On random fixed point theorems of random monotone operators

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Abstract

In this paper, we investigate the existence of random fixed point for random mixed monotone operators and random increasing (decreasing) operators and obtain some new random fixed point theorems.

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1. Introduction

Some random fixed point theorems play a main role in the developing theory of random differential and random integral equations [1]. The study of random fixed point theorems was initiated by Špacček [2] and Hanš [3]. They proved the random contraction mapping theorem. Mukherjea [4] proved the random Schauder fixed point theorem. Sehgal and Waters [5] proved the random Rothe fixed point theorem. The random fixed point theory and applications have been developed rapidly in recent years (see, e.g. [7–10]).

In this paper, we investigate some new problems: the existence of a random fixed point for random monotone operators.

Let $E$ be a separable real Banach space, $(\Omega, \Sigma, \mu)$ be a complete measure space, $(E, \beta)$ a measurable space, where $\beta$ denotes the $\sigma$-algebra of all Borel subsets generated by all open subsets in $E$. $D$ is a nonempty subset of $E$. Let $P$ be a cone on $E$ [6], and hence $P$ defines a partial ordering “$\leq$” as follows: $y - x \in P$, for each $x, y \in E$, $x \leq y$. A cone $P$ in $E$ is said to be normal if there exists a

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constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. If it contains interior points, i.e., $i(P) \neq \emptyset$, then $P$ is called a solid cone. Assume that $u_0, v_0 \in E,$ $u_0 < v_0$ ($u_0 \leq v_0$ but $u_0 \neq v_0$), then the set $\{u_0, v_0\} = \{u \in E \mid u_0 \leq u \leq v_0\}$ is said to a ordered interval in $E$.

**Definition 1.1.** A mapping $T : \Omega \times D \rightarrow E$ is called a random increasing (decreasing) operator if for any fixed $x \in D,$ $T(\cdot, x) : \Omega \rightarrow E$ is measurable, and for any fixed $\omega \in \Omega,$ $T(\omega, \cdot) : D \rightarrow E$ is increasing (decreasing) operator, i.e., $x, y \in D, x \leq y \Rightarrow A(\omega, x) \leq A(\omega, y)$ (or, $A(\omega, x) \geq A(\omega, y)$).

**Definition 1.2.** A mapping $T : \Omega \times D \times D \rightarrow E$ is called a random mixed monotone operator if for any fixed $(x, y) \in D,$ $T(\cdot, x, y) : \Omega \rightarrow E$ is measurable, and for any fixed $\omega \in \Omega,$ $T(\omega, \cdot, \cdot) : D \times D \rightarrow E$ is mixed monotone operator, i.e., $x_1 \leq x_2, y_1 \leq y_2 \Rightarrow A(\omega, x_1, y_1) \leq A(\omega, x_2, y_2)$.

**Definition 1.3** ([1]). A random operator $T : \Omega \times D \rightarrow E$ is said to be continuous if for any fixed $\omega \in \Omega,$ $T(\omega, \cdot) : D \rightarrow E$ is continuous.

**Definition 1.4** ([1]). A mapping $A(\omega) : \Omega \rightarrow \mathcal{L}(E)$ is said to be a random endomorphism of $E$ if $A(\omega)$ is an $\mathcal{L}(E)$-valued random variable, where $\mathcal{L}(E)$ denotes linear bounded operator space of $E$.

**Definition 1.5** ([1]). Assume $A : \Omega \times D \rightarrow E$ be a random operator. If $\xi(\omega) : \Omega \rightarrow E$ is a $E$-valued measurable vector function such that $A(\omega, \xi(\omega)) = \xi(\omega)_{a.e.}$, then $\xi(\omega)$ is called a random fixed point of the random operator $A$.

