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Heat Kernel Bounds on Manifolds with Cusps

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We describe a method of obtaining pointwise upper bounds for the heat kernel of a Riemannian manifold with cusps. We apply our results to a class of approximately hyperbolic manifolds, by which we mean manifolds which have bounded geometry with respect to a hyperbolic structure with cusps (these manifolds include all asymptotically hyperbolic manifolds). For such manifolds we obtain upper bounds on the heat kernels which we believe to be nearly optimal.

1. INTRODUCTION

Although there is by now a substantial literature on heat kernel bounds for Riemannian manifolds (see [1] and references therein), many of the results obtained for non-compact manifolds apply only to manifolds with bounded geometry, by which one usually means that the Ricci curvature and injectivity radius are bounded below. For the type of bounds usually obtained, both conditions are necessary, but the rather different behaviour occurring when the injectivity radius goes to zero at infinity is of great importance in hyperbolic geometry. One particular case of this type was analyzed in detail by Müller [10], and our purpose in this article is to obtain similar results in more general situations where the method of separation of variables is not available.

Our method will be to derive all the required bounds from suitable Sobolev inequalities and logarithmic Sobolev inequalities as pioneered by Gross [9]. The fact that Gross' method can be sharpened to obtain pointwise upper bounds was first shown by Davies and Simon [7], and the further device needed to obtain sharp Gaussian upper bounds was discovered in [4]. In this paper we do not discuss lower bounds but refer the reader to [2, 5, 8].

For simplicity of presentation we assume that the manifold M has dimension at least three. The case where M has dimension one or two can

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be dealt with in several ways. Davies and Simon [7] use modified Sobolev inequalities, while Fabes and Stroock [8] use Nash inequalities; yet another procedure is to pass from M to $M \times \mathbb{R}^2$ and to return to M at the very end of the calculation, using the fact that the heat kernel of $M \times \mathbb{R}^2$ splits as a direct product.

In Section 2 we show how to pass from a diagonal upper bound on the heat kernel to a Gaussian upper bound. We subsequently show how to prove the diagonal upper bound for a manifold which is topologically of the form $M = X \times (0, \infty)$, with a "cuspidal" metric generalizing that studied by Müller [10]. We have reversed the logical order of presentation because the material in Section 2 is of a more general character and will be used in a later paper [6] for hyperbolic manifolds of the form $M = \Gamma \backslash \mathbb{H}^{N+1}$, where Γ is a Kleinian group acting on the $(N+1)$-dimensional hyperbolic space $\mathbb{H}^{N+1}$.

2. Gaussian Upper Bounds

We suppose that M is a non-compact and possibly incomplete Riemannian manifold of dimension $(N+1)$ where $N \geqslant 2$, with volume element dm. The Laplace–Beltrami operator H on M is defined initially on $C_c^\infty(M)$ and is made self-adjoint by taking its quadratic form closure [3]. We suppose that E is the bottom of the spectrum of H, so that $E \geqslant 0$, but particularly wish to allow the case $E > 0$. We let $K(t, m, n)$ denote the heat kernel of e^{-Ht} for $0 < t < \infty$. It is known that K is a strictly positive C^∞ function on $(0, \infty) \times M \times M$ and we make the fundamental assumption that

$$0 \leqslant K(t, m, n) \leqslant c t^{-(N+1)/2} \sigma(m)^2 \tag{2.1}$$

for all $0 < t \leqslant 1$ and all $m \in M$, where σ is a positive C^∞ function on M satisfying the following condition: there exists a constant F such that the potential $V = -\sigma^{-1} \Delta \sigma$ satisfies

$$V \geqslant F. \tag{2.2}$$

In manifolds with bounded geometry these conditions are satisfied with $\sigma \equiv 1$, but we shall give a non-trivial example in Section 3. We refer to [6] for an application to the study of Kleinian groups, and to [7] for an analysis of the case where σ is a L^2 eigenfunction of H.

