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ELSEVIER
Journal of Computational and Applied Mathematics 189 (2006) 324–340
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Mathematics for generative processes: Living and non-living systems

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Received 23 September 2004; received in revised form 10 February 2005

Abstract

The traditional Differential Calculus often shows its limits when describing living systems. These in fact present such a richness of characteristics that are, in the majority of cases, much wider than the description capabilities of the usual differential equations. Such an aspect became particularly evident during the research (completed in 2001) for an appropriate formulation of Odum's *Maximum Em-Power Principle* (proposed by the Author as a possible *Fourth* Thermodynamic Principle). In fact, in such a context, the particular *non-conservative* Algebra, adopted to account for both Quality and quantity of generative processes, suggested we introduce a faithfully corresponding concept of "derivative" (of both integer and fractional order) to describe dynamic conditions however variable. The new concept not only succeeded in pointing out the corresponding differential bases of all the rules of Emergy Algebra, but also represented the preferential guide in order to recognize the most profound *physical nature* of the basic processes which mostly characterize self-organizing Systems (co-production, co-injection, inter-action, feed-back, splits, etc.).

From a mathematical point of view, the most important novelties introduced by such a new approach are: (i) the derivative of any integer or fractional order can be obtained *independently* from the evaluation of its lower order derivatives; (ii) the exponential function plays an extremely *hinge role*, much more marked than in the case of traditional differential equations; (iii) wide classes of differential equations, traditionally considered as being non-linear, become "intrinsically" linear when reconsidered in terms of "incipient" derivatives; (iv) their corresponding *explicit* solutions can be given in terms of new classes of functions (such as "binary" and "duet" functions); (v) every solution shows a sort of "persistence of form" when representing the product *generated* with respect to the agents of the *generating* process; (iv) and, at the same time, an intrinsic "genetic" *ordinality* which reflects the fact that any product "generated" is *something more* than the sum of the generating elements. Consequently all these

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doi:10.1016/j.cam.2005.03.032

properties enable us to follow the evolution of the “product” of any *generative* process from the very beginning, in its “rising”, in its “incipient” act of being born. This is why the new “operator” introduced, specifically apt when describing the above-mentioned aspects, was termed as “incipient” (or “spring”) derivative.

In addition, even if the considered approach was suggested by the analysis of *self-organizing living* Systems, some specific examples of non-living Systems will also be mentioned. In fact, what is much more surprising is that such an approach is even more valid (than the traditional one) to describe non-living Systems too. In fact the resulting “drift” between traditional solutions and “incipient” solutions led us to reconsider the phenomenon of Mercury’s precessions. The satisfactory agreement with the astronomical data suggested, as a consequential hypothesis, a different interpretation of its *physical origin*, substantially based on the *Maximum Em-Power Principle*.

Keywords: Self-organizing systems; Linear and non-linear differential equations; Integer and fractional “incipient” derivatives; Explicit solutions

1. Introduction—Thermodynamic context of the new linguistic-mathematical approach

From a conceptual point of view the paper can substantially be articulated in the following four aspects: (i) thermodynamic context of the *linguistic-mathematical* approach here proposed; (ii) introduction of the new concept of “incipient” derivative; (iii) its basic properties and related consequences in the field of *linear* and *non-linear* differential equations; (iv) applications to *living* and *non-living* processes.

As far as the first aspect is concerned, Fig. 1 schematically shows the well-known Thermodynamic Principles progressively discovered over the last two centuries, with some authors and dates of reference. As is

1st PRINCIPLE: Joule & Mayer (1841 -1848)	$dU = \xi Q \;\; \xi L$	*Conservation of ENERGY (En)*	*EXERGY (Ex)* Z. Rant (1955)
2nd PRINCIPLE: Carnot (1824)	$dS = \dfrac{\delta Q_{tot}}{T}$	*Increasing of ENTROPY (S)*	
3rd PRINCIPLE: Nernst (1906)	$\lim_{T \to 0} \Delta S = 0$	(Validity at low absolute Temperatures)	
		Onsager’s *Reciprocal Relations* (1931) Prigogine’s *Excess Entropy Production* (1971) Georgescou-Roegen’s *Matter Entropy* (1972)	
4th PRINCIPLE : Boltzmann - Lotka (1886 -1922)	**several proposals**	Odum’s *Maximum Em-Power Principle*	*EMERGY (Em)* H. Odum (1984)
		Ulanowicz’s *Maximum Ascendency* (1986, 1997) Jorgensen’s *Ecological Law of Thermodynamics* (1992, 2000) ………………………………………………	

Fig. 1. Principles of thermodynamics with some authors and dates of *reference*.[1]

[1] The formulation of the Maximum Em-Power Principle to which we are referring to, as far as the mathematical aspects we are concerned with, is given in [7,8].

well-known, the First Principle leads to the Conservation of Energy, whereas the Second Principle asserts the continuous increasing in Entropy. This latter quantity can more usefully be replaced, especially in *practical applications*, by the physical quantity named Exergy, whose definition can be obtained through a linear combination of the first two Principles. The Third Principle (1906), which has a great relevance only at very low absolute temperatures, completes Classical Thermodynamics which, however, had already reached its almost definitive and systematic theoretical structure 30 years before (in the 1870s).

The possibility of an ulterior *Fourth* Thermodynamic Principle emerged soon after this period, especially as a consequence of the first problematic aspects which appeared in the application of the first two Principles to the analysis of *Biological Systems*.

Hence it was immediately clear that those Principles (although *globally* valid even in the case of living systems) could not be considered as being Laws sufficient to explain, by themselves, how and why organisms develop through *self-organization* processes, during which they lose Entropy (by increasing their own order), in open contrast with the surrounding universe. Those Principles in actual fact are only able to "tell us that certain things cannot happen, but they do not tell us what does happen" (Lotka) [13].