Let $Z_i$ be separable Banach spaces, $(Z_i, \beta_i)(i = 1, 2)$ measurable spaces. Set $Z = Z_1 \times Z_2$, $\|x\| = \max\{\|x_1\|, \|x_2\|\}$ for each $x = (x_1, x_2)$ in $Z$. Obviously, $(Z, \|\cdot\|)$ is also a separable Banach space and $(Z, \beta_1 \times \beta_2)$ is a measurable space. Moreover, we have the following lemma.

**Lemma 1.6.** Assume that $d_i : \Omega \rightarrow Z_i(i = 1, 2);$ let

$$d_1 \times d_2 : \omega \rightarrow d_1(\omega) \times d_2(\omega) = (d_1(\omega), d_2(\omega)) \in Z.$$   

Then $d_1 \times d_2$ is measurable $\iff d_i(1 = 1, 2)$ are measurable, i.e., $x(\omega) = (x_1(\omega), x_2(\omega)) : \Omega \rightarrow Z$ is measurable $\iff x_i(\omega) : \Omega \rightarrow Z_i$ are measurable, $i = 1, 2$.

### 2. Main Results

**Theorem 2.1.** Let $A, B : \Omega \times [u_0, v_0] \times [u_0, v_0] \rightarrow E$ be two random continuous mixed monotone operators satisfying the following conditions:

(a) there exists a random endomorphism $\beta(\omega) : \Omega \rightarrow \mathcal{L}(E)$, $\|\beta(\omega)\| < 1$ such that

$$A(\omega, v, u) - B(\omega, u, v) \leq \beta(\omega)(v - u), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0;$$

(b) $B(\omega, v, u) \leq A(\omega, u, v), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0;$

(c) there exist random endomorphisms $a(\omega) : \Omega \rightarrow \mathcal{L}(E)$ and $b(\omega) : \Omega \rightarrow \mathcal{L}(E)$ and $\|a(\omega) + b(\omega) + \beta(\omega)\| < 1$ such that

$$u_0 + a(\omega)(v_0 - u_0) \leq B(\omega, u_0, v_0), A(\omega, v_0, u_0) \leq v_0 - b(\omega)(v_0 - u_0).$$

Then the system of random operator equations

$$\begin{cases}
A(\omega, u, u) = u, \\
B(\omega, u, u) = u
\end{cases}$$  

(1)
has a random common unique solution \( u^*(\omega) \) in \([u_0, v_0]\) and the iterative sequences

\[
\begin{align*}
  u_{n+1}(\omega) &= B(\omega, u_n(\omega), v_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\
  v_{n+1}(\omega) &= A(\omega, v_n(\omega), u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)),
\end{align*}
\]

both converge to \( u^*(\omega) \) and have the convergence rate

\[
\|u^*(\omega) - u_n(\omega)(\omega), v_n(\omega)\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n\|v_0 - u_0\|,
\]

where \( N \) is the normal constant of \( P \). Moreover, for any initial \( x_0 \in [u_0, v_0] \), \( x_{n+1}(\omega) = B(\omega, x_n(\omega), x_n(\omega)) \), we have \( u^*(\omega) = \lim_{n \to \infty} x_n(\omega) \).

**Proof.** (i) First, by induction, we can prove that

\[
u_{n-1}(\omega) \leq u_n(\omega) \leq v_n(\omega) \leq v_{n-1}(\omega), \quad \forall \omega \in \Omega, n = 1, 2, \ldots.
\]

(ii) By (a), (2) and (4), we have

\[
\theta \leq u_n(\omega) - u_n(\omega) = A(\omega, v_{n-1}(\omega), u_{n-1}(\omega)) - B(\omega, u_{n-1}(\omega), v_{n-1}(\omega)) \\
+ (a(\omega) + b(\omega))(v_{n-1}(\omega) - u_{n-1}(\omega)) \\
\leq \beta(\omega)(v_{n-1}(\omega) - u_{n-1}(\omega)) + (a(\omega) + b(\omega))(v_{n-1}(\omega) - u_{n-1}(\omega)) \\
= (a(\omega) + b(\omega) + \beta(\omega))(v_{n-1}(\omega) - u_{n-1}(\omega)) \\
\leq \cdots \leq (a(\omega) + b(\omega) + \beta(\omega))^n(v_0 - u_0).
\]

