J. Differential Equations 252 (2012) 5613–5661

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

The wave equation on hyperbolic spaces ☆

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A R T I C L E I N F O

Article history:
Received 14 August 2011
Revised 27 November 2011
Available online 10 February 2012

MSC:
35L05
43A85
22E30
35L71
43A90
47J35
58D25
58J45

Keywords:
Hyperbolic space
Semilinear wave equation
Dispersive estimate
Strichartz estimate
Local well-posedness

A B S T R A C T

We study the dispersive properties of the wave equation associated with the shifted Laplace–Beltrami operator on real hyperbolic spaces and deduce new Strichartz estimates for a large family of admissible pairs. As an application, we obtain local well-posedness results for the nonlinear wave equation.

1. Introduction

The aim of this paper is to study the dispersive properties of the linear wave equation on real hyperbolic spaces and their application to nonlinear Cauchy problems.

☆ This work was mostly carried out while the third author was a CNRS postdoc at the *Fédération Denis Poisson* Orléans–Tours. It was further supported by the Progetto GNAMPA 2010 *Analisi armonica su gruppi di Lie e varietà Riemanniane* and by the PHC Galilée 25970QB *VAMP*.
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2012.01.031

This theory is well established for the wave equation on $\mathbb{R}^n$:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x), \\ u(0,x) = f(x), \\ \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{1}$$

for which the following Strichartz estimates hold:

$$\|u\|_{L^p(I;L^q)} + \|u\|_{L^\infty(I;\dot{H}^\sigma)} + \|\partial_t u\|_{L^\infty(I;\dot{H}^{\sigma-1})} \lesssim \|f\|_{\dot{H}^\sigma} + \|g\|_{\dot{H}^{\sigma-1}} + \|F\|_{L^{\tilde{p}'}(I;\dot{H}^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})} \tag{2}$$

on any (possibly unbounded) interval $I \subseteq \mathbb{R}$, under the assumptions that

$$\sigma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \qquad \tilde{\sigma} = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right),$$

and separately the couples $(p,q), (\tilde{p},\tilde{q}) \in (2,\infty] \times [2, 2\frac{n-1}{n-3})$ satisfy the admissibility conditions

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad \frac{2}{\tilde{p}} + \frac{n-1}{\tilde{q}} = \frac{n-1}{2}.$$

The estimate (2) holds also at the endpoint $(2, 2\frac{n-1}{n-3})$ when $n \geqslant 4$. When $n = 3$ this endpoint is $(2,\infty)$ and the estimate (2) fails in this case without additional assumptions (see [13] and [25] for more details).

These estimates yield existence results for the nonlinear wave equation in the Euclidean setting. The problem of finding minimal regularity on initial data ensuring local well-posedness for semilinear wave equation was addressed for higher dimensions and nonlinearities in [24], and then almost completely answered in [27,12,25,8].

Once the Euclidean case was more or less settled, several attempts have been made in order to establish Strichartz estimates for dispersive equations in other settings. Here we consider real hyperbolic spaces $\mathbb{H}^n$, which are the most simple examples of noncompact Riemannian manifolds with negative curvature. For geometric reasons, we expect better dispersive properties hence stronger results than in the Euclidean setting.

It is well known that the spectrum of the Laplace–Beltrami operator $-\Delta_{\mathbb{H}^n}$ on $L^2(\mathbb{H}^n)$ is the half-line $[\rho^2, +\infty)$, where $\rho = \frac{n-1}{2}$. Thus one may study either the *non-shifted* wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_{\mathbb{H}^n} u(t,x) = F(t,x), \\ u(0,x) = f(x), \\ \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{3}$$

or the *shifted* wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \big(\Delta_{\mathbb{H}^n} + \rho^2\big) u(t,x) = F(t,x), \\ u(0,x) = f(x), \\ \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{4}$$

In [29] Pierfelice derived Strichartz estimates for the wave equation (3) with radial data on a class of Riemannian manifolds containing all hyperbolic spaces. The wave equation (3) was also investigated on the 3-dimensional hyperbolic space by Metcalfe and Taylor [28], who proved dispersive and Strichartz estimates with applications to small data global well-posedness for the semilinear wave equation. This result was recently generalized by Anker and Pierfelice [2] to other dimensions. Another recent work [15] by Hassani contains a first study of (3) on general Riemannian symmetric spaces of noncompact type.

To our knowledge, the semilinear wave equation (4) was first considered by Fontaine [9,10] in dimensions $n=3$ and $n=2$. The most famous work involving (4) is due to Tataru. In [30] he obtained dispersive estimates for the operators $\frac{\sin(t\sqrt{\Delta_{\mathbb{H}^n}+\rho^2})}{\sqrt{\Delta_{\mathbb{H}^n}+\rho^2}}$ and $\cos(t\sqrt{\Delta_{\mathbb{H}^n}+\rho^2})$ acting on inhomogeneous Sobolev spaces and then transferred them from $\mathbb{H}^n$ to $\mathbb{R}^n$ in order to get well-posedness results for the Euclidean semilinear wave equation (see also [11]). Though Tataru proved dispersive estimates with exponential decay in time, these are not sufficient to obtain actual Strichartz estimates on hyperbolic spaces. Complementary results were obtained by Ionescu [22], who investigated $L^q \to L^q$ Sobolev estimates for the above operators on all hyperbolic spaces.

In this paper we pursue our study of dispersive equations on hyperbolic spaces, initiated with the Schrödinger equation [1], by considering the shifted wave equation (4) on $\mathbb{H}^n$. We obtain a wider range of Strichartz estimates than in the Euclidean setting and deduce stronger well-posedness results. More precisely, in Section 4 we use spherical harmonic analysis on hyperbolic spaces to estimate the kernel of the operator $W_t^{(\sigma,\tau)} = D^{-\tau}\widetilde{D}^{\tau-\sigma}e^{itD}$, where $D=(-\Delta_{\mathbb{H}^n}-\rho^2)^{1/2}$, $\widetilde{D}=(-\Delta_{\mathbb{H}^n}+\tilde{\rho}^2-\rho^2)^{1/2}$ with $\tilde{\rho}>\rho$, and σ, τ are suitable exponents. In Section 5 we first deduce dispersive $L^{q'} \to L^q$ estimates for $W_t^{(\sigma,\tau)}$, when $2<q<\infty$, by using interpolation and the Kunze–Stein phenomenon [5,6,23]. In Section 6 we next deduce the following strong Strichartz estimates for solutions to the Cauchy problem (4):

$$\begin{aligned}
&\|u\|_{L^p(I;L^q)} + \|u\|_{L^\infty(I;H^{\sigma-\frac{1}{2},\frac{1}{2}})} + \|\partial_t u\|_{L^\infty(I;H^{\sigma-\frac{1}{2},-\frac{1}{2}})} \\
&\quad \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \|F\|_{L^{\tilde{p}'}(I;H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})},
\end{aligned} \tag{5}$$

where I is any (possibly unbounded) interval in $\mathbb{R}$, $(p,q),(\tilde{p},\tilde{q}) \in [2,\infty)\times[2,\infty)$ are admissible couples such that separately

$$\frac{2}{p}+\frac{n-1}{q} \geqslant \frac{n-1}{2}, \qquad \frac{2}{\tilde{p}}+\frac{n-1}{\tilde{q}} \geqslant \frac{n-1}{2},$$

and $\sigma \geqslant \frac{n+1}{2}(\frac{1}{2}-\frac{1}{q})$, $\tilde{\sigma} \geqslant \frac{n+1}{2}(\frac{1}{2}-\frac{1}{\tilde{q}})$. Notice that the Sobolev spaces involved in (5) are naturally related to the conservation laws of the shifted wave equation (see Section 3). We conclude in Section 7 with an application of (5) to local well-posedness of the nonlinear wave equation for initial data with low regularity. While we obtain the same regularity curve as in the Euclidean case for subconformal power-like nonlinearities, we prove local well-posedness for superconformal powers under lower regularity assumptions on the initial data.

In order to keep down the length of this paper, we postpone applications of the Strichartz estimates to global well-posedness of the nonlinear wave equation and generalizations of the previous results to Damek–Ricci spaces.

2. Spherical analysis on real hyperbolic spaces

In this paper, we consider the simplest class of Riemannian symmetric spaces of the noncompact type, namely real hyperbolic spaces $\mathbb{H}^n$ of dimension $n \geqslant 2$ (we shall restrict to $n \geqslant 3$ in Section 7). We refer to Helgason's books [16–18] and to Koornwinder's survey [26] for their algebraic structure and geometric properties, as well as for harmonic analysis on these spaces, and we shall be content with the following information. $\mathbb{H}^n$ can be realized as the symmetric space G/K, where $G=\mathrm{SO}(1,n)_0$ and $K=\mathrm{SO}(n)$. In geodesic polar coordinates on $\mathbb{H}^n$, the Riemannian volume writes

$$dx = \text{const.}(\sinh r)^{n-1}\,dr\,d\sigma$$

and the Laplace–Beltrami operator

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\coth r\,\partial_r + \sinh^{-2} r\,\Delta_{\mathbb{S}^{n-1}}.$$

The spherical functions φ_λ on $\mathbb{H}^n$ are normalized radial eigenfunctions of $\Delta_{\mathbb{H}^n}$:

$$\begin{cases} \Delta_{\mathbb{H}^n}\varphi_\lambda = -(\lambda^2+\rho^2)\varphi_\lambda, \\ \varphi_\lambda(0) = 1, \end{cases}$$

where $\lambda \in \mathbb{C}$ and $\rho = \frac{n-1}{2}$. They can be expressed in terms of special functions:

$$\varphi_\lambda(r) = \phi_\lambda^{(\frac{n}{2}-1,-\frac{1}{2})}(r) = {}_2F_1\left(\frac{\rho}{2}+i\frac{\lambda}{2}, \frac{\rho}{2}-i\frac{\lambda}{2}; \frac{n}{2}; -\sinh^2 r\right),$$

where $\phi_\lambda^{(\alpha,\beta)}$ denotes the Jacobi functions and ${}_2F_1$ the Gauss hypergeometric function. In the sequel we shall use the integral representations

$$\begin{aligned} \varphi_\lambda(r) &= \int_K dk\, e^{-(\rho+i\lambda)\mathrm{H}(a_{-r}k)} \\ &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})} \int_0^\pi d\theta\, (\sin\theta)^{n-2}(\cosh r - \sinh r\cos\theta)^{-\rho-i\lambda} \\ &= \pi^{-\frac{1}{2}} 2^{\frac{n-3}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} (\sinh r)^{2-n} \int_{-r}^{+r} du\, (\cosh r - \cosh u)^{\frac{n-3}{2}} e^{-i\lambda u}, \end{aligned} \tag{6}$$

which imply in particular that

$$\left|\varphi_\lambda(r)\right| \leqslant \varphi_0(r) \lesssim (1+r)e^{-\rho r} \quad \forall \lambda \in \mathbb{R},\ r \geqslant 0. \tag{7}$$

We shall also use the Harish–Chandra expansion

$$\varphi_\lambda(r) = \mathbf{c}(\lambda)\Phi_\lambda(r) + \mathbf{c}(-\lambda)\Phi_{-\lambda}(r) \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{Z},\ r > 0, \tag{8}$$

where the Harish–Chandra $\mathbf{c}$-function is given by

$$\mathbf{c}(\lambda) = \frac{\Gamma(2\rho)}{\Gamma(\rho)} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda+\rho)} \tag{9}$$

and

$$\begin{aligned} \Phi_\lambda(r) &= (2\sinh r)^{i\lambda-\rho} {}_2F_1\left(\frac{\rho}{2}-i\frac{\lambda}{2}, -\frac{\rho-1}{2}-i\frac{\lambda}{2}; 1-i\lambda; -\sinh^{-2} r\right) \\ &= (2\sinh r)^{-\rho} e^{i\lambda r} \sum_{k=0}^{+\infty} \Gamma_k(\lambda) e^{-2kr} \\ &\sim e^{(i\lambda-\rho)r} \quad \text{as } r \to +\infty. \end{aligned} \tag{10}$$

