of the dual \( A^* \) admits a countable weak\(^*\)-dense subset, then the state space \( \mathfrak{S}_A \) of \( A \) admits a countable weak\(^*\)-dense subset.

**Proof.** Let \( C \subset S \) be a countable subset of the unit ball \( S \) of the dual space \( A^* \) which is weak\(^*\)-dense in \( S \). Let \( C_{R_1} = \{ \frac{\varphi_{++} + \varphi_{--}}{2} : \varphi \in C \} \cap A_{sa}^* \subset S_{R_1} \). Then \( C_{R_1} \) is weak\(^*\)-dense in the unit ball of the Banach space dual \( A_{sa}^* \) of the self-adjoint part \( A_{sa} \) of \( A \). Let \( D_0 \) be the algebraic convex hull of \( C_{R_1} \) which is still countable. Then \( D_0 \) is weak\(^*\)-dense convex subset of \( S_{R_1} \). Replacing \( D_0 \) by \( \overline{\text{conv}}(D_0 \cup (-D_0)) \) if necessary, we may and do assume that \( D_0 = -D_0 \). Each point \( \varphi \in D_0 \) has a unique decomposition:

\[
\varphi = \lambda \varphi_{++} - \mu \varphi_{--}, \quad \| \varphi \| = \lambda + \mu, \quad \varphi_{++}, \varphi_{--} \in \mathfrak{S}_A.
\]

Set \( E_0 = \{ \varphi_{++}, \varphi_{--} : \varphi \in D_0 \} \). Let \( F_0 \) be the algebraic convex hull of \( E_0 \subset \mathfrak{S}_A \). The weak\(^*\) closure \( K \) of \( F_0 \) is a weak\(^*\) compact subset of \( \mathfrak{S}_A \). The convex hull of \( K \cup (-K) \) contains \( D_0 \) and weak\(^*\) compact, so that \( S_{R_1} = \overline{\text{conv}}(K \cup (-K)) \). The extreme boundary of \( S_{R_1} \) is then contained in \( K \cup (-K) \). Namely \( K \) contains all pure states of \( A \) by the above lemma and hence it contains \( \mathfrak{S}_A \). That is, \( \mathfrak{S}_A = K \). \( \square \)

If the unit ball of the dual \( A^* \) is separable in the weak\(^*\)-topology, then clearly, so does \( A \oplus A \oplus A \oplus A \). The latter is isomorphic, as a Banach space, to \( M_2(A) \), although these are very different \( C^* \)-algebras.

Suppose we knew that when two Banach spaces are isomorphic and one of them has a separable dual ball, then the other does as well. Then this would imply that \( M_2(A) \) has a separable dual ball.

But the following ingenious example by Haydon tells us that having a separable dual ball is a property which is not preserved under Banach space isomorphisms. Let \( A \) be the set of all real numbers. Then \( \text{Card}(A) \geq \aleph_1 \). Let \( e_A^t \) be the Banach space of all functions \( f : A \to \mathbb{C} \) such that \( \sum_{t \in A} |f(t)| < \infty \) and \( \| f \| = \sum_{t \in A} |f(t)| \). Then \( f \) is 0 except for a countable set of values of \( t \).

Let \( K = [-1, 1]^A \) with the weak topology. Then \( K \) is a separable \( [7] \) and compact group with a unique normalized Haar measure \( \mu \).

