On Operators Commuting with Toeplitz Operators
Modulo the Compact Operators

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We prove that an operator on $H^2$ of the disc commutes modulo the compacts with all analytic Toeplitz operators if and only if it is a compact perturbation of a Toeplitz operator with symbol in $H^\infty + C$. Consequently, the essential commutant of the whole Toeplitz algebra is the algebra of Toeplitz operators with symbol in $QC$. The image in the Calkin algebra of the Toeplitz operators with symbol in $H^\infty + C$ is a maximal abelian algebra. These results lead to a characterization of automorphisms of the algebra of compact perturbations of the analytic Toeplitz operators.

Johnson and Parrot [7] showed that if $M$ is an abelian von Neumann algebra on a Hilbert space $H$, $M'$ is its commutant, and $L^\infty(H)$ is the ideal of compact operators on $H$, then the essential commutant of $M$ is $M' + L^\infty(H)$. About the same time, Sarason [8] showed that a Toeplitz operator $T_g$ on $H^2$ of the unit circle commutes modulo the compacts with all analytic Toeplitz operators if and only if $g$ is in $H^\infty + C$. Here $C$ denotes the space of continuous functions on the unit circle. From this, Douglas [6] showed that the essential center of the Toeplitz algebra is the algebra of Toeplitz operators with symbol in $QC = H^\infty + C \cap \overline{H^\infty + C}$. Douglas [4] raised the natural question of which operators in $\mathcal{F}(H^2)$ essentially commute with all Toeplitz operators.

In this paper we prove the following theorem, which gives a complete answer to this question.

**Theorem 1.** An operator $S$ on $H^2$ commutes modulo compacts with all analytic Toeplitz operators if and only if $S = T_g + K$, where $g$ is in $H^\infty + C$ and $K$ is compact.

From this we get two immediate corollaries.

**Corollary 1.** The essential commutant of the Toeplitz algebra $\mathcal{T}(L^\infty)$ is $\mathcal{T}(QC)$.

**Corollary 2.** The image of $\mathcal{T}(H^\infty + C)$ in the Calkin algebra is a maximal abelian algebra.
As far as the author knows, Corollary 2 gives the first concrete example of a maximal abelian algebra in the Calkin algebra which is not an image of a maximal abelian von Neumann algebra. (That the latter is an example is a consequence of [7].)

The author would like to point out that, although the techniques of this paper are quite different from those used in [7], several important conceptual ideas for the proof were gleaned from the Johnson and Parrot paper.

Let $\mathcal{D}$ be the unit disc, and let $\partial \mathcal{D}$ be its boundary. Let $H^2$ be the subspace of $L^2$ of those functions with all negative Fourier coefficients equal to zero. Let $M_f$ denote the operator on $L^2$ of multiplication by the $L^\infty$ function $f$. The Toeplitz operator $T_f$ with symbol $f$ is the compression of $M_f$ to $H^2$. If $A$ is a subset of $L^\infty$, let $\mathcal{T}(A)$ be the norm closed algebra generated by $\{T_f : f \in A\}$.

Let $\mathcal{L}(H^2)$ be the bounded operators on $H^2$, and let $\mathcal{L}^\infty(H^2)$ be its ideal of compact operators. Let $\pi: \mathcal{L}(H^2) \to \mathcal{L}(H^2)/\mathcal{L}^\infty(H^2)$ be the canonical homomorphism onto the Calkin algebra. If $S$ is an operator, let $D(X) : SS - S'S$ be the derivation on $\mathcal{L}(H^2)$ induced by $S$.

We will prove the following theorem, which is somewhat stronger than Theorem 1.

**Theorem 2.** If an operator $S$ in $\mathcal{L}(H^2)$ is not the sum of a Toeplitz operator and a compact operator, then there is a function $h \in H^\infty$ such that $T_fS - ST_f$ is not compact. The function $h$ may be taken to have at most one discontinuity.

The proof will follow from a series of lemmas, but first we will outline the main ideas. If $T_fS - ST_f$ is not compact, we can take $h = z$. So for the remainder of the proof, we will assume that $S$ essentially commutes with $T_f$. From this, it follows that $S$ commutes modulo compacts with every Toeplitz operator with continuous symbol. We show that there is a subsequence $A$ of the positive integers and a function $f$ in $L^\infty$, such that in the $\omega^*$ topology on $\mathcal{L}(H^2)$,

$$T_f = \lim_{n \in A} T_{z^n}ST_{z^n}.$$  

Next, we find a countable collection of disjoint closed intervals $\{\chi_n\}$ of the unit circle such that $\|T_{\chi_n}S - T_{\chi_n'}S\| > \delta > 0$. (Here we take the liberty of denoting both $\chi_n$ and its characteristic function by the same symbol.) Combining the two preceding results, we obtain functions $p_n$ in $H^\infty$ such that $\|D(T_{w_n})\| > \delta$ and so that the partial sums of the series $\Sigma p_n$ are uniformly bounded. It follows that $T_{w_n}$ and $D(T_{w_n})$ converge strongly to zero, so we are able to extract a subsequence $\Gamma$ such that $h = \sum_{n \in \Gamma} p_n$ is in $H^\infty$ with the operators $D(T_{w_n})$ almost mutually orthogonal. This will allow us to conclude that $D(T_h)$ is not compact. By choosing the functions $p_n$ so that their closed supports cluster at only one point, we ensure that $h$ has only one discontinuity.

