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On random fixed point theorems of random monotone operators

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Abstract

In this paper, we investigate the existence of random fixed point for random mixed monotone operators and random increasing (decreasing) operators and obtain some new random fixed point theorems.

Keywords: Random mixed monotone operators; Random increasing (decreasing) operators; Random fixed point

1. Introduction

Some random fixed point theorems play a main role in the developing theory of random differential and random integral equations [1]. The study of random fixed point theorems was initiated by Špacček [2] and Hanš [3]. They proved the random contraction mapping theorem. Mukherjea [4] proved the random Schauder fixed point theorem. Sehgaland and Waters [5] proved the random Rothe fixed point theorem. The random fixed point theory and applications have been developed rapidly in recent years (see, e.g. [7–10]).

In this paper, we investigate some new problems: the existence of a random fixed point for random monotone operators.

Let E be a separable real Banach space, (Ω, Σ, μ) be a complete measure space, (E, β) a measurable space, where β denotes the σ-algebra of all Borel subsets generated by all open subsets in E. D is a nonempty subset of E. Let P be a cone on E [6], and hence P defines a partial ordering "$\leq$" as follows: $y - x \in P$, for each $x, y \in E \iff x \leq y$. A cone P in E is said to be normal if there exists a

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constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. If it contains interior points, i.e., $i(P) \ne \emptyset$, then P is called a solid cone. Assume that $u_0, v_0 \in E, u_0 < v_0$ ($u_0 \le v_0$ but $u_0 \ne v_0$), then the set $[u_0, v_0] = \{u \in E \mid u_0 \le u \le v_0\}$ is said to a ordered interval in E.

Definition 1.1. A mapping $T : \Omega \times D \to E$ is called a random increasing (decreasing) operator if for any fixed $x \in D$, $T(\cdot, x) : \Omega \to E$ is measurable, and for any fixed $\omega \in \Omega$, $T(\omega, \cdot) : D \to E$ is increasing (decreasing) operator, i.e., $x, y \in D, x \le y \Rightarrow A(\omega, x) \le A(\omega, y)(or, A(\omega, x) \ge A(\omega, y))$.

Definition 1.2. A mapping $T : \Omega \times D \times D \to E$ is called a random mixed monotone operator if for any fixed $(x, y) \in D$, $T(\cdot, x, y) : \Omega \to E$ is measurable, and for any fixed $\omega \in \Omega$, $T(\omega, \cdot, \cdot) : D \times D \to E$ is mixed monotone operator, i.e., $x_1 \le x_2, y_2 \le y_1 \Rightarrow A(\omega, x_1, y_1) \le A(\omega, x_2, y_2)$.

Definition 1.3 ([1]). A random operator $T : \Omega \times D \to E$ is said to be continuous if for any fixed $\omega \in \Omega$, $T(\omega, \cdot) : D \to E$ is continuous.

Definition 1.4 ([1]). A mapping $A(\omega) : \Omega \to \mathcal{L}(E)$ is said to be a random endomorphism of E if $A(\omega)$ is an $\mathcal{L}(E)$-valued random variable, where $\mathcal{L}(E)$ denotes linear bounded operator space of E.

Definition 1.5 ([1]). Assume $A : \Omega \times D \to E$ be a random operator. If $\xi(\omega) : \Omega \to E$ is a E-valued measurable vector function such that $A(\omega, \xi(\omega)) = \xi(\omega)_{a.e.}$, then $\xi(\omega)$ is called a random fixed point of the random operator A.

Let Z_i be separable Banach spaces, $(Z_i, \beta_i)(i = 1, 2)$ measurable spaces. Set $Z = Z_1 \times Z_2$, $\|x\| = \mathrm{Max}\{\|x_1\|, \|x_2\|\}$ for each $x = (x_1, x_2)$ in Z. Obviously, $(Z, \|\cdot\|)$ is also a separable Banach space and $(Z, \beta_1 \times \beta_2)$ is a measurable space. Moreover, we have the following lemma.

Lemma 1.6. *Assume that $d_i : \Omega \to Z_i(i = 1, 2)$; let*

$$d_1 \times d_2 : \omega \to d_i(\omega) \times d_i(\omega) = (d_i(\omega), d_i(\omega)) \in Z.$$

Then $d_1 \times d_2$ is measurable $\iff$ $d_i(i = 1, 2)$ are measurable, i.e., $x(\omega) = (x_1(\omega), x_2(\omega)) : \Omega \to Z$ is measurable $\iff$ $x_i(\omega) : \Omega \to Z_i$ are measurable, $i = 1, 2$.

