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Sliding piece puzzles with oriented tiles[1]

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Abstract

In this paper, we consider n identical tiles which are placed on the $n+1$ vertices of a graph and which move along the edges of the graph. The tiles come with an "orientation", an element of an arbitrary finite group H. Moving a tile along a given edge into the empty vertex changes the orientation of the tile in a prescribed way. We study the group of oriented positions of the tiles achievable from an initial position which fix the empty vertex. It may be thought of as a subgroup of the semidirect product $H^n \rtimes S_n$ or the wreath product $H \operatorname{wr} S_n$.

In a certain type of sliding piece puzzle, tiles are located on n of the $n+1$ vertices of a graph and are allowed to move along an edge into the empty vertex. The famous 14/15 puzzle [10] is the best known representative of such a sliding piece puzzle. The most basic question is to classify the possible rearrangements. The question is often normalized by requiring that the original empty vertex is again empty at the end of all the moves. We may then consider the possible rearrangements as a subgroup of the symmetric group S_n. This question has been completely answered by the following theorem. Before stating the theorem, we define θ_0 to be the special graph illustrated in Fig. 1.

Theorem (Wilson [12, Theorem 2]). *Let W be the group of rearrangements of a sliding piece puzzle on a finite, simple, nonseparable graph $\mathcal{G}$ with $n+1$ vertices.*

(1) *If $\mathcal{G}$ is a polygon, then $W \cong \mathbb{Z}_n$.*

(2) *If $\mathcal{G}$ is neither a polygon nor bipartite, then $W \cong A_n$.*

(3) *If $\mathcal{G}$ is not a polygon and is not bipartite, then $W \cong S_n$, except when $\mathcal{G} \cong \theta_0$, in which case the group is isomorphic to $\langle(1,2,3,4),(1,4,5,6)\rangle$, a group of order 120 isomorphic to S_5.*

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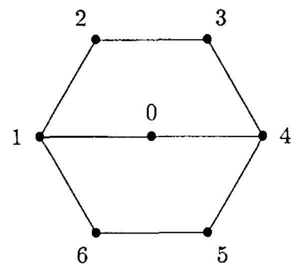

Fig. 1.

A brief discussion of Wilson's theorem is included as part of a broader treatment of similar puzzles in [1, pp. 756–760]. For the graph theory needed in the above theorem and in what follows, see [11].

We generalize the problem by considering n identical tiles with orientations which may change as the tiles move along a graph $\mathscr{G}$. Throughout, we assume that $\mathscr{G}$ is connected. Specifically, number the vertices of the graph from 0 to n, with 0 denoting the empty vertex, and let H be a finite group. When an edge exists between vertices i and j, let η_{ij} and η_{ji} be elements of H such that $\eta_{ji} = \eta_{ij}^{-1}$. To each tile assign an element of H, initially the identity. We call this the *orientation* of the tile. As a tile moves from vertex i to vertex j, its orientation is multiplied on the right by η_{ij}. Thus, a rearrangement of the tiles fixing the empty vertex may be specified by how the tile beginning in each position is reoriented, collectively an element of H^n, followed by how the tiles are permuted, an element of S_n. We see that the possible rearrangements form a group G under composition. Our main result, Theorem 3, is in the same spirit as the result of Wilson stated above, namely that under a suitable normalization the set of attainable states of the puzzle on a nonseparable, nonpolygonal graph is nearly as large as the set of all rearrangements of the tiles and their orientations. In particular, one can nearly always achieve either half or all of the possible permutations of the tiles and half or all of the possible reorientations of the set of tiles. In fact, the group G of possible rearrangements may be viewed as a subgroup of the semidirect product $H^n \rtimes S_n$. Here we compose on the right, i.e. $(\boldsymbol{h}_1, \sigma_1)(\boldsymbol{h}_2, \sigma_2) = (\boldsymbol{h}_1 \boldsymbol{h}_2^{\sigma_1^{-1}}, \sigma_1\sigma_2)$, where $\sigma_1\sigma_2$ denotes σ_1 followed by σ_2 and the action of S_n on H^n is defined by $\boldsymbol{h}^{\sigma^{-1}} = (h_{1^\sigma}, h_{2^\sigma}, \ldots, h_{n^\sigma})$. Alternately, one may view G as a subgroup of the wreath product $H \operatorname{wr} S_n$. We note that G is independent of the location of the empty vertex up to conjugacy in $H^{n+1} \rtimes S_{n+1}$, here viewing S_{n+1} as the group of permutations of $\{0, 1, \ldots, n\}$.