Let $\Sigma$ be the net of inner functions ordered by divisibility. If $\omega$ is an inner
function, let $\sigma_\omega = S - T_\omega ST_\omega$. Let $\sigma_n = \sigma_{z^n}$. Consider the sequence $\{\sigma_n\}$ and the set $\{\sigma_\omega\}$. Both are norm bounded, and hence lie in a $\sigma^*$ compact set. Consequently, they have $\sigma^*$ limit points.

**Lemma 1.** Let $S$ be in $\mathcal{L}(H^2)$ with $T_zS - ST_z$ compact, and let $S'$ be a $\sigma^*$ limit point of the sequence $\{\sigma_n\}$. Then,

1. $T_gS - ST_g$ is compact for all continuous functions $g$.
2. $S' = \sigma^* \lim_{n \in \mathbb{N}} \sigma_n$ for a subsequence $A$ of $\mathbb{N}$.
3. $S - S' = T_f$ for some $f$ in $L^\infty$.
4. $S' - T_bS'T_b = \sigma_b$ for all continuous inner functions $b$.
5. $\sigma^* \lim_{n \in \mathbb{N}} T_{z^n}T_gST_{z^n} = T_{gf}$ for all continuous functions $g$.

The operator $S'$ is compact if and only if $S$ is a compact perturbation of a Toeplitz operator. In this case, in the norm topology,

$$\lim_{n \to \infty} T_{z^n}S' = S' \quad \text{and} \quad \lim_{n \to \infty} T_{z^n}S'T_{z^n} = 0.$$

Let $S - T_\omega ST_\omega$ be compact for all inner functions $\omega \in \Sigma$, and let $S' = \sigma^* \lim_{\omega \in \Lambda} \sigma_\omega$ for a subsequence $\Lambda$ of $\Sigma$. Then,

- (4') $S' - T_\omega ST_\omega = \sigma_\omega$ for all inner functions $\omega$.
- (5') $\sigma^* \lim_{\omega \in \Lambda} T_{\omega}T_hST_{\omega} = T_{hf}$ for all $h$ in $L^\infty$.

If $S = T_f + X$ is in the Toeplitz algebra, where $X$ belongs to the commutator ideal of $\mathcal{F}(L^\infty)$, then

$$\lim_{\omega \in \Sigma} \sigma_{\omega} = X \quad \text{in norm}.$$  

Proof. The set of operators $\{\sigma_n\}$ lies in the ball of radius $2 \| S \|$, which is $\sigma^*$ compact and metrizable. Hence the $\sigma^*$ limit point $S'$ can be taken to be the limit of a subsequence $A$ of $\mathbb{N}$. Since $\pi(S)$ and $\pi(T_z)$ commute in the Calkin algebra, $\pi(S)$ commutes with $\pi(T_z) - \pi(T_z)^{-1}$. Hence $S$ commutes modulo compacts with $T_z$, and, therefore, also with $\mathcal{F}(C)$, the $C^*$ algebra generated by $T_z$.

If $K$ is compact, $\lim_{n \to \infty} T_{z^n}KT_{z^n} = 0$ in norm since $T_{z^n} \to 0$ in the weak operator topology. Let $b$ be a continuous inner function. Then,

$$S' - T_bS'T_b = \sigma^* \lim_{n \in \mathbb{N}} (S - T_{z^n}ST_{z^n}) - T_b(S - T_{z^n}ST_{z^n})T_b$$

$$= \sigma^* \lim_{n \in \mathbb{N}} (S - T_bS'T_b) - T_{z^n}(S - T_bS'T_b)T_{z^n}.$$  

But, $S - T_bS'T_b = T_b(T_bS - ST_b)$ is compact, so the second term tends to zero in norm. So, $S' - T_bS'T_b = T_bS'T_b = \sigma_b$. In particular, for $b = z$, $S - S' = T_z(S - S')T_z$. This is the functional equation determining the Toeplitz operators [1]. So there is a function $f$ in $L^\infty$ such that $S = T_f + S'$.  


