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A note on Marcinkiewicz integral operators

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Abstract

This paper is primarily concerned with proving the L^p boundedness of Marcinkiewicz integral operators with kernels belonging to certain block spaces. We also show the optimality of our condition on the kernel for the L^2 boundedness of the Marcinkiewicz integral.

Keywords: Marcinkiewicz integrals; Oscillatory integrals; Fourier transform; L^p boundedness; Rough kernels; Block spaces

1. Introduction and results

Let $\mathbf{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let Ω be a homogeneous function of degree 0 satisfying $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x| \in \mathbf{S}^{n-1}$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension corresponding to the Littlewood–Paley g function is defined by

$$\mathcal{M}_\Omega f(x) = \left(\int_0^\infty |K_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

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where

$$K(x) = |x|^{1-n}\Omega(x)\chi_{(0,1]}(|x|), \qquad K_t(x) = t^{-n}K(t^{-1}x)$$

and χ_A denotes the characteristic function of a set A.

E.M. Stein introduced the operator $\mathcal{M}_\Omega$ and showed that if $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ $(0 < \alpha \leqslant 1)$, then $\mathcal{M}_\Omega$ is of type (p, p) for $p \in (1, 2]$ and of weak type $(1, 1)$ (see [10]). Subsequently, A. Benedek, A. Calderón, and R. Panzone proved that $\mathcal{M}_\Omega$ is of type (p, p) for $p \in (1, \infty)$ if $\Omega \in C^1(\mathbf{S}^{n-1})$ (see [BCP]).

On the other hand, the related Calderón–Zygmund singular integral operator T_Ω, which is given by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbf{R}^n} \Omega(y')|y|^{-n} f(x-y)\,dy, \tag{1.2}$$

where $y' = y/|y|$ for $y \neq 0$, is known to be bounded on L^p under much weaker conditions on Ω. For example, if $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$, Calderón–Zygmund showed that T_Ω is bounded on L^p for all $p \in (1, \infty)$ and the condition $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ is essentially the weakest possible size condition on Ω for the L^p boundedness of T_Ω to hold [4]. Another condition on Ω was given by Jiang and Lu who introduced a special class of block spaces $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1})$ and proved the following L^2 boundedness result.

Theorem 1.1 [8]. *Let Ω and T_Ω be given as in* (1.1)–(1.2). *Then if $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ with $q > 1$, T_Ω is a bounded operator on $L^2(\mathbf{R}^n)$.*

Some years later, the L^p boundedness of the operator T_Ω was proved for all $p \in (1, \infty)$ under the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ (see, for example, [1,3]). Also, it was proved in [2] that the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ is the best possible for the L^p boundedness of T_Ω to hold. Namely, the L^p boundedness of T_Ω may fail for any p if it is replaced by a weaker condition $\Omega \in B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for any $-1 < \upsilon < 0$ and $q > 1$. The definition of the block space $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1})$ will be recalled in Section 2.

The results cited above on singular integrals give rise to the problem whether similar results hold for the Marcinkiewicz integral operator $\mathcal{M}_\Omega$. More precisely, we have the following:

Problem. Determine whether the L^p boundedness of the operator $\mathcal{M}_\Omega$ holds under a condition in the form of $\Omega \in B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$, $-1 < \upsilon$, and, if so, what is the best possible value of υ.

The main focus of this paper is to obtain a complete solution to the above problem. Moreover, we present a systematic treatment of Marcinkiewicz integrals with kernels belonging to certain block spaces. This method is presented mainly in Theorem 2.1 whose proof is based in part on a combination of ideas from [3,5,6,9], among others. We remark that this method has also found applications for other problems in this area that will appear in forthcoming papers.

Our main result in this paper is the following:

Theorem 1.2. (a) *If* $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$, $q > 1$, *and satisfies* (1.1), *then*

$$\|\mathcal{M}_\Omega\|_p \leqslant C_p \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})} \|f\|_p \tag{1.3}$$

for all $f \in L^p(\mathbf{R}^n)$ *and* $p \in (1, \infty)$.

(b) *There exists an* Ω *which lies in* $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ *for all* $-1 < \upsilon < -1/2$ *and satisfies* (1.1) *such that* $\mathcal{M}_\Omega$ *is not bounded on* $L^2(\mathbf{R}^n)$.

We point out here that part (a) represents an improvement over the results of Stein, and Benedek–Calderón–Panzone while part (b) shows that the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ is nearly optimal.

Throughout the rest of the paper the letter C will stand for a positive constant not necessarily the same one at each occurrence.