Before proceeding, we look at the particular sliding piece puzzle which led to this paper. The puzzle, which we nicknamed HEX, was created by us and, to the best of our knowledge, has never been physically realized. One of the referees of our original manuscript brought to our attention the similar "rolling cubes" puzzle [6], [4] or [5, p. 118], and [3, pp. 58–59]. HEX is illustrated in Fig. 2 below.

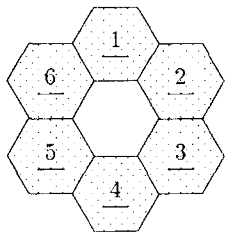

Fig. 2.

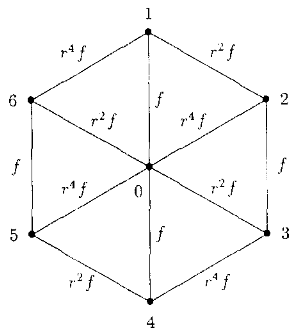

Fig. 3.

The six pieces are moved by flipping the hexagonal tiles over an edge into the empty space. The natural group of orientations H is D_6, the group of symmetries of a hexagon. Letting r denote a clockwise rotation of a tile by 60° and f denote a flip in the "horizontal" axis of the hexagon, we see that each η_{ij} is either f, r^2f, or r^4f. Thus, the group H may actually be taken to be a subgroup of D_6 isomorphic to $D_3 \cong S_3$. The graph of the HEX game, with edges labeled with η_{ij}, is given in Fig. 3 above. In general, we would need to represent a puzzle as a directed graph since $\eta_{ji} = \eta_{ij}^{-1}$, but here each η_{ij} is of order 2.

Returning to the general situation, define W to be the image of the canonical homomorphism from G to S_n, i.e. the permutation group associated to the puzzle without considering orientation. To the path in the graph through vertices $i_1, \ldots, i_\ell$, we associate the element $\eta_{i_1 i_2}\eta_{i_2 i_3} \ldots \eta_{i_{\ell-1} i_\ell}$ of H. Let H_i be the subgroup of H consisting of those elements associated to the closed paths beginning and ending at vertex i.

Consider a spanning tree of the original graph (see [11, p. 20]). If the graph is not a polygon, we further assume one or more vertices of the spanning tree has valence at least 3. For each vertex i of the graph, we define τ_i to be the element of H associated to the unique path from vertex 0 to vertex i in the tree. Observe that the set $\{\tau_i \eta_{ij} \tau_j^{-1}\}$, where the pair (i,j) runs over the edges of the graph $\mathscr{G}$ which are not in the spanning tree, form a set of generators for H_0, called Schreier generators of H_0 in H [8, pp. 164–165].

We have the following result.

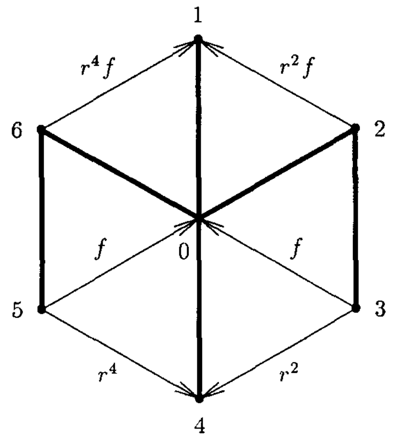

Fig. 4.

Theorem 1. *The map* ϕ: $G \to H_0^n \rtimes W$ *defined by*

$$\phi((\boldsymbol{h}, \sigma)) = ((\tau_1 h_1 \tau_{1^\sigma}^{-1}, \ldots, \tau_n h_n \tau_{n^\sigma}^{-1}), \sigma)$$

is an injective homomorphism.