We define the unitary operator U from $L^2(M, \sigma^2\, dm)$ to $L^2(M, dm)$ by $Uf = \sigma f$, and define the operator $\tilde{H}$ on $L^2(M, \sigma^2\, dm)$ by

$$\tilde{H} = U^{-1}(H - F)\, U.$$

The operator $\tilde{H}$ is associated in a standard way [3] with the closure of the quadratic form $\tilde{Q}$ defined initially on $C_c^\infty(M)$ by

$$\tilde{Q}(f) = \int_M |\nabla(\sigma f)|^2 \, dm - \int_M F|f|^2 \sigma^2 \, dm$$

$$= \int_M |\nabla f|^2 \sigma^2 \, dm + \int (V - F)|f|^2 \sigma^2 \, dm. \tag{2.3}$$

A use of Beurling–Deny conditions and the Trotter product formula shows that $e^{-\tilde{H}t}$ is a symmetric Markov semigroup, that is

$$\|e^{-\tilde{H}t} f\|_p \leqslant \|f\|_p \tag{2.4}$$

for all $t \geqslant 0$, $1 \leqslant p \leqslant \infty$ and $f \in L^p(M, \sigma^2 \, dm)$. Putting $p = 2$ we deduce that $E \geqslant F$.

The heat kernel $\tilde{K}$ of $e^{-\tilde{H}t}$ is given by

$$\tilde{K}(t, m, n) = \sigma(m)^{-1} K(t, m, n) \, \sigma(n)^{-1} e^{Ft}, \tag{2.5}$$

where $m, n \in M$. Since $\tilde{K}$ is a kernel of positive type, (2.1) implies that

$$0 \leqslant \tilde{K}(t, m, n) \leqslant c_0 t^{-(N+1)/2}$$

or equivalently

$$\|e^{-\tilde{H}t}\|_{\infty, 1} \leqslant c_0 t^{-(N+1)/2} \tag{2.6}$$

for all $0 < t \leqslant 1$, where $\|\cdot\|_{q, p}$ denotes the norm of an operator from L^p to L^q.

It was shown in [7] that (2.6) follows from a logarithmic Sobolev inequality of the form

$$\int f^2 \log f \, \sigma^2 \, dm \leqslant \varepsilon \tilde{Q}(f) + \left(c_1 - \frac{N+1}{4} \log \varepsilon \right) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{2.7}$$

for all $0 < \varepsilon \leqslant 1$ and all $0 \leqslant f \in \mathrm{Quad}(\tilde{H}) \cap L^1 \cap L^\infty$. Indeed (2.6) and (2.7) are equivalent to each other and also to a Sobolev inequality of the form

$$\|f\|_{2(N+1)/(N-1)}^2 \leqslant c_2 (\tilde{Q}(f) + \|f\|_2^2) \tag{2.8}$$

for all $0 \leqslant f \in \mathrm{Quad}(\tilde{H})$ by [11].

By interpolation and the fact that the spectrum of $\tilde{H}$ lies in $[E-F, \infty)$ one sees that $c_3 = \|e^{-\tilde{H}/2}\|_{2,1}$ is finite and

$$\|e^{-\tilde{H}t}\|_{\infty,1} \leqslant c_3^2 e^{(F-E)(t-1)}$$

for all $t \geqslant 1$. Therefore

$$\|e^{-\tilde{H}t}\|_{\infty,1} \leqslant c_\delta t^{-(N+1)/2} e^{(\delta+F-E)(t-1)} \tag{2.9}$$

for all $0 < t < \infty$ and $0 < \delta < 1$. This bound is equivalent, apart from a change in the constant c_δ, to the validity of

$$\int f^2 \log f \, \sigma^2 \, dm \leqslant \varepsilon \tilde{Q}(f) + \beta(\varepsilon)\|f\|_2^2 + \|f\|_2^2 \log\|f\|_2 \tag{2.10}$$

for all $0 < \varepsilon < \infty$ and $0 \leqslant f \in \mathrm{Quad}(\tilde{H}) \cap L^1 \cap L^\infty$, where

$$\beta(\varepsilon) = c_\delta - \frac{N+1}{4} \log \varepsilon + (\delta + F - E)\, \varepsilon. \tag{2.11}$$

We now obtain Gaussian upper bounds for the kernels $\tilde{K}$ by adapting the method of [4]. The extra difficulty of our present analysis arises from the possibility that $F \neq E$, which prevents us from simply putting $\mu = \frac{1}{2}$ below as in [4]. We let $\phi = e^{\alpha\psi}$, where $\alpha \in \mathbb{R}$ and ψ is a C^∞ bounded function on M satisfying $|\nabla\psi| \leqslant 1$ everywhere. Our fundamental lemma is as follows.