Boltzmann (1886) first had the original idea of looking for a direct relationship between Classical Thermodynamics and the Evolutionary Theory of the organic world [1]. Lotka in 1922 reconsidered Boltzmann's initial ideas and, on the basis of a thorough analysis of wide classes of living systems, formulated the *Maximum Power Principle* and contemporaneously proposed that it was the Fourth Thermodynamic Principle [13]. Subsequently Odum, in the early 1990s, after having introduced the new physical quantity termed Emergy [14], gave a more general formulation of Lotka's Principle in the form of the "*Maximum Em-Power Principle*" [15,16].[2] This Principle asserts that "*Every System tends to maximize the Flow of Processed Emergy*", where Emergy (contraction of the words *Em*bodied En*ergy*) is defined as the product of Energy *Quality* (expressed by *Transformity*) by Energy *Quantity* (expressed by *Exergy*).

In order to account for Energy Quality or, better, for the Transformity associated to any form of Exergy, some *special* algebraic rules have to be taken into account. Such rules, illustrated in detail in [2], refer to the basic processes (co-production, co-injection, inter-action, feed-back, splits, etc.) which characterize living systems and, in so doing, give rise to a *non-conservative* Algebra. This particular aspect is the one that stimulated a *new mathematical approach* to the Differential Calculus (both of integer and fractional order). In fact, the research for a Mathematical Formulation of the Maximum Em-Power Principle (faithfully respectful of the verbal enunciation mentioned above) required, as a preliminary step, the statement of an *Emergy Balance Equation*, under *dynamic conditions*, in perfect adherence to the rules of Emergy Algebra (valid in steady state or stationary conditions).

The first approach to the problem was based on the traditional *Fractional Calculus* [17]. This, in turn, led to the introduction of a *new concept* of *Intensive* Fractional Derivative [6]. Finally, the Mathematical

[2] Parallel to this line of thought (see Fig. 1), some other scientists discovered new Thermodynamic aspects and systematically proposed them as an expression of a "Fourth" Thermodynamic Principle. In fact "*Reciprocal Relations*", discovered by Onsager in 1931 [18], represented a substantial novelty with respect to Classical Thermodynamics. The same can be said with reference to "*Excess Entropy Production Principle*" discovered by Prigogine in 1971 [19]. Even from Economics a new Fourth Principle was suggested: the "*Matter Entropy Principle*", proposed by Georgescou–Roegen in 1972 [20]. More recently another "tentative Fourth Thermodynamic Principle" was proposed by Jorgensen in 1992, termed as "*the Ecological Law of Thermodynamics*" [9]. A wide analysis of these and other proposals (such as, for instance, Ulanowicz's *Maximum Ascendency* [21,22]) is given by Jorgensen et al. [10]. See also Ulgiati et al. [23].

Formulation of the Maximum Em-Power Principle [4,7] was achieved on the basis of a *generalized version* of the previous concept, now renamed as "Incipient Derivative" and illustrated here below.

2. Definition of "*incipient*" derivative

The definition of the "incipient" derivative is based on the *direct priority* of the sequence of the three operators which appear in the traditional definition of the derivative.

Such a *different perspective* starts from the consideration of the fact that the traditional definition of the derivative of a function $f(t)$ given in Mathematical Analysis

$$\lim_{\Delta t \to 0} \frac{\Delta}{\Delta t} f(t) \tag{2.1}$$

may be considered as being an "a posteriori" definition (e.g., let us think of the definition of velocity). In fact, although it is usually read from left to right, it is vice versa interpreted from right to left. In other words its meaning is based on a *reverse priority* of the order of the three elements that constitute its definition: (i) the concept of function (which is assumed to be a primary concept); (ii) the incremental ratio (of the supposedly known function); (iii) the operation of limit (referred to the result of the previous two steps).

Now we may ask: what happens if we interpret the sequence of symbols in expression (2.1) according to the same order as they are written (that is from left to right)?

Such a *direct perspective* gives rise to a new concept of derivative, indicated by $\frac{\tilde{d}}{\tilde{d}t}$ and defined as follows (for further details see also Appendix A):

$$\frac{\tilde{d}^q}{\tilde{d}t^q} f(t) = \tilde{\lim_{\tilde{\Delta}t:0 \to 0^+}} \widetilde{\left(\frac{\tilde{\delta} - 1}{\tilde{\Delta}t} \right)}^q f(t) \quad \text{for any } q \in Q, \tag{2.2}$$

where the sequence of symbols is interpreted according to the same order as they are written (from left to right). It can be named "incipient" because of some special characteristics that will be illustrated later on (especially through the derivatives of the exponential function $e^{\varphi(t)}$).

3. Basic properties and main consequences of the previous concept

Let us now examine the basic properties of such a *new concept* of derivative and its related consequences on *linear* and *non-linear* differential equations.

3.1. Incipient derivatives of integer order

(i) The integer-order incipient derivatives of the exponential function present a *persistence of form*: Each derivative maintains the same exponential structure (apart from an amplification factor)

$$\frac{\tilde{d}^n}{\tilde{d}t^n} e^{\varphi(t)} = \widetilde{\left(\frac{\tilde{d}}{\tilde{d}t} \varphi(t) \right)}^n e^{\varphi(t)} = (\tilde{\varphi}')^n e^{\varphi(t)}. \tag{3.1}$$

Such a property is particularly useful in modeling genetic processes. In fact living systems always show a "generative" *activity*, in which the generated "product" presents a *persistence of the form* with respect to its ancestors' genetic characteristics (although the generated being contemporaneously constitutes something new);