Proof. Given the connection with Schreier generators, it is not surprising that the theorem is a purely group theoretic result. When W is transitive, H_0 is replaced by the stabilizer of some i_0, and τ_i is an arbitrary element such that there exists an element in G of the form $((h_1, \ldots, h_{i-1}, \tau_i, h_{i+1}, \ldots, h_n), \sigma_i)$, where $i_0^{\sigma_i} = i$.

Because h_i is an element of H associated to a path from vertex i to vertex i^σ, it follows that $\tau_i h_i \tau_{i^\sigma}^{-1} \in H_0$. Let e denote the identity of H and id denote the identity in S_{n+1}. If we view G and $H_0^n \rtimes W$ as subgroups of $H^{n+1} \rtimes S_{n+1}$, then the map ϕ is simply conjugation by $((e, \tau_1^{-1}, \ldots, \tau_n^{-1}), \mathrm{id})$, completing the proof. $\square$

The image of ϕ is the group for the relabeled graph where an edge from vertex i to vertex j is assigned the element $\tau_i \eta_{ij} \tau_j^{-1}$. In particular, edges on the relabeled tree are the identity e. We illustrate this in Fig. 4 above for HEX. The darker edges form the spanning tree. We compute the new $\eta_{34} = (r^4 f \cdot f) \cdot r^4 f \cdot f^{-1} = r^2$, and so forth. Rather than referring to $\phi(G)$ in what follows, we shall refer to the group G of this relabeled graph and consider H_0 to be its group of orientations. This normalization of the puzzle also provides us with a canonical set of generators, corresponding to the Schreier generators of H_0, for its group G. Namely, for each edge of the graph which is not in the spanning tree, say that between vertices i and j, consider the element of the puzzle group obtained by sliding all tiles along the path of the spanning tree from j to 0 one position, then sliding the tile at vertex i to vertex j, and finally sliding tiles along the path of the spanning tree along the path from 0 to i one position. For convenience, we order i and j to avoid $i = 0$. This element has the form

$g_{ij} = ((e,\ldots,e,\eta_{ij},e,\ldots,e),\sigma_{ij})$, where the η_{ij} is in the ith position and σ_{ij} is a cycle in S_n. We shall return to these generators below, in Theorem 2, in Lemma 3, and when we describe how one may compute the group G in practice, following Theorem 3.

Our main result for nonseparable, nonpolygonal graphs, Theorem 3, is that G is usually of index 1 or 2, and always of index dividing 12, in $H_0^n \rtimes S_n$. We begin toward our goal with the following lemma.

Lemma 1. *For k, m, and n positive integers with $k < n$,*

$$\left\langle\left\{(\varepsilon_1,\ldots,\varepsilon_n)\in\mathbb{Z}_m^n\colon\ \varepsilon_i = 0 \text{ or } 1 \text{ and } \sum_{i=1}^n \varepsilon_i = k\right\}\right\rangle = \left\{(z_1,\ldots,z_n)\in\mathbb{Z}_m^n\colon\ \sum_{i=1}^n z_i \equiv 0 \bmod (k,m)\right\}.$$

Proof. Let $M = \langle\{(\varepsilon_1,\ldots,\varepsilon_n) \in \mathbb{Z}_m^n\colon\ \varepsilon_i = 0 \text{ or } 1 \text{ and } \sum_{i=1}^n \varepsilon_i = k\}\rangle$. The inclusion

$$M \subset \left\{(z_1,\ldots,z_n)\in\mathbb{Z}_m^n\colon\ \sum_{i=1}^n z_i \equiv 0 \bmod (k,m)\right\}$$

is clear. By subtracting two elements of M that differ only in positions i and j, it follows that $(0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0)\in M$, where 1 and -1 are in positions i and j. Thus, whether $(z_1,\ldots,z_n)\in\mathbb{Z}_m^n$ is in M depends only on $\sum_{i=1}^n z_i$. Because the multiples of k in $\mathbb{Z}_m$ are also generated by (k,m), the reverse inclusion holds, proving the lemma. $\square$

The critical step in our development is to show that one may choose the orientation (in H_0) for $n-1$ of the n tiles arbitrarily in the puzzle for a nonseparable, nonpolygonal graph. Letting id denote the identity in S_n, define $K = \{\boldsymbol{k}\in H_0^n\colon\ (\boldsymbol{k},\mathrm{id})\in G\}$.