It is well known that there exist $\nu > 0$, $\varepsilon > 0$ and $C > 0$ such that, for every $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > -\varepsilon$,

$$\left|\Gamma_k(\lambda)\right| \leqslant C(1+k)^{\nu}.$$

We need to improve upon this estimate, by enlarging the domain, by estimating the derivatives of Γ_k and by gaining some additional decay in λ for $k \in \mathbb{N}^*$. The following recurrence formula holds:

$$\begin{cases} \Gamma_0(\lambda) = 1, \\ \Gamma_k(\lambda) = \dfrac{\rho(\rho-1)}{k(k-i\lambda)} \displaystyle\sum_{j=0}^{k-1} (k-j)\Gamma_j(\lambda). \end{cases}$$

Lemma 2.1. *Let $0 < \varepsilon < 1$ and $\Omega_\varepsilon = \{\lambda \in \mathbb{C} \mid |\operatorname{Re}\lambda| \leqslant \varepsilon|\lambda|,\ \operatorname{Im}\lambda \leqslant -1+\varepsilon\}$. Then, for every $\ell \in \mathbb{N}$, there exists $C_\ell > 0$ such that*

$$\left|\partial_\lambda^\ell \Gamma_k(\lambda)\right| \leqslant C_\ell k^{\nu} \big(1+|\lambda|\big)^{-\ell-1} \quad \forall k \in \mathbb{N}^*,\ \lambda \in \mathbb{C} \smallsetminus \Omega_\varepsilon. \tag{11}$$

Proof. Consider first the case $\ell = 0$. There exists $A = A(\varepsilon) > 0$ such that $|k - i\lambda| \geqslant A \max\{k, 1+|\lambda|\}$. Choose $\nu \geqslant 1$ such that $\frac{\rho^2}{A}\frac{1}{\nu+1} \leqslant \frac{1}{2}$ and $C > 0$ such that $\frac{\rho^2}{A} \leqslant \frac{C}{2}$. For $k = 1$, we have $\Gamma_1(\lambda) = \frac{\rho(\rho-1)}{1-i\lambda}$, hence

$$\left|\Gamma_1(\lambda)\right| \leqslant \frac{\rho^2}{A}\frac{1}{1+|\lambda|} \leqslant C\frac{1}{1+|\lambda|},$$

as required. For $k > 1$, we have

$$\Gamma_k(\lambda) = \frac{\rho(\rho-1)}{k-i\lambda} + \frac{\rho(\rho-1)}{k(k-i\lambda)} \sum_{0<j<k} (k-j)\Gamma_j(\lambda),$$

hence

$$\begin{aligned} \left|\Gamma_k(\lambda)\right| &\leqslant \frac{\rho^2}{A}\frac{1}{1+|\lambda|} + \frac{\rho^2}{A}\frac{1}{k^2} \sum_{0<j<k} (k-j)\frac{Cj^{\nu}}{1+|\lambda|} \\ &\leqslant \frac{C}{2}k^{\nu}\frac{1}{1+|\lambda|} + C\frac{k^{\nu}}{1+|\lambda|}\frac{\rho^2}{A}\frac{1}{k}\sum_{0<j<k}\left(\frac{j}{k}\right)^{\nu} \\ &\leqslant C\frac{k^{\nu}}{1+|\lambda|}. \end{aligned}$$

Derivatives are estimated by the Cauchy formula. □

Under suitable assumptions, the spherical Fourier transform of a bi-K-invariant function f on G is defined by

$$\mathcal{H}f(\lambda) = \int_G dg\, f(g)\varphi_\lambda(g)$$

and the following formulae hold:

- Inversion formula:

$$f(x) = \text{const.} \int_0^{+\infty} d\lambda \, \left|\mathbf{c}(\lambda)\right|^{-2} \mathcal{H}f(\lambda)\varphi_\lambda(x).$$

- Plancherel formula:

$$\|f\|_{L^2}^2 = \text{const.} \int_0^{+\infty} d\lambda \, \left|\mathbf{c}(\lambda)\right|^{-2} \left|\mathcal{H}f(\lambda)\right|^2.$$

Here is a well-known estimate of the Plancherel density:

$$\left|\mathbf{c}(\lambda)\right|^{-2} \lesssim |\lambda|^2 \left(1+|\lambda|\right)^{n-3} \quad \forall \lambda \in \mathbb{R}. \tag{12}$$

In the sequel we shall use the fact that $\mathcal{H} = \mathcal{F} \circ \mathcal{A}$, where $\mathcal{A}$ denotes the Abel transform and $\mathcal{F}$ the Fourier transform on the real line. Actually we shall use the factorization $\mathcal{H}^{-1} = \mathcal{A}^{-1} \circ \mathcal{F}^{-1}$. Recall the following expression of the inverse Abel transform:

$$\mathcal{A}^{-1} g(r) = \text{const.} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{\frac{n-1}{2}} g(r). \tag{13}$$

If n is odd, the right hand side involves a plain differential operator while, if n is even, the fractional derivative must be interpreted as follows:

$$\left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{\frac{n-1}{2}} g(r) = \text{const.} \int_r^{+\infty} ds \, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(-\frac{1}{\sinh s} \right)^{\frac{n}{2}} g(s). \tag{14}$$

3. Sobolev spaces and conservation of energy

Let us first introduce inhomogeneous Sobolev spaces on hyperbolic spaces $\mathbb{H}^n$, which will be involved in the conservation laws, in the dispersive estimates and in the Strichartz estimates for the shifted wave equation. We refer to [31] for more details about functions spaces on Riemannian manifolds.

Let $1 < q < \infty$ and $\sigma \in \mathbb{R}$. By definition, $H_q^\sigma(\mathbb{H}^n)$ is the image of $L^q(\mathbb{H}^n)$ under $(-\Delta_{\mathbb{H}^n})^{-\frac{\sigma}{2}}$ (in the space of distributions on $\mathbb{H}^n$), equipped with the norm

$$\|f\|_{H_q^\sigma} = \left\| (-\Delta_{\mathbb{H}^n})^{\frac{\sigma}{2}} f \right\|_{L^q}.$$

In this definition, we may replace $-\Delta_{\mathbb{H}^n}$ by $-\Delta_{\mathbb{H}^n} - \rho^2 + \tilde{\rho}^2$, where $\tilde{\rho} > |\frac{1}{2} - \frac{1}{q}| 2\rho$. For simplicity, we choose $\tilde{\rho} > \rho$ independently of q and we set

$$\widetilde{D} = \left(-\Delta_{\mathbb{H}^n} - \rho^2 + \tilde{\rho}^2 \right)^{\frac{1}{2}}.$$

Thus $H_q^\sigma(\mathbb{H}^n) = \widetilde{D}^{-\sigma} L^q(\mathbb{H}^n)$ and $\|f\|_{H_q^\sigma} \sim \|\widetilde{D}^\sigma f\|_{L^q}$. If $\sigma = N$ is a nonnegative integer, then $H_q^\sigma(\mathbb{H}^n)$ coincides with the Sobolev space

$$W^{N,q}(\mathbb{H}^n) = \{ f \in L^q(\mathbb{H}^n) \mid \nabla^j f \in L^q(\mathbb{H}^n)\ \forall 1 \leqslant j \leqslant N \}$$

defined in terms of covariant derivatives and equipped with the norm

$$\|f\|_{W^{N,q}} = \sum_{j=0}^{N} \|\nabla^j f\|_{L^q}.$$

Proposition 3.1 *(Sobolev embedding theorem). Let* $1 < q_1 < q_2 < \infty$ *and* $\sigma_1, \sigma_2 \in \mathbb{R}$ *such that* $\sigma_1 - \frac{n}{q_1} \geqslant \sigma_2 - \frac{n}{q_2}$.[1] *Then*

$$H^{\sigma_1}_{q_1}(\mathbb{H}^n) \subset H^{\sigma_2}_{q_2}(\mathbb{H}^n).$$

By this inclusion, we mean that there exists a constant $C > 0$ *such that*

$$\|f\|_{H^{\sigma_2}_{q_2}} \leqslant C \|f\|_{H^{\sigma_1}_{q_1}} \quad \forall f \in C^\infty_c(\mathbb{H}^n).$$

Proof. We sketch two proofs. The first one is based on the localization principle for Lizorkin–Triebel spaces [31] and on the corresponding result in $\mathbb{R}^n$. More precisely, given a tame partition of unity $1 = \sum_{j=0}^{\infty} \varphi_j$ on $\mathbb{H}^n$, we have

$$\|f\|_{H^{\sigma_2}_{q_2}(\mathbb{H}^n)} \asymp \left\{ \sum_{j=0}^{\infty} \left\| (\varphi_j f) \circ \exp_{x_j} \right\|^{q_2}_{H^{\sigma_2}_{q_2}(\mathbb{R}^n)} \right\}^{\frac{1}{q_2}}.$$

Using the inclusions $H^{\sigma_1}_{q_1}(\mathbb{R}^n) \subset H^{\sigma_2}_{q_2}(\mathbb{R}^n)$ and $\ell^{q_1}(\mathbb{N}) \subset \ell^{q_2}(\mathbb{N})$, we conclude that

$$\|f\|_{H^{\sigma_2}_{q_2}(\mathbb{H}^n)} \lesssim \left\{ \sum_{j=0}^{\infty} \left\| (\varphi_j f) \circ \exp_{x_j} \right\|^{q_1}_{H^{\sigma_1}_{q_1}(\mathbb{R}^n)} \right\}^{\frac{1}{q_1}} \asymp \|f\|_{H^{\sigma_1}_{q_1}(\mathbb{H}^n)}.$$

The second proof is based on the $L^{q_1} \to L^{q_2}$ mapping properties of the convolution operator $\widetilde{D}^{\sigma_2 - \sigma_1}$ (see [7] and the references cited therein). □

Beside the L^q Sobolev spaces $H^\sigma_q(\mathbb{H}^n)$, our analysis of the shifted wave equation on $\mathbb{H}^n$ involves the following L^2 Sobolev spaces:

$$H^{\sigma,\tau}(\mathbb{H}^n) = \widetilde{D}^{-\sigma} D^{-\tau} L^2(\mathbb{H}^n),$$

where $D = (-\Delta_{\mathbb{H}^n} - \rho^2)^{\frac{1}{2}}$, $\sigma \in \mathbb{R}$ and $\tau < \frac{3}{2}$ (actually we are only interested in the cases $\tau = 0$ and $\tau = \pm\frac{1}{2}$). Notice that

$$\begin{cases} H^{\sigma,\tau}(\mathbb{H}^n) = H^\sigma_2(\mathbb{H}^n) & \text{if } \tau = 0, \\ H^{\sigma,\tau}(\mathbb{H}^n) \subset H^{\sigma+\tau}_2(\mathbb{H}^n) & \text{if } \tau < 0, \\ H^{\sigma,\tau}(\mathbb{H}^n) \supset H^{\sigma+\tau}_2(\mathbb{H}^n) & \text{if } 0 < \tau < \dfrac{3}{2}. \end{cases}$$

[1] Notice that $\sigma_1 - \sigma_2 \geqslant \frac{n}{q_1} - \frac{n}{q_2} > 0$.

Lemma 3.2. *If* $0<\tau<\frac{3}{2}$, *then*

$$H^{\sigma,\tau}(\mathbb{H}^n)\subset H_2^{\sigma+\tau}(\mathbb{H}^n)+H_{2+}^{\infty}(\mathbb{H}^n),$$

where $H_{2+}^{\infty}(\mathbb{H}^n)=\bigcap_{s\in\mathbb{R},\,q>2}H_q^s(\mathbb{H}^n)$ (*recall that* $H_q^s(\mathbb{H}^n)$ *is decreasing as* $q\searrow 2$ *and* $s\nearrow+\infty$).