From (4) and (5), we obtain, for any positive integer \( m \),

\[
\theta \leq u_{n+m}(\omega) - u_n(\omega) \leq (a(\omega) + b(\omega) + \beta(\omega))^n(v_0 - u_0),
\]

\[
\theta \leq u_n(\omega) - v_{n+m}(\omega) \leq (a(\omega) + b(\omega) + \beta(\omega))^n(v_0 - u_0).
\]

It follows from (5), (6), (7) and the normality of \( P \) that

\[
\|v_n(\omega) - u_n(\omega)\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n\|v_0 - u_0\|,
\]

\[
\|u_{n+m}(\omega) - u_n(\omega)\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n\|v_0 - u_0\|,
\]

\[
\|v_n(\omega) - v_{n+m}(\omega)\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n\|v_0 - u_0\|.
\]

(9) and (10) imply that \{\( u_n(\omega) \)\} and \{\( v_n(\omega) \)\} are Cauchy sequences in \( E \), hence there exists \( u'(\omega), v'(\omega) \in E \) such that \( \lim_{n \to \infty} u_n(\omega) = u'(\omega), \lim_{n \to \infty} v_n(\omega) = v'(\omega) \) and \( u_n(\omega) \leq u'(\omega) \leq v'(\omega) \leq v_n(\omega) \). By the normality of \( P \) and from (8), we have \( u^*(\omega) \triangleq u'(\omega) = v'(\omega) \in [u_0, v_0] \), and so

\[
u_n(\omega) \leq u^*(\omega) \leq v_n(\omega), \quad n = 0, 1, \ldots.
\]

(iii) Next we prove that \( u^*(\omega) : \Omega \to [u_0, v_0] \) is a random variable.

By (2), we have \( u_1(\omega) = B(\omega, u_0(\omega), v_0(\omega)) - a(\omega)(v_0(\omega) - u_0(\omega)) \). Since \( B(\omega, u_0, v_0) \) is measurable and \( a(\omega)x \) is a random linear continuous operator, \( u_1(\omega) : \Omega \to [u_0, v_0] \) is also measurable. By the measurable theorem of complex operators and Lemma 1.6, it is not difficult to prove that \( u_{n+1}(\omega) \) is measurable. Similarly, we can obtain that \( v_{n+1}(\omega) \) is also measurable. From [1, Theorem 1.6], we have \( u^*(\omega) = \lim_{n \to \infty} u_n(\omega) \) is measurable.

(iv) Now we prove that \( u^*(\omega) \) is the unique common solution of (1) in \([u_0, v_0]\). By hypothesis, noticing (11), we have

\[
u_n(\omega) \leq u_{n+1}(\omega) \leq u_{n+1}(\omega) + a(\omega)(v_n(\omega) - u_n(\omega)) = B(\omega, u_n(\omega), v_n(\omega))
\]
satisfying the following conditions:

\[ u_n(\omega) \leq B(\omega, u^*(\omega), u^*(\omega)) \leq A(\omega, u^*(\omega), u^*(\omega)) \leq A(\omega, v_n(\omega), u_n(\omega)) \]

\[ \leq A(\omega, v_n(\omega), u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)) = v_{n+1}(\omega) \leq v_n(\omega). \]

That is

\[ u_n(\omega) \leq B(\omega, u^*(\omega), u^*(\omega)) \leq A(\omega, u^*(\omega), u^*(\omega)) \leq v_n(\omega). \] (12)

Since \( u_n(\omega) \to u^*(\omega), v_n(\omega) \to u^*(\omega) (n \to \infty) \), we obtain

\[ A(\omega, u^*(\omega), u^*(\omega)) = B(\omega, u^*(\omega), u^*(\omega)). \]

And hence \( u^*(\omega) \) is the random common solution of (1) in \([u_0, v_0]\). Now suppose \( v^*(\omega) \in [u_0, v_0] \) is another solution of (1). By induction, it is easy to prove that

\[ u_n(\omega) \leq v^*(\omega) \leq v_n(\omega), \quad n = 0, 1, \ldots. \] (13)

Since \( u_n(\omega) \to u^*(\omega), v_n(\omega) \to u^*(\omega) (n \to \infty) \), so we obtain \( u^*(\omega) = v^*(\omega) \).