LEMMA 1. *If* $0 \leqslant f \in C_c^\infty(M)$, $2 \leqslant p < \infty$, *and* $0 < \mu < 1$ *then*

$$(1-\mu)\langle \tilde{H}f, f^{p-1}\rangle \leqslant \langle \phi^{-1}\tilde{H}\phi f, f^{p-1}\rangle + \alpha^2(1 + (p-2)\,\mu^{-1})\|f\|_p^p.$$

Proof. We have

$$\begin{aligned}
&\langle \phi^{-1}\tilde{H}\phi f, f^{p-1}\rangle \\
&\quad = \int_M \nabla(\sigma\phi f)\cdot\nabla(\sigma\phi^{-1}f^{p-1})\, dm - F\langle \phi f, \phi^{-1} f^{p-1}\rangle \\
&\quad = \int_M \nabla(\sigma f)\cdot\nabla(\sigma f^{p-1})\, dm - \alpha^2 \int_M |\nabla\psi|^2\, \sigma^2 f^p\, dm \\
&\qquad + \alpha \int_M [\sigma f \nabla\psi\cdot\nabla(\sigma f^{p-1}) - \sigma f^{p-1}\,\nabla\psi\cdot\nabla(\sigma f)]\, dm - F\|f\|_p^p \\
&\quad \geqslant \langle \tilde{H}f, f^{p-1}\rangle - \alpha^2\|f\|_p^p - |\alpha|(p-2)\int_M f^{p-1}|\nabla f|\, \sigma^2\, dm.
\end{aligned}$$

Also if $0 < s < \infty$ then

$$\begin{aligned}\int_M f^{p-1}|\nabla f|\, \sigma^2\, dm &= \frac{2}{p}\int_M f^{p/2}|\nabla f^{p/2}|\, \sigma^2\, dm \\ &\leqslant \frac{s}{p}\int_M |\nabla f^{p/2}|^2\, \sigma^2\, dm + \frac{1}{sp}\int_M f^p \sigma^2\, dm \\ &= \frac{sp}{4(p-1)}\int_M \nabla f \cdot \nabla f^{p-1}\sigma^2\, dm + \frac{1}{sp}\|f\|_p^p \\ &= \frac{sp}{4(p-1)}\{\langle \tilde{H}f, f^{p-1}\rangle - \langle (V-F) f, f^{p-1}\rangle\} + \frac{1}{sp}\|f\|_p^p \\ &\leqslant \frac{sp}{4(p-1)}\langle \tilde{H}f, f^{p-1}\rangle + \frac{1}{sp}\|f\|_p^p.\end{aligned}$$

Therefore

$$\begin{aligned}\langle \phi^{-1}\tilde{H}\phi f, f^{p-1}\rangle &\geqslant \langle \tilde{H}f, f^{p-1}\rangle - \alpha^2\|f\|_p^p \\ &\quad - |\alpha|(p-2)\left\{\frac{sp}{4(p-1)}\langle \tilde{H}f, f^{p-1}\rangle + \frac{1}{sp}\|f\|_p^p\right\}.\end{aligned}$$

If we put

$$s = \frac{4(p-1)\,\mu}{|\alpha|\; p(p-2)}$$

then we obtain the stated bound easily.