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2. Main theorem

For a given a family of measures $\{\sigma_t\colon t \in \mathbf{R}^+\}$, we define the maximal operator σ^* by $\sigma^*(f) = \sup_{t \in \mathbf{R}^+} ||\sigma_t| * f|$. Also, we write $t^{\pm\alpha} = \inf\{t^\alpha, t^{-\alpha}\}$ and $|\sigma|$ for the total variation of σ, which is a positive measure. The proof of Theorem 1.2(a) will rely heavily on the following theorem:

Theorem 2.1. *Let* $a \geqslant 2$, $B > 1$, $C > 0$, *and* $q > 1$. *Suppose that the family of measures* $\{\sigma_t\colon t \in \mathbf{R}^+\}$ *satisfies the following*:

(i) $\|\sigma_t\| \leqslant 1$;
(ii) $\int_{a^{kB}}^{a^{(k+1)B}} |\hat{\sigma}_t(\xi)|^2\, dt/t \leqslant CB(a^{kB}|\xi|)^{\pm\alpha/B}$;
(iii) $\|\sigma^*(f)\|_q \leqslant C\|f\|_q$ *for* $f \in L^q(\mathbf{R}^n)$.

Then for any $p \in (2q/(q+1), 2q/(q-1))$, *there exists a positive constant* C_p *such that*

$$\left\| \left(\int_0^\infty |\sigma_t * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \leqslant C_p B^{1/2} \|f\|_p \tag{2.1}$$

for all $f \in L^p(\mathbf{R}^n)$. *The constant* C_p *is independent of* B.

Proof. The argument of the proof mainly follows the ideas given in [6,9] and keeping track of various constants at each step. Let $\mathcal{M}f(x) = (\int_0^\infty |\sigma_t * f(x)|^2\, dt/t)^{1/2}$ and $\{\Phi_j\}_{-\infty}^\infty$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $E_j = [2^{-(j+1)B}, 2^{-(j-1)B}]$. More precisely, we require the following:

$$\Phi_j \in C^\infty, \quad 0 \leqslant \Phi_j \leqslant 1, \quad \sum_j \Phi_j(t) = 1, \quad \operatorname{supp} \Phi_j \subseteq E_j, \quad \left| \frac{d^s \Phi_j(t)}{dt^s} \right| \leqslant \frac{C}{t^s},$$

where C can be chosen to be independent of B. Let $\widehat{\Psi_k}(\xi) = \Phi_k(|\xi|)$. Decompose

$$f * \sigma_t(x) = \sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} (\Psi_{k+j} * \sigma_t * f)(x)\chi_{[2^{kB},2^{(k+1)B})}(t) := \sum_{j\in\mathbf{Z}} F_j(x,t)$$

and define

$$\mathcal{M}_j f(x) = \left(\int_0^\infty |F_j(x,t)|^2 \frac{dt}{t}\right)^{1/2}.$$

Then

$$\mathcal{M}(f) \leqslant \sum_{j\in\mathbf{Z}} \mathcal{M}_j(f)$$

holds for $f \in \mathcal{S}(\mathbf{R}^n)$.

First, by Plancherel's theorem

$$\begin{aligned}
\|\mathcal{M}_j(f)\|_2^2 &= \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^{kB}}^{2^{(k+1)B}} |\Psi_{k+j} * \sigma_t * f(x)|^2 \frac{dt}{t}\, dx \\
&\leqslant \sum_{k\in\mathbf{Z}} \int_{E_{j+k}} \left(\int_{2^{kB}}^{2^{(k+1)B}} |\hat{\sigma}_t(\xi)|^2 \frac{dt}{t}\right) |\hat{f}(\xi)|^2\, d\xi \\
&\leqslant CB \sum_{k\in\mathbf{Z}} \int_{E_{j+k}} \min(|2^{kB}\xi|^{\alpha/B}, |2^{kB}\xi|^{-\alpha/B}) |\hat{f}(\xi)|^2\, d\xi \\
&\leqslant CB 2^{-\alpha|j|} \sum_{k\in\mathbf{Z}} \int_{E_{j+k}} |\hat{f}(\xi)|^2\, d\xi \leqslant CB 2^{-\alpha|j|} \|f\|_2^2.
\end{aligned}$$

Therefore,

$$\|\mathcal{M}_j(f)\|_2 \leqslant C(B)^{1/2} 2^{-\alpha|j|/2} \|f\|_2. \tag{2.2}$$