Theorem 2. *Let $\mathscr{G}$ be a nonseparable, nonpolygonal graph. For $\sigma\in W$ and $h_1,\ldots,h_{i-1}$, $h_{i+1},\ldots,h_n\in H_0$, there exists $h_i\in H_0$ such that $((h_1,\ldots,h_{i-1},h_i,h_{i+1},\ldots,h_n),\sigma)\in G$.*

Proof. Since we may multiply on the left by elements of the subgroup $K\rtimes\{\mathrm{id}\}$, it suffices to prove the theorem for $\sigma = \mathrm{id}$.

We write $|\sigma_{ij}|$ for the order of σ_{ij}, which appears as the permutation component of the generator $g_{ij} = ((e,\ldots,e,\eta_{ij},e,\ldots,e),\sigma_{ij})$ of the group G. Because σ_{ij} is actually a cycle, we observe that $|\sigma_{ij}|$ is also the number of tiles moved under g_{ij}. The construction of our spanning tree ensures $|\sigma_{ij}| < n$. Furthermore, $g_{ij}^{|\sigma_{ij}|} = (\boldsymbol{h},\mathrm{id})\in K\rtimes\{\mathrm{id}\}$, with $h_k = \eta_{ij}$ when k is moved by σ_{ij} and $h_k = e$ otherwise. Thus, it suffices to show that there exists an element g of G which moves any $|\sigma_{ij}|$ designated tiles into the positions permuted by σ_{ij} while changing their orientations by at most a power of η_{ij}; for then the set of $gg_{ij}^{|\sigma_{ij}|}g^{-1}$ for such g enables us to apply Lemma 1 to conclude that we can multiply the orientation of $n-1$ of the tiles by arbitrary powers of η_{ij}, with

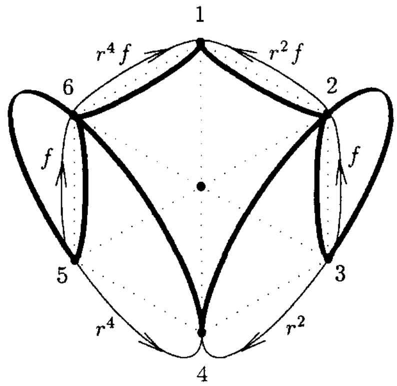

Fig. 5.

the orientation of the final tile multiplied by a suitable power of η_{ij}, while fixing the position of all tiles. Since $\{\eta_{ij}\}$ generates H_0, the theorem will follow.

While the physical realization of the element g_{ij} involves all of the tiles in the paths of the spanning tree from vertex j to vertex 0 and from vertex 0 to vertex i, no change results to the tiles in the intersection of both paths. For this reason, we find it convenient to superpose ellipses or similar curves on the graph of the puzzle and its spanning tree which represent the cycles σ_{ij}, illustrated for the six generators of our normalization of HEX back in Fig. 4,

$$((e,r^2f,e,e,e,e),(1,2)), \quad ((e,e,f,e,e,e),(2,3)), \quad ((e,e,r^2,e,e,e),(2,3,4))$$
$$((e,e,e,e,f,e),(5,6)), \quad ((e,e,e,e,r^4,e),(4,6,5)), \quad ((e,e,e,e,e,r^4f),(1,6)),$$

in Fig. 5 above. In a manner similar to that used in the spanning tree, we use thicker edges to represent paths which do not change orientation and dotted lines to represent the original graph. We find it helpful to imagine these "ellipses" of tiles as overlapping bicycle chains.

Call the set of positions permuted by σ_{ij} the "target cycle". The proof of our theorem may be further reduced to showing that, if at least one of the designated $|\sigma_{ij}|$ tiles is not in the target cycle, then there is an element of the puzzle group that increases the number of designated tiles in the target cycle while changing the orientations of the designated tiles by at most a power of η_{ij}.