(v) In (9) and (10), taking \( m \to \infty \), we get convergence rate (3).

(vi) For any initial \( x_0 \in [u_0, v_0] \), by hypothesis and induction, it is easy to prove that

\[ u_n(\omega) \leq x_n(\omega) \leq v_n(\omega), \quad n = 0, 1, \ldots. \] (14)

Similarly, since \( u_n(\omega) \to u^*(\omega), v_n(\omega) \to u^*(\omega) \) we have \( \lim_{n \to \infty} x_n(\omega) = u^*(\omega) \). This completes the proof of Theorem 2.1. \( \square \)

**Theorem 2.2.** Let \( A : \Omega \times [u_0, v_0] \times [u_0, v_0] \to E \) be a random continuous mixed monotone operator satisfying the following conditions:

(a) there exists a random endomorphism \( \beta(\omega) : \Omega \to \mathcal{L}(E), \| \beta(\omega) \| < 1 \) such that

\[ A(\omega, v, u) - A(\omega, u, v) \leq \beta(\omega)(v-u), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0; \]

(b) there exist random endomorphisms \( a(\omega) : \Omega \to \mathcal{L}(E) \) and \( b(\omega) : \Omega \to \mathcal{L}(E) \) and \( \| a(\omega) + b(\omega) + \beta(\omega) \| < 1 \) such that

\[ u_0 + a(\omega)(v_0 - u_0) \leq A(\omega, u_0, v_0), A(\omega, v_0, u_0) \leq v_0 - b(\omega)(v_0 - u_0). \]

Then random operator \( A \) has a unique random fixed point \( u^*(\omega) \) in \([u_0, v_0]\) and the iterative sequences

\[
\begin{cases}
  u_{n+1}(\omega) = A(\omega, u_n(\omega), v_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\
  v_{n+1}(\omega) = A(\omega, v_n(\omega), u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)),
\end{cases}
\]

both converge to \( u^*(\omega) \) and have the convergence rate

\[ \| u^*(\omega) - u_0(\omega)(or, v_n(\omega)) \| \leq N \| a(\omega) + b(\omega) + \beta(\omega) \| \| v_0 - u_0 \|, \]

where \( N \) is the normal constant of \( P \). Moreover, for any initial \( x_0 \in [u_0, v_0] \), \( x_{n+1}(\omega) = A(\omega, x_n(\omega), x_n(\omega)) \), we have \( u^*(\omega) = \lim_{n \to \infty} x_n(\omega) \).

**Proof.** We only need to set \( A = B \) in Theorem 2.1. \( \square \)

**Theorem 2.3.** Let \( A : \Omega \times [u_0, v_0] \to E \) be a random continuous increasing operator satisfying the following conditions:

(a) there exists a random endomorphism \( \beta(\omega) : \Omega \to \mathcal{L}(E), \| \beta(\omega) \| < 1 \) such that

\[ A(\omega, v) - A(\omega, u) \leq \beta(\omega)(v-u), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0; \]
(b) there exist random endomorphisms \(a(\omega) : \Omega \to \mathcal{L}(E)\) and \(b(\omega) : \Omega \to \mathcal{L}(E)\) and \(\|a(\omega) + b(\omega) + \beta(\omega)\| < 1\) such that

\[
u_0 + a(\omega)(v_0 - u_0) \leq A(\omega, u_0), A(\omega, v_0) \leq v_0 - b(\omega)(v_0 - u_0).
\]