Now it follows that $T_f = \omega^* \lim_{n \to \infty} T_{\xi^n} S T_{\xi^n}$. Let $h_1, h_2$ be functions in $H^\infty \cap C$. Then,

$$
\omega^* \lim_{n \to \infty} T_{\xi^n} T_{h_1} S T_{\xi^n} = \omega^* \lim_{n \to \infty} T_{\xi^n} T_{h_1} \left[ S T_{\xi^n} D(T_{h_2}) \right] T_{\xi^n}
$$

$$
= \omega^* \lim_{n \to \infty} \left[ T_{h_1} T_{\xi^n} S T_{\xi^n} h_2 + T_{\xi^n} D(T_{h_2}) T_{\xi^n} \right]
$$

$$
= T_{h_1} T_1 T_{h_2} := T_{h_1 h_2}, \quad \text{since } D(T_{h_2}) \text{ is compact}.
$$

Since $\{h_1, h_2 : h_1, h_2 \in H^\infty \cap C\}$ is dense in $C$, it follows that

$$
\omega^* \lim_{n \to \infty} T_{\xi^n} T_f S T_{\xi^n} = T_f
$$

for all continuous functions $g$.

If $S'$ is compact, then $S = T_f + S'$ and $\sigma_n := S - T_{\xi^n} S T_{\xi^n}$. Hence by the above remarks, we have $\lim_{n \to \infty} \sigma_n = S'$ in norm.

The proofs of (4') and (5') are identical to the calculations for (4) and (5). In (5'), we note that $\{h_1, h_2 : h_1, h_2 \in H^\infty\}$ is dense in $L^\infty$. If $S = T_f + X$ with $X$ in the commutator ideal of $\mathcal{F}(L^\infty)$, then $\sigma_n = X - T_{\xi^n} X T_{\xi^n}$. By Douglas [4], for any $\epsilon > 0$, there is an inner function $\omega$ such that $\|XT_{\omega}\| < \epsilon$. Hence $\lim_{n \to \infty} \sigma_n = X$.

The next lemma is a basic element in the proof of main theorem. Let $\mathcal{P}\xi$ be the space of piecewise continuous functions in $L^\infty$. Consider the function $F: \mathcal{P}\xi \to \mathcal{L}(H^2)$ defined by $F(X) = T_f S - T_{\xi^f}$, where $f$ is defined in terms of $S$ as in Lemma 1. It is clear that if $g_n$ are in $\mathcal{P}\xi$, uniformly bounded in the sup norm, such that $g_n \to g_0$ pointwise, then $F(g_n) \to F(g_0)$ in the strong operator topology. We have $F(1) = S - T_f = S'$, which is not compact unless $S$ is a compact perturbation of a Toeplitz operator. Keep this function in mind to motivate the following lemma.

**Lemma 2.** Suppose $F: \mathcal{P}\xi \to \mathcal{L}(H^2)$ is a linear map such that:

- (P1) If $g_n$ are in $\mathcal{P}\xi$ with $\|g_n\|_\infty \leq M$ and $g_n \to g_0$ pointwise, then $\omega - \lim F(g_n) = F(g_0)$ in the weak operator topology;
- (P2) $F(1)$ is not compact, say $\|\pi F(1)\| > \alpha > 0$;
- (P3) if $f, g$ are in $\mathcal{P}\xi$ and have disjoint closed supports, then $F(f) F(g)$ and $F(f) F(g)^\ast$ are compact.

Then, there exist characteristic functions $\{\chi_n : n \geq 1\}$ in $\mathcal{P}\xi$ of disjoint closed support such that $\|F(\chi_n)\| > \alpha / 4$. These sets can be chosen to cluster at only one point.

Consequently, there exist trigonometric polynomials $h_n$ such that $\|h_n\| \leq 2$, $\|h_n(1 - \chi_n)\| < 2^{-n}$, and $\|F(h_n)\| > \alpha / 4$. 


Remark. Since $\chi_n$ is in $\mathcal{P}^c$, it is the finite union of closed intervals. We can suppose, in fact, that $\chi_n$ is a closed interval if we change the constant to $\alpha/8$.

Proof. Let $\mathcal{E}$ be the collection of all characteristic functions $\chi$ of closed intervals such that $\|F(\chi)\| > \alpha/4$. Let

$$E = \bigcap_{n \geq 1} \left( \bigcup \{ \chi \in \mathcal{E} : |\chi| \leq 1/n \} \right)^{cl}.$$ 

Here $|\chi|$ is the linear measure of $\chi$ as a subset of the circle. We claim that $E$ is nonempty. For if $E$ is empty, then, since it is the intersection of nested closed sets, one of these sets is empty. That is, there exists an integer $n$ such that $|\chi| \leq 1/n$ implies that $\|F(\chi)\| \leq \alpha/4$.