With this goal in mind, consider the least integer m such that

$$g_{i_1j_1}^{k_1}g_{i_2j_2}^{k_2}\cdots g_{i_mj_m}^{k_m}, \quad k_1,k_2,\ldots,k_m\in\mathbb{Z},$$

moves one of the designated tiles from outside the target cycle into it. Furthermore, we assume that $k_1,\ldots,k_m$ are chosen to minimize $|k_1|$, then $|k_2|$, ..., and finally $|k_m|$ under the condition that the tile moved into the target cycle undergoes no changes in orientation along the way. When $m>1$, the minimality of m and k_1 implies that the only designated tiles moved by any of $g_{i_2j_2}^{k_2},\ldots,g_{i_mj_m}^{k_m}$ are those moved by both $g_{i_mj_m}$

and g_{ij}. Observe that the minimality condition also guarantees that if the tiles permuted by $g_{i_r j_r}$ and $g_{i_s j_s}$ intersect, then $s = r \pm 1$. In this case the intersection consists of tiles contiguous in the spanning tree.

This preliminary sequence of moves may suffer two potential problems. Designated tiles already in the target cycle could be moved out, or other designated tiles lying in the initial cycle could be reoriented. To avoid the first problem, before multiplying by any $g_{i_m j_m}^{\pm 1}$ we multiply by a suitable power of g_{ij} so that no designated tile already in the target cycle leaves it. Once this step is taken, the potential for reorienting designated tiles, other than by a power of η_{ij}, exists only when two or more designated tiles are in the initial cycle. Suppose this is the case. By minimizing $|k_1|$, we have chosen a tile closest to the second cycle, or the target cycle if $m = 1$, along the spanning tree. To reverse any orientation due to $g_{i_1 j_1}^{k_1}$, after multiplying by $g_{i_1 j_1}^{k_1} g_{i_2 j_2}^{k_2} \cdots g_{i_m j_m}^{k_m}$ and any intermediate powers of g_{ij}, we multiply by $g_{i_1 j_1}^{-k_1}$ (again subject to the above procedure of multiplying by a suitable power of g_{ij} before multiplying by $g_{i_1 j_1}^{\pm 1}$ if $m = 1$). This shows that if at least one of the designated $|\sigma_{ij}|$ tiles is not in the target cycle, then there is an element of the puzzle group that increases the number of designated tiles in the target cycle while changing the orientations of the designated tiles by at most a power of η_{ij}. Hence there exists an element g of G which moves any $|\sigma_{ij}|$ tiles into the positions permuted by σ_{ij} while changing their orientations by at most a power of η_{ij}, hence the theorem holds for $\sigma = \mathrm{id}$, hence in general. $\square$

Let $U = \{\sigma \in W\colon\ (\mathbf{e}, \sigma) \in G\}$, where $\mathbf{e} = (e, \ldots, e)$. Letting W' denote the commutator subgroup of W, we have the following.

Lemma 2. *If $\mathscr{G}$ is a nonseparable graph, then $U \supset W'$. Consequently, U is normal in W and, if $\mathscr{G}$ is nonpolygonal, U is transitive.*

Proof. If $\mathscr{G}$ is polygonal, then W is cyclic and the lemma is trivial. Recall that the commutator subgroup of S_n is A_n, and that of A_n is A_n for $n \geqslant 5$ and the Klein 4-group, K_4, for $n = 4$.

For $W \cong S_n$, $n \geqslant 3$, Theorem 2 guarantees the existence of elements of the form $((h, e, e, \ldots, e), (1,3))$ and $((h', e, e, \ldots, e), (1,2))$ in G. The calculation

$$\begin{aligned}&((h, e, e, \ldots, e), (1,3)) \cdot ((h', e, e, \ldots, e), (1,2)) \\ &\quad \cdot ((h, e, e, \ldots, e), (1,3))^{-1} \cdot ((h', e, e, \ldots, e), (1,2))^{-1} \\ &\quad = ((e, e, e, \ldots, e), (1,2,3))\end{aligned}$$

leads to $U \supset A_n$ in this case, since the 3-cycles generate A_n.