On the other hand, we compute the L^p-norm of $\mathcal{M}_j(f)$. For $2 \leqslant p < 2q/(q-1)$, there exists a function g in $L^{(p/2)'}$ with $\|g\|_{(p/2)'} \leqslant 1$ such that

$$\begin{aligned}
\|\mathcal{M}_j(f)\|_p^2 &= \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^{kB}}^{2^{(k+1)B}} |\Psi_{k+j} * \sigma_t * f(x)|^2 \frac{dt}{t} |g(x)|\, dx \\
&\leqslant \Big[\sup_t \|\sigma_t\|\Big] \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^{kB}}^{2^{(k+1)B}} |\sigma_t| * |\Psi_{k+j} * f(x)|^2 \frac{dt}{t} |g(x)|\, dx \\
&\leqslant CB \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |\Psi_{k+j} * f(x)|^2 \sigma^*(\tilde{g})(-x)\, dx \quad (\text{with } \tilde{g}(x) = g(-x)),
\end{aligned}$$

$$\leqslant CB\left\|\sum_{k\in\mathbf{Z}}|\Psi_{k+j}*f|^2\right\|_{p/2}\|\sigma^*(\tilde g)\|_{(p/2)'}.$$

By using (iii), the Littlewood–Paley theory and Theorem 3 along with the remark that follows its statement in [11, p. 96], we have

$$\|\mathcal{M}_j(f)\|_p\leqslant C(B)^{1/2}\|f\|_p\quad\text{for } 2\leqslant p<\infty.$$

To handle the case $2q/(q+1)<p<2$, we need to prove the following lemma.

Lemma 2.2. *Let $g_k(x,t)$ be arbitrary functions on $\mathbf{R}^n\times\mathbf{R}^+$. If $p>2$, then*

$$\left\|I=:\left(\sum_{k\in\mathbf{Z}}\int_{2^{kB}}^{2^{(k+1)B}}|g_k(\cdot,t)*\sigma_t|^2\frac{dt}{t}\right)^{1/2}\right\|_p\leqslant C\left\|\left(\sum_{k\in\mathbf{Z}}\int_{2^{kB}}^{2^{(k+1)B}}|g_k(\cdot,t)|^2\frac{dt}{t}\right)^{1/2}\right\|_p.$$

Proof. As above, if $p>2$, there exists a function $h\in L^{(p/2)'}(\mathbf{R}^n)$ such that

$$I=\left(\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{2^{kB}}^{2^{(k+1)B}}|g_k(\cdot,t)*\sigma_t|^2\frac{dt}{t}h(x)\,dx\right)^{1/2}.$$

By the same argument as above, we have

$$\begin{aligned}I&\leqslant\left(\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{2^{kB}}^{2^{(k+1)B}}|\sigma_t|*|g_k(\cdot,t)|^2\frac{dt}{t}h(x)\,dx\right)^{1/2}\\&\leqslant\left(\left\|\sum_{k\in\mathbf{Z}}\int_{2^{kB}}^{2^{(k+1)B}}|g_k(\cdot,t)|^2\frac{dt}{t}\right\|_{p/2}\|\sigma^*(\tilde h)\|_{(p/2)'}\right)^{1/2}\end{aligned}$$

which ends the proof of the lemma. □

Now we are ready to prove (2.1) for the case $2q/(q+1)<p<2$. Let $I_k=[2^{kB},2^{(k+1)B})$. By a duality argument, there exist functions $g_k(x,t)$ defined on $\mathbf{R}^n\times\mathbf{R}^+$ with $\|\|\|g_k\|_{L^2(\chi_{I_k},dt/t)}\|_{l^2}\|_{L^{p'}}\leqslant 1$ such that

$$\begin{aligned}\|\mathcal{M}_j(f)\|_p&=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{2^{kB}}^{2^{(k+1)B}}(\Psi_{k+j}*\sigma_t*f(x))g_k(x,t)\frac{dt}{t}\,dx\\&\leqslant(B)^{1/2}\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}(|\Psi_{k+j}*f(x)|^2)^{1/2}\left(\int_{2^{kB}}^{2^{(k+1)B}}|\sigma_t*g_k(x,t)|^2\frac{dt}{t}\right)^{1/2}dx\\&\leqslant C(B)^{1/2}\left\|\left(\sum_{k\in\mathbf{Z}}|\Psi_{k+j}*f|^2\right)^{1/2}\right\|_p\left\|\left\|\|g_k\|_{L^2(\chi_{I_k},dt/t)}\right\|_{l^2}\right\|_{L^{p'}}\\&\leqslant C(B)^{1/2}\|f\|_p.\end{aligned}$$

Here as above, the last inequality follows by the Littlewood–Paley theory and Theorem 3 along with the remark that follows its statement in [11, p. 96]. □

3. Some definitions and lemmas

We start by the following definition.