Proof. Let $f\in L^2(\mathbb{H}^n)$. We have $\widetilde{D}^{-\sigma}D^{-\tau}f=f*k_{\sigma,\tau}$, where

$$k_{\sigma,\tau}(x)=\text{const.}\int_0^{+\infty}d\lambda\,\bigl|\mathbf{c}(\lambda)\bigr|^{-2}|\lambda|^{-\tau}\bigl(\lambda^2+\tilde{\rho}^2\bigr)^{-\frac{\sigma}{2}}\varphi_\lambda(x)$$

by the inversion formula for the spherical Fourier transform on $\mathbb{H}^n$. Let us split up the integral

$$\int_0^{+\infty}=\int_0^1+\int_1^{+\infty}$$

and the kernel

$$k_{\sigma,\tau}=k_{\sigma,\tau}^0+k_{\sigma,\tau}^{\infty},$$

accordingly. On the one hand,

$$\mathbb{1}_{(1,+\infty)}(D)\widetilde{D}^{-\sigma}D^{-\tau}f=f*k_{\sigma,\tau}^{\infty}$$

maps $L^2(\mathbb{H}^n)$ into $H_2^{\sigma+\tau}(\mathbb{H}^n)$. On the other hand, $k_{\sigma,\tau}^0$ is a radial kernel in $H_2^{\infty}(\mathbb{H}^n)$, hence

$$\mathbb{1}_{[0,1]}(D)\widetilde{D}^{-\sigma}D^{-\tau}f=f*k_{\sigma,\tau}^0$$

maps $L^2(\mathbb{H}^n)$ into $H_{2+}^{\infty}(\mathbb{H}^n)$ by the Kunze–Stein phenomenon. Thus $\widetilde{D}^{-\sigma}D^{-\tau}f=f*k_{\sigma,\tau}$ belongs to $H_2^{\sigma+\tau}(\mathbb{H}^n)+H_{2+}^{\infty}(\mathbb{H}^n)$, as required. □

Let us next introduce the energy

$$E(t)=\frac{1}{2}\int_{\mathbb{H}^n}dx\,\bigl\{\bigl|\partial_t u(t,x)\bigr|^2+\bigl|D_x u(t,x)\bigr|^2\bigr\} \tag{15}$$

for solutions to the homogeneous Cauchy problem

$$\begin{cases}\partial_t^2 u-\bigl(\Delta_{\mathbb{H}^n}+\rho^2\bigr)u=0,\\ u(0,x)=f(x),\\ \partial_t|_{t=0}u(t,x)=g(x).\end{cases} \tag{16}$$

It is easily verified that $\partial_t E(t)=0$, hence (15) is conserved. In other words, for every time t in the interval of definition of u,

$$\bigl\|\partial_t u(t,x)\bigr\|_{L_x^2}^2+\bigl\|D_x u(t,x)\bigr\|_{L_x^2}^2=\|g\|_{L^2}^2+\|Df\|_{L^2}^2.$$

Let $\sigma \in \mathbb{R}$ and $\tau < \frac{3}{2}$. By applying the operator $\widetilde{D}^{\sigma} D^{\tau}$ to (16), we deduce that

$$\left\| \partial_t \widetilde{D}_x^{\sigma} D_x^{\tau} u(t,x) \right\|_{L_x^2}^2 + \left\| \widetilde{D}_x^{\sigma} D_x^{\tau+1} u(t,x) \right\|_{L_x^2}^2 = \left\| \widetilde{D}^{\sigma} D^{\tau} g \right\|_{L^2}^2 + \left\| \widetilde{D}^{\sigma} D^{\tau+1} f \right\|_{L^2}^2,$$

which can be rewritten in terms of Sobolev norms as follows:

$$\left\| \partial_t u(t,\cdot) \right\|_{H^{\sigma,\tau}}^2 + \left\| u(t,\cdot) \right\|_{H^{\sigma,\tau+1}}^2 = \|g\|_{H^{\sigma,\tau}}^2 + \|f\|_{H^{\sigma,\tau+1}}^2. \tag{17}$$

4. Kernel estimates

In this section we derive pointwise estimates for the radial convolution kernel $w_t^{(\sigma,\tau)}$ of the operator $W_t^{(\sigma,\tau)} = D^{-\tau} \widetilde{D}^{\tau-\sigma} e^{itD}$, for suitable exponents $\sigma \in \mathbb{R}$ and $\tau \in [0, \frac{3}{2})$. By the inversion formula of the spherical Fourier transform,

$$w_t^{(\sigma,\tau)}(r) = \text{const.} \int_0^{+\infty} d\lambda \left|\mathbf{c}(\lambda)\right|^{-2} \lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} \varphi_\lambda(r) e^{it\lambda}.$$

Contrarily to the Euclidean case, this kernel has different behaviors, depending whether t is small or large, and therefore we cannot use any rescaling. Let us split up

$$\begin{aligned} w_t^{(\sigma,\tau)}(r) &= w_{t,0}^{(\sigma,\tau)}(r) + w_{t,\infty}^{(\sigma,\tau)}(r) \\ &= \text{const.} \int_0^{2} d\lambda\, \chi_0(\lambda) \left|\mathbf{c}(\lambda)\right|^{-2} \lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} \varphi_\lambda(r) e^{it\lambda} \\ &\quad + \text{const.} \int_1^{+\infty} d\lambda\, \chi_\infty(\lambda) \left|\mathbf{c}(\lambda)\right|^{-2} \lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} \varphi_\lambda(r) e^{it\lambda} \end{aligned}$$

using smooth cut-off functions χ_0 and χ_∞ on $[0,+\infty)$ such that $1 = \chi_0 + \chi_\infty$, $\chi_0 = 1$ on $[0,1]$ and $\chi_\infty = 1$ on $[2,+\infty)$. We shall first estimate $w_{t,0}^{(\sigma,\tau)}$ and next a variant of $w_{t,\infty}^{(\sigma,\tau)}$. The kernel $w_{t,\infty}^{(\sigma,\tau)}$ has indeed a logarithmic singularity on the sphere $r = t$ when $\sigma = \frac{n+1}{2}$. We bypass this problem by considering the analytic family of operators

$$\widetilde{W}_{t,\infty}^{(\sigma,\tau)} = \frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2} - \sigma)} \chi_\infty(D) D^{-\tau} \widetilde{D}^{\tau-\sigma} e^{itD}$$

in the vertical strip $0 \leqslant \operatorname{Re}\sigma \leqslant \frac{n+1}{2}$ and the corresponding kernels

$$\widetilde{w}_{t,\infty}^{(\sigma,\tau)}(r) = \frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2} - \sigma)} \int_1^{+\infty} d\lambda\, \chi_\infty(\lambda) \left|\mathbf{c}(\lambda)\right|^{-2} \lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} e^{it\lambda} \varphi_\lambda(r). \tag{18}$$

Notice that the Gamma function, which occurs naturally in the theory of Riesz distributions, will allow us to deal with the boundary point $\sigma = \frac{n+1}{2}$, while the exponential function yields boundedness at infinity in the vertical strip. Notice also that, once multiplied by $\chi_\infty(D)$, the operator $D^{-\tau} \widetilde{D}^{\tau-\sigma}$ behaves like $\widetilde{D}^{-\sigma}$.

4.1. Estimate of $w_t^0 = w_{t,0}^{(\sigma,\tau)}$

Theorem 4.1. *Let $\sigma \in \mathbb{R}$ and $\tau < 2$. The following pointwise estimates hold for the kernel $w_t^0 = w_{t,0}^{(\sigma,\tau)}$:*

(i) *Assume that $|t| \leqslant 2$. Then, for every $r \geqslant 0$,*

$$\left|w_t^0(r)\right| \lesssim \varphi_0(r).$$

(ii) *Assume that $|t| \geqslant 2$.*
 (a) *If $0 \leqslant r \leqslant \frac{|t|}{2}$, then*

$$\left|w_t^0(r)\right| \lesssim |t|^{\tau-3}\varphi_0(r).$$

 (b) *If $r \geqslant \frac{|t|}{2}$, then*

$$\left|w_t^0(r)\right| \lesssim \left(1+\left|r-|t|\right|\right)^{\tau-2} e^{-\rho r}.$$

Proof. Recall that

$$w_t^0(r) = \text{const.} \int_0^2 d\lambda\, \chi_0(\lambda)\left|\mathbf{c}(\lambda)\right|^{-2}\lambda^{-\tau}\left(\lambda^2+\tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}}\varphi_\lambda(r)e^{it\lambda}. \tag{19}$$

By symmetry we may assume that $t > 0$.

(i) It follows from the estimates (7) and (12) that

$$\left|w_t^0(r)\right| \lesssim \int_0^2 d\lambda\, \lambda^{2-\tau}\varphi_0(r) \lesssim \varphi_0(r).$$

(ii) We prove first (a) by substituting in (19) the first integral representation of φ_λ in (6) and by reducing this way to Fourier analysis on $\mathbb{R}$. Specifically,

$$w_t^0(r) = \int_K dk\, e^{-\rho \mathrm{H}(a_{-r}k)} \int_0^2 d\lambda\, \chi_0(\lambda)a(\lambda)e^{i\{t-\mathrm{H}(a_{-r}k)\}\lambda},$$

where $a(\lambda) = |\mathbf{c}(\lambda)|^{-2}\lambda^{-\tau}(\lambda^2+\tilde{\rho}^2)^{\frac{\tau-\sigma}{2}}$, up to a positive constant. According to the estimate (12) and to Lemma A.1 in Appendix A, the inner integral is bounded above by

$$\left\{t-\mathrm{H}(a_{-r}k)\right\}^{\tau-3} \leqslant (t-r)^{\tau-3} \asymp t^{\tau-3}.$$

Since

$$\int_K dk\, e^{-\rho\mathrm{H}(a_{-r}k)} = \varphi_0(r),$$

we conclude that

$$\left|w_t^0(r)\right| \lesssim t^{\tau-3}\varphi_0(r).$$

We prove next (b) by substituting in (19) the Harish–Chandra expansion (8) of φ_λ and by reducing again to Fourier analysis on $\mathbb{R}$. Specifically,

$$w_t^0(r) = (2\sinh r)^{-\rho}\sum_{k=0}^{+\infty} e^{-2kr}\left\{I_k^{+,0}(t,r) + I_k^{-,0}(t,r)\right\}, \tag{20}$$

where

$$I_k^{\pm,0}(t,r) = \int_0^2 d\lambda\, \chi_0(\lambda)a_k^{\pm}(\lambda)e^{i(t\pm r)\lambda}$$

and

$$a_k^{\pm}(\lambda) = \mathbf{c}(\mp\lambda)^{-1}\lambda^{-\tau}\left(\lambda^2+\tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}}\Gamma_k(\pm\lambda).$$

By applying Lemma A.1 and by using the estimates (11) for Γ_k and its derivatives, we obtain

$$\left|I_k^{+,0}(t,r)\right| \lesssim (1+k)^{\nu}(t+r)^{\tau-2} \leqslant (1+k)^{\nu} r^{\tau-2}$$

and

$$\left|I_k^{-,0}(t,r)\right| \lesssim (1+k)^{\nu}\left(1+|r-t|\right)^{\tau-2}.$$

We conclude the proof by summing up these estimates in (20). □

4.2. Estimate of $\widetilde{w}_t^{\infty} = \widetilde{w}_{t,\infty}^{(\sigma,\tau)}$

Theorem 4.2. *The following pointwise estimates hold for the kernel $\widetilde{w}_t^{\infty} = \widetilde{w}_{t,\infty}^{(\sigma,\tau)}$, for any fixed $\tau \in \mathbb{R}$ and uniformly in $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{n+1}{2}$:*

(i) *Assume that $0 < |t| \leqslant 2$.*
  (a) *If $0 \leqslant r \leqslant 3$, then*

$$\left|\widetilde{w}_t^{\infty}(r)\right| \lesssim \begin{cases} |t|^{-\frac{n-1}{2}} & \text{if } n \geqslant 3, \\ |t|^{-\frac{1}{2}}(1-\log|t|) & \text{if } n = 2. \end{cases}$$

  (b) *If $r \geqslant 3$, then $\widetilde{w}_t^{\infty}(r) = \mathrm{O}(r^{-\infty}e^{-\rho r})$.*

(ii) *Assume that $|t| \geqslant 2$. Then*

$$\left|\widetilde{w}_t^{\infty}(r)\right| \lesssim \left(1+\left|r-|t|\right|\right)^{-\infty}e^{-\rho r} \quad \forall r \geqslant 0.$$

