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Another Proof of Jackson's Theorem

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Lebesgue's proof of the Weierstrass approximation theorem is based on the approximation of the single function $|x|$. Newman [3] has pointed out that Jackson's theorem [1], on the order of approximation of continuous functions, can be derived by a suitable polynomial approximation to $|x|$. Such a proof has not appeared in the literature, and in this paper, we carry out the details of Newman's statement. For convenience, we prove the theorem on $[-1, 1]$, but the proof carries over to an arbitrary interval.

Denote by P_n the space of polynomials of degree $\leqslant n$.

For $f \in C[-1, 1]$, denote by $\omega_f(\delta)$ the modulus of continuity of f.

THEOREM. *Let $f \in C[-1, 1]$. Then there exists a $p(x) \in P_n$ such that $|f(x) - p(x)| \leqslant c\omega_f(1/n)$, for all $x \in [-1, 1]$, where c is an absolute constant.*

Proof. Divide $[-1, 1]$ into $2n$ subintervals, $[k/n, (k+1)/n]$, $-n \leqslant k \leqslant n-1$. Let $L(k/n) = f(k/n)$ for all k, and let $L(x)$ be linear in each of the intervals $[k/n, (k+1)/n]$. Then $|L(x) - f(x)| \leqslant \omega_f(1/n)$, for all $x \in [-1, 1]$.

Let S_k be the slope of $L(x)$ in $[(k-1)/n, k/n]$, and let

$$a_k = (S_{k+1} - S_k)/2, \qquad -n+1 \leqslant k \leqslant n-1,$$
$$a_n = (-S_n - S_{-n+1})/2.$$

Then

$$L(x) = A + \sum_{k=-n+1}^{n} a_k |x - k/n| = A + \int_{-1}^{1} |x - t| \, dg(t),$$

where $g(t)$ is a step function having jumps at $x = k/n$ equal to a_k, $g(-1) = 0$, A a constant.

LEMMA 1. *If $p(x)$ is a polynomial satisfying $p(0) = 0$,*

$$\int_{-2}^{2} |d\{|x| - p(x)\}| \leqslant b/n,$$

then

$$\left| f(x) - A - \int_{-1}^{1} p(x-t)\, dg(t) \right| \leqslant (2b+1)\, \omega_f(1/n).$$

Proof.

$$\left| f(x) - A - \int_{-1}^{1} p(x-t)\, dg(t) \right| \leqslant \left| f(x) - A - \int_{-1}^{1} | x - t |\, dg(t) \right|$$
$$+ \left| \int_{-1}^{1} \{| x - t | - p(x-t)\}\, dg(t) \right|$$
$$\leqslant \omega_f(1/n) + \left| \{| x - t | - p(x-t)\}\, g(t) |_{-1}^{1} \right.$$
$$\left. - \int_{-1}^{1} g(t)\, d\{| x - t | - p(x-t)\} \right|$$
$$\leqslant \omega_f(1/n) + | g(1) |\, b/n + \max_{-1 \leqslant t \leqslant 1} | g(t) |\, b/n.$$

Now,

$$\max_{-1 \leqslant t \leqslant 1} | g(t) | = \max_j \left| \sum_{k=-n+1}^{j} a_k \right| \leqslant \max_j | S_j | \leqslant n\omega_f(1/n).$$

Thus,

$$\left| f(x) - A - \int_{-1}^{1} p(x-t)\, dg(t) \right| \leqslant (2b+1)\, \omega_f(1/n).$$

LEMMA 2. *There exists a* $p(x) \in P_{2n}$, *such that* $p(0) = 0$, *and* $\int_{-2}^{2} | d\{| x | - p(x)\}| \leqslant 2\pi/n$.

Proof. $\int_{-2}^{2} | d\{| x | - p(x)\}| = \int_{-2}^{2} | s(x) - p'(x) |\, dx$, where

$$s(x) = \begin{cases} -1, & -2 \leqslant x < 0, \\ 0, & x = 0, \\ 1, & 0 < x \leqslant 2. \end{cases}$$

Let $x_k = 2\cos[k\pi/(2n+2)]$, $k = 1, 2,..., n$, and let $q(x)$ be the odd polynomial of degree $\leqslant 2n - 1$, satisfying $q(x_k) = 1$, $k = 1, 2,..., n$. Then $q(x) - 1$ has simple zeros at the x_k, and no other zeros in [0, 2] (by Descartes' rule of signs); hence, $q(x) - 1$ changes sign precisely at the x_k. Therefore, $q(x) - s(x)$ changes sign precisely at the points $y_k = 2\cos(k\pi/(2n+2))$, $k = 1, 2,..., 2n+1$, and hence, [2], $q(x)$ is the polynomial of best L^1 approximation to $s(x)$ of degree $\leqslant 2n$, and the degree of approximation of $s(x)$ is given by

$$\left| \sum_{k=1}^{2n+2} (-1)^{k+1} \int_{y_k}^{y_{k-1}} s(x)\, dx \right|, \qquad (*)$$

where $y_0 = 2$, and $y_{2n+2} = -2$. Explicitly,

$$(*) = \left| 2 \sum_{k=1}^{n+1} (-1)^{k+1} \int_{y_k}^{y_{k-1}} dx \right| = 4 \left| 1 + 2 \sum_{k=1}^{n} (-1)^k \cos(k\pi/(2n+2)) \right|$$

$$= 4 \tan(\pi/(4n+4)) \leqslant 2\pi/n.$$

Now choose $p(x) = \int_0^x q(t)\,dt$. Then $p(x)$ satisfies the conditions of Lemma 2, thus concluding the proof of the theorem.

References

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2. G. G. Lorentz, "Approximation of Functions," pp. 112–113, Holt, Rinehart, and Winston, 1966.
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