**Example 6** (Richard Haydon). Let \( \mathcal{C}(K) \) be the \( C^* \)-algebra of all complex-valued continuous functions on \( K \). Then the closed unit ball \( \mathfrak{B} \) of the dual space \( \mathcal{C}(K)^* \) is weak\(^*\)-separable. We can think of \( L^1(K, \mu) \) as a subspace of \( \mathcal{C}(K)^* \). The coordinate functions \( r_a : K \to [-1, 1] \) (\( a \in A \)) are independent Rademacher functions in \( L^1(K, \mu) \) (that is, \( r_a \) is defined by, for each \( a \in A \), \( r_a(k) = k(a) \) \( (k \in K) \)) and let \( \text{Rad}_A^1 \) be the set \( \{ \sum_{a \in A} f(a)r_a \mid f \in e_A^1 \} \) with \( \| f \|_1 \leq 1 \). Then each element of \( \text{Rad}_A^1 \) is an element of \( \mathfrak{B} \) and \( \text{Rad}_A^1 \) is weak\(^*\)-compact. Fix \( \delta > 0 \) and consider the set \( \mathfrak{B}_1 \) defined to be the absolute convex hull of \( \mathfrak{B} \cup (1 + \delta) \text{Rad}_A^1 \). This is weak\(^*\)-compact and not weak\(^*\)-separable. So the dual unit ball \( \mathfrak{B} \) for \( \mathcal{C}(K) \) is weak\(^*\)-separable for its usual supremum norm but not for the \((1 + \delta)\)-equivalent norm with dual ball \( \mathfrak{B}_1 \).
We show that $\text{Rad}^1_\Lambda$ is a weak* compact subset of $\mathcal{B}$. As $\mathcal{C}(K) \subset L^\infty(K, \mu)$, we may assume that $L^1(K, \mu) \subset \mathcal{C}(K)^*$. Since $\varphi = \sum_{a \in A} f(a) r_a \in L^1(K, \mu)$ with $\|\varphi\| \leq \|f\|_1$ for each $f \in \ell^1_\Lambda$, it follows that $\text{Rad}^1_\Lambda \subset \mathcal{B}$ and so, to prove that $\text{Rad}^1_\Lambda$ is weak* compact, we only have to check that it is weak* closed. We show, for any different $a_1, \ldots, a_n \in \Lambda$,

$$
\int_K r_{a_1} r_{a_2} \cdots r_{a_n} d\mu(k) = 0.
$$

(2)

Note that $r_{a_1} r_{a_2} \cdots r_{a_n}$ are in $L^1(K, \mu)$. On the other hand, if we take $i \in \{1, 2, \ldots, n\}$ and let $k_i(a) = -1$ if $a = a_i$ and $1$ if $a \neq a_i$, then $k_i$ is an element of the compact topological group $K$ and for each $i \neq j$, we have $r_{a_i}(k_i) = (k_i) k_i(a) = k_i(k_i(a)) = k_i(k_i(a)) = r_{a_i}$ for all $k_i \in K$. Moreover, we have $r_{a_i}(k)(k_i) = (k_i) k_i(a_i) = k_i(k_i(a)) = -k_i(a) = -r_{a_i}(k)$ for all $k_i \in K$. Let $E_i = \{ k_i \in K | k_i(a_i) = 1 \} = \{ k_i \in K | r_{a_i}(k) = 1 \}$. Then $E_i$ is a closed and open subset of $K$ such that $k_i k_i \in E_i$ if and only if $k_i \in E_i^c = K \setminus E_i$ for each $i$. Hence it follows that $E_i = k_i(E_i^c)$ for each $i$. Since we have that $r_{a_i} = \chi_{E_i} - \chi_{E_i^c}$, for any $f \in L^1(K, \mu)$ with $f(k_i) = f(k)$ for all $k_i \in K$, we have

$$
\int_K r_{a_1} f \mu = \int_K \chi_{E_1} f \mu - \int_K \chi_{E_1^c} f \mu = \int_K \chi_{E_1} f \mu - \int_K \chi_{E_1} f(k_i) \mu(k).
$$

Since $f = r_{a_1} \cdots r_{a_n}$ satisfies $f(k_i) = f(k)$ for all $k_i \in K$, (2) follows. Let $\{\varphi_{a_1} \subset \text{Rad}^1_\Lambda$ such that $\varphi_{a_1} \rightarrow \varphi$ (weak*) for some $\varphi \in \mathcal{B}$. Since each $r_a \in \mathcal{C}(K)$, it follows that $\varphi_a(r_a) \rightarrow \varphi(r)$ for each $a \in A$. Since each $\varphi_a$ can be written in the form $\varphi_a = \sum_{b \in A} f^a(b) r_b$ with $\sum_{b \in A} |f^a(b)| \leq 1$. As was noted before, there exists a countable subset $\Lambda(\alpha) = \{ b \in A | f^a(b) \neq 0 \}$. So we have $\varphi_a = \sum_{b \in \Lambda(\alpha)} f^a(b) r_b$. Hence, for each $\alpha$, we have

$$
\varphi_a(r_a) = \int_K \left( \sum_{b \in \Lambda(\alpha)} f^a(b) r_b(k) \right) r_a(k) \mu(k) = \sum_{b \in \Lambda(\alpha)} f^a(b) \int_K r_b(k) r_a(k) \mu(k) = f^a(\alpha) \int_K r_a(k)^2 \mu(k) \quad \text{(by (2))}
$$