2. Main results

Theorem 2.1. *Let $A, B : \Omega \times [u_0, v_0] \times [u_0, v_0] \to E$ be two random continuous mixed monotone operators satisfying the following conditions:*

(a) *there exists a random endomorphism $\beta(\omega) : \Omega \to \mathcal{L}(E)$, $\|\beta(\omega)\| < 1$ such that*

$$A(\omega, v, u) - B(\omega, u, v) \le \beta(\omega)(v - u), \forall \omega \in \Omega, u_0 \le u \le v \le v_0;$$

(b) $B(\omega, v, u) \le A(\omega, u, v), \forall \omega \in \Omega, u_0 \le u \le v \le v_0$;

(c) *there exist random endomorphisms $a(\omega) : \Omega \to \mathcal{L}(E)$ and $b(\omega) : \Omega \to \mathcal{L}(E)$ and $\|a(\omega) + b(\omega) + \beta(\omega)\| < 1$ such that*

$$u_0 + a(\omega)(v_0 - u_0) \le B(\omega, u_0, v_0), A(\omega, v_0, u_0) \le v_0 - b(\omega)(v_0 - u_0).$$

Then the system of random operator equations

$$\begin{cases} A(\omega, u, u) = u, \\ B(\omega, u, u) = u \end{cases} \tag{1}$$

has a random common unique solution $u^*(\omega)$ in $[u_0, v_0]$ and the iterative sequences

$$\begin{cases} u_{n+1}(\omega) = B(\omega, u_n(\omega), v_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\ v_{n+1}(\omega) = A(\omega, v_n(\omega), u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)), n = 0, 1, \ldots \end{cases} \tag{2}$$

both converge to $u^*(\omega)$ and have the convergence rate

$$\|u^*(\omega) - u_n(\omega)(or, v_n(\omega))\| \le N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|, \tag{3}$$

where N is the normal constant of P. Moreover, for any initial $x_0 \in [u_0, v_0]$, $x_{n+1}(\omega) = B(\omega, x_n(\omega), x_n(\omega))$, we have $u^*(\omega) = \lim_{n\to\infty} x_n(\omega)$.

Proof. (i) First, by induction, we can prove that

$$u_{n-1}(\omega) \le u_n(\omega) \le v_n(\omega) \le v_{n-1}(\omega), \qquad \forall \omega \in \Omega, n = 1, 2, \ldots. \tag{4}$$

(ii) By (a), (2) and (4), we have

$$\begin{aligned} \theta \le v_n(\omega) - u_n(\omega) &= A(\omega, v_{n-1}(\omega), u_{n-1}(\omega)) - B(\omega, u_{n-1}(\omega), v_{n-1}(\omega)) \\ &\quad + (a(\omega) + b(\omega))(v_{n-1}(\omega) - u_{n-1}(\omega)) \\ &\le \beta(\omega)(v_{n-1}(\omega) - u_{n-1}(\omega)) + (a(\omega) + b(\omega))(v_{n-1}(\omega) - u_{n-1}(\omega)) \\ &= (a(\omega) + b(\omega) + \beta(\omega))(v_{n-1}(\omega) - u_{n-1}(\omega)) \\ &\le \cdots \le (a(\omega) + b(\omega) + \beta(\omega))^n (v_0 - u_0). \end{aligned} \tag{5}$$

From (4) and (5), we obtain, for any positive integer m,

$$\theta \le u_{n+m}(\omega) - u_n(\omega) \le (a(\omega) + b(\omega) + \beta(\omega))^n (v_0 - u_0), \tag{6}$$

$$\theta \le v_n(\omega) - v_{n+m}(\omega) \le (a(\omega) + b(\omega) + \beta(\omega))^n (v_0 - u_0). \tag{7}$$

It follows from (5), (6), (7) and the normality of P that

$$\|v_n(\omega) - u_n(\omega)\| \le N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|, \tag{8}$$

$$\|u_{n+m}(\omega) - u_n(\omega)\| \le N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|, \tag{9}$$

$$\|v_n(\omega) - v_{n+m}(\omega)\| \le N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|. \tag{10}$$

(9) and (10) imply that $\{u_n(\omega)\}$ and $\{v_n(\omega)\}$ are Cauchy sequences in E, hence there exists $u'(\omega), v'(\omega) \in E$ such that $\lim_{n\to\infty} u_n(\omega) = u'(\omega)$, $\lim_{n\to\infty} v_n(\omega) = v'(\omega)$ and $u_n(\omega) \le u'(\omega) \le v'(\omega) \le v_n(\omega)$. By the normality of P and from (8), we have $u^*(\omega) \triangleq u'(\omega) = v'(\omega) \in [u_0, v_0]$, and so

$$u_n(\omega) \le u^*(\omega) \le v_n(\omega), \qquad n = 0, 1, \ldots. \tag{11}$$

(iii) Next we prove that $u^*(\omega) : \Omega \to [u_0, v_0]$ is a random variable.