(ii) The same happens for *any* function $f(t)$ when represented in the exponential form [8]

$$\frac{\tilde{d}^n}{\tilde{d}t^n} f(t) = \frac{\tilde{d}^n}{\tilde{d}t^n} e^{\ln f(t)} = \left(\frac{\widetilde{\tilde{f}'(t)}}{f(t)} \right)^n \quad e^{\ln f(t)} = \left(\frac{\widetilde{\tilde{f}'(t)}}{f(t)} \right) f(t) = (\tilde{\beta}_f(t))^n f(t). \tag{3.2}$$

3.2. Main consequences on LDEs of integer order

Linear Differential Equations of order $n \leqslant 4$, even if with *variable* coefficients, *always* have explicit solutions.[3]

As an example we can consider a second order LDE with variable coefficients

$$\frac{\tilde{d}^2}{\tilde{d}t^2} f(t) + a_1(t) \frac{\tilde{d}}{\tilde{d}t} f(t) + a_0(t) f(t) = 0 \tag{3.3}$$

with its well-posed initial conditions

$$f^{(k)}(0) = f_{k0} \quad \text{for } k = 0, 1, \tag{3.4}$$

whose solution can be given in *finite terms and quadratures*. In fact, if we assume $f(t) = e^{\varphi(t)}$, we get an algebraic equation

$$(\tilde{\varphi}'(t))^2 + a_1(t)(\tilde{\varphi}'(t)) + a_0(t) = 0, \tag{3.5}$$

whose solutions $\tilde{\alpha}_i(t)$ $(i = 1, 2)$ enable us to write the explicit solution to Eq. (3.3) in the form

$$\tilde{f}(t) = \sum_{i=1}^{2} c_i e^{\int_0^{\overset{1}{\tilde{t}}} \tilde{\alpha}_i(u) \tilde{d}u}, \tag{3.6}$$

where the constant coefficients c_i are defined by means of the initial conditions (3.4).

If the solution $\tilde{\alpha}_1(t)$ has a multiplicity $v_1 = 2$, an additional independent solution is given by

$$\tilde{y}_2(t) = \int_0^{\overset{1}{\tilde{t}}} e^{\int_\xi^{\overset{1}{\tilde{t}}} \tilde{\alpha}_i(u) \tilde{d}u} \tilde{d}\xi. \tag{3.7}$$

[3] Such solutions can be easily obtained on the basis of the well-known algebraic methods. For $n > 4$ they can be researched for by means of *non-strictly* algebraic methods.

3.3. Basic properties of the incipient derivatives of fractional order

The fractional-order incipient derivative offers a multiplicity of results

$$\frac{\tilde{\mathrm{d}}^{m/n}}{\tilde{\mathrm{d}}t^{m/n}}\mathrm{e}^{\varphi(t)} = (\tilde{\varphi}')^{m/n}\mathrm{e}^{\varphi(t)} \tag{3.8}$$

according to the *multiplicity* of the *roots* of a complex number. For example

$$\frac{\tilde{\mathrm{d}}^{1/2}}{\tilde{\mathrm{d}}t^{1/2}}\mathrm{e}^{\alpha t} = \alpha^{1/2}\mathrm{e}^{\alpha t} = \pm\sqrt{\alpha}\mathrm{e}^{\alpha t}. \tag{3.9}$$

Such a property is particularly useful to describe some living systems. For instance, the topological distribution of the petals of a daisy (usually 12 in number) and the regular growth of their parts in accordance with Fibonacci's series (at discrete times) or better, at any time, according to a logarithmical spiral.

3.4. Main consequences on fractional LDEs

When introduced into linear differential equations, fractional incipient derivatives give rise to a new kind of functions: the "binary" functions equations [5,6,8].

Let us consider, for example, the following linear differential equation of the first order with constant coefficients

$$\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t}f(t) + A\frac{\tilde{\mathrm{d}}^{1/2}}{\tilde{\mathrm{d}}t^{1/2}}f(t) + Bf(t) = 0, \tag{3.10}$$

which contains a derivative of order $\frac{1}{2}$ in addition to the traditional derivative of order one.

If we take into account that the fractional derivative of order $\frac{1}{2}$ of the exponential function $\mathrm{e}^{\alpha t}$ has *two distinct values* defined by Eq. (3.9), it is possible to express the general solution to Eq. (3.10) by means of *two distinct* functions:

$$f_1(t) = c_{11}\mathrm{e}^{\alpha_{11}^2 t} + c_{12}\mathrm{e}^{\alpha_{12}^2 t}, \tag{3.11}$$

$$f_2(t) = c_{21}\mathrm{e}^{\alpha_{21}^2 t} + c_{22}\mathrm{e}^{\alpha_{22}^2 t}, \tag{3.12}$$

where the function $f_1(t)$ is carried out by assuming the structure $+\sqrt{\alpha}\mathrm{e}^{\alpha t}$, whereas $f_2(t)$ corresponds to the structure $-\sqrt{\alpha}\mathrm{e}^{\alpha t}$, while the pertinent exponential coefficients are derived from the *two* associated characteristic equations [5,6,8]. We may now observe that, although the derivative $(\tilde{\mathrm{d}}/\tilde{\mathrm{d}}t)^{1/2}f(t)$ presents *two distinct values,* it conceptually constitutes *one sole entity*. Consequently the general solution $\tilde{f}(t)$ may be written in a *compact form* as follows:

$$\tilde{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{22} \end{pmatrix}\mathrm{e}^{\begin{pmatrix} \alpha_{11} \\ \alpha_{22} \end{pmatrix}^2 t} + \begin{pmatrix} c_{12} \\ c_{21} \end{pmatrix}\mathrm{e}^{\begin{pmatrix} \alpha_{12} \\ \alpha_{21} \end{pmatrix}^2 t}, \tag{3.13}$$

where the upper and lower exponents α_{ij} are the pertinent solutions to the two associated characteristic equations respectively,y, while the coefficients c_{ij} depend on the initial conditions

$$\tilde{f}(0) = \begin{pmatrix} f_{10} \\ f_{20} \end{pmatrix}, \tag{3.14}$$

and

$$\tilde{f}^{(1/2)}(0) = \begin{pmatrix} f_{10}^{(1/2)} \\ f_{20}^{(1/2)} \end{pmatrix}. \tag{3.15}$$

The *comprehensive* solution $\tilde{f}(t)$ may be thus termed as a "binary" function, not only because it is made up of *two* distinct functions, but also (and especially) because the two components are so strictly related that they form *one sole entity*.

