Persistence of a Class of Periodic
Kolmogorov Systems

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Submitted by J. Eisenfeld
Received July 29, 1998

In this paper, we study the persistence of a periodic Kolmogorov system which models the growth of \( n \) interacting biological species. The system is assumed to have a simple dynamic, in the sense that each coordinate subsystem possesses a global attractor. Our research is mainly addressed to the case in which there exists at least a predator among the \( n \) species. However, we also obtain a result about persistence for a class of competitive systems.

1. INTRODUCTION

This research was motivated by a paper by Freedman and Waltman [5] about the persistence of a three interacting species system including a predator and a prey. The basic idea of that paper is simply to make all attractors on the faces of the three-dimensional positive cone, unstable.

Another important motivation comes from the fact that almost all papers about persistence are devoted to autonomous systems ([2–4, 6, 7]) and very little is known about the persistence of the periodic systems ([1, 10]).

In this paper, we improve the main result in [5] in two different directions. First, we consider periodic (nonautonomous) systems and second, we obtain some results in dimension greater than three.

To be more precise, we consider the system

\[
x'_i = x_i f_i(t, x_1, \ldots, x_n), \quad 1 \leq i \leq n,
\]

1. This paper was sponsored by the Spanish MEC, DGFPC, SAB 1995-0675.
where \( f_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is continuous and \( f_i(t, x) \) is \( T \)-periodic in \( t \) and locally Lipschitz continuous in \( x \), for \( 1 \leq i \leq n \).

In the following, we write \( I_n = \{1, \ldots, n\} \). Given a nonempty subset \( I \) of \( I_n \), we denote by
\[
x'_i = x_i f_i \left( t, \sum_{i \in I} x_i e_j \right), \quad i \in I,
\]
(1.2)
the subsystem of Eq. (1.1) that describes the growth of the species \( (x_i)_{i \in I} \) in the absence of the other ones. Here and henceforth, \( (e_1, \ldots, e_n) \) denotes the canonical vector basis of \( \mathbb{R}^n \).

Our basic assumption is that for each proper and nonempty subset \( I \) of \( I_n \), system (1.2) has a global attractor \( U^I := (U^I_t)_{t \in I} \), which is hyperbolic in the following sense:
\[
\int_0^T f_i \left( t, \sum_{i \in I} U^I_t (t) e_j \right) dt \neq 0, \quad \forall j \not\in I. \tag{1.3}
\]
We also assume that \( U^I \) is unstable in the sense that for some \( j \not\in I \), we have
\[
\int_0^T f_i \left( t, \sum_{i \in I} U^I_t (t) e_j \right) dt > 0. \tag{1.4}
\]
Finally, we say that \( U^I \) is positive if \( U^I_t > 0 \) for all \( i \in I \). Our main results are the following two theorems.

**Theorem 1.1.** Suppose that \( U^I \) is positive for each nonempty and proper subset \( I \) of \( I_n \) and that Eq. (1.4) holds for all \( j \not\in I \). Assume further that there exists \( 1 \leq i \leq n \) such that
\[
\int_0^T f_i (t, 0) dt \neq 0 < \int_0^T f_j (t, 0) dt, \quad \forall j \not\in i.
\]
If \( u = (u_1, \ldots, u_n) \) is a positive solution of Eq. (1.1) which is defined and bounded on \([0, \infty)\), then \( u \) is persistent. That is, \( \liminf_{i \to +\infty} u_i(t) > 0 \) for all \( i \).

**Theorem 1.2.** Let \( n = 3 \) and suppose that Eqs. (1.3)–(1.4) hold. Assume further that there exist \( 1 \leq j, k \leq 3 \) such that \( U^{(k)} \equiv 0 < U^{(j)} \) and
\[
\int_0^T f_i (t, 0) dt \neq 0, \quad \forall i.
\]
Then, the conclusion of Theorem 1.1 remains true.
Theorem 1.1 will be proved in Section 2, while Theorem 1.2 will be proved in Section 4. Our main tool is Proposition 2.5 below, which plays the role of Lemma A1 in [5], about the stable and unstable manifolds of a hyperbolic equilibrium belonging to an \( \omega \)-limit set.