Then \(A(\omega, x)\) has a unique random fixed point \(x^*(\omega)\) in \([u_0, v_0]\) and the iterative sequences

\[
\begin{align*}
&u_{n+1}(\omega) = A(\omega, u_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\
v_{n+1}(\omega) = A(\omega, v_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)),
\end{align*}
\]

both converge to \(x^*(\omega)\) and have the convergence rate

\[
\|x^*(\omega) - u_n(\omega)(\text{or}, v_n(\omega))\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n\|v_0 - u_0\|,
\]

where \(N\) is the normal constant of \(P\). Moreover, for any initial \(x_0 \in [u_0, v_0]\), \(x_{n+1}(\omega) = A(\omega, x_n(\omega))\), we have \(x^*(\omega) = \lim_{n \to \infty} x_n(\omega)\).

**Proof.** We only need to set \(A(\omega, u, v) = A(\omega, u)\) in Theorem 2.2. \(\square\)

**Theorem 2.4.** Let \(A : \Omega \times [u_0, v_0] \to E\) be a random continuous decreasing operator satisfying the following conditions:

(a) there exists a random endomorphism \(\beta(\omega) : \Omega \to \mathcal{L}(E)\), \(\|\beta(\omega)\| < 1\) such that

\[
A(\omega, u) - A(\omega, v) \leq \beta(\omega)(v - u), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0;
\]

(b) there exist random endomorphisms \(a(\omega) : \Omega \to \mathcal{L}(E)\) and \(b(\omega) : \Omega \to \mathcal{L}(E)\) and \(\|a(\omega) + b(\omega) + \beta(\omega)\| < 1\) such that

\[
u_0 + a(\omega)(v_0 - u_0) \leq A(\omega, v_0), A(\omega, u_0) \leq v_0 - b(\omega)(v_0 - u_0).
\]

Then \(A(\omega, x)\) has a unique random fixed point \(x^*(\omega)\) in \([u_0, v_0]\) and the iterative sequences

\[
\begin{align*}
&u_{n+1}(\omega) = A(\omega, u_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\
v_{n+1}(\omega) = A(\omega, v_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)),
\end{align*}
\]

both converge to \(x^*(\omega)\) and have the convergence rate

\[
\|x^*(\omega) - u_n(\omega)(\text{or}, v_n(\omega))\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n\|v_0 - u_0\|,
\]

where \(N\) is the normal constant of \(P\). Moreover, for any initial \(x_0 \in [u_0, v_0]\), \(x_{n+1}(\omega) = A(\omega, x_n(\omega))\), we have \(x^*(\omega) = \lim_{n \to \infty} x_n(\omega)\).

**Proof.** We only need to set \(A(\omega, u, v) = A(\omega, v)\) in Theorem 2.2. \(\square\)

**Remark 1.** In particular, if \(\beta(\omega), a(\omega), b(\omega)\) in Theorems 2.1–2.4 are measurable functions mapping \(\Omega\) to \([0, 1]\), our conclusions also hold. Indeed, we only need to let \(\beta(\omega)I, a(\omega)I, b(\omega)I\) be corresponding random endomorphisms in Theorems 2.1–2.4, where \(I\) is the identity operator in \(E\).

**Theorem 2.5.** Let \(P\) be a normal and solid cone of \(E\), and let \(A : \Omega \times i(P) \times i(P) \to i(P)\) be a random continuous mixed monotone operator; suppose that

(a) for fixed \((\omega, y)\), \(A(\omega, \cdot, y) : i(P) \to i(P)\) satisfies:

\[
A(\omega, tx, y) \geq t^\alpha A(\omega, x, y), \quad 0 < t < 1, \forall x \in i(P),
\]

and for fixed \((\omega, x)\), \(A(\omega, x, \cdot) : i(P) \to i(P)\) satisfies:

\[
A(\omega, x, sy) \geq s^{-\alpha} A(\omega, x, y), \quad s > 1, \forall y \in i(P),
\]
where $0 < \alpha < \frac{1}{2}$.