3.5. Main consequences on NLDEs of integer order

These equations are particularly frequent when modeling *living* Systems. Their analysis (in terms of incipient derivatives) started from Riccati's equation because this represents the most elementary nonlinear equation in the field of *self-organization* processes (in fact it models an interaction with feedback):

$$\frac{\mathrm{d}f}{\mathrm{d}t} + Q(t)f(t) + R(t)f^2(t) = P(t). \tag{3.16}$$

In this respect, it is well-known that the substitution

$$y(t) = \frac{1}{f(t)R(t)}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{3.17}$$

transforms such a nonlinear equation into a second-order *linear* differential equation with variable coefficients [3]:

$$R\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) - (R' - QR)y(t) - PR^2y(t) = 0. \tag{3.18}$$

Consequently, if the latter is interpreted in terms of *incipient* derivatives, it presents an *explicit solution* in *finite terms and quadratures* (see par. 3.2).

However it is worth pointing out that, when Eq. (3.16) is *immediately* and *directly* interpreted in terms of *incipient* derivatives

$$\frac{\tilde{\mathrm{d}}f}{\tilde{\mathrm{d}}t} + Q(t)f(t) + R(t)f^2(t) = P(t) \tag{3.19}$$

it presents a solution in the form of a "duet" function [8]. This new type of function can be represented as follows

$$f(t) = f_0[\beta_1^{2/2}(t), \beta_2^{2/2}(t)], \tag{3.20}$$

(where f_0 is the initial condition) to indicate that the two distinct solutions $\beta_1^{2/2}(t)$ and $\beta_2^{2/2}(t)$ are joined together in such a way as to form a *new sole entity* of *superior order* (or superior hierarchical structure). In fact the substitution

$$f = \left(\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t}\right)^{2/2} F \cdot \frac{1}{F} \tag{3.21}$$

transforms Eq. (3.19) into an algebraic equation

$$\frac{5+Q(t)}{4}[\beta^{2/2}(t)]^2 + P(t)[\beta^{2/2}(t)] + R(t) = 0, \tag{3.22}$$

where

$$\beta = \frac{1}{F}\frac{\tilde{\mathrm{d}}F}{\tilde{\mathrm{d}}t}, \tag{3.23}$$

and

$$\beta^{2/2} = \left[\begin{pmatrix} +\sqrt{\beta} \\ -\sqrt{\beta} \end{pmatrix}, \begin{pmatrix} +\sqrt{\beta} \\ -\sqrt{\beta} \end{pmatrix}\right]. \tag{3.24}$$

Similar considerations can be extended to other non-linear differential equations generally adopted in modeling *self-organizing* processes. For example, Abel's equations of any order (which model multiple interactions and feedbacks)

$$\frac{\mathrm{d}f}{\mathrm{d}t} + Q(t)f(t) + R(t)f^2(t) + S(t)f^3(t) = P(t), \tag{3.25}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t}f(t) = P_n[t, f(t)] = \sum_{k=0}^{n} A_k(t)[f(t)]^k. \tag{3.26}$$

They can analogously be either *preliminarily* reduced to a linear form [3] (and then interpreted in terms of incipient derivatives) or *directly* understood in terms of *incipient* derivatives. In the latter case they present solutions in the form of "n-et" functions [8]:

$$\tilde{f}(t) = f_0[\beta_1^{n/n}(t), \beta_2^{n/n}(t), \ldots, \beta_n^{n/n}(t)]. \tag{3.27}$$

Such results could also be progressively generalized to more complex non-linear differential equations. However, at this stage, it is of fundamental importance to underline a particular phenomenon: the reciprocal "drift" (between traditional and "incipient" solutions) which be analyzed in the next paragraph.

4. The "drift" phenomenon

For the sake of clarity such a fundamental aspect will be analyzed with reference to LDE of integer order. In such a case the solutions obtained by means of the *incipient* differential approach coincide exactly with the traditional solutions *only when the latter are available in finite terms and quadratures*, that is: (i) in the case of linear differential equations (of *any* integer order) with *constant coefficients*; (ii) *first order* linear differential equations with *variable coefficients*. *In all the other cases* the traditional solutions which are obtained, for instance, through expansion series, gradually *differ* from the explicit solutions obtained in terms of incipient derivatives (and vice versa). In other words *traditional solutions* present *a sort of a* "*drift*" which is even more marked according to the increasing order of the involved derivatives. This is evidently due to the differences in the derivatives of order $n > 1$, as schematically shown in Table 1, where the traditional derivative of order n is represented by the well-known Faà di Bruno formula [17].