**Remark.** In the autonomous case, Theorem 1.2 does not improve the main result in [5]. However, we can show that the result in that paper contains only a single case which is not covered by Theorem 1.2. In fact, using the notations in Theorem 1.2, the above mentioned case occurs when \( U_{i,k}, U_{j,k} \) are positive global attractors and \( U_{i,j} \) is positive and unstable. The proof of this assertion is not trivial and too long to be explained here.

### 2. THE PROOF OF THEOREM 1.1

Given \( p \in \mathbb{R}_+^n \), we denote by \( \nu(t, p) = (\nu_1(t, p), \ldots, \nu_n(t, p)) \) the solution of Eq. (1.1) determined by the initial condition \( \nu(0, p) = p \). We say that \( p \in \mathbb{R}_+^n \) is **periodic** if \( \nu(t, p) \) is a \( T \)-periodic solution of Eq. (1.1). This system is termed GAS (globally asymptotically stable) if \( \nu(t, p) \) is defined on \( [0, \infty) \) for all \( p > 0 \) and there exists a stable periodic point \( p_0 \in \mathbb{R}^n \) such that

\[
\nu(t, p) - \nu(t, p_0) \to 0 \quad \text{as } t \to +\infty,
\]

for all \( p > 0 \). In this case, we say that \( \nu(t, p_0) \) is a **global attractor** of Eq. (1.1).

The existence of global attractors was considered in [8]. See also [9, 10] for the case \( n = 1 \).

**Remark 2.1.** Suppose \( n = 1 \) and that \( U \) is a global attractor of Eq. (1.1). If \( u \) is a solution of this system and \( u(0) > 0 \), then it is defined and bounded on \( [0, \infty) \). Moreover, if \( u(0) < U(0) \), then \( u \) is defined on \( \mathbb{R} \) and \( u(t) \to 0 \) as \( t \to -\infty \), while, if \( u(0) > U(0) \), then \( u \) is unbounded. In particular, if \( U \equiv 0 \), then each positive solution of Eq. (1.1) is unbounded.

In the following, \( u = (u_1, \ldots, u_n) \) denotes a positive solution of Eq. (1.1) which is defined and bounded on \( [0, \infty) \). As usual, the (discrete) \( \omega \)-limit set of \( u \) is defined as the set \( \Omega = \Omega(u) \) consisting of all points \( p \in \mathbb{R}_+^n \) such that

\[
u(n_k T) \to p,  \quad \text{(2.1)}
\]

for some strictly increasing sequence \( (n_k) \) of \( \mathbb{N} \). If \( u \) is bounded, we also define the (discrete) \( \alpha \)-limit set of \( u \) as the set \( \Lambda = \Lambda(u) \) consisting of all
points \( p \in \mathbb{R}^n_+ \) such that Eq. (2.1) holds for some strictly decreasing sequence \( (n_k) \) in \( \mathbb{Z} \). We recall that \( \omega \) and \( \alpha \) limit sets are compact and nonempty.

**Remark 2.2.** If \( p \in \Omega = \Omega(u) \) (resp. \( p \in \Lambda = \Lambda(u) \)) and Eq. (2.1) holds, then \( u(n_k T + t) \to v(t, p) \) as \( k \to +\infty \) for all \( t \) in the domain of \( v(t, p) \). From this, \( v(t, p) \) is defined and bounded on \( \mathbb{R} \).

Finally, we say that \( u \) is (strong) persistent if \( \liminf_{t \to +\infty} u_i(t) > 0 \), for all \( i \). In the following, \( \Omega \) denotes the \( \omega \)-limit set of \( u \).

**Proposition 2.3.** If \( \Omega \cap \partial \mathbb{R}^n_+ = \emptyset \), then \( u \) is persistent.

**Proof.** Assume that our result is false and fix a sequence \( t \to +\infty \) such that \( u(t_i) \to p = (p_1, \ldots, p_n) \), where \( p_i = 0 \) for some \( i \). Let us write \( t_i = n_i T + r_i \), where \( n_i \) is an integer and \( r_i \in [0, T) \). Without loss of generality, we can suppose that \( (n_i) \) is a strictly increasing sequence of \( \mathbb{N} \) and that \( (r_i) \) converges to a point \( r \in [0, T] \).