$$
= f^a(\alpha), \quad \text{because } \mu \text{ is normalized.}
$$

Hence it follows that $f^a(\alpha) \rightarrow \varphi(r)$ for each $a \in \Lambda$. Take any finite subset $J$ of $\Lambda$ and we have $\sum_{b \in J} |f^a(b)| \leq 1$ for all $\alpha$. On putting $f(a) = \varphi(r_a)$ ($a \in \Lambda$), we get that $\sum_{b \in J} |f(b)| \leq 1$ for every finite subset of $\Lambda$. Hence it follows that $f \in \ell^1_\Lambda$ and $\|f\|_1 \leq 1$. We show that $\varphi = \sum_{b \in A} f(b) r_b \in \text{Rad}^1_\Lambda$. Let $\varphi_0 = \sum_{b \in \Lambda} f(b) r_b \in \text{Rad}^1_\Lambda$. As was shown above we have $\varphi_a(r_a) \rightarrow \varphi_0(r_b)$ for each $b \in A$. We note also that, by (2),

$$
\int_K \varphi_a \mu = \sum_{b \in \Lambda(\alpha)} f^a(b) \int_K r_b \mu(k) = 0.
$$

Similarly we have $\int_K \varphi_0(k) \mu(k) = 0$. Hence it follows that $\varphi_0(1) \rightarrow \varphi_0(1)$.

Take any $a_1, \ldots, a_n \in A$ with $a_i \neq a_j$ if $i \neq j$. Put $f_{1,2,\ldots,n} = r_{a_1} r_{a_2} \cdots r_{a_n}$. Calculation shows

$$
\int_K \varphi_{a_1} f_{1,2,\ldots,n} \mu = \int_K \left( \sum_{b \in \Lambda(\alpha)} f^a(b) r_b \right) r_{a_1} f_{2,\ldots,n} \mu = \sum_{b \in \Lambda(\alpha)} f^a(b) \int_K r_b r_{a_1} f_{2,\ldots,n} \mu = f^a(\alpha_1) \int_K f_{2,\ldots,n} \mu + \cdots + f^a(\alpha_n) \int_K f_{1,\ldots,n-1} \mu \rightarrow \int_K \varphi_0 f_{1,2,\ldots,n} \mu.
$$

Let $\mathcal{A}_0$ be the linear span of elements of the form $r_{a_1} r_{a_2} \cdots r_{a_n}$ where $a_1, \ldots, a_n$ runs through $\Lambda$ and the constant function 1. It is easy to check that $\mathcal{A}_0$ is a unital *-subalgebra of $\mathcal{C}(K)$ which separates the points of $K$. So by the Stone–Weierstrass theorem, it follows that $\mathcal{A}_0$ is norm dense in $\mathcal{C}(K)$. By our previous argument, we have $\varphi_b(g) \rightarrow \varphi_0(g)$ for every $g \in \mathcal{A}_0$. Since $\|\varphi_a\|_n > 0$ is bounded by 1, it follows that $\varphi_a \rightarrow \varphi_0$ with respect to the weak* topology. Hence we have $\varphi = \varphi_0 \in \text{Rad}^1_\Lambda$. So, Rad$^1_\Lambda$ is an absolutely convex weak* compact subset of $\mathcal{B}$. Then, see [3], if $X$ and $Y$ are absolutely convex weak* compact sets, the absolute convex hull of $X \cup Y$ is compact and can be identified with $\text{co}(X \cup Y) = \{ sb + tz | b \in X, z \in Y, |s| + |t| \leq 1 \}$. So the weak* closed absolute convex hull of $\mathcal{B}$ and $(1 + \delta) \text{Rad}^1_\Lambda$ is