COROLLARY 2. *If* $0 \leqslant f \in C_c^\infty(M)$, $0 < \delta < 1$, $0 < \mu < 1$, $0 < \varepsilon < \infty$, *and* $2 \leqslant p < \infty$ *then*

$$\begin{aligned}\int f^p \log f\, \sigma^2\, dm &\leqslant \varepsilon\langle \phi^{-1}\tilde{H}\phi f, f^{p-1}\rangle \\ &\qquad + \gamma(\varepsilon, p)\|f\|_p^p + \|f\|_p^p \log\|f\|_p,\end{aligned}$$

where

$$\begin{aligned}\gamma(\varepsilon, p) &= \frac{2}{p}\left\{c(\delta, \mu) - \frac{N+1}{4}\log\varepsilon + (\delta + F - E)(1-\mu)\,\varepsilon\right\} \\ &\quad + \varepsilon\alpha^2(1 + (p-2)\,\mu^{-1}).\end{aligned}$$

Proof. We replace f by $f^{p/2}$ in (2.10) as in [7, 9] to obtain

$$\int f^p \log f\sigma^2 \, dm \leqslant \varepsilon(1-\mu)\langle \tilde{H}f, f^{p-1}\rangle$$
$$+\frac{2}{p}\beta(\varepsilon(1-\mu))\|f\|_p^p + \|f\|_p^p \log\|f\|_p$$

and then apply Lemma 1.

THEOREM 3. *For all* $0<\delta<1$, *there exists a constant* c_δ *such that*

$$\|\phi^{-1}e^{-\tilde{H}t}\phi f\|_\infty \leqslant c_\delta t^{-(N+1)/4}\exp\{(2\delta+F-E)\,t+\alpha^2 t(1+\delta)\}\|f\|_2$$

for all $f\in L^2$ *and* $0<t<\infty$.

Proof. We proceed exactly as in Theorem 3 of [4]. If $1<\lambda<\infty$, we obtain

$$\|\phi^{-1}e^{-\tilde{H}t}\phi f\|_\infty \leqslant e^{M(t)}\|f\|_2,$$

where

$$\begin{aligned} M(t) &= \int_2^\infty \gamma(\lambda 2^\lambda t p^{-\lambda}, p)\, p^{-1}\, dp \\ &= \int_2^\infty \frac{2}{p^2}\left\{c - \frac{N+1}{4}\log(\lambda 2^\lambda t p^{-\lambda}) \right. \\ &\quad \left. + (\delta+F-E)(1-\mu)\,\lambda 2^\lambda t p^{-\lambda}\right\} dp \\ &\quad + \int_2^\infty \lambda 2^\lambda t p^{-\lambda-1}\alpha^2(1+(p-2)\,\mu^{-1})\, dp \\ &= c(\delta,\mu,\lambda) - \frac{N+1}{4}\log t + (\delta+F-E)(1-\mu)\frac{\lambda t}{\lambda+1} \\ &\quad + \alpha^2 t\left(1+\frac{2}{(\lambda-1)\,\mu}\right). \end{aligned}$$

By making μ small and then λ large we obtain

$$\begin{aligned} M(t) &\leqslant c(\delta) - \frac{N+1}{4}\log t \\ &\quad + (2\delta+F-E)\,t + \alpha^2 t(1+\delta). \end{aligned}$$

THEOREM 4. *If* $0<\delta<1$ *and* $0<t<\infty$ *then*

$$0\leqslant K(t, x, y)\leqslant c_\delta\sigma(x)\,\sigma(y)\,t^{-(N+1)/2}\times\exp\{(2\delta-E)\,t-d(x, y)^2/4(1+\delta)\,t\}$$

where E *is the bottom of the spectrum of* H *and* $d(x, y)$ *is the Riemannian distance between* x *and* y.

Proof. We prove that

$$0\leqslant \tilde{K}(t, x, y)\leqslant c_\delta t^{-(N+1)/2}\exp\{(2\delta+F-E)\,t-d(x, y)^2/4(1+\delta)\,t\}$$

exactly as in Corollary 6 of [4] and then apply (2.5).

3. MANIFOLDS WITH CUSP-LIKE SINGULARITIES

We show that the results of Section 2 are applicable to a class of approximately hyperbolic manifolds (defined below). For simplicity we treat only the case where M is topologically of the form $X\times(0,\infty)$, where the manifold X need not be compact, but comment that the extension to several cusps is straightforward. Our bounds improve and extend those obtained by Müller [10] using an entirely different method; in our notation Müller assumes that X is compact and that γ below equals y^{-2}, and proceeds by using the method of separation of variables.