Proof of Theorem 4.2(ii). Recall that, up to a positive constant,

$$\widetilde{w}_t^\infty(r)=\frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2}-\sigma)}\int_1^{+\infty}d\lambda\,\chi_\infty(\lambda)\left|\mathbf{c}(\lambda)\right|^{-2}\lambda^{-\tau}\left(\lambda^2+\tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}}\varphi_\lambda(r)e^{it\lambda}.$$

By symmetry we may assume again that $t>0$. If $0\leqslant r\leqslant\frac{t}{2}$, we resume the proof of Theorem 4.1(ii)(a), using Lemma A.2 instead of Lemma A.1, and estimate this way

$$\left|\widetilde{w}_t^\infty(r)\right|\lesssim(t-r)^{-\infty}\varphi_0(r)\lesssim t^{-\infty}e^{-\rho r}. \tag{21}$$

If $r\geqslant\frac{t}{2}$, we resume the proof of Theorem 4.1(ii)(b) and expand this way

$$\widetilde{w}_t^\infty(r)=\frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2}-\sigma)}(\sinh r)^{-\rho}\sum_{k=0}^{+\infty}e^{-2kr}\left\{I_k^{+,\infty}(t,r)+I_k^{-,\infty}(t,r)\right\}, \tag{22}$$

where

$$I_k^{\pm,\infty}(t,r)=\int_0^{+\infty}d\lambda\,\chi_\infty(\lambda)a_k^\pm(\lambda)e^{i(t\pm r)\lambda}$$

and

$$a_k^\pm(\lambda)=\mathbf{c}(\mp\lambda)^{-1}\lambda^{-\tau}\left(\lambda^2+\tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}}\Gamma_k(\pm\lambda).$$

It follows from the expression (9) of the **c**-function and from the estimates (11) of the coefficients Γ_k that $\chi_\infty a_k^\pm$ is a symbol of order

$$d=\begin{cases}-1 & \text{if } k=0,\\ -2 & \text{if } k\in\mathbb{N}^*.\end{cases}$$

By applying Lemma A.2, we obtain the following estimates of the expressions $I_k^{\pm,\infty}(t,r)$, except for $I_0^{-,\infty}(t,r)$: $\forall N\in\mathbb{N}^*$, $\exists C_N\geqslant 0$,

$$\left|I_k^{+,\infty}(t,r)\right|\leqslant C_N|\sigma|^N(1+k)^\nu(t+r)^{-N}\leqslant C_N|\sigma|^N(1+k)^\nu r^{-N}, \tag{23}$$

$$\left|I_k^{-,\infty}(t,r)\right|\leqslant C_N|\sigma|^N(1+k)^\nu\left(1+|r-t|\right)^{-N}. \tag{24}$$

As far as $I_0^{-,\infty}(t,r)$ is concerned, Lemma A.2 yields the estimates

$$\left|I_0^{-,\infty}(t,r)\right|\leqslant\begin{cases}C_N|\sigma|^N|r-t|^{-N} & \text{if } |r-t|\geqslant 1,\\ C(1+\log\frac{1}{|r-t|}) & \text{if } |r-t|\leqslant 1.\end{cases} \tag{25}$$

The second one can be improved by applying Lemma A.3 instead of Lemma A.2. For this purpose, let us establish the asymptotic behavior of the symbol $a_0^-(\lambda)$, as $\lambda\to+\infty$. On the one hand,

$$\mathbf{c}(\lambda)^{-1} = \frac{\Gamma(\rho)}{\Gamma(2\rho)} \frac{\Gamma(i\lambda+\rho)}{\Gamma(i\lambda)} = \frac{\Gamma(\rho)}{\Gamma(2\rho)} e^{-\rho} \left(\frac{i\lambda+\rho}{i\lambda}\right)^{i\lambda-\frac{1}{2}} (i\lambda+\rho)^{\rho} \left\{1 + \mathrm{O}\left(\lambda^{-1}\right)\right\}$$
$$= e^{i\frac{\rho\pi}{2}} \lambda^{\rho} \left\{1 + \mathrm{O}\left(\lambda^{-1}\right)\right\},$$

according to Stirling's formula

$$\Gamma(\xi) = \sqrt{2\pi}\, \xi^{\xi-\frac{1}{2}} e^{-\xi} \left\{1 + \mathrm{O}\left(|\xi|^{-1}\right)\right\}.$$

On the other hand,

$$\lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} = \lambda^{-\sigma} \left\{1 + \mathrm{O}\left(|\sigma|\lambda^{-2}\right)\right\}.$$

Hence

$$a_0^-(\lambda) = c_0 \lambda^{-1-i\,\mathrm{Im}\,\sigma} + b_0(\lambda) \quad \text{with } \left|b_0(\lambda)\right| \leqslant C|\sigma|\lambda^{-2}.$$

By applying Lemma A.3 with $m = 0$ and $d = -2$, we obtain

$$\left|I_0^{-,\infty}(t,r)\right| \leqslant C \frac{|\sigma|^2}{|\mathrm{Im}\,\sigma|} \quad \text{if } |r-t| \leqslant 1. \tag{26}$$

Instead of the singularity $\log \frac{1}{|r-t|}$ in (25), the estimate (26) of $I_0^{-,\infty}(t,r)$ involves this time the singularity $\frac{1}{\mathrm{Im}\,\sigma}$, which cancels with the denominator of the front expression

$$\frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2} - \sigma)} \tag{27}$$

in (22). Notice moreover that the numerator of (27) yields enough decay to get uniform bounds in σ. In conclusion, by combining (22), (23), (24), (25), (26), we obtain

$$\left|\widetilde{w}_t^{\infty}(r)\right| \lesssim \left(1 + |r-t|\right)^{-\infty} e^{-\rho r} \quad \forall r \geqslant \frac{t}{2}. \qquad \square$$

Remark 4.3. The kernel $w_t^{\infty}(r)$ can be estimated in the same way, except that

$$\left|w_t^{\infty}(r)\right| \lesssim e^{-\rho t} \log \frac{1}{|r - |t||}$$

when r is close to $|t|$.

Let us turn to the small time estimates in Theorem 4.2. The estimate (i)(a) is of local nature and thus similar to the Euclidean case. For the sake of completeness, we include a proof in Appendix C. It remains for us to prove the estimate (i)(b).

Proof of Theorem 4.2(i)(b). Here $0 < |t| \leqslant 2$ and $r \geqslant 3$. By symmetry we may assume again that $t > 0$. We use now the inverse Abel transform given by formulae (13) and (14). Up to positive constants, the inverse spherical Fourier transform (18) can be rewritten in the following way:

$$\widetilde{w}_t^\infty(r) = \frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2}-\sigma)} \mathcal{A}^{-1} g_t(r),$$

where

$$g_t(r) = 2 \int_1^{+\infty} d\lambda\, \chi_\infty(\lambda) \lambda^{-\tau} \big(\lambda^2 + \tilde{\rho}^2\big)^{\frac{\tau-\sigma}{2}} e^{it\lambda} \cos\lambda r.$$

Let us split up $2\cos\lambda r = e^{i\lambda r} + e^{-i\lambda r}$ and $g_t(r) = g_t^+(r) + g_t^-(r)$ accordingly, so that

$$g_t^\pm(r) = \int_1^{+\infty} d\lambda\, \chi_\infty(\lambda) \lambda^{-\tau} \big(\lambda^2 + \tilde{\rho}^2\big)^{\frac{\tau-\sigma}{2}} e^{i(t\pm r)\lambda}.$$

Case 1: Assume that $n = 2m+1$ is odd. First of all, let us expand

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m = \sum_{\ell=1}^m \alpha_\ell^\infty(r) \left(\frac{\partial}{\partial r}\right)^\ell.$$

Since the coefficients $\alpha_\ell^\infty(r)$ are linear combinations of products

$$\left(\frac{1}{\sinh r}\right) \times \left(\frac{\partial}{\partial r}\right)^{\ell_2} \left(\frac{1}{\sinh r}\right) \times \cdots \times \left(\frac{\partial}{\partial r}\right)^{\ell_m} \left(\frac{1}{\sinh r}\right),$$

with $\ell_2 + \cdots + \ell_m = m - \ell$, and $\frac{1}{\sinh r} = 2\sum_{j=0}^{+\infty} e^{-(2j+1)r}$ is $\mathrm{O}(e^{-r})$, as well as its derivatives, we deduce that $\alpha_\ell^\infty(r)$ is $\mathrm{O}(e^{-mr})$ as $r \to +\infty$. Consider next

$$\left(\frac{\partial}{\partial r}\right)^\ell g_t^\pm(r) = \int_1^{+\infty} d\lambda\, \chi_\infty(\lambda) \lambda^{-\tau} \big(\lambda^2 + \tilde{\rho}^2\big)^{\frac{\tau-\sigma}{2}} (\pm i\lambda)^\ell e^{i(t\pm r)\lambda}.$$

According to Lemma A.2, for every $N \in \mathbb{N}^*$, there exists $C_N \geqslant 0$ such that

$$\left|\left(\frac{\partial}{\partial r}\right)^\ell g_t^\pm(r)\right| \leqslant C_N |\sigma|^N (r \pm t)^{-N}.$$

As a conclusion,

$$\big|\widetilde{w}_t^\infty(r)\big| = C \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \big(g_t^+ + g_t^-\big)(r) \leqslant C_N r^{-N} e^{-\frac{n-1}{2}r} \quad \forall N \in \mathbb{N}^*.$$

Case 2: Assume that $n = 2m$ is even. According to Case 1, for every $N \in \mathbb{N}^*$, there exists $C_N \geqslant 0$ such that

$$\left|\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^m g_t(s)\right| \leqslant C_N |\sigma|^N s^{-N} e^{-ms} \quad \forall s \geqslant 3.$$

By estimating

$$\cosh s - \cosh r = 2\sinh\frac{s+r}{2}\sinh\frac{s-r}{2} \gtrsim e^r \sinh\frac{s-r}{2},$$
$$\sinh s \lesssim e^s, \qquad e^{-(m-1)s} \leqslant e^{-(m-1)r}, \qquad s^{-N} \leqslant r^{-N},$$

and performing the change of variables $s = r + u$, we deduce that

$$\begin{aligned}
\left|\widetilde{w}_t^\infty(r)\right| &\lesssim \frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2}-\sigma)} \int_r^{+\infty} ds\, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left|\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^m g_t(s)\right| \\
&\leqslant C_N \int_r^{+\infty} ds\, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} s^{-N} e^{-ms} \\
&\leqslant C_N r^{-N} e^{-(m-\frac{1}{2})r} \int_0^{+\infty} \frac{du}{\sqrt{\sinh\frac{u}{2}}} \leqslant C_N r^{-N} e^{-\frac{n-1}{2}r}. \qquad \square
\end{aligned}$$

5. Dispersive estimates

In this section we obtain $L^{q'} \to L^q$ estimates for the operator $D^{-\tau}\widetilde{D}^{\tau-\sigma}e^{itD}$, which will be crucial for our Strichartz estimates in next section. Let us split up its kernel $w_t = w_t^0 + w_t^\infty$ as before. We will handle the contribution of w_t^0, using the pointwise estimates obtained in Section 4.1 and the following criterion based on the Kunze–Stein phenomenon.

Lemma 5.1. *There exists a constant $C > 0$ such that, for every radial measurable function κ on $\mathbb{H}^n$, for every $2 \leqslant q, \tilde{q} < \infty$ and $f \in L^{q'}(\mathbb{H}^n)$,*

$$\|f * \kappa\|_{L^q} \leqslant C\|f\|_{L^{\tilde{q}'}} \left\{ \int_0^{+\infty} dr\, (\sinh r)^{n-1} \varphi_0(r)^\mu \left|\kappa(r)\right|^Q \right\}^{\frac{1}{Q}},$$

where $\mu = \frac{2\min\{q,\tilde{q}\}}{q+\tilde{q}}$ and $Q = \frac{q\tilde{q}}{q+\tilde{q}}$.[2]

Proof. This estimate is obtained by complex multilinear interpolation between the following version [19] of the Kunze–Stein phenomenon

$$\|f * \kappa\|_{L^2} \lesssim \|f\|_{L^2} \int_0^{+\infty} dr\, (\sinh r)^{n-1} \varphi_0(r) \left|\kappa(r)\right|$$

and the elementary inequalities

$$\|f * \kappa\|_{L^q} \leqslant \|f\|_{L^1} \|\kappa\|_{L^q}, \qquad \|f * \kappa\|_{L^\infty} \leqslant \|f\|_{L^{\tilde{q}'}} \|\kappa\|_{L^{\tilde{q}}}.$$

[2] Notice that $\frac{1}{Q} = \frac{1}{q} + \frac{1}{\tilde{q}}$ and $\mu + Q > 2$.