Let \( v = (v_1, \ldots, v_n) \) be the solution of Eq. (1.1) determined by the initial condition \( v(r) = p \). Then, \( u(n_i T + t) \to v(t) \) as \( j \to +\infty \) and hence, \( v \) is bounded. In particular, it is defined on \( \mathbb{R} \) and \( v(0) \in \Omega \). On the other hand, \( v_i = 0 \) and so, \( v(0) \in \partial \mathbb{R}^n_+ \). This contradicts our assumption and the proof is complete.

**Proposition 2.4.** Suppose that Eq. (1.1) has a global attractor \( U \) and that \( u \) is bounded. If \( u \neq U \), then \( \Lambda(u) \subset \partial \mathbb{R}^n_+ \).

**Proof.** Assume on the contrary that there exists a strictly increasing sequence \( (n_k) \) in \( \mathbb{N} \) such that \( u(-n_k T) \) converges to a positive point of \( \mathbb{R}^n \) and fix a compact set \( K \) in the interior of \( \mathbb{R}^n_+ \) such that \( u(-n_k T) \in K \) for all \( k \in \mathbb{N} \). Since \( U(0) \) is a stable periodic point of Eq. (1.1) and \( U \) is a global attractor of this system, then, for each \( \epsilon > 0 \), there exists \( \tau > 0 \) such that

\[
\| v(t, p) - U(t) \| \leq \epsilon \quad \text{if} \quad t \geq \tau; \quad p \in K.
\]

If we write \( u(s) = v(s, q) \), then

\[
\| v(t, v(-n_k T, q)) - U(t) \| \leq \epsilon \quad \forall t \geq \tau; \quad k \in \mathbb{N}.
\]

Now, let us fix \( k \in \mathbb{N} \) such that \( n_k T > \tau \). By the above relation, we have

\[
\| q - U(0) \| = \| v(n_k T, v(-n_k T, q)) - U(n_k T) \| \leq \epsilon,
\]

and hence, \( q = U(0) \). Thus, \( u \equiv U \), and this contradiction ends the proof.
Now, we shall show our basic tool.

**Proposition 2.5.** Suppose that \( p = (p_1, \ldots, p_n) \in \Omega \) is periodic and let \( I = \{i \in I_n; p_i > 0\} \). If there exists a proper subset \( J \) of \( I_n \) containing \( I \) such that

\[
\alpha_i := \frac{1}{T} \int_0^T f_i(t, \nu(t, p)) \, dt > 0 \quad \forall i \notin J,
\]

then \( \Omega \) contains a point \( q = (q_1, \ldots, q_n) \neq p \) such that \( q_i > 0 \) if \( i \in I \) and \( q_i = 0 \) if \( i \notin J \).

**Proof.** We first assume that \( \nu(t, p) = 1 \) for all \( i \in I \) and that \( f_i(t, \nu(t, p)) > 0 \) for all \( i \notin J \). (Note that, in this case, \( \nu(t, p) \equiv p \).) From this, there exist \( \alpha_i > 0, \rho \in (0, 1) \) such that

\[
f_i(t, x) \geq \alpha_i \quad \text{if } i \notin J \quad \text{and} \quad \|x - p\| \leq \rho.
\] (2.2)

Given \( \epsilon \in (0, \rho] \) and \( \sigma \in \mathbb{R} \), we shall show that the following holds.

**Claim.** There exists \( \tau > \sigma \) such that \( u_i(\tau) < \epsilon \) if \( i \notin J \) and either \( |u_i(\tau) - |1 = \rho \) for some \( i \in I \) or \( u_i(\tau) = \rho \) for some \( i \in J \setminus I \). To show this, let us define \( R = I_1 \times \cdots \times I_n \), where \( I_i = (1 - \rho, 1 + \rho) \) if \( i \in I \), \( I_i = (0, \rho) \) if \( i \in J \setminus I \), and \( I_i = (0, \epsilon) \) if \( i \notin J \). Since \( p \in \Omega \) and Eq. (2.2) holds, there exist \( t_2 > t_1 > t_0 > \sigma \) such that \( u(t_0) \in R \), \( u(t_1) \notin R \), and \( u(t_2) \in R \); thus, there exists \( \tau \in (t_1, t_2) \) such that \( u(t) \in R \) for all \( t \in (\tau, t_2) \) and \( u(\tau) \notin R \). Using Eq. (2.2) once again, we conclude by contradiction that \( u_i(\tau) < \epsilon \) if \( i \notin J \) and the proof of our special case follows easily.