By (2), we have $u_1(\omega) = B(\omega, u_0, v_0) - a(\omega)(v_0 - u_0)$. Since $B(\omega, u_0, v_0)$ is measurable and $a(\omega)x$ is a random linear continuous operator, $u_1(\omega) : \Omega \to [u_0, v_0]$ is also measurable. By the measurable theorem of complex operators and Lemma 1.6, it is not difficult to prove that $u_{n+1}(\omega)$ is measurable. Similarly, we can obtain that $v_{n+1}(\omega)$ is also measurable. From [1, Theorem 1.6], we have $u^*(\omega) = \lim_{n\to\infty} u_n(\omega)$ is measurable.

(iv) Now we prove that $u^*(\omega)$ is the unique common solution of (1) in $[u_0, v_0]$. By hypothesis, noticing (11), we have

$$u_n(\omega) \le u_{n+1}(\omega) \le u_{n+1}(\omega) + a(\omega)(v_n(\omega) - u_n(\omega)) = B(\omega, u_n(\omega), v_n(\omega))$$

$$\leq B(\omega, u^*(\omega), u^*(\omega)) \leq A(\omega, u^*(\omega), u^*(\omega)) \leq A(\omega, v_n(\omega), u_n(\omega))$$
$$\leq A(\omega, v_n(\omega), u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)) = v_{n+1}(\omega) \leq v_n(\omega).$$

That is

$$u_n(\omega) \leq B(\omega, u^*(\omega), u^*(\omega)) \leq A(\omega, u^*(\omega), u^*(\omega)) \leq v_n(\omega). \tag{12}$$

Since $u_n(\omega) \to u^*(\omega)$, $v_n(\omega) \to u^*(\omega)(n \to \infty)$, we obtain

$$A(\omega, u^*(\omega), u^*(\omega)) = B(\omega, u^*(\omega), u^*(\omega)).$$

And hence $u^*(\omega)$ is the random common solution of (1) in $[u_0, v_0]$. Now suppose $v^*(\omega) \in [u_0, v_0]$ is another solution of (1). By induction, it is easy to prove that

$$u_n(\omega) \leq v^*(\omega) \leq v_n(\omega), \qquad n = 0, 1, \ldots. \tag{13}$$

Since $u_n(\omega) \to u^*(\omega)$, $v_n(\omega) \to u^*(\omega)(n \to \infty)$, so we obtain $u^*(\omega) = v^*(\omega)$.

(v) In (9) and (10), taking $m \to \infty$, we get convergence rate (3).

(vi) For any initial $x_0 \in [u_0, v_0]$, by hypothesis and induction, it is easy to prove that

$$u_n(\omega) \leq x_n(\omega) \leq v_n(\omega), \qquad n = 0, 1, \ldots. \tag{14}$$

Similarly, since $u_n(\omega) \to u^*(\omega)$, $v_n(\omega) \to u^*(\omega)$ we have $\lim_{n\to\infty} x_n(\omega) = u^*(\omega)$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $A : \Omega \times [u_0, v_0] \times [u_0, v_0] \to E$ be a random continuous mixed monotone operator satisfying the following conditions:*

(a) *there exists a random endomorphism $\beta(\omega) : \Omega \to \mathcal{L}(E)$, $\|\beta(\omega)\| < 1$ such that*

$$A(\omega, v, u) - A(\omega, u, v) \leq \beta(\omega)(v - u), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0;$$

(b) *there exist random endomorphisms $a(\omega) : \Omega \to \mathcal{L}(E)$ and $b(\omega) : \Omega \to \mathcal{L}(E)$ and $\|a(\omega) + b(\omega) + \beta(\omega)\| < 1$ such that*

$$u_0 + a(\omega)(v_0 - u_0) \leq A(\omega, u_0, v_0), A(\omega, v_0, u_0) \leq v_0 - b(\omega)(v_0 - u_0).$$