Definition 3.1. (1) For $x_0' \in \mathbf{S}^{n-1}$ and $0 < \theta_0 \leqslant 2$, the set

$$B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1}\colon\ |x' - x_0'| < \theta_0\}$$

is called a cap on $\mathbf{S}^{n-1}$.

(2) For $1 < q \leqslant \infty$, a measurable function b is called a q-block on $\mathbf{S}^{n-1}$ if b is a function supported on some cap $I = B(x_0', \theta_0)$ with $\|b\|_{L^q} \leqslant |I|^{-1/q'}$ where $|I| = \sigma(I)$ and $1/q + 1/q' = 1$.

(3) $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})\colon\ \Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and $M_q^{(\kappa,\upsilon)}(\{c_k\},\{I_k\}) = \sum_{\mu=1}^{\infty} |c_\mu|(1 + \phi_{\kappa,\upsilon}(|I_\mu|)) < \infty\}$, where

$$\phi_{\kappa,\upsilon}(t) = \chi_{(0,1)}(t) \int_t^1 u^{-1-\kappa} \log^{\upsilon}(u^{-1})\, du. \tag{3.1}$$

We remark that the definition of $B_q^{(\kappa,\upsilon)}([a,b]), a, b \in \mathbf{R}$ will be the same as that of $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1})$ except for minor modifications. One observes that

$$\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1}) \quad \text{as } t \to 0 \text{ for } \kappa > 0,\ \upsilon \in \mathbf{R},$$

and

$$\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}(t^{-1}) \quad \text{as } t \to 0 \text{ for } \upsilon > -1.$$

The following properties of $B_q^{(\kappa,\upsilon)}$ can be found in [7]:

(i) $B_q^{(\kappa,\upsilon_2)} \subset B_q^{(\kappa,\upsilon_1)}$ if $\upsilon_2 > \upsilon_1 > -1$ and $\kappa \geqslant 0$; (3.2)

(ii) $B_q^{(\kappa_2,\upsilon_2)} \subset B_q^{(\kappa_1,\upsilon_1)}$ if $\upsilon_1, \upsilon_2 > -1$ and $0 \leqslant \kappa_1 < \kappa_2$; (3.3)

(iii) $B_{q_2}^{(\kappa,\upsilon)} \subset B_{q_1}^{(\kappa,\upsilon)}$ if $1 < q_1 < q_2$; (3.4)

(iv) $L^q(\mathbf{S}^{n-1}) \subset B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) \subset L^1(\mathbf{S}^{n-1})$ for $\upsilon > -1$ and $\kappa \geqslant 0$. (3.5)

The following result due to Keitoku and Sato which can be found in [7]:

Lemma 3.2. (i) *If $1 < p \leqslant q \leqslant \infty$, then for $\kappa > 1/p'$ we have*

$$B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) \subseteq L^p(\mathbf{S}^{n-1}) \quad \textit{for any } \upsilon > -1;$$

(ii)

$$B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) = L^q(\mathbf{S}^{n-1}) \quad \textit{if and only if } \kappa \geqslant \frac{1}{q'} \textit{ and } \upsilon \geqslant 0;$$

(iii) *for any* $\upsilon > -1$, *we have*

$$\bigcup_{q>1} B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) \nsubseteq \bigcup_{q>1} L^q(\mathbf{S}^{n-1}).$$

Let $N_q^{(0,\upsilon)}(\Omega) = \inf\{M_q^{(0,\upsilon)}(\{c_k\},\{I_k\})\colon\ \Omega = \sum_{k=1}^{\infty} c_k b_k$ and each b_k is a q-block function supported on a interval $I_k\}$.

To prove part (b) of Theorem 1.2 we shall rely heavily on the following lemma from [2].