By the above claim, there exists a sequence \( t_j \to +\infty \) such that \( (u(t_j)) \) converges to a point \( x = (x_1, \ldots, x_n) \) with the following properties: \( x_i = 0 \) if \( i \notin J \) and either \( x_i = \rho \) for some \( i \in J \setminus I \) or \( |x_i - 1| = \rho \) for some \( i \in I \). In particular, \( x_i > 0 \) for all \( i \in I \) and \( x \neq p \).

Without loss of generality, we can suppose that \( t_j = n_j T + r_j \), where \( (n_j) \) is a strictly increasing sequence of \( \mathbb{N} \) and \( (r_j) \) is a sequence of \( [0, T) \) converging to a point \( r \in [0, T) \). Let \( v = (v_1, \ldots, v_n) \) be the solution of Eq. (1.1) determined by the initial condition \( v(r) = x \) and let \( q = (q_1, \ldots, q_n) = v(0) \). Obviously, \( q_i = 0 \) if \( i \notin J \) and \( q_i > 0 \) if \( i \in I \). On the other hand, if \( v(0) = p \), then, by uniqueness, \( v(t) \equiv v(t, p) \equiv p \) and hence \( x = p \). This contradiction proves that \( q \neq p \) and the proof of our special case is complete.

To show the general case, let us fix, for \( i \notin J \), a \( T \)-periodic differentiable function \( B_i : \mathbb{R} \to (0, \infty) \) such that

\[
f_i(t, \nu(t, p)) = \alpha_i + \frac{B_i'(t)}{B_i(t)}.
\]
Define also $B_i(t) = v_i(t, p)$ if $i \in I$ and $B_i(t) \equiv 1$ if $i \in J \setminus I$. By the change of variables $y_i = x_i/B_i$, system (1.1) becomes
\begin{equation}
y'_i = y_i g_i(t, y_1, \ldots, y_n), \quad 1 \leq i \leq n,
\end{equation}
where $g_i(t, y_1, \ldots, y_n) = f_i(t, B_i(t)y_1, \ldots, B_n(t)y_n) - B'_i(t)/B_i(t)$.

Let $u^* = (u^*_1, \ldots, u^*_n)$ be the solution of Eq. (2.3) defined by $u^*_i = u_i/B_i$ and let $\Omega^*$ be the $\omega$-limit set of $u^*$. Then, $p^* := (p^*_1/B_i(0), \ldots, p^*_n/B_n(0))$ is a periodic point of Eq. (2.3) in $\Omega^*$ and the assumptions of our special case are satisfied. The proof follows now easily.

**Remark 2.6.** Proposition 2.5 remains true if $u$ is bounded and we replace $\Omega$ by the $\alpha$-limit set $\Lambda = \Lambda(u)$.

**Proof.** As above, we can suppose that $v_i(t, p) \equiv 1$ for all $i \in I$ and that Eq. (2.2) holds. Given $\epsilon \in (0, \rho]$ and $\sigma \in \mathbb{R}$, we shall prove the existence of $\tau < \sigma$ such that the assertions in the above claim are valid. To this end, define $R$ as in Proposition 2.5 and fix $t_i < t_0 < \sigma$ such that $u(t_0) \in R$ and $u(t_0) \notin R$. Now it suffices to define $\tau$ as the point in $(t_i, t_0)$ determined by the conditions $u(t) \in R$ for $t \in (\tau, t_0)$ and $u(t) \not\in R$.

**Proposition 2.7.** Suppose that the trivial solution is a global attractor of Eq. (1.2) for each nonempty subset $I$ of $I_n$. Then, each positive solution of Eq. (1.1) is unbounded.

**Proof.** If $n = 1$, the proof follows from Remark 2.1. By induction, we can suppose that Eq. (1.2) has no positive bounded solutions for any proper and nonempty subset $I$ of $I_n$.

Assume now that $u = (u_1, \ldots, u_n)$ is a positive bounded solution of Eq. (1.1) and let $\Lambda$ be its $\alpha$-limit set. It is clear that $0 \notin \Lambda$ since the trivial solution is a global attractor of Eq. (1.1). Let us fix $p \in \Lambda$ and note that by Proposition 2.4, $I := \{i \in I_n: p_i > 0\}$ is a proper subset of $I_n$. Note also that $I$ is nonempty since $0 \notin \Lambda$. From this and Remark 2.2, $(v_i(t, p))_{i \in I}$ is a bounded positive solution of Eq. (1.2) and this contradiction ends the proof.