Then random operator A has a unique random fixed point $u^*(\omega)$ in $[u_0, v_0]$ and the iterative sequences

$$\begin{cases} u_{n+1}(\omega) = A(\omega, u_n(\omega), v_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\ v_{n+1}(\omega) = A(\omega, v_n(\omega), u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)), n = 0, 1, \ldots \end{cases}$$

both converge to $u^*(\omega)$ and have the convergence rate

$$\|u^*(\omega) - u_n(\omega)(or, v_n(\omega))\| \leq N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|,$$

where N is the normal constant of P. Moreover, for any initial $x_0 \in [u_0, v_0]$, $x_{n+1}(\omega) = A(\omega, x_n(\omega), x_n(\omega))$, we have $u^*(\omega) = \lim_{n\to\infty} x_n(\omega)$.

Proof. We only need to set $A = B$ in Theorem 2.1. □

Theorem 2.3. *Let $A : \Omega \times [u_0, v_0] \to E$ be a random continuous increasing operator satisfying the following conditions:*

(a) *there exists a random endomorphism $\beta(\omega) : \Omega \to \mathcal{L}(E)$, $\|\beta(\omega)\| < 1$ such that*

$$A(\omega, v) - A(\omega, u) \leq \beta(\omega)(v - u), \forall \omega \in \Omega, u_0 \leq u \leq v \leq v_0;$$

(b) *there exist random endomorphisms* $a(\omega) : \Omega \to \mathcal{L}(E)$ *and* $b(\omega) : \Omega \to \mathcal{L}(E)$ *and* $\|a(\omega) + b(\omega) + \beta(\omega)\| < 1$ *such that*

$$u_0 + a(\omega)(v_0 - u_0) \le A(\omega, u_0),\ A(\omega, v_0) \le v_0 - b(\omega)(v_0 - u_0).$$

Then $A(\omega, x)$ *has a unique random fixed point* $x^*(\omega)$ *in* $[u_0, v_0]$ *and the iterative sequences*

$$\begin{cases} u_{n+1}(\omega) = A(\omega, u_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\ v_{n+1}(\omega) = A(\omega, v_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)), n = 0, 1, \ldots \end{cases}$$

both converge to $x^*(\omega)$ *and have the convergence rate*

$$\|x^*(\omega) - u_n(\omega)(or, v_n(\omega))\| \le N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|,$$

where N *is the normal constant of* P*. Moreover, for any initial* $x_0 \in [u_0, v_0]$*,* $x_{n+1}(\omega) = A(\omega, x_n(\omega))$*, we have* $x^*(\omega) = \lim_{n\to\infty} x_n(\omega)$.

Proof. We only need to set $A(\omega, u, v) = A(\omega, u)$ in Theorem 2.2. □

Theorem 2.4. *Let* $A : \Omega \times [u_0, v_0] \to E$ *be a random continuous decreasing operator satisfying the following conditions:*

(a) *there exists a random endomorphism* $\beta(\omega) : \Omega \to \mathcal{L}(E)$*,* $\|\beta(\omega)\| < 1$ *such that*

$$A(\omega, u) - A(\omega, v) \le \beta(\omega)(v - u), \forall\omega \in \Omega, u_0 \le u \le v \le v_0;$$

(b) *there exist random endomorphisms* $a(\omega) : \Omega \to \mathcal{L}(E)$ *and* $b(\omega) : \Omega \to \mathcal{L}(E)$ *and* $\|a(\omega) + b(\omega) + \beta(\omega)\| < 1$ *such that*

$$u_0 + a(\omega)(v_0 - u_0) \le A(\omega, v_0),\ A(\omega, u_0) \le v_0 - b(\omega)(v_0 - u_0).$$

Then $A(\omega, x)$ *has a unique random fixed point* $x^*(\omega)$ *in* $[u_0, v_0]$ *and the iterative sequences*

$$\begin{cases} u_{n+1}(\omega) = A(\omega, v_n(\omega)) - a(\omega)(v_n(\omega) - u_n(\omega)), \\ v_{n+1}(\omega) = A(\omega, u_n(\omega)) + b(\omega)(v_n(\omega) - u_n(\omega)), n = 0, 1, \ldots \end{cases}$$

both converge to $x^*(\omega)$ *and have the convergence rate*

$$\|x^*(\omega) - u_n(\omega)(or, v_n(\omega))\| \le N\|a(\omega) + b(\omega) + \beta(\omega)\|^n \|v_0 - u_0\|,$$

where N *is the normal constant of* P*. Moreover, for any initial* $x_0 \in [u_0, v_0]$*,* $x_{n+1}(\omega) = A(\omega, x_n(\omega))$*, we have* $x^*(\omega) = \lim_{n\to\infty} x_n(\omega)$.

