JOURNAL OF APPROXIMATION THEORY **60**, 360–363 (1990)

Letter to the Editor

Hilbert Transforms and Lagrange Interpolation*

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Communicated by the Editors

Received October 7, 1989

This letter is about my paper "*Mean Convergence of Lagrange Interpolation*, III," see [4]. In that paper I show that many theorems on weighted mean convergence of Lagrange interpolation can be proved by reducing the problem to the boundedness of certain Hilbert transforms. As it is well known, the Hilbert transform $H(f)$ of a function $f \in L_1(\mathbf{R})$ is defined by

$$H(f, x) = \lim_{\varepsilon \downarrow 0+} \int_{|x-t| > \varepsilon} \frac{f(t)}{x-t}\, dt.$$

Reference [4] was quoted and was applied in a number of papers dealing with convergence of various types of interpolation and quadrature procedures. I was really surprised when *Giuseppe Mastroianni* noted in September 1989 that in the proof of Theorem 1 in [2], I apply the formula

$$\int_{\mathbf{R}} H(F)G = -\int_{\mathbf{R}} H(G)F \tag{1}$$

without proper justification. Of course, (1) is well known for $F \in L_p$ and $G \in L_q$ for $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, see, e.g., [1, pp. 1059–1060] or [6, Theorem 102, p. 138]. I use (1) on page 682 of [4], in lines 12 and 13 from the top and in lines 4 and 5 from the bottom of the page.[1] Unfortunately, however, in those particular contexts the functions F and G do

* This material is based upon research supported by the National Science Foundation under Grant DMS-8814488 and by NATO.

[1] I use this opportunity to correct a misprint: please insert "1_n^c" once between "w_*" and "|" in [4, p. 682, line 7^-] and four times between "w_*" and "G_5" in [4, p. 682, lines 1^-–5^-].

0021-9045/90 $3.00

not belong to conjugate L_p and L_q spaces with finite p and q. Instead, they both have compact supports and satisfy $F \in L(\log^+ L)^r$ with some $r > 1$ and $G \in L_\infty$. Therefore, to fully justify the formula right after [4, p. 682, formula (56)], we need the following

THEOREM. *Let F and G have compact supports. If $F \in L \log^+ L$ and $G \in L_\infty$ then* (1) *holds.*

Although I could not find a direct reference to this theorem in the literature, it is not so difficult to prove it at all. The following very simple proof was given by *Harold Widom.*

Proof of the Theorem. In what follows, we will assume that both F and G are supported in a compact interval K, and we will use the notation $\|F\|_{L\log^+ L} = \int_{\mathbf{R}} |F| \log^+ |F|$.

First we will prove that

$$\int_K |H(F)G| + \int_K |H(G)F| \leqslant C(K) \, \|G\|_\infty \, (\|F\|_{L\log^+ L} + 1) \tag{2}$$

for every $F \in L \log^+ L$ and $G \in L_\infty$ supported in K, where the constant $C(K)$ depends on K only. The inequality, due to A. Zygmund,

$$\int_K |H(F)| \leqslant C(K)(\|F\|_{L\log^+ L} + 1), \qquad F \in L \log^+ L, \tag{3}$$

is well known, see, e.g., [7, Theorem 2.8, p. 254; 3, Theorem, p. 135; 2, Further Results 7, p. 127; 5, Further Results 6.2b, p. 48]. To be honest, in [7, 3] the case of conjugate functions is treated only, but it is clear that conjugate functions and Hilbert transforms are essentially the same operators (with proper normalization their difference is a bounded operator from L_1 into L_1) as long as we consider finite intervals only. On the other hand, neither [2] (for conjugate functions) nor [5] (for Hilbert transforms) contains a proof nor is the explicit inequality (3) given. Unfortunately, I could not find (3) in any book I am familiar with, and in this letter I want to avoid giving references to journal articles. Following Harold Widom's idea, we prove

$$\int_K |H(G)F| \leqslant C(K) \, \|G\|_\infty \, (\|F\|_{L\log^+ L} + 1) \tag{4}$$

by observing that it is sufficient to prove it for, say, $F \in L_\infty$, since then we can use approximation to extend it for all $F \in L \log^+ L$. But for F, $G \in L_\infty$

we can use (1) (recall both F and G are supported in a compact interval K) so that by (3)

$$\int_K |H(G)F| = \int_K H(G)\,\rho F = -\int_K GH(\rho F) \leqslant \|G\|_\infty \int_K |H(\rho F)|$$
$$\leqslant C(K)\,\|G\|_\infty(\|F\rho\|_{L\log^+ L}+1) \leqslant C(K)\|G\|_\infty(\|F\|_{L\log^+ L}+1),$$

where ρ is some function with $|\rho| = 1$. Hence (4) holds, and then (2) follows from (3) and (4).

Having established (2) we can prove (1) as follows. Let us define the bilinear functional A by

$$A(F, G) = \int_{\mathbf{R}} H(F)G + \int_{\mathbf{R}} H(G)F.$$

Then by (1) we have

$$A(F, G) = 0, \qquad F \in L_2 \text{ and } G \in L_\infty, \tag{5}$$

and by (2) the inequality

$$|A(F, G)| \leqslant C(K)\,\|G\|_\infty\,(\|F\|_{L\log^+ L}+1), \qquad F \in L\log^+ L \text{ and } G \in L_\infty \tag{6}$$

holds. In view of (5) we can use approximation in inequality (6) to obtain

$$|A(F, G)| \leqslant C(K)\,\|G\|_\infty, \qquad F \in L\log^+ L \text{ and } G \in L_\infty,$$

as well. Finally, replacing here F by σF and letting $\sigma \to \infty$ we get

$$A(F, G) = 0, \qquad F \in L\log^+ L \text{ and } G \in L_\infty,$$

that is we have proved the theorem. ▮

I herewith express my gratitude to *Alphonse Magnus*, *Gerald Edgar*, *Boris Mityagin*, and *Walter Van Assche* for providing me with much needed references, and, above all, to *Harold Widom* for his idea to prove (4) so elegantly. I also thank *Giuseppe Mastroianni* for having read my paper [4] so carefully and for having noted the lack of proper justification in my reasoning which is quite an accomplishment on its own merit.

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