Such a particular aspect led us to think about some unsolved problems in the past which could possibly be interpreted on the basis of such a "drift" effect due to the different derivatives adopted in modeling

Table 1
Comparison between traditional and incipient derivatives of the exponential function $e^{\varphi(t)}$

1st order	$\frac{d}{dt}e^{\varphi(t)} = \varphi' e^{\varphi(t)}$	$\frac{\tilde{d}}{\tilde{d}t}e^{\varphi(t)} = \tilde{\varphi}' e^{\varphi(t)}$
2nd order	$\frac{d^2}{dt^2}e^{\varphi(t)} = (\varphi')^2 e^{\varphi(t)} + \varphi'' e^{\varphi(t)}$	$\frac{\tilde{d}^2}{\tilde{d}t^2}e^{\varphi(t)} = (\tilde{\varphi}')^2 e^{\varphi(t)}$
	...	...
nth order	$\frac{d^n}{dt^n}e^{\varphi(t)} = n! \sum_{m=1}^{n} (\varphi')^m e^{\varphi(t)} \sum \prod_{k=1}^{n} \frac{1}{P_k!}\left[\frac{\varphi^{(k)}}{k!}\right]^{P_k}$	$\frac{\tilde{d}^n}{\tilde{d}t}e^{\varphi(t)} = (\tilde{\varphi}')^n e^{\varphi(t)}$

the *same* physical system. One of those is the well-known problem of Celestial Mechanics represented by the *precession of the planets* (especially Mercury), which, as we will see later on, could be of great relevance for Classical Mechanics and its fundamental Laws.

5. Application of incipient derivatives to Classical Mechanics

The application of the incipient derivatives to the analysis of Mercury's precessions was also stimulated by the fact that the same Maximum Em-Power Principle suggests we reconsider the Fundamental Laws of Classical Mechanics in terms of *incipient derivatives* [8]:

$$\vec{F} = \frac{\tilde{d}}{\tilde{d}t}\vec{p}, \tag{5.1}$$

$$\vec{M} = \frac{\tilde{d}}{\tilde{d}t}\vec{b}. \tag{5.2}$$

Such a reformulation allows much wider degrees of freedom in the motion of the mechanical system analyzed because of the fact that the condition

$$\frac{\tilde{d}}{\tilde{d}t} f(t) = 0 \tag{5.3}$$

formulated in terms of incipient derivatives, is *not*, *by itself*, *a sufficient condition* for asserting that

$$f(t) = \text{const} \tag{5.4}$$

(as in the case of traditional derivatives). This allows us to consider, in principle, *two distinct* cases when analyzing the precessions of the planets.

5.1. *First case: assumption of $\vec{b} = const$ (planar orbits)*

Under this hypothesis, the *classical* variation ($\Delta\varphi_c$) of the angular anomaly per each revolution is given, according to Newton's Laws, by the following expression [12]

$$\Delta\varphi_c = 2\int_{r_{\min}}^{r_{\max}} \frac{(b/r^2)\,dr}{\sqrt{2m(E-U)-(b/r)^2}}, \tag{5.5}$$

which, however, gives rise to a global effect which vanishes after an integer number of revolutions because $\Delta\varphi_c$ is always a rational fraction of 2π. This is due to the fact that the central force field is characterized by a potential energy which is proportional to $1/r$ (ib.).

If, vice versa, we start our analysis by taking the *incipient derivative* of the angular momentum

$$b = mr^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \text{const}, \tag{5.6}$$

we get the following differential equation in the unknown angular anomaly

$$\frac{\tilde{\mathrm{d}}^2\varphi}{\tilde{\mathrm{d}}t^2} + \frac{2}{r}\frac{\tilde{\mathrm{d}}r}{\tilde{\mathrm{d}}t}\frac{\tilde{\mathrm{d}}\varphi}{\tilde{\mathrm{d}}t} = 0, \tag{5.7}$$

whose integration will give the *incipient* angular variation $\Delta\tilde{\varphi}_{\text{inc}}$ (per each revolution).

In order to facilitate the comparison between the latter and the classical variation $\Delta\varphi_c$, Eq. (5.7) can be very well approximated and usefully restructured in two "branches" as follows:

$$\frac{\tilde{\mathrm{d}}^2\varphi}{\tilde{\mathrm{d}}t^2} \pm \chi\frac{b}{mr^2}\frac{\tilde{\mathrm{d}}\varphi}{\tilde{\mathrm{d}}t} = 0, \tag{5.8}$$

where:

(i) the signs $\pm$ account for the fact that $(\mathrm{d}r/\mathrm{d}t) \geqslant 0$ when $r(t)$ increases from $r_{\min}$ (perihelion) to $r_{\max}$ (aphelion), whereas $(\tilde{\mathrm{d}}r/\tilde{\mathrm{d}}t) \leqslant 0$ when $r(t)$ decreases from $r_{\max}$ to $r_{\min}$;

(ii) the term

$$2(\tilde{\mathrm{d}}r/\tilde{\mathrm{d}}t)/r \tag{5.9}$$

is preliminarily expressed by means of the relationship $r(t) = p/[1 + \mathrm{e}\cos\varphi(t)]$, which is valid in the case of an elliptic orbit;

(iii) the coefficient χ is the mean value of term (5.9) in the range $[r_{\min}, r_{\max}]$:

$$\chi \cong \frac{2}{\pi}\frac{r_{\max} - r_{\min}}{r_{\min}}. \tag{5.10}$$

The explicit solutions to Eqs. (5.8) (with their pertinent initial conditions) enable us to express the *incipient* variations of the angular anomaly, in the two considered parts of the orbit, as follows

$$\Delta\tilde{\varphi}_1 = \frac{1}{\chi}[1 - \mathrm{e}^{-\chi\frac{\Delta\varphi_c}{2}}] \quad \text{for the branch from } r_{\min} \text{ to } r_{\max} \tag{5.11}$$

and

$$\Delta\tilde{\varphi}_2(t) = \frac{1}{\chi}\mathrm{e}^{\chi\frac{\Delta\varphi_c}{2}}[\mathrm{e}^{\chi\frac{\Delta\varphi_c}{2}} - 1] \quad \text{for the branch from } r_{\max} \text{ to } r_{\min}. \tag{5.12}$$

Expressions (5.11) and (5.12) can be easily obtained by making use of the relationship