In the case $W \cong A_n$, $n \geqslant 4$, we begin similarly with the calculation

$$\begin{aligned}&((e, h, e, \ldots, e), (1,2,3)) \cdot ((e, h', e, \ldots, e), (1,2,4)) \\ &\quad \cdot ((e, h, e, \ldots, e), (1,2,3))^{-1} \cdot ((e, h', e, \ldots, e), (1,2,4))^{-1} \\ &\quad = ((e, e, e, \ldots, e), (1,2)(3,4)).\end{aligned}$$

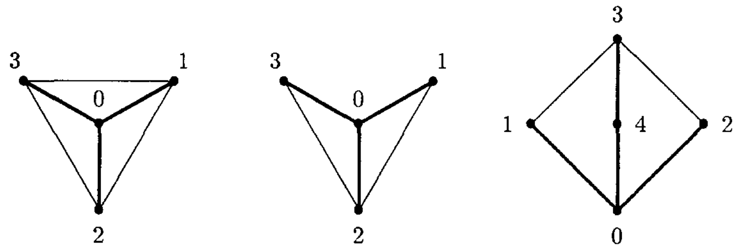

Fig. 6.

If $n=4$, this and similar expressions generate K_4, hence $U \supset K_4$. When $n>4$, we see that $(1,2)(3,4) \cdot (1,2)(3,5) = (3,4,5) \in U$, and again go on to conclude $U \supset A_n$.

The special case of the θ_0 graph is in the same spirit. Use of GAP [7] makes this last computation less tedious. □

We analyze the structure of the subgroup $K \rtimes \{\mathrm{id}\} \subset G$ in the next two lemmas. Recall that an action of S_n on H_0^n arises in the definition of the multiplication law for $H_0^n \rtimes S_n$, given by $\boldsymbol{h}^{\sigma^{-1}} = (h_{1^\sigma}, h_{2^\sigma}, \ldots, h_{n^\sigma})$.

Lemma 3. *If $\mathscr{G}$ is a nonseparable graph, then K is invariant under the action of S_n on H_0^n.*

Proof. In the polygonal case, $K = \{(h,\ldots,h)\colon\ h \in H_0\}$. Otherwise, since Lemma 2 implies U is doubly transitive except possibly when $W \cong S_3$ or $W \cong A_4$, the subset of K consisting of n-tuples with at most two components not equal to the identity of H is invariant under the action of S_n. By Theorem 2 an arbitrary element of K may be broken down into such a product, and Lemma 3 follows up to the two exceptions.

To treat the exceptions, we see that there are two nonseparable graphs on four vertices with $W \cong S_3$ and one bipartite, nonseparable graph on five vertices with $W \cong A_4$. Up to numbering of the vertices, there turns out to be a unique spanning tree with at least one vertex of valence 3 on each. We illustrate them in Fig. 6 above.

Sliding tiles along the spanning tree to change the location of the empty vertex corresponds to conjugation of G in $H_0^{n+1} \rtimes S_{n+1}$ by an element of the form $(\mathbf{e}, \sigma)$. Therefore, we may assume the vertices labeled 0 are the empty vertices in the three graphs above.

The group G for the first graph above is generated by $g_{12} = ((\eta_{12}, e, e), (1,2))$, $g_{23} = ((e, \eta_{23}, e), (2,3))$, and $g_{31} = ((e, e, \eta_{31}), (1,3))$. From g_{12}^2, g_{23}^2, and g_{31}^2, we see that $(\eta_{12}, \eta_{12}, e)$, $(e, \eta_{23}, \eta_{23})$, and $(\eta_{31}, e, \eta_{31})$ are in K, respectively. Computation of $g_{23}g_{12}$ along with the result $U \supset A_3$ from Lemma 2 let us conclude that $(\eta_{12}, \eta_{23}, e) \in K$. Conjugating $((\eta_{12}, \eta_{23}, e), \mathrm{id})$ by g_{31} yields $(e, \eta_{23}, \eta_{12}) \in K$. Similarly, $(e, \eta_{23}, \eta_{31})$, $(\eta_{23}, e, \eta_{31})$, $(\eta_{12}, e, \eta_{31})$, and $(\eta_{12}, \eta_{31}, e)$ are all in K. Because $U \supset A_3$, it follows that the images of these elements under the action of S_3 also lie in K. On the other hand, any element of $K \rtimes \{\mathrm{id}\}$ is the product of an even number of the three generators.