**Corollary 2.8.** Suppose that there exists a nonempty and proper subset $J$ of $I_n$ such that
\[ \int_0^T f_i(t, 0) \, dt > 0 \quad \text{if } i \notin J. \]
Assume further that the trivial solution is a global attractor of Eq. (1.2) for each nonempty subset $I$ of $J$. Then $0 \notin \Omega$. 

**Proof.** Assume on the contrary that $0 \in \Omega$. By Proposition 2.5 (with $p = 0$ and $I = \emptyset$), there exists a point $q = (q_1, \ldots, q_n) \in \Omega$ such that $q_i = 0$ if $i \notin J$ and $q \neq 0$. Thus, $I := \{i \in I_n: q_i > 0\}$ is a nonempty subset of $J$.
Note that by our assumption and by Proposition 2.7, system (1.2) has no positive bounded solutions. On the other hand, \((v_i(t), q)_{i \in I}\) is a positive bounded solution of this system, and this contradiction ends the proof.

Remark 2.9. Corollary 2.8 remains valid if \(u\) is bounded and \(\Omega\) is replaced by the \(\alpha\)-limit set of \(u\).

Given a subset \(I\) of \(I_n\), we define \(R^I = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n: x_i = 0 \ \forall i \notin I\}\). Note that \(R^\emptyset = \{0\}\).

**Corollary 2.10.** Let \(I\) be a proper and nonempty subset of \(I_n\) and suppose that \(U_{i \in I}\) is a positive global attractor of Eq. (1.2) such that
\[
\int_0^T f_i\left(t, \sum_{i \in I} U_i(t)e_i\right) dt > 0 \quad \text{if} \ j \notin I.
\]

If \(\Omega \cap R^I = \emptyset\) for each proper subset \(J\) of \(I\), then \(\Omega \cap R^I = \emptyset\).

**Proof.** Assume on the contrary that there exists \(p = (p_1, \ldots, p_n) \in \Omega\) such that \(p_i = 0\) for all \(i \notin I\) and note that \(p_i > 0\) for all \(i \in I\) since \(\Omega \cap R^I = \emptyset\) for each proper subset \(J\) of \(I\). From this, \(v_\sharp(t) = (v_i(t), p)_{i \in I}\) is a positive solution of Eq. (1.2) and by Proposition 2.4, applied to system (1.2), we have \(U_\sharp = v_\sharp\). In particular, \(p = \sum_{i \in I} U_{i}(0)e_i\), and so, \(p\) is periodic. By Proposition 2.5, with \(J = I\), there exists \(q = (q_1, \ldots, q_n) \in \Omega\) such that \(q_i > 0\) for \(i \in I\), \(q_i = 0\) if \(i \notin I\) and \(q \neq p\). From this, \(q \in \Omega \cap R^I\), and replacing \(p\) by \(q\) in the above argument, we obtain \(q = \sum_{i \in I} U_i(0)e_i\). Thus, \(q = p\) and this contradiction ends the proof.

**Theorem 2.11.** Let \(\mathcal{P}\) be the family of all proper and nonempty subsets of \(I_n\), and suppose that there exists a subfamily \(\mathcal{F}\) of \(\mathcal{P}\) such that

1. \(\emptyset \in \mathcal{F}\).
2. \(R^I \cap \Omega = \emptyset\) for all \(I \in \mathcal{F}\).
3. For each \(I \in \mathcal{P} \setminus \mathcal{F}\), system (1.2) has a positive global attractor \(U^I = (U^I_i)_{i \in I}\) such that
\[
\int_0^T f_i\left(t, \sum_{i \in I} U^I_i(t)e_i\right) dt > 0 \quad \forall j \notin I.
\]

Then \(u\) is persistent.

**Proof.** Assume on the contrary that the result is false. By Proposition 2.3, \(\Omega \cap R^I \neq \emptyset\) for some \(I \in \mathcal{P} \setminus \mathcal{F}\), and we can suppose, without loss of generality, that the cardinality of \(I\) is minimal with respect to this property.
Let $J$ be a proper subset of $I$. By our choice of $I$ and assumption (ii), $R^J \cap \Omega = \emptyset$. Moreover, by (i), $0 \notin \Omega$ and by Corollary 2.10, $R^I \cap \Omega = \emptyset$. This contradiction ends the proof.