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \sqrt{\frac{2}{m}[E - U(r)] - \frac{b^2}{m^2 r^2}}, \tag{5.13}$$

which is a direct consequence of the Energy Conservation Principle.[4]

The total incipient variation $\Delta\tilde{\varphi}_{\text{inc}}$ (per each revolution) can be then expressed as follows:

$$\Delta\tilde{\varphi}_{\text{inc}} = \Delta\tilde{\varphi}_1 + \Delta\tilde{\varphi}_2 = \frac{1}{\chi}[1 - \mathrm{e}^{-\chi\frac{\Delta\varphi_c}{2}}] + \frac{1}{\chi}\mathrm{e}^{\chi\frac{\Delta\varphi_c}{2}}[\mathrm{e}^{\chi\frac{\Delta\varphi_c}{2}} - 1] \tag{5.14}$$

and, consequently, the contribution to the secular variation $\Delta\varphi_{\text{sec}}$ (per each revolution) is

$$\Delta\varphi_{\text{sec}} = \tilde{\Delta}\tilde{\varphi}_{\text{inc}} - \Delta\varphi_c = \psi\left(\frac{\Delta\varphi_c}{2}\right) + \psi\frac{\chi}{2!}\left(\frac{\Delta\varphi_c}{2}\right)^2 + (2+\psi)\frac{\chi^2}{3!}\left(\frac{\Delta\varphi_c}{2}\right)^3 + \cdots, \tag{5.15}$$

where

$$\psi = \mathrm{e}^{\chi\frac{\Delta\varphi_c}{2}} - 1 = \chi\left(\frac{\Delta\varphi_c}{2}\right) + \frac{\chi^2}{2!}\left(\frac{\Delta\varphi_c}{2}\right)^2 + \cdots. \tag{5.16}$$

In the case of Mercury, the pertinent value of $\chi \cong 0.332$ (see Eq. (5.10)) leads (through Eqs. (5.15) and (5.16)) to a secular precession of $42.45''$ per century.

The result obtained is in almost perfect agreement with the most recent available data.[5]

This shows that Newton's Laws, when reinterpreted in terms of *incipient* derivatives, can *still* be considered as being substantially *adequate* to describe even such an effect, which has always remained inexplicable in terms of Classical Mechanics (whose basic Laws were, and still are, formulated in terms of traditional derivatives).

On the other hand, the explanation of the same effect given by General Relativity (which still makes use of *a posteriori* derivatives as a basic language) was made possible only on the hypothesis of a *completely different physical* model.

5.2. *Second case: assumption of* $\vec{b} \neq \text{const}$

Such a hypothesis is perfectly compatible with the basic equations of "Incipient" Mechanics (5.1) and (5.2), now rewritten in the form

$$\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t}\vec{p} = -\frac{\mathrm{d}U}{\mathrm{d}r}\frac{\vec{r}}{r}, \tag{5.17}$$

$$\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t}\vec{b} = 0. \tag{5.18}$$

[4] In Eq. (5.13) (as well as in Eq. (3.9)) the "tilde" notation was omitted only for the sake of simplicity. On the other hand the *first order* incipient derivatives, differentials and integrals (see also Eqs. (3.6) and (3.7)), which appear in the above-mentioned equations, quantitatively coincide with the corresponding traditional concepts.

[5] Astronomical measurements give $42.6'' \pm 0.9''$ per century [11]. The value predicted by Relativity Theory is $43.0''$ per century [ib.].

In particular it is compatible with Eq. (5.18), as a consequence of the considerations made at the end of par. 5.

If we then integrate Eqs. (5.17) and (5.18), we obtain the following main results: (i) there exists a multiplicity of motions able to satisfy both Eqs. (5.17) and (5.18); (ii) among the various possibilities, we searched for a possibly *stable* gyroscopic motion of Mercury's orbital plane; (iii) such a motion, under particular hypotheses (specified immediately below), results as being perfectly compatible with the basic equations adopted.

In fact, by expressing Eq. (5.18) in its three fundamental components in an *xyz*-space, we get the two following equations:

$$\frac{\tilde{d}^2 \widetilde{\varphi_x}}{\tilde{d}t^2} + \frac{2}{r}\frac{\tilde{d}r}{\tilde{d}t}\frac{\tilde{d}\widetilde{\varphi_x}}{\tilde{d}t} = 0, \tag{5.19}$$

$$\frac{\tilde{d}^2 \widetilde{\varphi_y}}{\tilde{d}t^2} + \frac{2}{r}\frac{\tilde{d}r}{\tilde{d}t}\frac{\tilde{d}\widetilde{\varphi_y}}{\tilde{d}t} = 0 \tag{5.20}$$

in addition to the traditional Eq. (5.7) (now characterized by a specific pedix z)

$$\frac{\tilde{d}^2 \widetilde{\varphi_z}}{\tilde{d}t^2} + \frac{2}{r}\frac{\tilde{d}r}{\tilde{d}t}\frac{\tilde{d}\widetilde{\varphi_z}}{\tilde{d}t} = 0. \tag{5.21}$$

If we now assume that

$$\varepsilon_x = \frac{b_x}{b_z} \ll 1, \tag{5.22}$$

$$\varepsilon_x = \frac{b_x}{b_z} \ll 1 \tag{5.23}$$

and integrate Eqs. (5.19) and (5.20) according to the same procedure adopted in the case of Eq. (5.7), we find a gyroscopic motion in which the angle between the vector $\vec{b}$ and the axis z is given by

$$\theta_{n+1} \cong \theta_n (1 + \beta_n), \tag{5.24}$$

where

$$\beta_n = \frac{k\chi_n \left(\frac{\Delta\phi_c}{2}\right)_n [\gamma_{x,n}^2 \varepsilon_{x,n}^2 b_{x,n} + \gamma_{y,n}^2 \varepsilon_{x,n}^2 b_{y,n}]}{b_{z,n}^2}, \tag{5.25}$$

n is the number of centuries; k is a constant factor to convert the quantity $(\frac{\Delta\phi_c}{2})_n$ into the corresponding angular momentum; while $\gamma_{x,n}$ and $\gamma_{x,n}$, explicitly defined as

$$\gamma_{x,n} = \left(\frac{\bar{r}_{z,\min}^2}{r_{x,\min}^2}\right)_n, \tag{5.26}$$

$$\gamma_{y,n} = \left(\frac{\overline{r^2_{z,\min}}}{r^2_{y,\min}} \right)_n , \tag{5.27}$$

represent two mean form factors which accounts for the non-perfect similarity between the elliptic orbits orthogonal to the axes x and y with respect to that orthogonal to the axis z.