Specifically, if $2<q<\infty$ and $2\leqslant\tilde{q}\leqslant q$, define $2\leqslant r\leqslant\infty$ by $\frac{1}{r}=\frac{1}{2}\frac{1/\tilde{q}-1/q}{1/2-1/q}$. By complex interpolation, let us deduce the intermediate estimate

$$\left|\int_{\mathbb{H}^n} f*\left(\varphi_0^{-\frac{2}{q}}g\right)(x)h(x)\,dx\right|\lesssim\|f\|_{L^{\tilde{q}'}}\|g\|_{L^Q}\|h\|_{L^{q'}} \tag{28}$$

from the endpoint estimates

$$\left|\int_{\mathbb{H}^n} f_0*g_0(x)h_0(x)\,dx\right|\leqslant\|f_0\|_{L^{r'}}\|g_0\|_{L^r}\|h_0\|_{L^1} \tag{29}$$

and

$$\left|\int_{\mathbb{H}^n} f_1*\left(\varphi_0^{-1}g_1\right)(x)h_1(x)\,dx\right|\lesssim\|f_1\|_{L^2}\|g_1\|_{L^1}\|h_1\|_{L^2}. \tag{30}$$

Here

$$f=\sum_{\text{finite}}\alpha_j\mathbb{1}_{A_j},\qquad g=\sum_{\text{finite}}\beta_k\mathbb{1}_{B_k},\qquad h=\sum_{\text{finite}}\gamma_\ell\mathbb{1}_{C_\ell}$$

are linear combinations with nonzero complex coefficients of characteristic functions of disjoints Borel sets in $\mathbb{H}^n$ with finite positive measure, the B_k's being moreover spherical. As in the proof of the Riesz–Thorin theorem (see for instance [3, §1.1]), we assume that $\|f\|_{L^{\tilde{q}'}}=\|g\|_{L^Q}=\|h\|_{L^{q'}}=1$, we consider the analytic families of simple functions

$$f_z=\sum_{\text{finite}}\alpha_j|\alpha_j|^{a(z)-1}\mathbb{1}_{A_j},\qquad g_z=\sum_{\text{finite}}\beta_k|\beta_k|^{b(z)-1}\mathbb{1}_{B_k},\qquad h_z=\sum_{\text{finite}}\gamma_\ell|\gamma_\ell|^{c(z)-1}\mathbb{1}_{C_\ell},$$

where

$$\frac{a(z)}{\tilde{q}'}=\left(\frac{1}{r}-\frac{1}{2}\right)z+\frac{1}{r'},\qquad\frac{b(z)}{Q}=\frac{1}{r'}z+\frac{1}{r},\qquad\frac{c(z)}{q'}=-\frac{1}{2}z+1,$$

and we apply the Hadamard three lines theorem to the holomorphic function

$$\psi(z)=\int_{\mathbb{H}^n} f_z*\left(\varphi_0^{-z}g_z\right)(x)h_z(x)\,dx$$

in the vertical strip $\{z\in\mathbb{C}\mid 0\leqslant\operatorname{Re}z\leqslant 1\}$. More precisely, if $\operatorname{Re}z=0$, then

$$\begin{cases}\operatorname{Re}a(z)=\dfrac{\tilde{q}'}{r'}\quad\Longrightarrow\quad\|f_z\|_{L^{r'}}^{r'}=\|f\|_{L^{\tilde{q}'}}^{\tilde{q}'}=1,\\[2mm]\operatorname{Re}b(z)=\dfrac{Q}{r}\quad\Longrightarrow\quad\|g_z\|_{L^r}^{r}=\|g\|_{L^Q}^{Q}=1,\\[2mm]\operatorname{Re}c(z)=q'\quad\Longrightarrow\quad\|h_z\|_{L^1}=\|h\|_{L^{q'}}^{q'}=1,\end{cases}$$

hence $|\psi(z)|\leqslant 1$, according to (29). Similarly, if $\operatorname{Re}z=1$, then

$$\begin{cases} \operatorname{Re} a(z) = \dfrac{\tilde{q}'}{2} & \Longrightarrow \quad \|f_z\|_{L^2}^2 = \|f\|_{L^{\tilde{q}'}}^{\tilde{q}'} = 1, \\ \operatorname{Re} b(z) = Q & \Longrightarrow \quad \|g_z\|_{L^1} = \|g\|_{L^Q}^Q = 1, \\ \operatorname{Re} c(z) = \dfrac{q'}{2} & \Longrightarrow \quad \|h_z\|_{L^2}^2 = \|h\|_{L^{q'}}^{q'} = 1, \end{cases}$$

hence $|\psi(z)| \leqslant C$, according to (30). The estimate (28) is obtained by applying the three lines theorem to $\psi(z)$ at the point $z = \frac{2}{q}$, where

$$\begin{cases} a(z) = 1 & \Longrightarrow \quad f_z = f, \\ b(z) = 1 & \Longrightarrow \quad g_z = g, \\ c(z) = 1 & \Longrightarrow \quad h_z = h. \end{cases}$$

Eventually, the symmetric case, where $2 < \tilde{q} < \infty$ and $2 \leqslant q \leqslant \tilde{q}$, is handled similarly. □

For the second part w_t^∞, we resume the Euclidean approach, which consists in interpolating analytically between $L^2 \to L^2$ and $L^1 \to L^\infty$ estimates for the family of operators

$$\widetilde{W}_{t,\infty}^{(\sigma,\tau)} = \frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2} - \sigma)} \chi_\infty(D) D^{-\tau} \widetilde{D}^{\tau-\sigma} e^{itD} \tag{31}$$

in the vertical strip $0 \leqslant \operatorname{Re}\sigma \leqslant \frac{n+1}{2}$.

5.1. Small time dispersive estimate

Theorem 5.2. *Assume that* $0 < |t| \leqslant 2$, $2 < q < \infty$, $0 \leqslant \tau < \frac{3}{2}$ *and* $\sigma \geqslant (n+1)(\frac{1}{2} - \frac{1}{q})$. *Then,*

$$\left\| D^{-\tau} \widetilde{D}^{\tau-\sigma} e^{itD} \right\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(n-1)(\frac{1}{2}-\frac{1}{q})} & \text{if } n \geqslant 3, \\ |t|^{-(\frac{1}{2}-\frac{1}{q})} (1 - \log|t|)^{1-\frac{2}{q}} & \text{if } n = 2. \end{cases}$$

Proof. We divide the proof into two parts, corresponding to the kernel decomposition $w_t = w_t^0 + w_t^\infty$. By applying Lemma 5.1 and by using the pointwise estimates in Theorem 4.1(i), we obtain on one hand

$$\begin{aligned} \left\| f * w_t^0 \right\|_{L^q} &\lesssim \left\{ \int_0^{+\infty} dr\, (\sinh r)^{n-1} \varphi_0(r) \left| w_t^0(r) \right|^{\frac{q}{2}} \right\}^{\frac{2}{q}} \|f\|_{L^{q'}} \\ &\lesssim \left\{ \int_0^{+\infty} dr\, (1+r)^{1+\frac{q}{2}} e^{-\rho r (\frac{q}{2}-1)} \right\}^{\frac{2}{q}} \|f\|_{L^{q'}} \\ &\lesssim \|f\|_{L^{q'}} \quad \forall f \in L^{q'}. \end{aligned}$$

For the second part, we consider the analytic family (31). If $\operatorname{Re}\sigma = 0$, then

$$\left\| f * \widetilde{w}_t^\infty \right\|_{L^2} \lesssim \|f\|_{L^2} \quad \forall f \in L^2.$$

If $\operatorname{Re}\sigma = \frac{n+1}{2}$, we deduce from the pointwise estimates in Theorem 4.2(i) that

$$\big\| f * \widetilde{w}_t^{\infty} \big\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \| f \|_{L^1} \quad \forall f \in L^1.$$

By interpolation we conclude for $\sigma = (n+1)(\frac{1}{2} - \frac{1}{q})$ that

$$\big\| f * w_t^{\infty} \big\|_{L^q} \lesssim |t|^{-(n-1)(\frac{1}{2}-\frac{1}{q})} \| f \|_{L^{q'}} \quad \forall f \in L^{q'}. \qquad \square$$

5.2. Large time dispersive estimate

Theorem 5.3. *Assume that $|t| \geqslant 2$, $2 < q < \infty$, $0 \leqslant \tau < \frac{3}{2}$ and $\sigma \geqslant (n+1)(\frac{1}{2} - \frac{1}{q})$. Then*

$$\big\| D^{-\tau} \widetilde{D}^{\tau-\sigma} e^{itD} \big\|_{L^{q'} \to L^q} \lesssim |t|^{\tau-3}.$$

Proof. We divide the proof into three parts, corresponding to the kernel decomposition

$$w_t = \mathbb{1}_{B(0,\frac{|t|}{2})} w_t^0 + \mathbb{1}_{\mathbb{H}^n \smallsetminus B(0,\frac{|t|}{2})} w_t^0 + w_t^{\infty}.$$

Estimate 1: By applying Lemma 5.1 and using the pointwise estimates in Theorem 4.1(ii)(a), we obtain

$$\begin{aligned} \big\| f * \big\{ \mathbb{1}_{B(0,\frac{|t|}{2})} w_t^0 \big\} \big\|_{L^q} &\lesssim \left\{ \int_0^{\frac{|t|}{2}} dr\, (\sinh r)^{n-1} \varphi_0(r) \big| w_t^0(r) \big|^{\frac{q}{2}} \right\}^{\frac{2}{q}} \| f \|_{L^{q'}} \\ &\lesssim \underbrace{\left\{ \int_0^{+\infty} dr\, (1+r)^{1+\frac{q}{2}} e^{-\rho r(\frac{q}{2}-1)} \right\}^{\frac{2}{q}}}_{<+\infty} |t|^{\tau-3} \| f \|_{L^{q'}} \quad \forall f \in L^{q'}. \end{aligned}$$

Estimate 2: By applying Lemma 5.1 and using the pointwise estimates in Theorem 4.1(ii)(b), we obtain

$$\begin{aligned} \big\| f * \big\{ \mathbb{1}_{\mathbb{H}^n \smallsetminus B(0,\frac{|t|}{2})} w_t^0 \big\} \big\|_{L^q} &\lesssim \left\{ \int_{\frac{|t|}{2}}^{+\infty} dr\, (\sinh r)^{n-1} \varphi_0(r) \big| w_t^0(r) \big|^{\frac{q}{2}} \right\}^{\frac{2}{q}} \| f \|_{L^{q'}} \\ &\lesssim \underbrace{\left\{ \int_{\frac{|t|}{2}}^{+\infty} dr\, r e^{-(\frac{q}{2}-1)\rho r} \right\}^{\frac{2}{q}}}_{\lesssim |t|^{-\infty}} \| f \|_{L^{q'}} \quad \forall f \in L^{q'}. \end{aligned}$$

Estimate 3: In order to estimate the $L^{q'} \to L^q$ norm of $f \mapsto f * w_t^{\infty}$, we may apply Lemma 5.1 and use pointwise estimates of w_t^{∞} (see Remark 4.3). While

$$\int_0^{|t|-1} dr\, (\sinh r)^{n-1} \varphi_0(r) \big| w_t^{\infty}(r) \big|^{\frac{q}{2}} \quad \text{and} \quad \int_{|t|+1}^{+\infty} dr\, (\sinh r)^{n-1} \varphi_0(r) \big| w_t^{\infty}(r) \big|^{\frac{q}{2}}$$

are $\mathrm{O}(|t|^{-\infty})$ for any $\sigma \in \mathbb{R}$, the integral

$$\int_{|t|-1}^{|t|+1} dr\,(\sinh r)^{n-1}\varphi_0(r)\big|w_t^{\infty}(r)\big|^{\frac{q}{2}}$$

is finite provided $\sigma > \frac{n+1}{2} - \frac{2}{q}$, which is too large compared with the critical exponent $(n+1)(\frac{1}{2}-\frac{1}{q})$. Instead we use again interpolation for the analytic family (31). If $\operatorname{Re}\sigma = 0$, then

$$\big\|f * \widetilde{w}_t^{\infty}\big\|_{L^2} \lesssim \|f\|_{L^2} \quad \forall f \in L^2.$$

If $\operatorname{Re}\sigma = \frac{n+1}{2}$, we deduce from Theorem 4.2(ii) that

$$\big\|f * \widetilde{w}_t^{\infty}\big\|_{L^\infty} \lesssim |t|^{-\infty}\|f\|_{L^1} \quad \forall f \in L^1.$$

By interpolation we conclude for $\sigma = (n+1)(\frac{1}{2}-\frac{1}{q})$ that

$$\big\|f * w_t^{\infty}\big\|_{L^q} \lesssim |t|^{-\infty}\|f\|_{L^{q'}} \quad \forall f \in L^{q'}. \qquad \square$$

By taking $\tau = 1$ in Theorems 5.2 and 5.3, we obtain in particular the following dispersive estimates.