Proof of Theorem 1.1. If $\int_0^T f(t,0) \, dt > 0$, then the trivial solution of Eq. (1.1) is a source and so $0 \notin \Omega$. On the other hand, if $\int_0^T f(t,0) \, dt < 0$, then $U^{(0)} = 0$, and by Corollary 2.8, $0 \notin \Omega$. The proof follows now from Theorem 2.11, with $\mathcal{F} = \{\emptyset\}$.

3. PLANAR SYSTEMS

In this section, we consider the planar system

$$
\begin{align*}
x' &= x f(t, x, y), \\
y' &= y g(t, x, y),
\end{align*}
$$

(3.1)

where $f, g: \mathbb{R} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ are continuous functions which are $T$-periodic in $t$ and locally Lipschitz continuous in $(x, y)$. We shall assume that the logistic equation

$$
x' = x f(t, x, 0)
$$

(3.2)

has a positive global attractor $U$ such that

$$
\int_0^T g(t, U(t), 0) \, dt > 0.
$$

(3.3)

We also assume that the logistic equation

$$
y' = y g(t, 0, y)
$$

(3.4)

has a global attractor $V$.

Theorem 3.1 ("Predator–prey"). If $V = 0$ and $\int_0^T f(t, 0, 0) \, dt > 0$, then each positive and bounded solution of Eq. (3.1) in forward time is persistent.

Proof. Let $\Omega$ be the $\omega$-limit set of a positive solution of Eq. (3.1) bounded on forward time. By Corollary 2.8, $0 \notin \Omega$. On the other hand, by Remark 2.1, $\Omega$ does not meet the $y$ axis and the proof follows from a suitable application of Theorem 2.11.

From Theorem 1.1, we also have the following.

Theorem 3.2 ("Competition"). Assume $V > 0$. If $\int_0^T f(t, 0, V(t)) \, dt > 0$ and

$$
\int_0^T f(t, 0, 0) \, dt, \int_0^T g(t, 0, 0) \, dt > 0,
$$

then the conclusion of Theorem 3.1 remains valid.
4. THE CASE \( n = 3 \)

In this section, we improve the main result in [5] to the periodic case. We shall assume that system (1.1) satisfies the following hypotheses:

\begin{align*}
\text{(H}_1\text{)} & \quad \text{For each proper and nonempty subset } I \text{ of } I_n, \text{ system (1.2) has a global attractor } U_I = (U_I^i)_{i \in I}. \\
\text{(H}_2\text{)} & \quad \int_T^t f_j(t, \sum_{i \in I} U_I^i(t) e_i) \, dt \neq 0 \text{ if } j \notin I. \\
\text{(H}_3\text{)} & \quad \int_0^T f_j(t, \sum_{i \in I} U_I^i(t) e_i) \, dt > 0 \text{ for some } j \notin I. \\
\text{(H}_4\text{)} & \quad \int_0^T f_j(t, 0) \, dt \neq 0 \text{ for all } i \text{ and there exists } (j, k) \text{ such that } \int_0^T f_j(t, 0) \, dt > 0 > \int_0^T f_k(t, 0) \, dt.
\end{align*}

Remark 4.1. Suppose that \( \mathbb{R}^{(\beta)} = 0 \) for some \( \beta \in I_n \). By Remark 2.1, it follows that if \( 0 \notin \Omega \), then \( \mathbb{R}^{(\beta)} \cap \Omega = \emptyset \).

**Proposition 4.2** \((n \geq 3)\). Suppose that there exists \( \alpha \neq \beta \) in \( I_n \) such that \( U^{(\alpha)} > 0 \) and \( U^{(\beta)} = U^{(\beta)}_{\alpha, \beta} = 0 \). If \( 0 \notin \Omega \) and

\[ \int_0^T f_j(t, U^{(\alpha)}(t) e_\alpha) \, dt > 0 \quad \forall j \neq \alpha, \beta, \]

then \( \mathbb{R}^{(\alpha, \beta)} \cap \Omega = \emptyset \).