$$
\mathcal{B}_1 = \{ sb + tz | b \in \mathcal{B}, z \in (1 + \delta) \text{Rad}^1_\Lambda, |s| + |t| \leq 1 \}.
$$
Suppose that $\mathcal{B}_1$ is weak$^*$ separable. Then a countable dense sequence will be of the form $\{snbn + tazn \mid n = 1, 2, \ldots\}$ where $b_n \in \mathcal{B}_1$, $z_n \in (1 + \delta)\mathcal{M}_1$, $|sn| + |azn| \leq 1$. But each $z_n$ is of the form $(1 + \delta)\sum_{a \in A} f(a)r_a$ where $f \in \ell^1$. So there is a countable set $\mathcal{A}_n \subseteq A$ such that $(1 + \delta)\sum_{a \in \mathcal{A}_n} f(a)r_a$ converges to $f \in \ell^1$. Hence each $z_n$ is of the form $(1 + \delta)\sum_{a \in \mathcal{A}} f(a)r_a$ where $f \in \ell^1$. Let $\mathcal{D}$ be the countable set $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. Hence each $z_n$ is of the form $(1 + \delta)\sum_{a \in \mathcal{D}} f(a)r_a$ where $f \in \ell^1$. Let $\mathcal{D}_n = \{\sum_{a \in \mathcal{D}} f(a)r_a \mid f \in \ell^1\}$. If $x \in \mathcal{D} \setminus \mathcal{D}_n$, we then have $(1 + \delta)r_{sn} = sb + t(1 + \delta)z$ for some $z = \sum_{a \in \mathcal{D}} f(a)r_a$, with $b$ in the unit ball of $\mathcal{B}$ and $z \in \mathcal{D}_n$. We note here that $\mathcal{A} \setminus \mathcal{D}_n \neq \emptyset$, because $\mathcal{A}$ is uncountable. Since the Rademacher functions are bounded functions, we can think of them as belonging to $L^\infty(K, \mu) \cap L^1(K, \mu)$. Hence $b$ can be identified with a function in the unit ball of $L^1(K, \mu)$. So $\mathcal{B}_1$ is a weak$^*$-compact convex subset of $C(K)^*$, which is not separable. But it corresponds to an equivalent norm on $C(K)$.

Hence we are forced back from a Banach space approach to adopting C*$^*$-methods. Wright [14] showed that if a C*$^*$-algebra is almost separably representable, then the regular $\sigma$-co-completion $\hat{\mathcal{A}}$ is a monotone complete algebra which is also almost separably representable. Hence, the first question in this section is equivalent to asking:

For any monotone complete C*$^*$-algebra $A$, if $\mathcal{G}_A$ is weak$^*$-separable, then are all $\mathcal{G}_{M_m(A)}$ weak$^*$-separable for $m \in \mathbb{N}$?

That is,

If $A$ is almost separably representable, then is it small?

We shall give a partial answer to this question:

Theorem 7. Let $\mathcal{A}$ be a monotone complete C*$^*$-algebra. Suppose that $\mathcal{A}$ does not have any direct summand that is isomorphic to a wild type II$_1$ algebra $B$ which satisfies $M_\infty(B) \not\cong B$ for some $n \in \mathbb{N}$. If $\mathcal{A}$ is almost separably representable, then $\mathcal{A}$ is small.

Remark. We shall consider the following statement:

(WS) Let $\mathcal{B}$ be a C*$^*$-algebra. If $\mathcal{G}_\mathcal{B}$ (or equivalently the unit ball of the dual $B^*$) is weak$^*$-separable, then $\mathcal{G}_{M_2(B)}$ (or equivalently the unit ball of the dual $M_2(B)^*$) is also weak$^*$-separable.

If (WS) holds true for every C*$^*$-algebra $\mathcal{B}$, then by mathematical induction, $\mathcal{G}_{M_m(B)}$ is weak$^*$-separable for every $m \in \mathbb{N}$. Suppose that for every $m \in \mathbb{N}$, $\mathcal{G}_{M_m(B)}$ is weak$^*$-separable. Since for every $n \in \mathbb{N}$, $\mathcal{G}_{M_n(B)}$ is $\ast$-isomorphic to a corner of $M_2\mathcal{B}$ for some $m \in \mathbb{N}$, by Lemma 8, $\mathcal{G}_{M_n(B)}$ is weak$^*$-separable for every $n \in \mathbb{N}$. (See Lemma 8 for details.)

Lemma 8. Let $\mathcal{A}$ be a C*$^*$-algebra and let $B$ be a C*$^*$-subalgebra with unit $e$ ($e$ is not necessarily the unit of $\mathcal{A}$ but $e \in \mathcal{A}$ is a projection). Suppose that $\mathcal{G}_A$ is weak$^*$-separable. Then $\mathcal{G}_B$ is also weak$^*$-separable.