Proof. We only need to set $A(\omega, u, v) = A(\omega, v)$ in Theorem 2.2. □

Remark 1. In particular, if $\beta(\omega), a(\omega), b(\omega)$ in Theorems 2.1–2.4 are measurable functions mapping Ω to $[0, 1]$, our conclusions also hold. Indeed, we only need to let $\beta(\omega)I, a(\omega)I, b(\omega)I$ be corresponding random endomorphisms in Theorems 2.1–2.4, where I is the identity operator in E.

Theorem 2.5. *Let* P *be a normal and solid cone of* E*, and let* $A : \Omega \times i(P) \times i(P) \to i(P)$ *be a random continuous mixed monotone operator; suppose that*

(a) *for fixed* (ω, y)*,* $A(\omega, \cdot, y) : i(P) \to i(P)$ *satisfies:*

$$A(\omega, tx, y) \ge t^\alpha A(\omega, x, y), \quad 0 < t < 1, \forall x \in i(P),$$

and for fixed (ω, x)*,* $A(\omega, x, \cdot) : i(P) \to i(P)$ *satisfies:*

$$A(\omega, x, sy) \ge s^{-\alpha} A(\omega, x, y), \quad s > 1, \forall y \in i(P),$$

where $0 < \alpha < \frac{1}{2}$.

(b) *there exist* $u_0, v_0 \in i(P)$ *and* $\epsilon > 0$ *such that, for every* $\omega \in \Omega$

$$\theta \ll u_0 \le v_0, u_0 \le A(\omega, u_0, v_0), A(\omega, v_0, u_0) \le v_0 \tag{15}$$

$$A(\omega, \theta, v_0) \ge \epsilon A(\omega, v_0, u_0). \tag{16}$$

Then A *has exactly one random fixed point* $x^*(\omega)$ *in* $[u_0, v_0]$, *and for any initial* x_0, $y_0 \in [u_0, v_0]$, *constructing successively the sequences*

$$x_n(\omega) = A(\omega, x_{n-1}(\omega), y_{n-1}(\omega)), \quad y_n(\omega) = A(\omega, y_{n-1}(\omega), x_{n-1}(\omega)) \tag{17}$$

both converge to $x^*(\omega)$.

Proof. Let

$$u_n(\omega) = A(\omega, u_{n-1}(\omega), v_{n-1}(\omega)), \quad v_n(\omega) = A(\omega, v_{n-1}(\omega), u_{n-1}(\omega))(n = 1, 2, \ldots).$$

By induction, it is easy to show

$$\theta \ll u_0 \le u_1(\omega) \le \cdots \le u_n(\omega) \le \cdots \le v_n(\omega) \le \cdots \le v_1(\omega) \le v_0. \tag{18}$$

Hence by (16)

$$u_n(\omega) \ge u_1(\omega) \ge \epsilon v_1(\omega) \ge \epsilon v_n(\omega). \tag{19}$$

Set

$$t_n(\omega) = \sup\{t(\omega) > 0 \mid u_n(\omega) \ge t(\omega) v_n(\omega)\} \quad (n = 1, 2, \ldots), \tag{20}$$

then

$$u_n(\omega) \ge t_n(\omega) v_n(\omega), \tag{21}$$

and on account of the fact $u_{n+1}(\omega) \ge u_n(\omega) \ge t_n(\omega) v_n(\omega) \ge t_n(\omega) v_{n+1}(\omega)$, we have

$$0 < \epsilon \le t_1(\omega) \le t_2(\omega) \le \cdots \le t_n(\omega) \le \cdots \le 1, \tag{22}$$

which implies that $\lim_{n\to\infty} t_n(\omega) = t^*(\omega)$ exists and $\epsilon \le t^*(\omega) \le 1$. By condition (a), it is not difficult to prove that $t^*(\omega) = 1$. From (18) and (21), we have

$$\theta \le u_{n+m}(\omega) - u_n(\omega) \le v_n(\omega) - u_n(\omega) \le (1 - t_n(\omega)) v_n(\omega) \le (1 - t_n(\omega)) v_0,$$

$$\theta \le v_n(\omega) - v_{n+m}(\omega) \le v_n(\omega) - u_n(\omega) \le (1 - t_n(\omega)) v_n(\omega) \le (1 - t_n(\omega)) v_0.$$