We assume that X has dimension $N\geqslant 2$ and is provided with the metric ds_X and volume element dx. We give M the metric

$$ds_M^2=\gamma(x, y)(ds_X^2+dy^2)$$

with corresponding volume element $dm=\gamma^{(N+1)/2}\,dx\,dy$. We assume that M is approximately hyperbolic in the sense that γ is a positive C^∞ function which satsifies

$$c^{-1}y^{-2}\leqslant\gamma(x, y)\leqslant cy^{-2} \tag{3.1}$$

$$\left|\frac{\partial\gamma}{\partial y}\right|\leqslant cy^{-3} \tag{3.2}$$

on M for some $c>0$. The Laplace–Beltrami operator H on $L^2(M, \gamma^{(N+1)/2}\,dx\,dy)$ is associated with the closure of the quadratic form

$$Q(f)=\int\gamma^{(N-1)/2}\left(|\nabla_x f|^2+\left|\frac{\partial f}{\partial y}\right|^2\right)dx\,dy$$

with initial domain $C_c^\infty(M)$. In operator terms we have

$$Hf = -\gamma^{-(N+1)/2}\nabla_x \cdot (\gamma^{(N-1)/2}\nabla_x f) \\ -\gamma^{-(N+1)/2}\frac{\partial}{\partial y}\left(\gamma^{(N-1)/2}\frac{\partial f}{\partial y}\right).$$

We next define the unitary operator U from $L^2(M, \sigma^2\, dm)$ to $L^2(M, dm)$ by $Uf = \sigma f$, where

$$\sigma(x, y) = (1+y)^{N/2}. \tag{3.3}$$

LEMMA 5. *There exists a constant F such that σ satisfies the bound* (2.2).

Proof. We have

$$V = -\sigma^{-1}\Delta\sigma \\ = -(1+y)^{-N/2}\gamma^{-(N+1)/2}\frac{\partial}{\partial y}\left(\gamma^{(N-1)/2}\frac{N}{2}(1+y)^{N/2-1}\right),$$

so V is bounded by the hypotheses (3.1) and (3.2) on γ.

If we define $\tilde{H}$ as in Section 2 we now obtain all of the results there provided we have the fundamental inequality (2.6). We choose to verify the equivalent bound (2.8).

LEMMA 6. *Let* $p = 2(N+1)/(N-1)$ *and suppose that the following conditions are satisfied for some* $c_1 > 0$:

(i) $$\left(\int_{X\times[1,\infty)} |g|^p y^{-1}\, dx\, dy\right)^{2/p} \\ \leqslant c_1 \int_{X\times[1,\infty)} \left[y\left(|\nabla_x g|^2 + \left|\frac{\partial g}{\partial y}\right|^2\right) + y^{-1}|g|^2\right] dx\, dy$$

for all $0 \leqslant g \in C_c^\infty(X\times[1,\infty))$.

(ii) $$\left(\int_{X\times(0,1]} |h|^p y^{-N-1}\, dx\, dy\right)^{2/p} \\ \leqslant c_2 \int_{X\times(0,1]} \left[y^{1-N}\left(|\nabla_x h|^2 + \left|\frac{\partial h}{\partial y}\right|^2\right) + y^{-1-N}|h|^2\right] dx\, dy$$

for all $0 \leqslant h \in C_c^\infty(X\times(0,1])$. *Then there exists a constant* $c_2 > 0$ *for which* (2.8) *holds.*

Proof. If $0 \leqslant f \in C_c^\infty(M)$ then we let g denote the restriction of f to $X \times [1, \infty)$ and h its restriction to $X \times (0, 1]$. Since $y \geqslant 1$ implies

$$\sigma^2 \gamma^{(N+1)/2} \sim y^{-1}$$

and

$$\sigma^2 \gamma^{(N-1)/2} \sim y$$

we see that (i) implies

$$\left(\int_{X \times [1, \infty)} |g|^p \, \sigma^2 \gamma^{(N+1)/2} \, dx \, dy \right)^{2/p} \leqslant c_3 \int_{X \times [1, \infty)} \left[\sigma^2 \gamma^{(N-1)/2} \left(|\nabla_x f|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) + \sigma^2 \gamma^{(N+1)/2} |f|^2 \right] dx \, dy.$$