(b) there exist $u_0, v_0 \in i(P)$ and $\epsilon > 0$ such that, for every $\omega \in \Omega$

\begin{align*}
\theta &\ll u_0 \leq v_0, u_0 \leq A(\omega, u_0, v_0), A(\omega, v_0, u_0) \leq v_0 \quad (15) \\
A(\omega, \theta, v_0) &\geq \epsilon A(\omega, v_0, u_0). \quad (16)
\end{align*}

Then $A$ has exactly one random fixed point $x^*(\omega)$ in $[u_0, v_0]$, and for any initial $x_0, y_0 \in [u_0, v_0], \ 
\text{constructing successively the sequences}
\begin{align*}
x_n(\omega) = A(\omega, x_{n-1}(\omega), y_{n-1}(\omega)), \quad y_n(\omega) = A(\omega, y_{n-1}(\omega), x_{n-1}(\omega)) \quad (17)
\end{align*}
both converge to $x^*(\omega)$.

\textbf{Proof.} Let
\begin{align*}
u_n(\omega) = A(\omega, u_{n-1}(\omega), v_{n-1}(\omega)), \quad v_n(\omega) = A(\omega, v_{n-1}(\omega), u_{n-1}(\omega))(n = 1, 2, \ldots).
\end{align*}

By induction, it is easy to show
\begin{align*}
\theta &\ll u_0 \leq u_1(\omega) \leq \cdots \leq u_n(\omega) \leq \cdots \leq v_n(\omega) \leq \cdots \leq v_1(\omega) \leq v_0. \quad (18)
\end{align*}

Hence by (16)
\begin{align*}
u_n(\omega) &\geq u_1(\omega) \geq \epsilon v_1(\omega) \geq \epsilon v_n(\omega). \quad (19)
\end{align*}

Set
\begin{align*}
t_n(\omega) = \sup\{t(\omega) > 0 \mid u_n(\omega) \geq t(\omega)v_n(\omega)\} \quad (n = 1, 2, \ldots), \quad (20)
\end{align*}
then
\begin{align*}
u_n(\omega) &\geq t_n(\omega)v_n(\omega), \quad (21)
\end{align*}
and on account of the fact $u_{n+1}(\omega) \geq u_n(\omega) \geq t_n(\omega)v_n(\omega) \geq t_n(\omega)v_{n+1}(\omega)$, we have
\begin{align*}
0 &\leq \epsilon \leq t_1(\omega) \leq t_2(\omega) \leq \cdots \leq t_n(\omega) \leq \cdots \leq 1, \quad (22)
\end{align*}
which implies that $\lim_{n \to \infty} t_n(\omega) = t^*(\omega)$ exists and $\epsilon \leq t^*(\omega) \leq 1$. By condition (a), it is not difficult to prove that $t^*(\omega) = 1$. From (18) and (21), we have
\begin{align*}
\theta &\leq u_{n+m}(\omega) - u_n(\omega) \leq v_n(\omega) - u_n(\omega) \leq (1 - t_n(\omega))v_n(\omega) \leq (1 - t_n(\omega))v_0, \\
\theta &\leq v_n(\omega) - v_{n+m}(\omega) \leq v_n(\omega) - u_n(\omega) \leq (1 - t_n(\omega))v_n(\omega) \leq (1 - t_n(\omega))v_0.
\end{align*}

Since $P$ is normal and $t_n(\omega) \to 1$, $\{u_n(\omega)\}$ and $\{v_n(\omega)\}$ are Cauchy sequences in $E$, hence there exists $u^*(\omega), v^*(\omega) \in E$ such that $\lim_{n \to \infty} u_n(\omega) = u^*(\omega), \lim_{n \to \infty} v_n(\omega) = v^*(\omega)$ and
\begin{align*}
u_n(\omega) &\leq u^*(\omega) \leq v^*(\omega) \leq u_n(\omega). \quad (23)
\end{align*}