Lemma 3.3. *For any* $\upsilon > -1$, $a, b \in \mathbf{R}$,

(i) $N_q^{(0,\upsilon)}$ *is a norm on* $B_q^{(0,\upsilon)}([a,b])$ *and* $(B_q^{(0,\upsilon)}([a,b]), N_q^{(0,\upsilon)})$ *is a Banach space*;

(ii) *If* $f \in B_q^{(0,\upsilon)}([a,b])$ *and* g *is a measurable on* $[a,b]$ *with* $|g| \leqslant |f|$, *then* $g \in B_q^{(0,\upsilon)}([a,b])$ *with*

$$N_q^{(0,\upsilon)}(g) \leqslant N_q^{(0,\upsilon)}(f);$$

(iii) *Let* I_1 *and* I_2 *be two disjoint intervals in* $[a,b]$ *with* $|I_1|, |I_2| < 1$ *and* $\alpha_1, \alpha_2 \in \mathbf{R}^+$. *Then*

$$N_q^{(0,\upsilon)}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \geqslant N_q^{(0,\upsilon)}(\alpha_1\chi_{I_1}) + N_q^{(0,\upsilon)}(\alpha_2\chi_{I_2});$$

(iv) *Let* I *be an interval in* $[a,b]$ *with* $|I| < 1$. *Then*

$$N_q^{(0,\upsilon)}(\chi_I) \geqslant |I|\left(1 + \log^{\upsilon+1}(|I|^{-1})\right).$$

4. Proof of Theorem 1.2(b)

It is clear that $\mathcal{M}_\Omega$ is bounded on $L^2(\mathbf{R}^n)$ if and only if the multiplier

$$m(\xi) = \left(\int_0^\infty \left|\int_{|y|\leqslant t} e^{-2\pi i \xi'\cdot y} \frac{\Omega(y)}{|y|^{n-1}}\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

is an L^∞ function, where $\xi' = \xi/|\xi|$.

It is easy to see that

$$m(\xi) = \lim_{N\to\infty,\,\varepsilon\to 0} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)} \times \int_0^1\int_0^1 \left(\int_\varepsilon^N e^{-2\pi i t \xi'\cdot(rx-sy)}\frac{dt}{t}\right) dr\,ds\,d\sigma(x)\,d\sigma(y).$$

Note that

$$\int_{\varepsilon}^{N} (e^{-2\pi i t \xi'\cdot(rx-sy)} - \cos(2\pi t))\frac{dt}{t}$$
$$\to \quad \log|\xi'\cdot(rx-sy)|^{-1} - i\frac{\pi}{2}\operatorname{sgn}(\xi'\cdot(rx-sy))$$

as $N\to\infty$ and $\varepsilon\to 0$, and the integral is bounded, uniformly in ε and N, by $C(\log|\xi'\cdot(rx-sy)|)$.

Thus, using (1.1) and the Lebesgue dominated convergence theorem, we get

$$m(\xi) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \int_0^1\int_0^1 \Omega(x)\overline{\Omega(y)}$$
$$\times\left(\log|\xi'\cdot(rx-sy)|^{-1} - i\frac{\pi}{2}\operatorname{sgn}(\xi'\cdot(rx-sy))\right)dr\,ds\,d\sigma(x)\,d\sigma(y).$$

Now, if Ω is a real-valued function, by (1.1) and a straightforward computations we get

$$m(\xi) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \left(\Omega(x)\Omega(y)\left(\frac{\xi'\cdot x}{\xi'\cdot y} - 1\right)\log|\xi'\cdot(x-y)|\right.$$
$$\left. - \left(\frac{\xi'\cdot x}{\xi'\cdot y}\right)\log|\xi'\cdot(x-y)|\right)d\sigma(x)\,d\sigma(y). \tag{4.1}$$

Now, we are ready to prove part (b) of Theorem 1.2. For the sake of simplicity we shall present the construction of our Ω only in the case $n=2$ and $q=\infty$. Other cases can be obtained by minor modifications. Also, we shall work on $[-1,1]$ instead of $\mathbf{S}^1$. For $u\in[-1,1]$, let

$$\Omega(u) = \sum_{k=1}^{\infty} C_k b_k(u) \tag{4.2}$$

where

$$C_1 = \sum_{k=2}^{\infty}\frac{1}{(k+1)(\log k)^{3/2}}, \qquad b_1(u) = -\chi_{[-1,0]}(u),$$
$$C_k = \frac{1}{(k+1)(\log k)^{3/2}},$$
$$b_k(u) = |D_k|^{-1}\chi_{D_k}(u) \quad \text{and} \quad D_k = \left[\frac{1}{k+1}, \frac{1}{k}\right) \text{ for } k \geqslant 2.$$