**Proof.** Assume on the contrary that there exists \( p = (p_1, \ldots, p_n) \in \Omega \) such that \( (p_\alpha, p_\beta) \neq (0, 0) \) and \( p_i = 0 \) if \( i \neq \alpha, \beta \). By Remark 4.1, we have \( p_\alpha > 0 \).

**Claim.** There exists \( q = (q_1, \ldots, q_n) \in \Omega \) such that \( q_i > 0 \) if \( i = \alpha, \beta \) and \( q_i = 0 \) otherwise. To show this, we can assume \( p_\beta = 0 \). (Otherwise, we take \( q = p \).) By the argument in Corollary 2.10, \( p = U^{(\alpha)}(0) e_\alpha \), and by Proposition 2.5 with \( J = (\alpha, \beta) \), there exists \( q = (q_1, \ldots, q_n) \neq p \) in \( \Omega \) such that \( q_\alpha > 0 \) and \( (q_\alpha, q_\beta) \neq (0, 0) \). If \( q_\beta = 0 \), then, by the above argument, \( q = U^{(\beta)}(0) e_\beta = p \) and this contradiction proves the claim.

Let \( \Lambda \) be the \( \alpha \)-limit set of \( \nu(t, q) \), where \( q \) is given by the claim above and let \( J = (\alpha, \beta) \). By a suitable application of Proposition 2.4 to the system (1.2), we conclude that \( \Lambda \subset \mathbb{R}^{(\alpha)} \cup \mathbb{R}^{(\beta)} \). But \( \Lambda \subset \Omega \), and by Remark 4.1, \( \Lambda \subset \mathbb{R}^{(\alpha)} \).

Note now that \( (U^{(\alpha)}(0), 0) \) and \( (0, 0) \) are the only \( T \)-periodic solutions of Eq. (1.2) and \( (0, 0) \) is not a global attractor since \( \int_0^T f_j(t, 0) \, dt > 0 > \int_0^T f_k(t, 0) \, dt \). From this, \( U^{(\alpha, \beta)} = U^{(\alpha)} \) and hence \( U^{(\alpha)}(0) e_\alpha \notin \Lambda \), since \( U^{(\alpha, \beta)} \) is the global attractor of Eq. (1.2).

Let us fix \( Q \in \Lambda \). By the argument in the above claim, we conclude that \( Q = U^{(\alpha)}(0) e_\alpha \) and this contradiction ends the proof.
PROPOSITION 4.3 \((n \geq 3)\). Suppose that there exists \(\alpha \neq \beta\) in \(I_n\) such that \(U^{(i)}\) is positive for \(i = \alpha, \beta, U^{[\alpha, \beta]} \equiv 0\), and
\[
\int_0^T f_i(t, U^{(\alpha)}(t)) e_\alpha + U^{(\beta)}(t) e_\beta \, dt > 0 \quad \text{if} \; j \neq \alpha, \beta.
\]
If \(\mathbb{R}^{(\beta)} \cap \Omega = \emptyset\), then \(\mathbb{R}^{(\alpha)} \cap \Omega = \emptyset\).

Proof. Assume on the contrary that \(\mathbb{R}^{(\alpha)} \cap \Omega\) contains a point \(p\). Since \(\Omega\) does not meet the \(\beta\) axis, then \(0 \not\in \Omega\), and by Remark 2.1, \(p = U^{(\alpha)}(0) e_\alpha\). By Proposition 2.5, there exists \(q = (q_1, \ldots, q_n) \in \Omega \cap \mathbb{R}^{(\alpha, \beta)}\) such that \(q \neq p\) and \(q_\alpha > 0\), and the proof follows as in Proposition 4.2.

Proof of Theorem 1.2. Without loss of generality, we can suppose that \(U^{(3)} \equiv 0 < U^{(1)}\). In particular, \(\int_0^T f_1(t, 0) \, dt > 0 > \int_0^T f_3(t, 0) \, dt\).

If \(U^{(2)} \equiv 0\), then \(\int_0^T f_2(t, 0) \, dt < 0\) and hence \(U^{(1,2)} \equiv 0\). From this and Corollary 2.8 with \(J = (2, 3)\), we have \(0 \not\in \Omega\). On the other hand, if \(U^{(2)}\) is positive, then \(\int_0^T f_2(t, 0) \, dt > 0\), and by Corollary 2.8 with \(J = (3)\), we once again have \(0 \not\in \Omega\). Anyway, \(R^2 \cap \Omega = \emptyset\).