Since $0 < y \leqslant 1$ implies

$$\sigma^2 \gamma^{(N+1)/2} \sim y^{-N-1}$$

and

$$\sigma^2 \gamma^{(N-1)/2} \sim y^{1-N}$$

we see that (ii) implies

$$\left(\int_{X \times (0, 1]} |h|^p \, \sigma^2 \gamma^{(N+1)/2} \, dx \, dy \right)^{2/p} \leqslant c_3 \int_{X \times (0, 1]} \left[\sigma^2 \gamma^{(N-1)/2} \left(|\nabla_x f| + \left| \frac{\partial f}{\partial y} \right|^2 \right) + \sigma^2 \gamma^{(N+1)/2} |f|^2 \right] dx \, dy.$$

Therefore

$$\begin{aligned} \|f\|_p^2 &\leqslant 2\|g\|_p^2 + 2\|h\|_p^2 \\ &\leqslant 2c_3 \int_M \left[\sigma^2 \gamma^{(N-1)/2} \left(|\nabla_x f|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) + \sigma^2 \gamma^{(N+1)/2} |f|^2 \right] dx \, dy \\ &= 2c_3 \int_M \sigma^2 |\nabla f|^2 \, dm + 2c_3 \|f\|_2^2 \\ &\leqslant 2c_3 (\tilde{Q}(f) + \|f\|_2^2) \end{aligned}$$

by (2.3).

LEMMA 7. *Condition* (i) *of Lemma* 6 *is true provided the heat kernel* K_X *of* X *satisfies*

$$0 \leqslant K_X(t, x_1, x_2) \leqslant c t^{-N/2} \tag{3.4}$$

for all $0 < t \leqslant 1$ *and* $x_i \in X$.

Proof. By (3.4) the heat kernel of $X \times [0, \infty)$ with the product metric and Neumann boundary conditions on $s=0$ satisfies

$$0 \leqslant K_{X \times [0, \infty)}(t; x_1, s_1; x_2, s_2) \leqslant c_1 t^{-(N+1)/2}$$

and this is equivalent to the bound

$$\left(\int_{X \times [0, \infty)} |g|^p \, dx \, ds\right)^{2/p} \leqslant c_1 \int_{X \times [0, \infty)} \left\{ |\nabla_x g|^2 + \left|\frac{\partial g}{\partial s}\right|^2 + |g|^2 \right\} dx \, ds$$

for all $0 \leqslant g \in C_c^\infty(X \times [0, \infty))$. This obviously implies

$$\left(\int_{X \times [0, \infty)} |g|^p \, dx \, ds\right)^{2/p} \leqslant c_1 \int_{X \times [0, \infty)} \left\{ e^{2s}\left(|\nabla_x g|^2 + \left|\frac{\partial g}{\partial s}\right|^2 \right) + |g|^2 \right\} dx \, ds$$

which is equivalent to (i) if we put $y = e^s$.

LEMMA 8. *The condition* (ii) *of Lemma* 6 *is true under the condition* (3.4).

Proof. If we make the change of variables $y^{-N} = s$ then (ii) becomes

$$\left(\int_{X \times [1, \infty)} |h|^p \, dx \, ds\right)^{2/p}$$
$$\leqslant c_2 \int_{X \times [1, \infty)} \left(s^{-2/N} |\nabla_x h|^2 + s^2 \left|\frac{\partial h}{\partial s}\right|^2 + |h|^2 \right) dx \, ds$$

for all $0 \leqslant h \in C_c^\infty(X \times [1, \infty))$. If we put $a(s) = 2^{-2r/N}$ for $2^{r-1} \leqslant s < 2^r$ and $b(s) = 2^{2r}$ for $2^{r-1} \leqslant s < 2^r$, then we shall actually prove the stronger fact that

$$\left(\int_{X \times [1, \infty)} |h|^p \, dx \, ds\right)^{2/p}$$
$$\leqslant c_3 \int_{X \times [1, \infty)} \left(a(s) |\nabla_x h|^2 + b(s) \left|\frac{\partial h}{\partial s}\right|^2 + |h|^2 \right) dx \, ds$$