Proof. By the Hahn–Banach theorem, $\mathcal{G}_\mathcal{B}$ can be identified with $\{\varphi|_\mathcal{B} \mid \varphi \in \mathcal{G}_\mathcal{A}, \varphi(e) = 1\}$. Let $\varphi_n \mid n \in \mathbb{N}$ be a weak$^*$-dense subset of $\mathcal{G}_\mathcal{A}$. We show that $\{\varphi_n(x) \mid \varphi_n(e) \neq 0\}$ is weak$^*$-dense in $\mathcal{G}_\mathcal{B}$.

Take any $\varphi \in \mathcal{G}_\mathcal{A}$. Note that $\varphi$ can be thought of as a state on $\mathcal{A}$ such that $\varphi(e) = 1$. Take any positive real number $\varepsilon \in (0, 1)$ and take any $x \in \varepsilon \mathcal{A}$. Then there exists $n \in \mathbb{N}$ such that $|\varphi(x) - \varphi_n(x)| < \frac{\varepsilon}{4}$ and $|\varphi_n(e) - 1| < \frac{\varepsilon}{4(|\varphi(x)| + 1)}$. Thus, $|\varphi_n(x)| < \frac{\varepsilon}{4}$. So it follows that

$$\frac{\varphi_n(x)}{\varphi_n(e)} = \frac{\varphi_n(x)}{\varphi_n(e)} \left|\frac{\varphi_n(x) - \varphi_n(x)}{\varphi_n(x)}\right| \leq \frac{1}{\varphi_n(e)} \left|\frac{\varphi(x) - \varphi_n(x)}{\varphi(x)} + \frac{\varphi(x)}{\varphi(x)} \cdot |1 - \varphi_n(e)|\right| \leq \frac{4}{3} |\varphi(x) - \varphi_n(x)| + \frac{4}{3} \frac{|\varphi(x)| \varepsilon}{4(|\varphi(x)| + 1)} < \varepsilon.$$ 

Thus, $\{\varphi_n \mid \varphi_n(e) \neq 0\}$ is weak$^*$-dense in $\mathcal{G}_\mathcal{B}$.
is a wild $AW^*$-algebra of type I$_1$, which satisfies that $C \otimes \mathcal{M}_n(C) \cong C$ for all $n \in \mathbb{N}$ or $[0]$. Suppose that $\mathcal{S}_A$ is weak$^*$-separable. Upon noting that the finite direct sum of small $C^*$-algebras is also small, by Lemma 8, we may argue each summand separately.

**Lemma 9.** Let $A$ be a $C^*$-algebra. Suppose that $A$ is stable under the tensor product by every $\mathcal{M}_n(C)$, that is, $A \cong A \otimes \mathcal{M}_n(C)$ for each $n \in \mathbb{N}$. If $\mathcal{S}_A$ (or equivalently the unit ball of the dual $A^*$) is weak$^*$-separable, then $A$ is small. In particular, every properly infinite $AW^*$-algebra which is almost separably representable is small.

**Proof.** Let $A$ be a properly infinite $AW^*$-algebra. Then, for every $n \in \mathbb{N}$, $A \cong A \otimes \mathcal{M}_n(C)$ and so $\mathcal{S}_{\mathcal{M}_n(A)}$ is weak$^*$-separable for every $n \in \mathbb{N}$. Hence by Theorem 1, $A$ is small. □

**Lemma 10.** Let $A$ be a commutative $C^*$-algebra. If $\mathcal{S}_A$ (or equivalently the unit ball of the dual $A$) is weak$^*$-separable, then $A$ is small.