Divide $\mathcal{E}$ into an even number of closed intervals $\chi_i$, $i = 1, \ldots, 2k$, which are disjoint except for their endpoints, so that $\mathcal{E} = \bigcup \chi_i$. Each of the two collections $\{\chi_i : i \text{ is odd}\}$, $\{\chi_i : i \text{ is even}\}$ consist of mutually disjoint closed intervals. We have

$$\alpha < \| \pi F(1) \| \leq \frac{1}{2} \| \pi (F(1) + F(1)^*) \| + \frac{1}{2} \| \pi (F(1) - F(1)^*) \|.$$ 

Therefore, for $\varepsilon = 1$ or $-1$, we have $\| \pi (F(1) + \varepsilon F(1)^*) \| > \alpha$. Let $A = \pi (F(1) + \varepsilon F(1)^*)$. Let $A_i = \pi (F(\chi_i) + \varepsilon F(\chi_i)^*)$. By (P3), if $\chi_i, \chi_j$ are disjoint, then $A_i A_j = 0$. Hence $\{A_i : i \text{ odd}\}$ (respectively, even) is a finite collection of normal, commuting, mutually annihilating operators. Therefore, by a simple estimate on the spectral radius, we get

$$\left\| \sum_{i \text{ odd}} A_i \right\| = \max_{i} \|A_i\| \leq 2 \max \|F(\chi_i)\| \leq \alpha/2.$$ 

The analogous inequality holds for $i$ even. Now by the linearity of $F$, $\sum_{i=1}^{2k} A_i = 1$. So we have

$$\alpha < \| A \| = \left\| \sum A_i \right\| \leq \left\| \sum_{\text{odd}} A_i \right\| + \left\| \sum_{\text{even}} A_i \right\| \leq \alpha.$$ 

This contradiction shows that $E$ is nonempty.

Let $x_0 \in E$. We proceed by induction. Suppose we have chosen disjoint characteristic functions $\chi_i \in \mathcal{E}$, $i = 1, \ldots, n$, with $\alpha_n = d(x_0, \bigcup \chi_i) > 0$. Since $x_0 \in E$, we have

$$x_0 \in \left( \bigcup \{ \chi \in \mathcal{E} : |\chi| \leq \alpha_n/3 \} \right)^{cl}.$$ 

Hence, there exists a $\mu$ in $\mathcal{E}$, $|\mu| \leq \alpha_n/3$ such that $d(x_0, \mu) < \alpha_n/3$. So, $d(\mu, \bigcup \chi_i) \geq \alpha_n/3$. If $x_0$ does not belong to $\mu$, let $\chi_{n+1} = \mu$. Otherwise, set $\mu_t = \mu \cap \{x : d(x, x_0) \geq t\}$. Then $\mu_t \to \mu$ pointwise, so, by (P1), $F(\mu_t) \to F(\mu)$.
in the weak operator topology. Since the norm is lower semicontinuous in the weak operator topology, and \( \| F(\mu) \| > \alpha/4 \), we see that there exists a \( t > 0 \) such that \( \| F(\mu_t) \| > \alpha/4 \). Let \( \chi_{n+1} = \mu_t \). Then \( \alpha_{n+1} - d(\chi_{n+1} \cup \chi_0) \geq t > 0 \). It is clear that the sets \( \{ \chi_n \} \) cluster only at \( x_0 \). We remark that the sets \( \mu_t \) may be the union of two intervals, say \( \mu^+ \) and \( \mu^- \). Then since \( F(\mu_t) = F(\mu^+) + F(\mu^-) \), we can choose one of these with norm greater than \( \alpha/8 \).

Fix \( \chi = \chi_n \), and choose continuous function \( g_i \) such that \( 0 \leq g_i \leq \chi \) and \( g_i \rightarrow \chi \) pointwise. We argue as above to find an integer \( i \) such that \( \| F(g_i) \| > \alpha/4 \).

Let \( k_j \) be the \( j \)th Fejer mean of \( g_i \). Then \( \| k_j - g_i \|_x \) tends to zero as \( j \) tends to infinity, so again by the above argument we choose an integer \( j \) such that \( \| F(k_j) \| > \alpha/4 \) and also \( \| k_j - g_i \| < 2^{-n} \). For \( \chi = \chi_n \), let \( h_n = k_j \). We compute

\[
\| h_n \| \leq \| g_i \| + 2^{-n} \leq 2,
\]

\[
\| h_n(1 - \chi_n) \| \leq \| g_i(1 - \chi_n) \| + 2^{-n} = 2^{-n}.
\]

Hence we see that the functions \( h_n \) satisfy the requirements of the lemma.

**Lemma 3.** Let \( \mathfrak{A} \) be a weakly closed subalgebra of \( \mathcal{L}(H^2) \). Let \( S \) be an operator on \( H^2 \) and \( D: \mathfrak{A} \rightarrow \mathcal{L}(H^2) \) be the derivation \( D(a) = aS - Sa \). Suppose there exist \( \delta > 0, M > 0 \), and elements \( a_n \) in \( \mathfrak{A} \) such that \( \| D(a_n) \| > \delta > 0 \) and \( \| \sum_{n \in J} a_n \| \leq M \) for all finite subsets \( J \) of \( \mathbb{N} \). Then, there exists an element \( b \) in \( \mathfrak{A} \) such that \( D(b) \) is not compact.