whenever $h = \sum_{r=1}^{R} h_r$ and $0 \leqslant h_r \in C_c^\infty(X \times [2^{r-1}, 2^r])$ and h_r need not satisfy any boundary conditions where $s = 2^{r-1}$ or $s = 2^r$. This new condition is equivalent by the arguments of Section 2 to

$$0 \leqslant \hat{K}(t, m_1, m_2) \leqslant c_4 t^{-(N+1)/2}$$

for all $0 < t \leqslant 1$ where $\hat{K}$ is the heat kernel for the operator $\hat{H}$ given when $2^{r-1} < s < 2^r$ by

$$\hat{H}f = -2^{-2r/N} \Delta_x f - 2^{2r} \frac{\partial^2 f}{\partial s^2}$$

with Neumann boundary conditions on the boundary of each strip. But $\hat{K}$ vanishes unless m_1 and m_2 lie in the same strip $X \times [2^{r-1}, 2^r]$, in which case we have

$$\hat{K}(t, m_1, m_2) = K_X(2^{-2r/N}t, x_1, x_2)\, K_r(2^{2r}t, s_1, s_2),$$

where K_r is the heat kernel for $-(d^2/ds^2)$ on $[2^{r-1}, 2^r]$ with Neumann boundary conditions at $s = 2^{r-1}$ and $s = 2^r$. But

$$0 \leqslant K_r(t, s_1, s_2) \leqslant c_5 t^{-1/2} \tag{3.5}$$

whenever $0 < t \leqslant 2^{2r}$ by a direct computation. Combining (3.4) and (3.5) we obtain

$$\begin{aligned} 0 \leqslant \hat{K}(t, m_1, m_2) &\leqslant c(2^{-2r/N}t)^{-N/2}\, c_5(2^{2r}t)^{-1/2} \\ &= cc_5 t^{-(N+1)/2} \end{aligned}$$

for all $0 < t \leqslant 1$, as required to complete the proof.

THEOREM 9. *If the heat kernel of X satisfies the bound* (3.4) *for all* $0 < t \leqslant 1$, *and if* $0 < \delta < 1$ *and* $0 < t < \infty$, *then the heat kernel of M satisfies the bound*

$$\begin{aligned} 0 \leqslant K(t, m_1, m_2) &\leqslant c_\delta (1 + y_1)^{N/2} (1 + y_2)^{N/2}\, t^{-(N+1)/2} \\ &\quad \times \exp\{(2\delta - E)\, t - d(m_1, m_2)^2/4(1+\delta)\, t\}, \end{aligned} \tag{3.6}$$

where E is the bottom of the spectrum of H and $d(m_1, m_2)$ is the Riemannian distance between m_1 and m_2.

Proof. By Lemma 6 the basic condition (2.8) is satisfied for the manifold M of this section, so we may apply Theorem 4.

Note 10. The above bound can be compared with the estimate [10, p. 226],

$$\begin{aligned} 0 \leqslant K(t, m_1, m_2) &\leqslant C(1 + y_1 y_2)^{N/2}\, t^{-(N+1)/2} \\ &\quad \times \exp\{ct - d(m_1, m_2)^2/8t\}. \end{aligned} \tag{3.7}$$

It is easy to see that (3.6) implies (3.7) for a specific constant c. Moreover we do not require that X is compact but only that (3.4) holds, and we treat a more general metric than [10]. On the other hand we have not attempted to control the derivatives of the kernel K as in Proposition 2.49 of [10].

Note 11. Since the kernel $G(\lambda, n, m_1, m_2)$ of $(H+\lambda)^{-n}$ is given by

$$G(\lambda, n, m_1, m_2) = \Gamma(n)^{-1} \int_0^\infty K(t, m_1, m_2)\, e^{-\lambda t} t^{n-1}\, dt$$

we can obtain bounds on G from (3.6). In particular the bound

$$0 \leqslant G(\lambda, n, m_1, m_2) \leqslant c(\lambda, n)(1+y_1)^{N/2}\,(1+y_2)^{N/2}$$

holds if $\lambda + E > 0$ and $n > (N+1)/2$. We conjecture that this result is optimal of its type.

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