Corollary 5.4. *Let $2 < q < \infty$ and $\sigma \geqslant (n+1)(\frac{1}{2}-\frac{1}{q})$. Then*

$$\bigg\|\widetilde{D}^{-\sigma+1}\frac{e^{itD}}{D}\bigg\|_{L^{q'}\to L^q} \lesssim \begin{cases} |t|^{-(n-1)(\frac{1}{2}-\frac{1}{q})} & \text{if } 0 < |t| \leqslant 2, \\ |t|^{-2} & \text{if } |t| \geqslant 2, \end{cases}$$

with $|t|^{-(n-1)(\frac{1}{2}-\frac{1}{q})}$ replaced by $|t|^{-(\frac{1}{2}-\frac{1}{q})}(1-\log|t|)^{1-\frac{2}{q}}$ in dimension $n = 2$.

Remark 5.5. Notice that Tataru [30] obtained dispersive estimates with exponential decay in time for the operators $\cos tD$ and $\frac{\sin tD}{D}$, but did not prove actual Strichartz estimates. Here we obtain dispersive estimates with polynomial decay in time for the operator e^{itD}. This difference reflects the fact that the Fourier multipliers associated with the operators $\cos tD$ and $\frac{\sin tD}{D}$ are analytic in a strip of the complex plane, which is not the case of e^{itD}.

By applying Lemma 5.1 in full generality, we obtain the following decoupled estimate for the operators

$$W_{t,0}^{(\sigma,\tau)} = \chi_0(D)D^{-\tau}\widetilde{D}^{\tau-\sigma}e^{itD}.$$

Proposition 5.6. *Let $2 < q, \tilde{q} < \infty$, $0 \leqslant \tau < \frac{3}{2}$ and $\sigma \in \mathbb{R}$. Then*

$$\big\|W_{t,0}^{(\sigma,\tau)}\big\|_{L^{\tilde{q}'}\to L^q} \lesssim \big(1+|t|\big)^{\tau-3} \quad \forall t \in \mathbb{R}.$$

6. Strichartz estimates

We shall assume $n \geqslant 3$ throughout this section and discuss the 2-dimensional case in the final remark. Consider the inhomogeneous linear wave equation on $\mathbb{H}^n$:

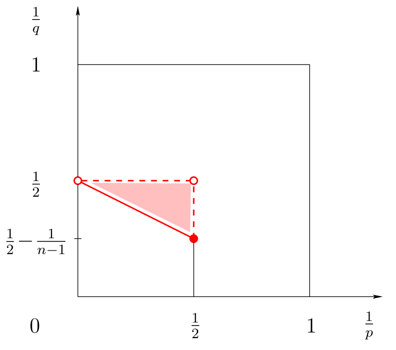

Fig. 1. Admissibility in dimension $n \geqslant 4$.

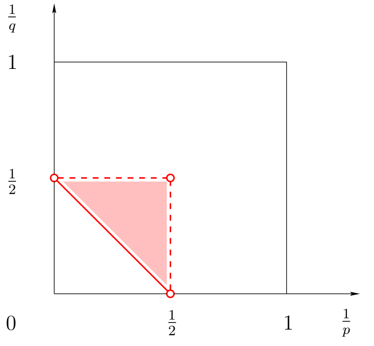

Fig. 2. Admissibility in dimension $n = 3$.

$$\begin{cases} \partial_t^2 u(t,x) - \big(\Delta_{\mathbb{H}^n} + \rho^2\big) u(t,x) = F(t,x), \\ u(0,x) = f(x), \\ \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{32}$$

whose solution is given by Duhamel's formula:

$$u(t,x) = (\cos tD_x) f(x) + \frac{\sin tD_x}{D_x} g(x) + \int_0^t ds\, \frac{\sin(t-s)D_x}{D_x} F(s,x).$$

Definition 6.1. A couple (p,q) is called *admissible* if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$T_n = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right] \times \left(0, \frac{1}{2}\right) \;\middle|\; \frac{2}{p} + \frac{n-1}{q} \geqslant \frac{n-1}{2} \right\} \tag{33}$$

(see Fig. 1).

Remark 6.2. Observe that the endpoint $(\frac{1}{2}, \frac{1}{2} - \frac{1}{n-1})$ is included in the triangle T_n in dimension $n > 3$ but not in dimension $n = 3$ (see Fig. 2).

Theorem 6.3. *Let (p,q) and $(\tilde{p}, \tilde{q})$ be two admissible couples. Then the following Strichartz estimate holds for solutions to the Cauchy problem* (32):

$$\|u\|_{L^p(\mathbb{R};L^q)} \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})}, \tag{34}$$

where $\sigma \geqslant \frac{(n+1)}{2}(\frac{1}{2}-\frac{1}{q})$ *and* $\tilde{\sigma} \geqslant \frac{(n+1)}{2}(\frac{1}{2}-\frac{1}{\tilde{q}})$. *Moreover,*

$$\|u\|_{L^\infty(\mathbb{R};H^{\sigma-\frac{1}{2},\frac{1}{2}})} + \|\partial_t u\|_{L^\infty(\mathbb{R};H^{\sigma-\frac{1}{2},-\frac{1}{2}})} \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})}. \tag{35}$$

Proof. Consider the operator

$$Tf(t,x) = \widetilde{D}_x^{-\sigma+\frac{1}{2}} \frac{e^{\pm itD_x}}{\sqrt{D_x}} f(x),$$

initially defined from $L^2(\mathbb{H}^n)$ into $L^\infty(\mathbb{R};H^{-\frac{1}{2},\frac{1}{2}}(\mathbb{H}^n))$, and its formal adjoint

$$T^*F(x) = \int_{-\infty}^{+\infty} ds\, \widetilde{D}_x^{-\sigma+1/2} \frac{e^{\mp isD_x}}{\sqrt{D_x}} F(s,x),$$

initially defined from $L^1(\mathbb{R};L^2(\mathbb{H}^n))$ into $H^{-\frac{1}{2},\frac{1}{2}}(\mathbb{H}^n)$. The TT^* method consists in proving first the $L^{p'}(\mathbb{R};L^{q'}(\mathbb{H}^n)) \to L^p(\mathbb{R};L^q(\mathbb{H}^n))$ boundedness of the operator

$$TT^*F(t,x) = \int_{-\infty}^{+\infty} ds\, \widetilde{D}_x^{-2\sigma+1} \frac{e^{\pm i(t-s)D_x}}{D_x} F(s,x)$$

and of its truncated version

$$\mathcal{T}F(t,x) = \int_{-\infty}^{t} ds\, \widetilde{D}_x^{-2\sigma+1} \frac{e^{\pm i(t-s)D_x}}{D_x} F(s,x),$$

for every admissible couple (p,q) and for every $\sigma \geqslant \frac{n+1}{2}(\frac{1}{2}-\frac{1}{q})$, and in decoupling next the indices.

Assume that the admissible couple (p,q) is different from the endpoint $(2, 2\frac{n-1}{n-3})$. Then we deduce from Corollary 5.4 that the norms $\|TT^*F(t,x)\|_{L^p_t L^q_x}$ and $\|\mathcal{T}F(t,x)\|_{L^p_t L^q_x}$ are bounded above by

$$\Bigg\| \int_{0<|t-s|<1} ds\, |t-s|^{-\alpha} \big\|F(s,x)\big\|_{L^{q'}_x} \Bigg\|_{L^p_t} + \Bigg\| \int_{|t-s|\geqslant 1} ds\, |t-s|^{-2} \big\|F(s,x)\big\|_{L^{q'}_x} \Bigg\|_{L^p_t}, \tag{36}$$

where $\alpha = (n-1)(\frac{1}{2}-\frac{1}{q}) \in (0,1)$. On the one hand, the convolution kernel $|t-s|^{-2}\mathbb{1}_{\{|t-s|\geqslant 1\}}$ defines obviously a bounded operator from $L^{p_1}(\mathbb{R})$ to $L^{p_2}(\mathbb{R})$, for all $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, in particular from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$, since $p \geqslant 2$. On the other hand, the convolution kernel $|t-s|^{-\alpha}\mathbb{1}_{\{0<|t-s|<1\}}$ with $0<\alpha<1$ defines a bounded operator from $L^{p_1}(\mathbb{R})$ to $L^{p_2}(\mathbb{R})$, for all $1 < p_1, p_2 < \infty$ such that $0 \leqslant \frac{1}{p_1} - \frac{1}{p_2} \leqslant 1-\alpha$, in particular from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$, since $p \geqslant 2$ and $\frac{2}{p} \geqslant \alpha$.

At the endpoint $(p,q) = (2, 2\frac{n-1}{n-3})$, we have $\alpha = 1$. Thus the previous argument breaks down and is replaced by the refined analysis carried out in [25]. Notice that the problem lies only in the first part of (36) and not in the second one, which involves an integrable convolution kernel on $\mathbb{R}$.

Thus TT^* and $\mathcal{T}$ are bounded from $L^{p'}(\mathbb{R};L^{q'}(\mathbb{H}^n))$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$, for every admissible couple (p,q). As a consequence, T^* is bounded from $L^{p'}(\mathbb{R};L^{q'}(\mathbb{H}^n))$ to $L^2(\mathbb{H}^n)$ and T is bounded from $L^2(\mathbb{H}^n)$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$. In particular,

$$\big\|(\cos tD_x)f(x)\big\|_{L^p_tL^q_x}\lesssim\big\|\widetilde{D}_x^{-\sigma+\frac12}D_x^{-\frac12}e^{\pm itD_x}\widetilde{D}_x^{\sigma-\frac12}D_x^{\frac12}f(x)\big\|_{L^p_tL^q_x}\lesssim\|f\|_{H^{\sigma-\frac12,\frac12}}$$

and

$$\bigg\|\frac{\sin tD_x}{D_x}g(x)\bigg\|_{L^p_tL^q_x}\lesssim\big\|\widetilde{D}_x^{-\sigma+\frac12}D_x^{-\frac12}e^{\pm itD_x}\widetilde{D}_x^{\sigma-\frac12}D_x^{-\frac12}g(x)\big\|_{L^p_tL^q_x}\lesssim\|g\|_{H^{\sigma-\frac12,-\frac12}}.$$