By the normality of $P$ and $v_n(\omega) - u_n(\omega) \leq 2(1 - t_n(\omega))v_0$, we get $x^*(\omega) = u^*(\omega) = v^*(\omega)$. Since $A(\omega, x, y)$ is continuous in $(x, y)$, we have $x^*(\omega) = A(\omega, x^*(\omega), x^*(\omega))$. Also since $A(\omega, x, y)$ is a random continuous operator, it follows from Lemma 1.6 and the measurable theorem of complex operators that $u_n(\omega), v_n(\omega)(n = 1, 2, \ldots)$ are all measurable, and hence $x^*(\omega)$ is also measurable. The fact that $u_0 \leq u_n(\omega) \leq x^*(\omega) \leq v_n(\omega) \leq v_0$ shows that $x^*(\omega)$ is a random fixed point of $A(\omega, x, y)$ in $[u_0, v_0]$. 

Next we prove that $x^*(\omega)$ is unique. Indeed, suppose $x'(\omega)$ is another random fixed point in $[u_0, v_0]$. By induction, it is easy to prove that

$$u_n(\omega) \leq x'(\omega) \leq v_n(\omega), \quad \forall \omega \in \Omega, \; n = 1, 2, \ldots.$$  \hspace{1cm} (24)

Since $u_n(\omega) \to x^*(\omega), v_n(\omega) \to x^*(\omega)$ and $P$ is normal, by (24) we obtain $x'(\omega) = x^*(\omega)$.

Finally, similar to (25), for every $(x_0, y_0) \in [u_0, v_0], \omega \in \Omega$, we have

$$u_n(\omega) \leq x_n(\omega) \leq v_n(\omega), \quad u_n(\omega) \leq y_n(\omega) \leq v_n(\omega), \quad n = 1, 2, \ldots.$$  \hspace{1cm} (24)

Since $u_n(\omega) \to x^*(\omega), v_n(\omega) \to x^*(\omega)$ and $P$ is normal, we have

$$\|x_n(\omega) - x^*(\omega)\| \to 0, \; \|y_n(\omega) - x^*(\omega)\| \to 0, \; n \in \infty.$$  \hspace{1cm} (24)

This completes the proof of Theorem 2.5. \hfill \Box

3. Applications

We consider the following random Hammerstein integral equation (\ast):

$$x(\omega, t) = Ax(\omega, t) = \int_{-\infty}^{+\infty} k(\omega, t, s)(1 + \sqrt{x(\omega, s)})ds.$$  \hspace{1cm} (25)

Suppose that

(i) the kernel $k(\omega, t, s)$ is non-negative, bounded and random continuous on $\Omega \times R^1 \times R^1$.

(ii) for any bounded continuous functions $u(t), v(t)$ satisfying the following condition:

$$1/9 \leq u(t) \leq v(t) \leq 1,$$

there exists $\beta \in (0, 1)$ such that for any $\omega \in \Omega$,

$$\int_{-\infty}^{+\infty} k(\omega, t, s)[\sqrt{u(s)} - \sqrt{v(s)}]ds \leq \beta[v(t) - u(t)].$$

(iii) there exists $a, b \in [0, 1]$ and $a + b + \beta < 1$, such that for any $\omega \in \Omega$,

$$\frac{3}{4} \left( \frac{1}{9} + \frac{8}{9}a \right) \leq \int_{-\infty}^{+\infty} k(\omega, t, s) \left( 1 + \sqrt{x(\omega, s)} \right)ds \leq \frac{1}{2} \left( 1 - \frac{8}{9}b \right).$$

Then for equation (\ast) there exists a unique random continuous solution $x^*(\omega, t)$ and $\frac{1}{9} \leq x^*(\omega, t) \leq 1$.

Proof. It is easy to prove the conclusion using Theorem 2.3. \hfill \Box

References

[9] H.K. Xu, Some random fixed point theorems for condensing and nonexpansive operators, Proc. Amer. Math. Soc. 110 (1990) 395–400.