**Proof.** We can assume that \( D(a_n) \) is compact for all \( n \). We claim that \( a_n \rightarrow 0 \) in the strong operator topology. If not, there is a unit vector \( h \) in \( H^2 \) such that \( k_n = a_nh \) has \( \| k_n \| \geq \delta \), for all \( n \) in an infinite set \( J \). It is an elementary exercise, left to the reader, to show that we can find a finite subset \( J' \) of \( J \) so that \( \| \sum_{n \in J'} k_n \| > M \). This contradicts \( \| \sum_{n \in J'} a_n \| \leq M \).

Let \( \{ z_n : n \geq 0 \} \) be an orthonormal basis for \( H^2 \). Let \( R_n \) be the orthogonal projection onto the span of \( \{ z_0, ..., z_n \} \). Now since \( a_n \rightarrow 0 \) strongly, we also have \( D(a_n) \rightarrow 0 \) strongly. Using this fact and the compactness of \( D(a_n) \), we can inductively choose a subsequence \( \Gamma \) of \( \mathbb{N} \) and corresponding projections \( Q_k \) which are finite dimensional and mutually orthogonal. These will be chosen so that for \( n \) in \( \Gamma \), we have:

1. \( \| Q_k D(a_{n_k}) Q_k \| > \delta \),
2. \( \| D(a_{n_k})(I - Q_k) \| < 3^{-k}\delta \),
3. \( \| R_k a_{n_k} R_k \| < 2^{-k} \).

If we have chosen \( n_1, ..., n_k \) and \( Q_1, ..., Q_k \), let \( Q = \sum Q_k \). Since \( Q \) is finite dimensional and \( D(a_n) \rightarrow 0 \) strongly, we can find an \( a_n \) such that \( \| R_{k+1} a_n R_{k+1} \| < 2^{-k-1} \) and \( \| D(a_n) Q \| < 3^{k-2}\delta \). Then since \( D(a_n) \) is compact, we can choose \( Q_{k+1} \) orthogonal to \( Q \), finite dimensional, so that (1) and (2) are satisfied. Set \( a_{n_{k+1}} = a_n \).
For convenience, we relabel so that $a_k = a_{n_k}$. Let $h_k = \sum_{n=1}^{k} a_n$. If $h_1, h_2$ are in $R_nH$ and $k \geq l \geq n$, then

$$
\left\langle (h_k - b_l), h_1, h_2 \right\rangle \leq \sum_{l=1}^{k} \left\langle (a_l, h_1, h_2) \right\rangle \leq \sum_{l=1}^{k} \left\| R_n a_l R_n \right\| \left\| h_1 \right\| \left\| h_2 \right\|
$$

$$
\leq \sum_{l=1}^{k} 2^{-l} \left\| h_1 \right\| \left\| h_2 \right\| < 2^{-l} \left\| h_1 \right\| \left\| h_2 \right\|.
$$

Since $\bigcup_n R_nH^2$ is dense in $H^2$, and $\| b_k \| \leq M$ for every $k$, we conclude that the sequence $\{b_k\}$ converges weakly to an element $b$ in $\mathfrak{M}$. Hence $D(b_k)$ converges weakly to $D(b)$. Therefore, since the $Q_n$ are finite dimensional, we have

$$
\lim_{k \to \infty} Q_n D(b_k) Q_n = Q_n D(b) Q_n \quad \text{in norm.}
$$

Hence,

$$
\left\| Q_n D(b) Q_n \right\| = \lim_{k \to \infty} \left\| Q_n \sum_{i=1}^{k} D(a_i) Q_n \right\|
$$

$$
\geq \lim_{k \to \infty} \left\| Q_n D(a_i) Q_n \right\| - \sum_{i \neq n} \left\| Q_n D(a_i) Q_n \right\|.
$$

But if $i \neq n$,

$$
\left\| Q_n D(a_i) Q_n \right\| = \left\| Q_n [D(a_i)(I - Q_i)] Q_n \right\| < 3^{-i} \delta.
$$

So,

$$
\left\| Q_n D(b) Q_n \right\| \geq \lim_{k \to \infty} \delta - \sum 3^{-i} \delta \geq \delta/2.
$$

This is true for all $n$, and the projections $\{Q_n\}$ are nonzero and mutually orthogonal. It follows that $D(b)$ is not compact.

**Proof of Theorem 2.** We are now ready to complete the proof of our main theorem. Suppose $S$ in $\mathcal{L}(H^2)$ is not the sum of a Toeplitz operator and a compact operator. We suppose that $T_z S - ST_z$ is compact, for otherwise we can take $h = z$. By Lemma 1, we choose a subsequence $A$ of $\mathbb{N}$, and a function $f$ in $L^2$ such that

$$
S' - S - T_f = w^* \lim_{n \to 1} S - T_{g_n} ST_{g_n}.
$$

The operator $S'$ is not compact, say $\| \pi(S') \| > \alpha > 0$.