Then Ω has the desired properties. More precisely, Ω satisfies the following:

$$\int_{-1}^{1}\Omega(u)\,du = 0; \tag{4.3}$$

$$\Omega \in B_\infty^{(0,\upsilon)}([-1,1]) \quad \text{for each } \upsilon, -1 < \upsilon < -\frac{1}{2}; \tag{4.4}$$

$$\Omega \notin B_\infty^{(0,-1/2)}([-1,1]); \tag{4.5}$$

$$\begin{aligned} \mathcal{J}_1 &= \int\limits_{[0,1]^2} \left(\Omega(u)\Omega(v)\left(1-\frac{u}{v}\right)\log|u-v|^{-1} + \left(\frac{u}{v}\right)\log|u|^{-1}\right) du\,dv \\ &= \infty; \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathcal{J}_2 &= \int\limits_{[-1,1]^2\setminus[0,1]^2} \left|\Omega(u)\Omega(v)\left(1-\frac{u}{v}\right)\log|u-v|^{-1} + \left(\frac{u}{v}\right)\log|u|^{-1}\right| du\,dv \\ &< \infty. \end{aligned} \tag{4.7}$$

The proof of (4.3)–(4.4) is straightforward. Now we turn to the proof of (4.5). We first notice that each b_k is an ∞-block supported on the interval D_k. So to prove (4.5), we only need to show that $N_\infty^{(0,-1/2)}(\Omega) = \infty$. To this end, by Lemma 3.3 we have for each l,

$$\begin{aligned} N_\infty^{(0,-1/2)}(\Omega + C_1\chi_{[-1,0]}) &\geqslant \sum_{k=2}^{l} |C_k||D_k|^{-1} N_\infty^{(0,-1/2)}(\chi_{D_k}) \\ &\geqslant \sum_{k=2}^{l} |C_k|\left(1+\log^{1/2}(|D_k|^{-1})\right). \end{aligned}$$

Letting $l \to \infty$, we get $N_\infty^{(0,-1/2)}(\Omega + C_1\chi_{[-1,0]}) = \infty$. Since, $N_\infty^{(0,-1/2)}(C_1\chi_{[-1,0]}) < \infty$ we get $N_\infty^{(0,-1/2)}(\Omega) = \infty$.

Now, we verify (4.6). Let

$$a_k = \frac{k}{(\log k)^{3/2}}, \quad k \geqslant 2;$$

$$I(u,v) = \left(1-\frac{u}{v}\right)\log|u-v|^{-1} + \left(\frac{u}{v}\right)\log|u|^{-1};$$

$$I^*(u,v) = \left(\frac{u}{v}\right)\left(\log|u|^{-1} - \log|u-v|^{-1}\right).$$

It is clear that

$$\mathcal{J}_1 \geqslant S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7,$$

where

$$S_1 = \sum_{j=2}^{\infty} a_j \sum_{k\geqslant 2(j+1)}^{\infty} a_k \int\limits_{D_k\times D_j} I(u,v)\,du\,dv;$$

$$S_2 = \sum_{j=2}^{\infty} a_j \sum_{k=j+1}^{2j-1} a_k \int\limits_{D_k\times D_j} I(u,v)\,du\,dv;$$

$$S_3 = \sum_{j=2}^{\infty} a_j a_{2j+1} \int\limits_{D_{2j+1}\times D_j} I^*(u,v)\,du\,dv;$$

$$S_4 = \sum_{j=2}^{\infty} a_j a_{2j} \int\limits_{Da_{2j}\times D_j} I^*(u,v)\,du\,dv;$$

$$S_5 = \sum_{j=2}^{\infty} (a_j)^2 \int\limits_{D_j\times D_j} I(u,v)\,du\,dv;$$

$$S_6 = \sum_{j=2}^{\infty} a_j a_{j-1} \int\limits_{D_{j-1}\times D_j} I(u,v)\,du\,dv;$$

$$S_7 = \sum_{j=2}^{\infty} a_j \sum_{k=2}^{j-2} a_k \int\limits_{D_k\times D_j} I^*(u,v)\,du\,dv.$$

It is clear that to prove (4.6), it suffices to prove the following:

(i) $S_1 = \infty$;
(ii) $S_2 \geqslant 0$; and
(iii) $|S_i| < \infty$ for $i = 3,\dots,7$.