**Proof.** We show that for every $n \in \mathbb{N}$, $\mathcal{S}_{\mathcal{M}_n(A)}$ is weak$^*$-separable. Since $\mathcal{M}_n(A) \cong A \otimes \mathcal{M}_n(C)$, we only have to show that $\mathcal{S}_{\mathcal{M}_n(A)}$ is weak$^*$-separable. Since $A$ is commutative, every pure state of $\mathcal{M}_n(A)$ is of the form $\omega_1 \otimes \omega_2$ for some pure states $\omega_1$ of $A$ and $\omega_2$ of $\mathcal{M}_n(C)$. (See Theorem 4.14 in [11].) So, we have $\hat{\mathcal{A}}(\mathcal{A}_0 \otimes \mathcal{M}_n(C)) \subset \hat{\mathcal{A}}(\hat{\mathcal{A}}(\mathcal{M}_n(C)) \subset \mathcal{S}_{\mathcal{M}_n(A)} \otimes \mathcal{M}_n(C) \subset \mathcal{S}_{\mathcal{M}_n(A)} \otimes \mathcal{M}_n(C)$. Since $\mathcal{S}_{\mathcal{M}_n(A)}$ is the weak$^*$-closure of the convex hull of $\hat{\mathcal{A}}(\mathcal{A}_0 \otimes \mathcal{M}_n(C))$, $\mathcal{S}_{\mathcal{M}_n(A)}$ is the weak$^*$-closed convex hull of $\mathcal{S}_{\mathcal{M}_n(A)} \otimes \mathcal{M}_n(C)$. Since $\mathcal{S}_A$ and $\mathcal{S}_{\mathcal{M}_n(A)}$ are weak$^*$-separable, it follows that $\mathcal{S}_{\mathcal{M}_n(A)}$ is also weak$^*$-separable. So, $A$ is small. □

**Remark.** Let us consider the small commutative $C^*$-algebra $L^\infty[0,1]$. The set $\{\delta_k \mathcal{G}_{L^\infty[0,1]}\}$ of all pure states is not weak$^*$-separable, because $C^\infty(\mathbb{N})$ is the only infinite dimensional $L^\infty(X)$ that satisfies $\delta_k \mathcal{G}_{L^\infty(X)}$ to be weak$^*$-separable.

**Lemma 11.** Let $A$ be a type I $AW^*$-algebra. If $\mathcal{S}_A$ (or equivalently the unit ball of the dual $A^*$) is weak$^*$-separable, then $A$ is small.

**Proof.** By our assumption, there exists a weak$^*$-dense subset $\{\varphi_n \mid n \in \mathbb{N}\}$ in $\mathcal{S}_A$. Let $\varphi = \sum_{n \geq 1} \frac{1}{n} \varphi_n$. Clearly $\varphi$ is a faithful state on $A$ and so $A$ must be $\sigma$-finite. Since there exists an orthogonal sequence $\{z_n \mid n \geq 1\}$ of central projections in $A$ such that $\sum_{n \geq 1} z_n = 1$ and for each $n \in \mathbb{N}$, $A z_n$ is $\ast$-isomorphic to $\mathcal{M}_{\ell_n}(\mathbb{Z}_{\ell_n})$ for some $\ell_n \in \mathbb{N}$ Here $\mathbb{Z}$ is the centre of $A$.

Take $n \in \mathbb{N}$. ByLemma 8, $\{\frac{\varphi_n(z_n)}{\varphi_m(z_n)} \mid \varphi_m(z_n) \neq 0\}$ is weak$^*$-dense in $\mathcal{S}_{A z_n}$. Since $\mathcal{M}_{\ell_n}(\mathbb{Z}_{\ell_n})$ has an abelian projection $e$ such that $e \mathcal{M}_{\ell_n}(\mathbb{Z}_{\ell_n}) e \cong \mathbb{Z}_{\ell_n}$, by Lemma 8, $\mathcal{S}_{\mathbb{Z}_{\ell_n}}$ is weak$^*$-separable. So also $\mathcal{S}_{\mathcal{M}_{\ell_n}(A z_n)}$ is weak$^*$-separable for every $p \in \mathbb{N}$ by Lemma 8, because $\mathcal{M}_{\ell_n}(A z_n) \cong \mathcal{M}_{\ell_n}(\mathbb{Z}_{\ell_n})$. Hence $A z_n$ is small for each $n \in \mathbb{N}$. So there exist a separable Hilbert space $\mathcal{H}_n$ and a unitary complete isometry $\Gamma_n : A z_n \hookrightarrow \mathcal{L}(\mathcal{H}_n)$. Let $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$. Clearly $\mathcal{H}$ is separable. Let