We apply Lemma 2 to the map $F(g) = T_{g'} S - T_{g''}$. Because of the remarks preceding Lemma 2, we need only show that $F$ satisfies P3. Let $f_1, f_2$ be piecewise continuous with disjoint closed supports. Let $g_1, g_2$ be functions in $C$ such that $g_i \equiv 1$ on the support of $f_i$, and $g_1 g_2 = 0$. By Lemma 1, $\pi T_{g_i}$ commutes with $\pi S$, and by [4], we have for every $h$ in $L^\infty$,

$$
\pi(T_{g_1}) \pi(T_h) = \pi(T_{g_1 h}) = \pi(T_h) \pi(T_{g_1}).
$$
It follows that
\[ \pi(T_{g_i}) \pi(F(f_i)) \cdots \pi F(f_i) \cdots \pi F(f_i) \circ \pi(T_{g_i}), \]
and the analogous relation holds for \( \pi F(f_i) \). Thus we have
\[
\pi F(f_1) \cdot \pi F(f_2) = \pi F(f_1) \cdot \pi T_{a_1} \cdot \pi T_{a_2} \cdot \pi F(f_2)
\]
\[ = \pi F(f_1) \cdot \pi T_{a_1} \cdot \pi F(f_2) = 0. \]

Hence \( F(f_1) F(f_2) \) is compact, and similarly \( F(f_1) F(f_2) \) is compact. So, from Lemma 2, there exist trigonometric polynomials \( h_n \) and characteristic functions \( \chi_n \) of disjoint closed sets such that \( \| h_n \| = 2 \), \( \| h_n(1 - \chi_n) \| \ll 2^{-n} \), and \( \| F(h_n) \| > \alpha / 4 \).

We compute for \( h = h_n \)
\[
T_{2h} T_h D(T_{zi}) = T_{2h} T_h (T_{zi} + ST) = T_{2h} T_h S - T_{2h} T_h ST_{zi}. \]

From the derivation identity, we have
\[
D(T_{zi}) - D(T_{zi} T_{zi}) = T_{2h} D(T_{zi}) + D(T_{h}) T_{zi}. \]

We get
\[
\omega^* \lim_{k \to \lambda} T_{2h} D(T_{zi}) = \omega^* \lim_{n \to \lambda} (T_{2h} S - T_{2h} T_h ST_{zi}) + T_{2h} D(T_{h}) T_{zi}
\]
\[ = T_{2h} S - T_{2h} T_{zi} \] by Lemma 1.

By lower semicontinuity of the norm, and the inequality \( \| T_{2h} S - T_{2h} T_{zi} \| > \alpha / 4 \), we can choose an integer \( k \) in \( \lambda \) such that \( \| T_{2h} D(T_{zi}) \| > \alpha / 4 \)
and \( p_n = z^{kh_n} \) belongs to \( H^\infty \). We then have \( \| D(T_{zi}) \| > \| T_{2h} D(T_{zi}) \| > \alpha / 4 \), and
\[
\| p_n \| = \| z^{kh_n} \| \ll 2, \quad \text{and} \quad \| p_n(1 - \chi_n) \| = \| z^{kh_n}(1 - \chi_n) \| \ll 2^{-n}. \]

Now, if \( J \) is a finite subset of \( \mathbb{N} \), let \( p_J = \sum_{j \in J} p_n \). Then
\[
\| p_J \chi_m \| \ll \sum_{j \in J} \| p_n \chi_m \| \ll \| p_m \| + \sum_{j \notin J} \| p_n(1 - \chi_n) \|
\]
\[ \ll 2 + \sum 2^{-n} = 3. \]

If \( \mu = 1 - \sum \chi_m \), then \( \| p_J \mu \| \ll \| p_n \mu \| \ll 2^{-n} = 1 \). Hence \( \| p_J \| \ll 3 \) for all finite subsets \( J \).

Therefore we can apply Lemma 3, with \( a_n = T_{zi} \), and \( \mathcal{U} = \mathcal{F}(H^\infty) \). This gives us a function \( h \) in \( H^\infty \) such that \( D(T_{zi}) \) is not compact.
We have $T_h = w - \lim_{k \to \infty} T_{b_k}$ where $b_k = -\sum_{i=1}^{n} \sigma_{n_i}$. The set $\{x: \sigma_{n_i} (x) \geq 2^{-n_i}\}$ is contained in $x_n$, and the sets $\{x_n\}$ cluster only at the point $x_0$. So the sequence $\{b_k\}$ converges uniformly on sets bounded away from $x_0$ to a function which is continuous except at $x_0$. Hence $h$ is continuous except at $x_0$. This concludes the proof.