To prove (i), we notice that, for $(u,v) \in D_k \times D_j$ with $k \geqslant 2(j+1)$, we have $v \geqslant 2u$ and hence $\log|1 - v/u| \geqslant 0$ which in turn leads to $I(u,v) \geqslant \log|u-v|^{-1}$. Also, $|u-v| \leqslant 1/j$ and hence $I(u,v) \geqslant \log j$. Therefore,

$$S_1 \geqslant C \sum_{j=2}^{\infty} \left(\sum_{k=2(j+1)}^{\infty} \left(\frac{1}{k(\log k)^{3/2}} \right) \frac{1}{j(\log j)^{1/2}} \right) = \infty.$$

The proof of (ii) is easy. In fact, for $(u,v) \in D_k \times D_j$ with $j+1 \leqslant k \leqslant 2j-1$, we get $0 \leqslant v/u \leqslant 2$ and hence $|v/u - 1| \leqslant 1$ which when combined with $(u/v - 1) \leqslant 0$ gives (ii).

Now, we turn to prove that $|S_3| < \infty$. To this end, we notice that if $(u,v) \in D_{2j+1} \times D_j$, we have

$$|u-v| = v - u \geqslant \frac{1}{j+1} - \frac{1}{2j+1} > \frac{1}{6j} \quad \text{and} \quad u \geqslant \frac{1}{6j}.$$

Also, by the mean-value theorem we have

$$\frac{u}{v}\big|\log|u-v| - \log|u|\big| \leqslant \frac{u}{\min\{|u-v|,|u|\}} \leqslant \frac{6j}{2j+1} \leqslant 3. \tag{4.8}$$

Therefore,

$$|S_3| \leqslant C \sum_{j=2}^{\infty} a_j a_{2j+1} \left(\frac{1}{j^2} \right) < \infty.$$

Similarly, $|S_4| < \infty$. The proof of $|S_5| < \infty$ is an immediate consequence of the observation that

$$\int\limits_{D_j \times D_j} |I(u,v)|\,du\,dv$$
$$\leqslant \int\limits_{D_j \times D_j} \left(\left| \left(\frac{u-v}{v}\right) \log|u-v|^{-1} \right| + \left| \left(\frac{u}{v}\right) \log|u|^{-1} \right| \right) du\,dv \leqslant \frac{C}{j^3}.$$

Similarly, $|S_6| < \infty$.

Now, we want to verify $|S_7| < \infty$. To this end, notice that since $k \leqslant j-2$, we get $v > u$. Thus, by (4.8) we have

$$|S_7| \leqslant \sum_{j=2}^{\infty} a_j \sum_{k=2}^{j-2} \frac{a_k}{k} \int\limits_{D_j} \log\left(\frac{k+1}{k} \left(\frac{1-kv}{1-(k+1)v} \right) \right) dv$$
$$\leqslant \sum_{j=2}^{\infty} a_j \sum_{k=2}^{j-2} \frac{a_k \log(2(k+1)/k)}{kj^2} < \infty.$$

The proof of (4.6) is complete.

Finally, we verify (4.7). To this end, we divide the integral domain $[-1,1]^2 \backslash [0,1]^2$ into three parts: $[-1,0] \times [0,1]$, $[0,1] \times [-1,0]$, and $[-1,0] \times [-1,0]$. First, the integral over $[-1,0] \times [0,1]$ is dominated from above by

$$\sum_{j=2}^{\infty} a_j \int\limits_{D_j} \int\limits_{-1}^{0} \left(\left| \left(\frac{u-v}{v}\right) \log|u-v|^{-1} \right| + \left| \left(\frac{u}{v}\right) \log|u|^{-1} \right| \right) du\,dv$$
$$\leqslant C \sum_{j=2}^{\infty} \frac{a_j}{j^3} < \infty.$$

On the other hand, the integral over $[0,1] \times [-1,0]$ is dominated from above by

$$\sum_{k=2}^{\infty} a_k \int\limits_{-1}^{0} \int\limits_{D_k} \left(\log|u-v|^{-1} + \left| \left(\frac{u}{v}\right) (\log|u-v| - \log|u|) \right| \right) du\,dv$$
$$\leqslant C \left(\sum_{k=2}^{\infty} \frac{a_k}{k^2} + \sum_{k=2}^{\infty} a_k \int\limits_{-1}^{0} \int\limits_{D_k} \log|u-v|^{-1}\,du\,dv \right) < \infty.$$

Finally, since $(C_1)^2 \chi_{[-1,0]\times[-1,0]} \in L^\infty$, the integral over $[-1,0] \times [-1,0]$ is finite.