Along with Lemma 1, this shows that K is generated by the elements produced above, proving that it is invariant under the action of S_3.

To prove the lemma for the second graph, note that $(1,3) \in W$, hence there is an element of the form $((e,e,h),(1,3))$ in G by Theorem 2. Thus, adding the edge between vertices 1 and 3 to the graph with $\eta_{31} = h$ does not change the group G, and the first case applies.

In the final case, the group G has generators $g_{13} = ((\eta_{13}, e, e, e), (1,3,4))$ and $g_{23} = ((e, \eta_{23}, e, e), (2,3,4))$. Explicit calculations similar to the first case show that $(\eta_{13}, \eta_{13}, \eta_{13}, e)$, $(\eta_{23}, \eta_{23}, \eta_{23}, e)$, $(\eta_{13}, \eta_{23}, e, e)$, and $(e, \eta_{23}, e, \eta_{13})$ all lie in K. Because $U \supset K_4$ by Lemma 2, the images of the above four elements under the action of S_4 also lie in K. Note that $(1,3,4)$ and $(2,3,4)$ represent the two nontrivial cosets of the Klein 4-group K_4 in A_4. If an element of G is in $K \rtimes \{\mathrm{id}\}$, it follows that the number of factors of the two generators in a factorization of the element must differ by a multiple of 3. It is again a consequence of Lemma 1 that K is generated by the above four elements and their images under the action of S_4. Hence K is invariant under the action of S_4. □

Let $K_0 = \{k \in H_0\colon\ (k, e, \ldots, e) \in K\}$.

Lemma 4. *For $\mathscr{G}$ nonseparable and nonpolygonal, whether $((h_1, \ldots, h_n), \sigma)$ is in G depends only on the coset $K_0(h_1 \cdots h_n)$. Furthermore, $K_0 \supset H_0'$ and $[H_0^n : K] = [H_0 : K_0]$.*

Proof. Given $h \in H_0$, Theorem 2 guarantees the existence of an element of K of the form $\boldsymbol{k} = (h, e, h_3, \ldots, h_n)$. From Lemma 3 we see that $\boldsymbol{k}(\boldsymbol{k}^{-1})^{(1,2)} = (h, h^{-1}, e, \ldots, e) \in K$. A second application of Lemma 3 implies

$$(e, \ldots, e, h, e, \ldots, e, h^{-1}, e \ldots, e) \in K,$$

where h and h^{-1} are the ith and jth coordinates, respectively. Thus $((h_1, \ldots, h_n), \sigma) \in G$ if and only if

$$\begin{aligned} ((h_1 \cdots h_n, e, \ldots, e), \sigma) &= ((h_1, \ldots, h_n), \sigma)((h_2, h_2^{-1}, e, \ldots, e)^\sigma, \mathrm{id}) \\ &\quad \cdots ((h_n, e, \ldots, e, h_n^{-1})^\sigma, \mathrm{id}) \in G, \end{aligned}$$

proving the first assertion. Moreover, essentially the same calculation shows that the product of the h_i's may be taken in any order, hence $K_0 \supset H_0'$.

Finally, Theorem 2 implies that the projection from K to its last $n-1$ coordinates is surjective onto H_0^{n-1}. This projection has the kernel $K_0 \times \{e\}^{n-1}$, from which it follows that $[H_0^n : K] = [H_0 : K_0]$, completing the proof of the lemma. □

The next lemma puts things together.

Lemma 5. *For a nonseparable, nonpolygonal graph $\mathscr{G}$, the map $\psi\colon W/U \to H_0/K_0$ defined by $\psi(U\sigma) = K_0 h_1 \cdots h_n$, where $((h_1, \ldots, h_n), \sigma) \in G$, is a group isomorphism.*

Proof. Lemma 2 asserts that U is normal in W and Lemma 4 asserts that K_0 is normal in H_0. Lemma 4 also implies $\sigma \mapsto K_0 h_1 \cdots h_n$ is well-defined, so the definition of U implies that ψ is well-defined. It is now immediate that ψ is injective. The map ψ sends the canonical generators of G to the canonical generators of H_0. Thus, ψ is surjective. Since $|W/U| = 1, 2$, or 3 and $\psi(U) = K_0$, it now follows that ψ is a group isomorphism. □