**Proof of Theorem 1.** We will now prove the results stated in the first section. If $S$ commutes modulo the compacts with all analytic Toeplitz operators, it follows from Theorem 2 that $S$ has the form $S = T_f + K$, where $K$ is compact. Therefore, we see from Sarason [8] that $f$ is in $H^\infty + C$.

We remark that if $S$ is in the Toeplitz algebra, this result follows in a more elementary way. For then $S = T_f + X$, where $X$ is in the commutator ideal of $\mathcal{T}(L^2)$. We have that $\sigma_n = T_{\sigma_n}(T_{\sigma_n}S - ST_{\sigma_n})$ is compact for every inner function $\sigma_n$. So be Lemma 1, $X = \lim_{n \to \infty} \sigma_n$ in norm, and hence $X$ is compact. We now apply Sarason's result as above.

To prove Corollary 1, we note that $\mathcal{T}(L^2)$ is generated by $\mathcal{T}(H^\infty)$ and $\mathcal{T}(H^2)$.

Hence

$$\mathcal{T}(L^2)^{ec} = \mathcal{T}(H^\infty)^{ec} \cap \mathcal{T}(H^2)^{ec} = \mathcal{T}(H^\infty + C) \cap \mathcal{T}(\overline{H^\infty + C}) = \mathcal{T}(Q\mathcal{C}).$$

Corollary 2 is immediate from Theorem 1 and the fact that $\pi \mathcal{T}(H^\infty + C)$ is abelian.

**Corollary 3.** If an operator $S$ is not in $\mathcal{T}(H^\infty + C)$, then there is an inner function $\omega$ such that $ST_{\omega} - T_{\omega}S$ is not compact.

**Proof.** By Theorem 1, there is an analytic function $h$ such that $ST_h - T_hS$ is not compact. A theorem of Marshall [10] shows that the linear span of the inner functions is norm dense in $H^\infty$. The set of noncompact operators is open, so we can approximate $h$ by a finite linear combination of inner functions $\sum \alpha_i \sigma_i$, so that $\sum \alpha_i (ST_{\sigma_i} - T_{\sigma_i}S)$ is not compact.

**Remark.** Marshall's results actually say more. If we take $h$ to be continuous except at $x_0$, we can approximate $h$ in norm by Blaschke products which are continuous except at $x_0$. So if $S$ is not a Toeplitz operator plus a compact, we can find a Blaschke product $b$ with zeros accumulating only at $x_0$ for which $ST_b - T_bS$ is not compact.

**Definition.** A derivation $D$ of an algebra $\mathcal{A}$ into itself is inner if $D(X) := XS - SX$ for some $S$ in $\mathcal{A}$.

**Corollary 4.** Every derivation $D$ of $\mathcal{T}(H^\infty + C)$ into $L^\infty(H^2)$ is inner.

**Proof.** The operator $D$ restricted to $L^\infty(H^2)$ is a derivation of the compacts into themselves. It is well known [2] that every derivation on the compacts has
the form $D(X) = XS - SX$ for some $S$ in $\mathcal{L}(H^2)$. If $A$ is in $\mathcal{F}(H^\infty - C)$ and $K$ is compact, then

$$D(AK) = AKS - SAK = (AKS - ASK) + (AS - SA)K$$

$$= AD(K) + D(A)K = AKS - SAK - D(A)K.$$ 

Therefore $D(A)K = (AS - SA)K$ for every compact operator $K$. Hence $D(A) = AS - SA$. Since $A$ commutes with all $A$ in $\mathcal{F}(H^\infty + C)$ modulo the compacts, we have that $S$ is in $\mathcal{F}(H^\infty + C)$ by Theorem 1.

An immediate consequence of this is the following.

**Corollary 5.** Every derivation of $\mathcal{F}(L^\infty)$ into the compact operators is of the form $D(X) = XS - SX$ with $S$ in $\mathcal{F}(QC)$.

We consider the matrix-valued case. The operator algebra $\mathcal{F}(L^\infty) \otimes M_n$ acts on $H^2 \otimes \mathbb{C}^n$, $M_n$ is the $n \times n$ matrix algebra over $\mathbb{C}$. A general reference for this is Douglas [5].

**Corollary 6.** An operator $S$ in $\mathcal{L}(H^2 \otimes \mathbb{C}^n)$ commutes modulo the compacts with all operators in $\mathcal{F}(H^\infty) \otimes M_n$ if and only if $S = T_f \otimes I_n + K$, where $f$ is in $H^\infty$ and $K$ is compact.

**Proof.** Let $\delta_{ij}$ be the $n \times n$ matrix zero everywhere except for a 1 in the $(i,j)$ entry. A simple computation of $D(T_h \otimes \delta_{ij})$ for $h$ in $H^\infty$ shows that $S$ has the desired form.