5. Proof of Theorem 1.2(a)

By assumption Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$, b_μ is a q-block with support on a cap I_μ on $\mathbf{S}^{n-1}$ and

$$M_q^{(0,-1/2)}(\{c_k\},\{I_k\}) = \sum_{\mu=1}^{\infty} |c_\mu|\left(1+\left(\log^{1/2}|I_\mu|^{-1}\right)\right) < \infty. \tag{5.1}$$

To each block function $b_\mu(\cdot)$, let $\tilde{b}_\mu(\cdot)$ be a function defined by

$$\tilde{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u)\,d\sigma(u). \tag{5.2}$$

Then one can easily verify that $\tilde{b}_\mu$ enjoys the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(u)\,d\sigma(u) = 0, \tag{5.3}$$

$$\|\tilde{b}_\mu\|_{L^q} \leqslant 2|I_\mu|^{-1/q'}, \tag{5.4}$$

$$\|\tilde{b}_\mu\|_{L^1} \leqslant 2. \tag{5.5}$$

Let $A = \{\mu \in \mathbf{N}\colon |I_\mu| \geqslant e^{-1}\}$ and $B = \{\mu \in \mathbf{N}\colon |I_\mu| < e^{-1}\}$. For $\mu \in \mathbf{N}$, we set

$$\mathcal{B}_\mu = \begin{cases} 1, & \text{if } \mu \in A, \\ \log(|I_\mu|^{-1}), & \text{if } \mu \in B. \end{cases}$$

Using the assumption that Ω has the mean zero property (1.1), and the definition of $\tilde{b}_\mu$, we deduce that Ω can be written as

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu,$$

which in turn gives

$$\mathcal{M}_\Omega(f) \leqslant \sum_{\mu=1}^{\infty} |c_\mu| \mathcal{M}_{\tilde{b}_\mu}(f). \tag{5.6}$$

Define the family of measures $\{\sigma_{t,\mu}\colon t \in \mathbf{R}^+\}$ and the corresponding maximal function on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} f\,d\sigma_{t,\mu} = \frac{1}{t}\int_{|y|\leqslant t} \frac{\tilde{b}_\mu(y)}{|y|^{n-1}} f(y)\,dy \quad \text{and} \quad \sigma_\mu^*(f) = \sup_{t\in\mathbf{R}^+} \big||\sigma_{t,\mu}| * f\big|.$$

Then the following holds for $t \in \mathbf{R}^+$, $\xi \in \mathbf{R}^n$ and $p > 1$:

$$\begin{cases} \text{(i)} & \|\sigma_{t,\mu}\| \leqslant C; \\ \text{(ii)} & \int_{2^{k\mathcal{B}_\mu}}^{2^{(k+1)\mathcal{B}_\mu}} |\hat{\sigma}_{t,\mu}(\xi)|^2 \frac{dt}{t} \leqslant C\mathcal{B}_\mu |2^{k\mathcal{B}_\mu}\xi|^{\alpha/\mathcal{B}_\mu}; \\ \text{(iii)} & \int_{2^{k\mathcal{B}_\mu}}^{2^{(k+1)\mathcal{B}_\mu}} |\hat{\sigma}_{t,\mu}(\xi)|^2 \frac{dt}{t} \leqslant C\mathcal{B}_\mu |2^{k\mathcal{B}_\mu}\xi|^{-\alpha/\mathcal{B}_\mu}; \\ \text{(iv)} & \|\sigma_\mu^*(f)\|_p \leqslant C_p\|f\|_p \quad \text{for all } f \in L^p. \end{cases} \tag{5.7}$$

First, the proof of (5.7)(i) is obvious, and (5.7)(ii) follows by (5.7)(i) and (1.1). Also, (5.7)(iv) follows easily by Proposition 1 on p. 477 of [12].

On the other hand, by the proof of Corollary 4.1 on p. 551 of [6],

$$|\hat{\sigma}_{t,\mu}(\xi)| \leqslant C\|\tilde{b}_\mu\|_q|t\xi|^{-1/6};$$

which when combined with (5.7)(i) and (5.4) gives

$$|\hat{\sigma}_{t,\mu}(\xi)|^2 \leqslant C|t\xi|^{-1/(3\mathcal{B}_\mu)}$$

and this leads easily to (5.7)(iii).

By (5.7) and Theorem 2.1 we get

$$\|\mathcal{M}_{\tilde{b}_\mu}(f)\|_p \leqslant C_p(\mathcal{B}_\mu)^{1/2}\|f\|_p \tag{5.8}$$

for all $f \in L^p$ and $1 < p < \infty$. By (5.1), (5.6), and (5.7) we get (1.3). This completes the proof of Theorem 1.2(a).

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Further reading

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