By the change of variables \((x_1, x_2, x_3) \rightarrow (x_2, x_1, x_3)\), it suffices to consider the following seven cases:

(i) \(U^{(2)}\) is positive and \(U^{(1,3)}, U^{(2,3)}\) are not positive.
(ii) \(U^{(2)}, U^{(2,3)}\) are positive and \(U^{(1,3)}, U^{(1,2)}\) are not positive.
(iii) \(U^{(2)}\) is positive, \(U^{(1,3)}\) is not positive, and \(U^{(2,3)}, U^{(1,2)}\) are positive.
(iv) \(U^{(2)}, U^{(1,3)}\) and \(U^{(2,3)}\) are positive and \(U^{(1,2)}\) is not positive.
(v) \(U^{(2)}, U^{(1,3)}, U^{(2,3)}\), and \(U^{(1,2)}\) are positive.
(vi) \(U^{(2)} \equiv 0\) and \(U^{(1,3)}\) is not positive.
(vii) \(U^{(2)} \equiv 0\) and \(U^{(1,3)}\) is positive.

We remark at once that (v) is a special case of Theorem 1.1.

Case i. It is easy to show that \(U^{(1,3)} = U^{(1)}\) and \(U^{(1,3)} \equiv 0\). In particular, \(\int_0^T f_3(t, U^{(1)}(t), 0, 0) \, dt < 0\), and by (H3),
\[
\int_0^T f_2(t, U^{(1)}(t), 0, 0) \, dt > 0. \tag{4.1}
\]
From this and Proposition 4.2, \(\Omega \cap \mathbb{R}^{(1,3)} = \emptyset\). Analogously,
\[
\int_0^T f_1(t, 0, U^{(2)}(t), 0) \, dt > 0 \quad \text{and} \quad \Omega \cap \mathbb{R}^{(2,3)} = \emptyset.
\]
It is clear now that $U^{(1,2)}$ is positive and the proof of case (i) follows from Theorem 2.11 with $\mathcal{F} = \mathcal{P} \setminus \{1, 2\}$.

Case iv. Since $U^{(1,2)}$ is not positive, we can assume without loss of generality that $U^{(1,2)}_{1, 2} = 0$. From this, $U^{(1,2)} = U^{(2)}$ and
\[
\int_0^T f_1(t, U^{(1)}(t), 0, 0) \, dt > 0 > \int_0^T f_2(t, 0, U^{(2)}(t), 0) \, dt.
\]
On the other hand, $U^{(2,3)}$ is positive and hence $\int_0^T f_3(t, U^{(1)}(t), 0, 0) \, dt > 0$. It follows from Corollary 2.10 (with $I = \{1\}$) that $\Omega \cap \mathbb{R}^{(1)} = \emptyset$, and by Proposition 4.3, $\Omega \cap \mathbb{R}^{(2)} = \emptyset$.

We shall prove now that $\Omega \cap \mathbb{R}^{(1,2)} = \emptyset$. To this end, assume by contradiction that there exists $p = (p_1, p_2, 0) \in \Omega$ such that $p_i > 0$ for $i = 1, 2$. Since $v(t, p) = U^{(2)}(t)e_2 \to 0$ as $t \to +\infty$ and $v(kT, p) \in \Omega$ for all $k \in \mathbb{N}$, then $U^{(2)}e_2 \in \Omega$ and this contradiction proves our assertion. The proof follows now from Theorem 2.11 with $\mathcal{F} = \mathcal{P} \setminus \{1, 3, 2, 3\}$.

Case vi. As in case (i), $\Omega \cap \mathbb{R}^{(1,3)} = \emptyset$ and Eq. (4.1) holds. In particular, $U^{(1,2)}$ is positive. On the other hand, $\int_0^T f_i(t, 0) \, dt < 0$ for $i = 2, 3$, and hence, $U^{(2,3)} = 0$. By Proposition 2.7 and Remark 2.2, $\Omega \cap \mathbb{R}^{(2,3)} = \emptyset$ and the proof follows as in case (i).

The other cases are proved in similar form, and the proof is complete. 

REFERENCES