From our sequence of lemmas, combined with the knowledge of the commutator subgroups of the permutation groups that arise as W, we now immediately obtain

Theorem 3. *The group of a sliding piece puzzle with oriented tiles on a nonseparable graph $\mathscr{G}$ is isomorphic to one of the following.*

If $\mathscr{G}$ is a polygon,

(1) *the cyclic group of order $n|H_0|$.*

If $\mathscr{G}$ is bipartite,

(2) $H_0^n \rtimes A_n$, *or*

(3) $\{((h_1, \ldots, h_4), \sigma) \in H_0^4 \rtimes A_4\colon\ h_1 \cdots h_4 \in K_0 h_0^r$ and $\sigma \in K_4(1,2,3)^r$, $r = 0$, 1, *or* 2$\}$, *for some normal subgroup K_0 of H_0 of index* 3 *and some $h_0 \in H_0 - K_0$.*

If $\mathscr{G}$ is isomorphic to θ_0,

(4) $H_0^6 \rtimes \langle (1,2,3,4), (1,4,5,6) \rangle$, *or*

(5) $\{((h_1, \ldots, h_n), \sigma) \in H_0^6 \rtimes \langle (1,2,3,4), (1,4,5,6) \rangle\colon\ h_1 \cdots h_n \in K_0$ *iff* $\sigma \in \langle (1,2,3,4), (1,4,5,6) \rangle \cap A_6\}$, *for some normal subgroup K_0 of H_0 of index* 2.

If $\mathscr{G}$ is not a polygon, bipartite, or isomorphic to θ_0,

(6) $H_0^n \rtimes S_n$, *or*

(7) $\{((h_1, \ldots, h_n), \sigma) \in H_0^n \rtimes S_n\colon\ h_1 \cdots h_n \in K_0$ *iff* $\sigma \in A_n\}$, *for some normal subgroup K_0 of H_0 of index* 2.

In particular, except for the cyclic case, the index of G in $H_0^n \rtimes S_n$ is 1, 2, 6, *or* 12, *with* 6 *possible only for the nonseparable, bipartite graph on* 5 *vertices and for θ_0, and* 12 *possible only for θ_0.*

To summarize our results, we describe how to compute the group G of a puzzle. Begin by constructing the graph $\mathscr{G}$ of the puzzle. The group for a separable graph may be broken down into the direct product of the groups for its components, so we assume $\mathscr{G}$ is nonseparable. Choose a spanning tree for $\mathscr{G}$, with at least one vertex of valence 3 in the nonpolygonal case, and normalize the puzzle via Theorem 1 and its subsequent discussion. Consider the canonical generators of the group G associated to this normalization. Temporarily ignoring orientation, compute the group W based on Wilson's theorem. Polygonal graphs and θ_0 are easily recognized. Usually one must determine whether the graph is bipartite or not, i.e. whether $W \cong A_n$ or $W \cong S_n$. Compute the commutator subgroup H_0' of H_0 and use it to determine whether there exists a subgroup $K_0 \supset H_0'$ such that all of the generators of G are in the subgroups given in the relevant case of (3), (5), or (7).

For example, using our normalization of HEX illustrated in Fig. 4, its puzzle group has the six generators

$$((e,r^2 f,e,e,e,e),(1,2)), \quad ((e,e,f,e,e,e),(2,3)), \quad ((e,e,r^2,e,e,e),(2,3,4))$$
$$((e,e,e,e,f,e),(5,6)), \quad ((e,e,e,e,r^4,e),(4,6,5)), \quad ((e,e,e,e,e,r^4 f),(1,6)).$$

We see that $W \cong S_6$ and $H_0 = \langle f, r^2 \rangle$ is of order 6, with commutator subgroup $\langle r^2 \rangle$. Checking each generator, we see that we are in case (5), with $K_0 = \langle r^2 \rangle$. Therefore, the puzzle group for HEX has order $6^6 \cdot 720/2 = 16,796,160$.

Similar results hold for different puzzles; see [9] and [2].

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