The structure of Eq. (5.25) allows us to assume, as a first approximation,

$$\beta_n \cong \beta_i \cong \beta_0 = \text{const} \tag{5.28}$$

and, consequently, Eq. (5.24) can be rewritten as follows:

$$\theta_{n+1} \cong \theta_0 (1 + \beta_0)^n. \tag{5.29}$$

The estimated magnitude of β_0 ($\cong 5 \times 10^{-7}$) is so small that, over a period of 2 or 3 centuries (that is from Newton's time to nowadays), the width of the gyroscopic cone can be considered as having remained practically constant. In this sense the gyroscopic motion can be considered as being not only *stable*, but also *stationary*. Over a medium period (from 10 to 20 centuries) Eq. (5.29) can be approximated by a linear trend

$$\theta_{n+1} \cong \theta_0 (1 + n\beta_0). \tag{5.30}$$

This means that the gyroscopic motion can still be considered as being substantially *stable*, but *not strictly stationary*. Over hundreds of millennia Eq. (5.29), now rewritten as

$$\theta_{n+1} \cong \theta_0 (1 + \beta_0)^n = \theta_0 \, e^{n \ln(1+\beta_0)} \cong \theta_0 e^{n\beta_0} \tag{5.31}$$

can be restructured in such a way as to show that the vertex of the vector $\vec{b}$ describes a spiraloid trajectory whose projection on the *xy*-plane is very well approximated by a logarithmical spiral.

In reality the latter result shows its relevance much more in terms of a *qualitative* trend than as a rigorously quantitative approximation. In fact Eq. (5.31) is only valid as a first order approximation and, in addition, has been carried out by modeling the considered planet as a single "material point", which interacts with the Sun in terms of mere *functional* relationships. This suggests we look for a more adequate model which could be even more valid over such long periods.

In this respect, a much more *adherent* solution to the indications of the Maximum Em-Power Principle resides on the possibility of interpreting the system made up of the Sun and Mercury as a *unique entity* (that is as a "binary system"). In such a case Eqs. (5.17) and (5.18) should be rewritten as follows:

$$\left(\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t} \right)^{\frac{2}{2}} \vec{p} + F \left[\left(\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}r} \right)^{\frac{2}{2}} \frac{\alpha}{r} \frac{\vec{r}}{r} \right] = 0, \tag{5.32}$$

$$\left(\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t} \right)^{\frac{2}{2}} \vec{b} = 0, \tag{5.33}$$

where in Eq. (5.32) the potential energy $U(r)$ is replaced by a *reciprocity action* expressed in *binary terms*.

In this paper we will not deal with the solution to Eqs. (5.32) and (5.33). These have only been mentioned here to show the possible *progressive development* of even more adherent models, formulated in terms of incipient derivatives, as a consequence of the increasingly wider potentialities of description offered by the latter. However, we can anticipate that Eqs. (5.32) and (5.33) are also satisfied by a gyroscopic motion but, in such a case, the “binary” initial conditions pertaining to the *two bodies* (the Sun and Mercury) show that the latter are so strictly related to each other that they seem to form a real *unique entity*. Consequently, even in this case, Mercury could possibly have a gyroscopic motion. This is especially due to the fact that both planar orbits and gyroscopic orbits are always compatible (in principle) with *Incipient Mechanics*.

6. Conclusions

The introduction of the concept of *incipient derivative* was mainly finalized to transform the enunciation of the *Maximum Em-Power Principle* into an adherent *Mathematical Formulation*. Subsequently, several applications showed that such a new mathematical concept seemed to be particularly indicated to model *self-organizing* (living) Systems. These results contemporaneously showed why the *basic* presuppositions of the traditional mathematical approach to the analysis of the same systems are rather restrictive and somewhat reductive.

In particular, the consideration of differential equations of *only* integer order (generally adopted to describe such systems) is rather limiting with respect to the much wider variety of their biodiversity characteristics. Vice versa, the introduction of the concept of *incipient* differential equations (of *integer* and *fractional* order) could represent a valid approach to describe and analyze the *spring-dynamism* of such systems. In fact, we have shown that: (i) the incipient derivative presents a *persistence of form* which is particularly indicated to describe a dynamic evolution in which every “product” generated, besides representing a substantial novelty, is always “faithful” to the presuppositions of its generation. That is, it is always in *consonance* with its “ancestors’ genetic characteristics”; (ii) on the other hand, *fractional* derivatives (and their associated differential equations) generate a new class of functions (“binary”, “ternary” functions, and so on) that are able to describe the new reality generated by a given process as being *one sole entity*; (iii) some non-linear differential equations (usually adopted to describe self-organizing systems), such as Riccati’s and Abel’s equations, when re-interpreted in terms of *incipient* derivatives, present *explicit* solutions in terms of “duet” functions, which are able to represent the “product” generated as a *unique* entity of *superior order* (or superior hierarchical structure); (iv) above all, when incipient differential equations have *explicit* solutions in *finite terms and quadratures*, these are not affected by that “drift” phenomenon which characterizes the traditional *a posteriori* derivatives; (v) this particular aspect allowed us to reconsider more clearly a well-known problem of Classical Mechanics (Mercury’s *precessions*); (vi) and also led us to foresee its *potential orbital gyroscopic effects* (never hypothesized before).