We next decouple the indices. Let $(p,q)\neq(\tilde p,\tilde q)$ be two admissible couples and let $\sigma\geqslant\frac{n+1}{2}(\frac12-\frac1q)$, $\tilde\sigma\geqslant\frac{n+1}{2}(\frac12-\frac1{\tilde q})$. Since T and T^* are separately continuous, the operator

$$TT^*F(t,x)=\int\limits_{-\infty}^{+\infty}ds\,\widetilde{D}_x^{-\sigma-\tilde\sigma+1}\frac{e^{\pm i(t-s)D_x}}{D_x}F(s,x)$$

is bounded from $L^{\tilde p'}(\mathbb{R};L^{\tilde q'}(\mathbb{H}^n))$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$. According to [4], this result remains true for the truncated operator

$$\mathcal{T}F(t,x)=\int\limits_{-\infty}^{t}ds\,\widetilde{D}_x^{-\sigma-\tilde\sigma+1}\frac{e^{\pm i(t-s)D_x}}{D_x}F(s,x)$$

and hence for

$$\widetilde{\mathcal{T}}F(t,x)=\int\limits_{0}^{t}ds\,\widetilde{D}_x^{-\sigma-\tilde\sigma+1}\frac{\sin(t-s)D_x}{D_x}F(s,x)$$

as long as p and $\tilde p$ are not both equal to 2. We handle the remaining case, where $p=\tilde p=2$ and $2<q\neq\tilde q\leqslant2\frac{n-1}{n-3}$, by combining the bilinear approach in [25] with our previous estimates. Specifically let us split up again $I=\chi_0(D)+\chi_\infty(D)^2$, using smooth cut-off functions, and $\mathcal{T}=\mathcal{T}^0+\mathcal{T}^\infty$ accordingly. On one hand, it follows from Proposition 5.6 that

$$\mathcal{T}^0F(t,x)=\int\limits_{-\infty}^{t}ds\,\chi_0(D_x)D_x^{-1}\widetilde{D}_x^{1-\sigma-\tilde\sigma}e^{\pm i(t-s)D_x}F(s,x)$$

is bounded from $L^{\tilde p'}(\mathbb{R};L^{\tilde q'}(\mathbb{H}^n))$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$, for every $2\leqslant p,\tilde p\leqslant\infty$ and $2<q,\tilde q<\infty$, in particular for $p=\tilde p=2$ and $2<q,\tilde q\leqslant2\frac{n-1}{n-3}$. As far as it is concerned, the $L^2L^{\tilde q'}\to L^2L^q$ boundedness of

$$\mathcal{T}^\infty F(t,x)=\int\limits_{-\infty}^{t}ds\,\chi_\infty(D_x)^2D_x^{-1}\widetilde{D}_x^{1-\sigma-\tilde\sigma}e^{\pm i(t-s)D_x}F(s,x)$$

amounts to estimating the Hermitian form

$$\mathcal{B}^\infty(F,G)=\iint\limits_{s<t}ds\,dt\int\limits_{\mathbb{H}^n}dx\,\chi_\infty(D_x)D_x^{-1/2}\widetilde{D}_x^{1/2-\tilde\sigma}e^{\mp isD_x}F(s,x)$$
$$\times\,\overline{\chi_\infty(D_x)D_x^{-1/2}\widetilde{D}_x^{1/2-\sigma}e^{\mp itD_x}G(t,x)}$$

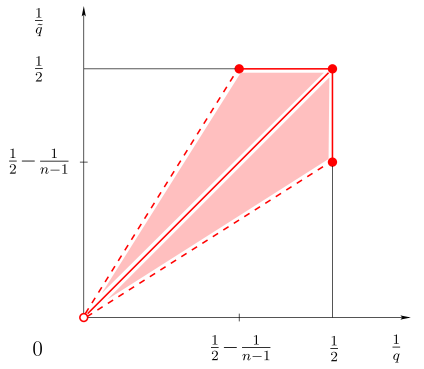

Fig. 3. Interpolation.

by $\|F\|_{L^2L^{\tilde{q}'}}\|G\|_{L^2L^{q'}}$. Let us split up dyadically

$$\iint\limits_{s<t} = \sum_{j=-\infty}^{+\infty} \iint\limits_{2^j\leqslant t-s<2^{j+1}}$$

and $\mathcal{B}^\infty = \sum_{j=-\infty}^{+\infty}\mathcal{B}_j^\infty$ accordingly. For every $j\in\mathbb{Z}$, let us further split up

$$F(s,x) = \sum_{k=-\infty}^{+\infty} \underbrace{\mathbb{1}_{[k2^j,(k+1)2^j)}(s)F(s,x)}_{F_k^{(j)}(s,x)} \quad \text{and} \quad G(t,x) = \sum_{\ell=-\infty}^{+\infty} \underbrace{\mathbb{1}_{[\ell 2^j,(\ell+1)2^j)}(t)G(t,x)}_{G_\ell^{(j)}(t,x)}.$$

Notice the orthogonality

$$\|F\|_{L^2L^{\tilde{q}'}} = \Biggl\{\sum_{k=-\infty}^{+\infty}\bigl\|F_k^{(j)}\bigr\|_{L^2L^{\tilde{q}'}}^2\Biggr\}^{1/2}, \qquad \|G\|_{L^2L^{q'}} = \Biggl\{\sum_{\ell=-\infty}^{+\infty}\bigl\|G_\ell^{(j)}\bigr\|_{L^2L^{q'}}^2\Biggr\}^{1/2}$$

and the almost orthogonality

$$\mathcal{B}_j^\infty(F,G) = \sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell-k\in\{1,2\}}} \mathcal{B}_j^\infty\bigl(F_k^{(j)},G_\ell^{(j)}\bigr).$$

We claim that

$$\bigl|\mathcal{B}_j^\infty\bigl(F_k^{(j)},G_\ell^{(j)}\bigr)\bigr| \lesssim \begin{cases} 2^{\kappa(q,\tilde{q})j}\bigl\|F_k^{(j)}\bigr\|_{L^2L^{\tilde{q}'}}\bigl\|G_\ell^{(j)}\bigr\|_{L^2L^{q'}} & \text{if } j\leqslant 0,\\ 2^{-\infty j}\bigl\|F_k^{(j)}\bigr\|_{L^2L^{\tilde{q}'}}\bigl\|G_\ell^{(j)}\bigr\|_{L^2L^{q'}} & \text{if } j>0, \end{cases} \tag{37}$$

when $2<q,\tilde{q}\leqslant 2\frac{n-1}{n-3}$ and $\kappa(q,\tilde{q}) = \frac{n-1}{2}(\frac{1}{q}+\frac{1}{\tilde{q}}) - \frac{n-3}{2}$. These estimates will be obtained by complex interpolation between the following cases (see Fig. 3):

(a) $q=2$ and $2\leqslant\tilde{q}\leqslant 2\frac{n-1}{n-3}$,
(b) $2\leqslant q\leqslant 2\frac{n-1}{n-3}$ and $\tilde{q}=2$,
(c) $2<q=\tilde{q}<\infty$.

Case (a): Assume that $q=2$, $2\leqslant\tilde q\leqslant 2\frac{n-1}{n-3}$ and $\operatorname{Re}\sigma=0$, $\operatorname{Re}\tilde\sigma=\frac{n+1}{2}(\frac12-\frac1q)$. Consider the operators

$$T^\infty f(t,x)=\chi_\infty(D_x)\widetilde D_x^{-\tilde\sigma+\frac12}\frac{e^{\pm itD_x}}{\sqrt{D_x}}f(x)$$

and

$$\left(T^\infty\right)^*F(x)=\int_{-\infty}^{+\infty}ds\,\chi_\infty(D_x)\widetilde D_x^{-\tilde\sigma+\frac12}\frac{e^{\mp isD_x}}{\sqrt{D_x}}F(s,x).$$

By resuming the proof of Theorem 5.2 and by applying the $T^\infty(T^\infty)^*$ argument, we obtain that $(T^\infty)^*$ is bounded from $L^{\tilde p'}(\mathbb R;L^{\tilde q'}(\mathbb H^n))$ to $L^2(\mathbb H^n)$, where $\frac1{\tilde p}=\frac{n-1}{2}(\frac12-\frac1{\tilde q})$. By combining this result with Hölder's inequality, we deduce that

$$\begin{aligned}\left|\mathcal B_j^\infty\left(F_k^{(j)},G_\ell^{(j)}\right)\right| &\lesssim\sup_{t\in\mathbb R}\left\|\int_{t-2^{j+1}<s\leqslant t-2^j}ds\,\chi_\infty(D_x)D_x^{-\frac12}\widetilde D_x^{\frac12-\tilde\sigma}e^{\mp isD_x}F_k^{(j)}(s,x)\right\|_{L_x^2}\\&\quad\times\left\|\chi_\infty(D_x)D_x^{-\frac12}\widetilde D_x^{\frac12}e^{\mp itD_x}G_\ell^{(j)}(t,x)\right\|_{L_t^1L_x^2}\\&\lesssim\sup_{t\in\mathbb R}\left\|\mathbb 1_{(t-2^{j+1},t-2^j)}(s)F_k^{(j)}(s,x)\right\|_{L_s^{\tilde p'}L_x^{\tilde q'}}\left\|G_\ell^{(j)}(t,x)\right\|_{L_t^1L_x^2}\\&\lesssim 2^{\frac{j}{\tilde p'}}\left\|F_k^{(j)}\right\|_{L^2L^{\tilde q'}}\left\|G_\ell^{(j)}\right\|_{L^2L^2},\end{aligned}$$

with $\frac1{\tilde p'}=\kappa(2,\tilde q)$.

Case (b): If $2<q\leqslant 2\frac{n-1}{n-3}$, $\tilde q=2$ and $\operatorname{Re}\sigma=\frac{n+1}{2}(\frac12-\frac1q)$, $\operatorname{Re}\tilde\sigma=0$, we have symmetrically

$$\left|\mathcal B_j^\infty\left(F_k^{(j)},G_\ell^{(j)}\right)\right|\lesssim 2^{\kappa(q,2)j}\left\|F_k^{(j)}\right\|_{L^2L^2}\left\|G_\ell^{(j)}\right\|_{L^2L^{q'}}.$$

Case (c): Assume that $2<q=\tilde q<\infty$ and $\operatorname{Re}\sigma=\operatorname{Re}\tilde\sigma=\frac{n+1}{2}(\frac12-\frac1q)$. Let us rewrite

$$\mathcal B_j^\infty\left(F_k^{(j)},G_\ell^{(j)}\right)=\iint_{2^j\leqslant t-s<2^{j+1}}ds\,dt\int_{\mathbb H^n}dx\left\{\chi_\infty(D_x)^2D_x^{-1}\widetilde D_x^{1-\sigma-\tilde\sigma}e^{\pm i(t-s)D_x}F_k^{(j)}(s,x)\right\}\overline{G_\ell^{(j)}(t,x)}.$$

By using the dispersive estimates

$$\left\|\chi_\infty(D)^2D^{-1}\widetilde D^{1-\sigma-\tilde\sigma}e^{\pm i(t-s)D}\right\|_{L^{\tilde q'}\to L^{\tilde q}}\lesssim\begin{cases}(t-s)^{-(n-1)(\frac12-\frac1{\tilde q})} & \text{if } 0<t-s<2,\\(t-s)^{-\infty} & \text{if } t-s\geqslant 2\end{cases}$$

(see the proofs of Theorems 5.2 and 5.3), we obtain

$$\left|\mathcal B_j^\infty\left(F_k^{(j)},G_\ell^{(j)}\right)\right|\lesssim\begin{cases}2^{-(n-1)(\frac12-\frac1q)j}\|F_k^{(j)}\|_{L^1L^{q'}}\|G_\ell^{(j)}\|_{L^1L^{q'}} & \text{if } j\leqslant 0,\\2^{-\infty j}\|F_k^{(j)}\|_{L^1L^{q'}}\|G_\ell^{(j)}\|_{L^1L^{q'}} & \text{if } j>0.\end{cases}$$

Hence, by Hölder's inequality,

$$\left|\mathcal{B}_j^{\infty}\left(F_k^{(j)}, G_\ell^{(j)}\right)\right| \lesssim \begin{cases} 2^{\kappa(q,q)j}\|F_k^{(j)}\|_{L^2L^{q'}}\|G_\ell^{(j)}\|_{L^2L^{q'}} & \text{if } j \leqslant 0, \\ 2^{-\infty j}\|F_k^{(j)}\|_{L^2L^{q'}}\|G_\ell^{(j)}\|_{L^2L^{q'}} & \text{if } j > 0. \end{cases}$$

Our claim (37) follows now by complex interpolation between the estimates obtained in Cases (a), (b) and (c) above. By summing up (37) and by using Hölder's inequality, we conclude that

$$\begin{aligned} \left|\mathcal{B}^{\infty}(F,G)\right| &\leqslant \sum_{j\in\mathbb{Z}}\left|\mathcal{B}_j^{\infty}(F,G)\right| \leqslant \sum_{\substack{j,k,\ell\in\mathbb{Z} \\ \ell-k\in\{1,2\}}}\left|\mathcal{B}_j^{\infty}\left(F_k^{(j)}, G_\ell^{(j)}\right)\right| \\ &\lesssim \left\{\sum_{j\leqslant 0}2^{\kappa(q,\tilde q)j} + \sum_{j>0}2^{-\infty j}\right\}\left\{\sum_{k\in\mathbb{Z}}\|F_k^{(j)}\|_{L^2L^{\tilde q'}}^2\right\}^{1/2}\left\{\sum_{\ell\in\mathbb{Z}}\|G_\ell^{(j)}\|_{L^2L^{q'}}^2\right\}^{1/2} \\ &\lesssim \|F\|_{L^2L^{\tilde q'}}\|G\|_{L^2L^{q'}} \end{aligned}$$

if $2 < q \neq \tilde q \leqslant 2\frac{n-1}{n-3}$. Notice that $\kappa(q,\tilde q) > 0$ under this assumption.