**Corollary 7.** An operator $S$ in $(H^2 \otimes \mathbb{C}^n)$ is in the essential commutant of $\mathcal{F}(L^\infty) \otimes M_n$ if and only if $S = T_f \otimes I_n + K$, where $f$ is in QC and $K$ is compact.

**Theorem 3.** Let $\alpha$ be an automorphism of $\mathcal{F}(H^\infty) + \mathcal{L}\mathcal{E}(H^2)$. Then $\alpha$ is spatial, and has the factorization $\alpha = \alpha_1 \alpha_2$, where

1. $\alpha_1(T_h) = T_{h_{ob}}$ for a Blaschke factor $b = \lambda(z - a)/(1 - \bar{a}z)$, $|a| < 1$, and $|\lambda| = 1$. More generally, $\alpha_1(A) = U_1^*AU_1$ for all $A$ in $\mathcal{F}(H^\infty) + \mathcal{L}\mathcal{E}(H^2)$, where $U_1^*f = e(f \circ b)$ for $f$ in $H^2$ and $e$ a unit vector in $H^2 \otimes bH^2$.

2. $\alpha_2(A) = U_2^*AU_2$ for a unitary $U_2$ in $\mathcal{F}(QC)$. So $U = T_f + K$, where $g$ is unimodular in QC and $K$ is compact.

**Proof.** Since $\mathcal{L}\mathcal{E}(H^2)$ is the unique minimal closed two-sided ideal in $\mathcal{F}(H^\infty) + \mathcal{L}\mathcal{E}(H^2)$, we must have $\alpha(\mathcal{L}\mathcal{E}(H^2)) = \mathcal{L}\mathcal{E}(H^2)$. So, by a well-known theorem [3], there is a unitary operator $W$ such that $\alpha(K) = WKW$ for all $K$ in $\mathcal{L}\mathcal{E}(H^2)$. If $A$ is in $\mathcal{F}(H^\infty) + \mathcal{L}\mathcal{E}(H^2)$, then $(W^*AW)(W^*KW)$

$W^*AKW = \alpha(AK) = \alpha(A)\alpha(K) = \alpha(A)W^*KW$ for all compact operators $K$. Hence $\alpha(A) = W^*AW$. 


There is a natural map from the automorphisms of $\mathcal{F}(H^\infty) + \mathcal{L}C(H^2)$ onto the automorphisms of $H^\infty$ by projecting into the Calkin algebra. The algebras $H^\infty$, $\mathcal{F}(H^\infty)$, and $\pi\mathcal{F}(H^\infty)$ are isometrically isomorphic as Banach algebras, so we can identify them here. The automorphisms of $H^\infty$ are known [9, p. 143] to be of the form $\alpha(h) = h \circ b$, where $b$ is a conformal map of the disc onto itself.

The kernel of this map is the set of automorphisms $\alpha$ such that $\alpha(T_h) = T_{h + K}$, with $K$ compact. Since $\alpha$ is spatial, it is induced by a unitary operator $U$. So $U$ and $U^*$ essentially commute with $\mathcal{F}(H^\infty)$. Hence $U$ belongs to $\mathcal{F}(Q\mathcal{C})$.

We now show that automorphisms of $\mathcal{F}(H^\infty)$ are spatial. Let $b$ be a conformal automorphism of the disc. Let $e$ be a unit vector in $H^2 \otimes bH^2$. Define an operator on $H^2$ by $U_1f \mapsto e(f \circ b)$ for $f$ in $H^2$. Since $e$ is in $H^\infty$, it follows that $e(f \circ b)$ is in $H^2$. A computation shows that $|db/dz| = |e|^2$. So $\|U_1f\|^2 = \int |e(f \circ b)|^2 dz = \int |f|^2 \cdot b \cdot db/dz |dz| = \int |f|^2 |dz| \cdot \|f\|^2$. The operator $U_1^*$ is clearly invertible, hence it is unitary on $H^2$. If $h$ is in $H^\infty$,

$$U_1^*T_hU_1(f \circ b) = U_1^*T_hf = e(fh \circ b) = (h \circ b)(e(f \circ b)) = T_{h \circ b} e(f \circ b).$$

Hence, $U_1^*T_hU_1 = T_{h \circ b}$.

Let $\alpha$ be an automorphism of $\mathcal{F}(H^\infty) + \mathcal{L}C(H^2)$. Then $\pi\alpha$ is an automorphism of $H^\infty$. This lifts to a spatial automorphism of $\mathcal{F}(H^\infty)$, $\alpha_1(A) = U_1^*AU_1$. Let $\alpha_2 = \alpha_1^{-1}\alpha$. Since $\pi\alpha_2$ is the identity, we have $\alpha_2(A) = U_2^*AU_2$, for some unitary $U_2$ in $\mathcal{F}(Q\mathcal{C})$.

Note added in proof. Theorem 3 is also valid for the automorphisms of $\mathcal{F}(H^\infty) \otimes \mathcal{C}$.

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