$$\Gamma(x) = \bigoplus_{n \in \mathbb{N}} \Gamma_n(z_nx) \quad (x \in A).$$

It is clear that $\Gamma$ is a unital CP-map from $A$ into $\mathcal{L}(\mathcal{H})$. We show that $\Gamma$ is a complete isometry. Take any $m \in \mathbb{N}$ and consider $\Gamma_m = \Gamma \otimes t_m$ where $t_m$ is the identity map of $\mathcal{M}_m(C)$ onto itself. Put $z_n = z_n \otimes 1_m$ for each $n \in \mathbb{N}$. Then, $\{z_n \mid n \in \mathbb{N}\}$ is an orthogonal sequence of central projections in $A \otimes \mathcal{M}_m(C)$ such that $\sum_{n \geq 1} z_n = 1$ and $\Gamma \otimes t_m(x) = \bigoplus_{n \geq 1} \Gamma_n(z_n \otimes t_m(z_nx))(x \in A \otimes \mathcal{M}_m(C))$. Since each $\Gamma_n \otimes t_m$ is an isometry, we have

$$\| \Gamma \otimes t_m(x) \| = \sup_{n \geq 1} \| \Gamma_n \otimes t_m(z_nx) \| = \sup_{n \geq 1} \| z_nx \| = \| x \|$$

for every $x \in A \otimes \mathcal{M}_m(C)$. Hence $\Gamma$ is a complete isometry. So, $A$ is small. □

**Corollary 12.** Let $A$ be a postliminary $C^*$-algebra (see [2] and [8]). If $A$ is almost separably representable, then it is small.

**Proof.** Let $A$ be an almost separably representable postliminary $C^*$-algebra. Since $A$ is almost separably representable, its regular $\sigma$-completions $\hat{A}$ has a faithful state (see [14]). Hence $\hat{A}$ is monotone complete (and so it is an $AW^*$-algebra). To show that $A$ is small, we have to show that $\hat{A}$ is of type I by the above lemma. For the case where $A$ is separable, see [9]. To show that $\hat{A}$ is of type I for the general $A$, it is sufficient to check that each direct summand of $\hat{A}$ has a non-zero abelian projection. To show this, take some non-zero central projection $p$ and put $\mathcal{I} = A \cap \hat{A}$. Then $\mathcal{I}$ is a closed two-sided ideal of $\hat{A}$ which is the kernel of the $*$-homomorphism $\pi : A \hookrightarrow \hat{A}(1 - z)$. We show that $\mathcal{I} \neq [0]$. If $\mathcal{I} = [0]$, then $\pi$ is a unital injective $*$-homomorphism which is the restriction of the $*$-homomorphism $\rho : \hat{A} \hookrightarrow \hat{A}(1 - z) \to A$. Take any self-adjoint element $b \in \rho^{-1}(0)$. Since $\mathcal{A}_{10}$ is order dense in $\hat{A}_{10}$, it follows that $b = \sup \{a \in \mathcal{A}_{10} \mid a \leq b\}$ in $\hat{A}_{10}$. So, $\rho(b) = 0$ implies $\pi(a) \leq 0$ for all $a \in \mathcal{A}_{10}$ with $a \leq b$. As $\pi$ is injective, this implies that $a \leq 0$ for all $a \in \mathcal{A}_{10}$ with $a \leq b$. Hence we have $b \leq 0$. Since $-b \in \rho^{-1}(0)$, we also have that $b \geq 0$ and so $b = 0$ follows. That is, $[0] = \rho^{-1}(0) = \hat{A}_{10}$ and hence $z = 0$. But this is a contradiction. Hence $\mathcal{I}$ is a non-zero closed two-sided ideal of $A$. Let $\{a_r\}$ be an increasing bounded approximate unit for $\mathcal{I}$. Since $\hat{A}$ is monotone complete, there exists a non-zero central projection $w$ in $\hat{A}$ with $w \leq z$ such
that the regular (\(\sigma\)-)completion of \(C^*(I, w)\) is nothing but \(\hat{A}w\). (See the proof of Theorem 2.1 in [9].) As \(C^*(I, w)\) is a postliminary \(C^*\)-algebra (see [2]) which is almost separably representable, to find out a non-zero abelian projection in \(\hat{A}z\) (and so in \(\hat{A}w\)), we may assume that \(z = 1\).