Since P is normal and $t_n(\omega) \to 1$, $\{u_n(\omega)\}$ and $\{v_n(\omega)\}$ are Cauchy sequences in E, hence there exists $u^*(\omega), v^*(\omega) \in E$ such that $\lim_{n\to\infty} u_n(\omega) = u^*(\omega)$, $\lim_{n\to\infty} v_n(\omega) = v^*(\omega)$ and

$$u_n(\omega) \le u^*(\omega) \le v^*(\omega) \le v_n(\omega). \tag{23}$$

By the normality of P and $v_n(\omega) - u_n(\omega) \le 2(1 - t_n(\omega)) v_0$, we get $x^*(\omega) = u^*(\omega) = v^*(\omega)$. Since $A(\omega, x, y)$ is continuous in (x, y), we have $x^*(\omega) = A(\omega, x^*(\omega), x^*(\omega))$. Also since $A(\omega, x, y)$ is a random continuous operator, it follows from Lemma 1.6 and the measurable theorem of complex operators that $u_n(\omega), v_n(\omega)(n = 1, 2, \ldots)$ are all measurable, and hence $x^*(\omega)$ is also measurable. The fact that $u_0 \le u_n(\omega) \le x^*(\omega) \le v_n(\omega) \le v_0$ shows that $x^*(\omega)$ is a random fixed point of $A(\omega, x, y)$ in $[u_0, v_0]$.

Next we prove that $x^*(\omega)$ is unique. Indeed, suppose $x'(\omega)$ is another random fixed point in $[u_0, v_0]$. By induction, it is easy to prove that

$$u_n(\omega) \le x'(\omega) \le v_n(\omega), \quad \forall \omega \in \Omega, n = 1, 2, \ldots. \tag{24}$$

Since $u_n(\omega) \to x^*(\omega)$, $v_n(\omega) \to x^*(\omega)$ and P is normal, by (24) we obtain $x'(\omega) = x^*(\omega)$.

Finally, similar to (25), for every $(x_0, y_0) \in [u_0, v_0]$, $\omega \in \Omega$, we have

$$u_n(\omega) \le x_n(\omega) \le v_n(\omega), u_n(\omega) \le y_n(\omega) \le v_n(\omega), \quad n = 1, 2, \ldots.$$

Since $u_n(\omega) \to x^*(\omega)$, $v_n(\omega) \to x^*(\omega)$ and P is normal, we have

$$\|x_n(\omega) - x^*(\omega)\| \to 0, \|y_n(\omega) - x^*(\omega)\| \to 0, n \in \infty.$$

This completes the proof of Theorem 2.5. □

3. Applications

We consider the following random Hammerstein integral equation $(*)$:

$$x(\omega, t) = Ax(\omega, t) = \int_{-\infty}^{+\infty} k(\omega, t, s)(1 + \sqrt{x(\omega, s)})\mathrm{d}s.$$

Suppose that

(i) the kernel $k(\omega, t, s)$ is non-negative, bounded and random continuous on $\Omega \times R^1 \times R^1$.

(ii) for any bounded continuous functions $u(t)$, $v(t)$ satisfying the following condition:

$$\frac{1}{9} \le u(t) \le v(t) \le 1,$$

there exists $\beta \in (0, 1)$ such that for any $\omega \in \Omega$,

$$\int_{-\infty}^{+\infty} k(\omega, t, s)[\sqrt{v(s)} - \sqrt{u(s)}]\mathrm{d}s \le \beta[v(t) - u(t)].$$

(iii) there exists $a, b \in [0, 1]$ and $a + b + \beta < 1$, such that for any $\omega \in \Omega$,

$$\frac{3}{4}\left(\frac{1}{9} + \frac{8}{9}a\right) \le= \int_{-\infty}^{+\infty} k(\omega, t, s)\left(1 + \sqrt{x(\omega, s)}\right)\mathrm{d}s \le \frac{1}{2}\left(1 - \frac{8}{9}b\right).$$

Then for equation $(*)$ there exists a unique random continuous solution $x^*(\omega, t)$ and $\frac{1}{9} \le x^*(\omega, t) \le 1$.

Proof. It is easy to prove the conclusion using Theorem 2.3. □

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