Considering such results, what is much more surprising in such an approach is especially the fact that it is not only more adequate to describe *living* systems, but even more valid to describe *non-living* systems too.

In this respect it is worth pointing out that the previous results do not lead us to assert that Mercury really has a gyroscopic motion. Such an assertion, in fact, requires some additional information which is, at the moment, unknown. The results only allow us to assert that a gyroscopic motion of Mercury is surely compatible with the basic equations of *Incipient Mechanics* and thus, in actual fact, it could even exist. In all cases, the effective *ascertainment* of such a possibility can be only made on the basis of more accurate astronomical data which, however, should not necessarily be affected by the preliminary assumption of planar revolution motions of the planets (as cogently established by Classical Mechanics as well as by Relativistic Mechanics).

Appendix A. Incipient derivative of integer and fractional order

The definition of the incipient derivative, first referred to any integer n, is given by (see par. 2 and also [8])

$$\frac{\tilde{d}^n}{\tilde{d}t^n} f(t) = \tilde{\lim_{\tilde{\Delta}t:0\to 0^+}} \left(\frac{\widetilde{\tilde{\delta} - 1}}{\tilde{\Delta}t} \right)^n f(t), \tag{A.1}$$

where the symbol $\tilde{\delta}$ represents an "operator" that *generates* a translation of a function, that is

$$\tilde{\delta} f(t) = f(t + \Delta t), \tag{A.2}$$

which has the following characteristics: (i) the time variation $\tilde{\Delta}t$ can *also* be real, but in general it is understood as being *virtual* (and the associated symbol $\tilde{\Delta}$ reminds us of such an assumption); (ii) the symbol $\tilde{\delta} f(t)$ is not only the representation of the second side of Eq. (A.2), because the "operator" $\tilde{\delta}$ is *prior* with respect to $f(t)$: it is the one that *originates* such a virtual translation; (iii) the "operator" $\tilde{\delta}$ may be thus better named as "generator" because, according to definition (A.2), it "acts" as *generator* of a translation; (iv) the name "generator" also reminds us that it acts in combination with something else: $\tilde{\delta}$ is in fact the *prior* "principle", $f(t)$ is the *posterior* "principle", and $f(t + \tilde{\Delta}t)$ is what "rises" from the combination of both. Such a result (or "product") is *something new*, but at the same time it retains the main genetic characteristics of its *generating* "principles".

Analogous considerations can be made with respect to the "operator" $\left(\frac{\widetilde{\tilde{\delta}-1}}{\tilde{\Delta}t} \right)^n$.

Finally, the operation of limit ($\tilde{\lim_{\tilde{\Delta}t:0\to 0^+}}$) is here also considered as a *prior* operator with respect to those that follow it in Eq. (A.1) but, at the same time, it is *posterior* to the very *primary* operation: the passage from the time t, initially prefixed, to the virtual time

$$\tilde{\delta}t = t + \tilde{\Delta}t \tag{A.3}$$

as a consequence of a virtual translation generated by the "generator" $\tilde{\delta}$. Such an operation is represented by the symbol $\tilde{\Delta}t : 0 \to 0^+$ to remind us that our concept of "limit" is a "*spring-concept*": it is the "source" of what rises as a consequence of an infinitesimal virtual variation, immediately after a given time t, which in turn activates the sequence of the successive "generators" in its "spring-perspective". As a basic example, the incipient derivative of order n of the exponential function $e^{\varphi(t)}$ is

$$\frac{\tilde{d}^n}{\tilde{d}t^n} e^{\varphi(t)} = \left(\widetilde{\frac{\tilde{d}}{\tilde{d}t}\varphi(t)} \right)^n e^{\varphi(t)} = (\tilde{\varphi}')^n e^{\varphi(t)}. \tag{A.4}$$

Such a result is always *formally* different from the one obtainable through traditional ordinary derivatives, even when both results coincide *quantitatively* (that is, for any order derivative, if $\varphi(t)=\alpha t+\beta$; otherwise, if $\varphi(t)$ is a non-linear function, only in the case of first-order derivative). Consequently the adopted symbology reminds us of the main differences: (i) the resulting expression refers to a *virtual* evolution, which may also become a *real* evolution, *but only* in dependence on particular boundary conditions; (ii) the comprehensive structure of Eq. (A.4) reminds us that the obtained result is due to a "generating process", the virtual (evolutive) possibilities of which are delineated in terms of its intrinsic *genetic* characteristics $(\tilde{\varphi}')^n$, which are essentially due to *both* the *generator* $\frac{\tilde{d}^n}{\tilde{d}t^n}$ (understood as a *prior* "operator") and the "fertile" co-operation of the considered function $e^{\varphi(t)}$; (iii) thus the final result represents an evolutive modality which is *completely new* with respect to the original function: it is not seen now as a "necessary" consequence (as in the case of operators interpreted *a posteriori*) but, because of the *a priori* interpretation of operators, it is conceived as an "adherent" consequence of its "generation" modalities: all the various functions resulting from the "generating process" represented by Eq. (A.4), for $n \in N$, are a similar to *harmonic evolutions* which are in "resonance" (as in a "musical chord") with the original function and at the same time with each other.

Such considerations are also valid with reference to any function $f(t)$ once represented in its exponential form (see Eq. (3.2)).

Eq. (A.4) and, consequently, Eq. (3.2), can be extended to any $q \in Q$ and then applied to the most common functions in Mathematical Analysis (such as, for instance, analytical functions).

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