Since \(A\) is postliminary, there exists a non-zero positive element \(x \in A\) such that \(\partial xAx\) is commutative (see Proposition 4.3.4 in [2] and Theorem 6.2.6 in [8]). Since \(A\) is the monotone closure of \(A\), \(xAXx\) is also commutative. Indeed let \(A = \{ y \in A_{sa} | xyw = wxyx \text{ for all } w \in xA_{sa}\} \). Then it is clear that \(A\) is a real subspace of \(A_{sa}\) which contains \(A_{sa}\). To show that \(A\) is monotone closed, take any increasing net \(\{ y_j \} \) in \(A\) such that \(y_j \not\rightarrow y\) in \(A_{sa}\) for some \(y \in A_{sa}\). We show \(y \in A\). Since \(xy_jxw \rightharpoonup xywx \) and \(wxyx_jx \rightharpoonup wxxyx \in A\) in the order (see [5]), it follows that \(xywx = wxxyx\) for all \(w \in xA_{sa}\). Hence we have \(y \in A\). So, \(A\) is monotone closed in \(A_{sa}\). Since \(A_{sa}\) is the smallest monotone closed set which contains \(A_{sa}\), it follows that \(A_{sa} \subset A\). Hence, \(wxxywx = wxxyxwxyx\) for every pair \(y \in A_{sa}\) and \(w \in A_{sa}\). Similarly, we can show that \(wxxywx = wxxyx\) for all \(y, w \in A\). So, it follows that \(xAXx\) is commutative.

Since \(A\) is an \(AW^*\)-algebra, there exists a non-zero projection \(e \in A\) and \(y \in A_{sa}\) such that \(e = xy\). So, we have \(eAe \subset A\) and hence it follows that \(eAe\) is commutative, that is, \(e\) is non-zero abelian projection in \(A\). This means that \(A\) is of type I. Hence by Lemma 11, \(A\) is small. \(\Box\)

A proof of Theorem 7: By Lemmas 9 and 11, it follows that \(A_1 \oplus A_2 \oplus A_3 \oplus A_4\) is small. Since \(A_2\) is a von Neumann algebra of type II1 and \(\mathcal{G}_{A_2}\) is weak*-separable, by Akemann’s theorem [1], \(A_2\) has a faithful separable representation. Since any faithful \(\ast\)-representation is completely isometric, \(A_2\) is small. Hence \(A\) is small. This completes the proof.

**Corollary 13.** Let \(A\) be a \(C^*\)-algebra. Suppose that \(A_{sa}\) has a countable order dense subset. Then, \(A\) is small.

**Proof.** Since \(A_{sa}\) itself has a countable order dense subset, \(\partial y\mathcal{G}_{A}\) is weak*-separable. Since \(\mathcal{G}_{A}\) is the weak*-closed convex hull of \(\partial y\mathcal{G}_{A}\), \(\mathcal{G}_{A}\) itself is weak*-separable. Note that \(A\) has no type II1-direct summand, by Lemma 9, \(A\) is small. So is \(A\). \(\Box\)

**Corollary 14.** Let \(A\) be any prime \(C^*\)-algebra. Suppose that the closed unit ball of the dual is weak*-separable. Then \(A\) is small. In particular, every simple \(C^*\)-algebra with the weak*-separable dual unit ball is small.

**Proof.** Let \(\hat{A}\) be the regular \(\sigma\)-completion of \(A\). Since \(\mathcal{G}_{\hat{A}}\) is weak*-separable by Proposition 5, \(\hat{A}\) has a faithful state and hence it is monotone complete. So it is the regular completion of \(A\). Since \(A\) is prime, \(\hat{A}\) is an \(AW^*\)-factor with a faithful state and hence it is either finite or else infinite. By Wright’s theorem [12], if \(\hat{A}\) is a finite factor with a faithful state, then it is a von Neumann factor. Since \(\mathcal{G}_{\hat{A}}\) is weak*-separable, by [1] \(\hat{A}\) acts on a separable Hilbert space. Since any \(\ast\)-isomorphism is completely isometric, \(\hat{A}\) is small. So is \(A\).

On the other hand if \(\hat{A}\) is infinite, then by Lemma 9, \(\hat{A}\) is small. So is \(A\). \(\Box\)

4. Open question

The following question remains open. Let \(A\) be a unital \(C^*\)-algebra and let \(M_2(A)\) be the \(2 \times 2\) matrix algebra over \(A\).

If the closed unit ball of the dual \(A^*\) is weak*-separable, is the closed unit ball of the dual \(M_2(A)^*\) weak*-separable?

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**References**