Let us turn to (35). On the one hand, the energy estimate (17) yields

$$\begin{aligned} &\left\|(\cos tD)f + \frac{\sin tD}{D}g\right\|_{H^{\sigma-\frac12,\frac12}} + \left\|-(\sin tD)Df + (\cos tD)g\right\|_{H^{\sigma-\frac12,-\frac12}} \\ &\quad\leqslant \sqrt{2}\left\{\|f\|_{H^{\sigma-\frac12,\frac12}} + \|g\|_{H^{\sigma-\frac12,-\frac12}}\right\} \end{aligned}$$

for every $t \in \mathbb{R}$. On the other hand, since T^* is bounded from $L^{\tilde p'}(\mathbb{R}; L^{\tilde q'}(\mathbb{H}^n))$ to $L^2(\mathbb{H}^n)$, both expressions

$$\left\|\int_0^t ds\,\frac{\sin(t-s)D_x}{D_x}F(s,x)\right\|_{H_x^{\sigma-\frac12,\frac12}} = \left\|\int_0^t ds\,\widetilde{D}_x^{\sigma-\frac12}D_x^{-\frac12}\sin(t-s)D_xF(s,x)\right\|_{L_x^2}$$

and

$$\left\|\int_0^t ds\,\cos(t-s)D_xF(s,x)\right\|_{H_x^{\sigma-\frac12,-\frac12}} = \left\|\int_0^t ds\,\widetilde{D}_x^{\sigma-\frac12}D_x^{-\frac12}\cos(t-s)D_xF(s,x)\right\|_{L_x^2}$$

are bounded above by

$$\begin{aligned} &\left\|e^{\pm itD_x}\int_{-\infty}^{+\infty} ds\,\widetilde{D}_x^{-\tilde\sigma+\frac12}D_x^{-\frac12}e^{\mp isD_x}\widetilde{D}_x^{\sigma+\tilde\sigma-1}\mathbb{1}_{(0,t)}(s)F(s,x)\right\|_{L_x^2} \\ &\quad\lesssim \left\|\mathbb{1}_{(0,t)}(s)\widetilde{D}_x^{\sigma+\tilde\sigma-1}F(s,x)\right\|_{L_s^{\tilde p'}L_x^{\tilde q'}} \lesssim \|F\|_{L^{\tilde p'}(\mathbb{R};H_{\tilde q'}^{\sigma+\tilde\sigma-1}(\mathbb{H}^n))}, \end{aligned}$$

uniformly in $t \in \mathbb{R}$. We conclude the proof of (35) by summing up the previous estimates and by taking the supremum over $t \in \mathbb{R}$. □

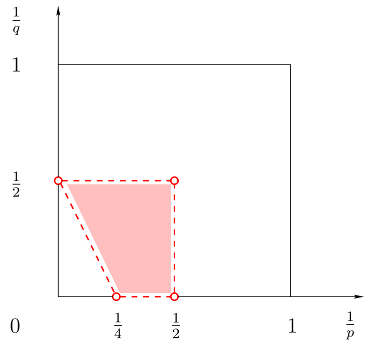

Fig. 4. Admissibility in dimension $n = 2$.

Remark 6.4. Observe that, in the statement of Theorem 6.3, we may replace $\mathbb{R}$ by any time interval I containing 0.

Remark 6.5. An analogous result holds in dimension $n = 2$ and its proof is similar, except for the first convolution kernel in (36), which becomes

$$|t-s|^{-\alpha}\big(1-\log|t-s|\big)^{\beta}\mathbb{1}_{\{0<|t-s|<1\}},$$

with $\alpha = \frac{1}{2} - \frac{1}{q}$ and $\beta = 2(\frac{1}{2} - \frac{1}{q})$. It turns out that, in this case, a couple (p, q) is *admissible* if $(\frac{1}{p}, \frac{1}{q})$ belongs to the region $T_2 = \{(\frac{1}{p}, \frac{1}{q}) \in (0, \frac{1}{2}] \times (0, \frac{1}{2}) \mid \frac{2}{p} + \frac{1}{q} > \frac{1}{2}\}$ (see Fig. 4).

7. LWP results for NLW equation on $\mathbb{H}^n$

We shall assume $n \geqslant 4$ throughout this section and discuss the lower-dimensional cases $n = 3$ and $n = 2$ in the final remarks. We apply Strichartz estimates for the inhomogeneous linear Cauchy problem associated with the wave equation to prove local well-posedness results for the following nonlinear Cauchy problem

$$\begin{cases} \partial_t^2 u(t,x) - \big(\Delta_{\mathbb{H}^n} + \rho^2\big)u(t,x) = F\big(u(t,x)\big), \\ u(0,x) = f(x), \\ \partial_t|_{t=0}u(t,x) = g(x), \end{cases} \tag{38}$$

with a power-like nonlinearity $F(u)$. By this we mean that

$$\big|F(u)\big| \leqslant C|u|^{\gamma} \quad \text{and} \quad \big|F(u)-F(v)\big| \leqslant C\big(|u|^{\gamma-1}+|v|^{\gamma-1}\big)|u-v| \tag{39}$$

for some $C \geqslant 0$ and $\gamma > 1$. Let us recall the definition of local well-posedness.

Definition 7.1. The NLW Cauchy problem (38) is *locally well-posed* in $H^{\sigma,\tau} \times H^{\sigma,\tau-1}$ if, for any bounded subset B of $H^{\sigma,\tau} \times H^{\sigma,\tau-1}$, there exist $T > 0$ and a Banach space X_T, continuously embedded into $C([-T,T]; H^{s,\tau}) \cap C^1([-T,T]; H^{s,\tau-1})$, such that

- for any initial data $(f, g) \in B$, (38) has a unique solution $u \in X_T$,
- the map $(f, g) \mapsto u$ is continuous from B into X_T.

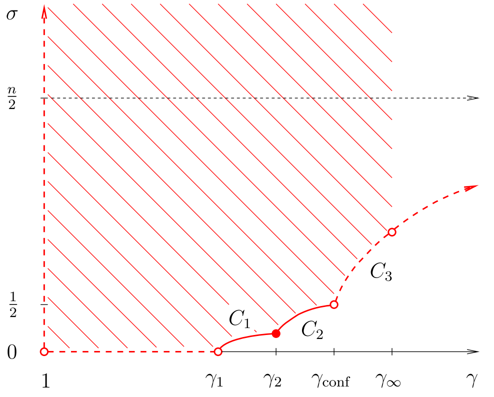

Fig. 5. Regularity in dimension $n \geqslant 4$.

The amount of smoothness σ requested for LWP of (38) in $H^{\sigma-\frac{1}{2},\frac{1}{2}} \times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ depends on γ and is represented in Fig. 5.

There

$$\gamma_1 = \frac{n+3}{n} = 1 + \frac{3}{n}, \qquad \gamma_2 = \frac{(n+1)^2}{(n-1)^2+4} = 1 + \frac{2}{\frac{n-1}{2} + \frac{2}{n-1}}, \qquad \gamma_{\text{conf}} = \frac{n+3}{n-1} = 1 + \frac{4}{n-1},$$

$$\gamma_3 = \frac{n^2+5n-2+\sqrt{n^4+2n^3+21n^2-12n+4}}{2n^2-2n} = 1 + \frac{\sqrt{4n + (\frac{n-6}{2} - \frac{2}{n-1})^2} - (\frac{n-6}{2} - \frac{2}{n-1})}{n},$$

$$\gamma_4 = \frac{n^2+2n-5}{n^2-2n-1} = 1 + \frac{2}{\frac{n-1}{2} - \frac{1}{n-1}}, \qquad \gamma_\infty = \min\{\gamma_3, \gamma_4\} = \begin{cases} \gamma_3 & \text{if } n = 4, 5, \\ \gamma_4 & \text{if } n \geqslant 6 \end{cases}$$

and the curves C_1, C_2, C_3 are given by

$$C_1(\gamma) = \frac{n+1}{4}\left(1 - \frac{n+5}{2n\gamma - n - 1}\right), \qquad C_2(\gamma) = \frac{n+1}{4} - \frac{1}{\gamma-1}, \qquad C_3(\gamma) = \frac{n}{2} - \frac{2}{\gamma-1}.$$

When $\gamma < \gamma_\infty$, we obtain the same regularity curve as in the Euclidean case. Since our Strichartz estimates hold for a large family of admissible pairs, they are sufficient to study the regularity problem via a fixed point argument; in the Euclidean setting this problem was solved by different methods, depending on the range of the power γ involved in the nonlinearity and on the regularity of initial data.

Theorem 7.2. *Let $n \geqslant 4$ and assume that $F(u)$ satisfies* (39). *Then the NLW* (38) *is locally well-posed in $H^{\sigma-\frac{1}{2},\frac{1}{2}} \times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ in the following cases*:

(A) $1 < \gamma \leqslant \gamma_1$ *and* $\sigma > 0$;
(B) $\gamma_1 < \gamma \leqslant \gamma_2$ *and* $\sigma \geqslant C_1(\gamma)$;
(C) $\gamma_2 \leqslant \gamma < \gamma_{\text{conf}}$ *and* $\sigma \geqslant C_2(\gamma)$;
(D) $\gamma_{\text{conf}} \leqslant \gamma < \gamma_\infty$ *and* $\sigma > C_3(\gamma)$.

More precisely, for all such nonlinearity power γ and regularity σ, there exists a positive T, depending on the initial data, and a unique solution u to NLW (38) *such that*

$$u \in C\big([-T,T]; H^{\sigma-\frac{1}{2},\frac{1}{2}}\big(\mathbb{H}^n\big)\big) \cap L^{p_0}\big([-T,T]; L^{q_0}\big(\mathbb{H}^n\big)\big),$$

for a suitable admissible couple (p_0, q_0)*, and*

$$\partial_t u \in C\big([-T,T]; H^{\sigma-\frac{1}{2},-\frac{1}{2}}\big(\mathbb{H}^n\big)\big).$$

Proof. We apply the standard fixed point method based on Strichartz estimates. Define $u = \Phi(v)$ as the solution to the following linear Cauchy problem

$$\begin{cases} \partial_t^2 u(t,x) - D_x^2 u(t,x) = F\big(v(t,x)\big), \\ u(0,x) = f(x), \\ \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{40}$$

which is given by the Duhamel formula

$$u(t,x) = (\cos t D_x) f(x) + \frac{\sin t D_x}{D_x} g(x) + \int_0^t ds\, \frac{\sin(t-s) D_x}{D_x} F\big(v(s,x)\big).$$

We deduce from the Strichartz estimates (34), (35) and from Remark 6.4 that

$$\begin{aligned} &\|u\|_{L^\infty([-T,T]; H^{\sigma-\frac{1}{2},\frac{1}{2}})} + \|\partial_t u\|_{L^\infty([-T,T]; H^{\sigma-\frac{1}{2},-\frac{1}{2}})} + \|u\|_{L^p([-T,T]; L^q)} \\ &\quad \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \big\|F(v)\big\|_{L^{\tilde{p}'}([-T,T]; H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})}, \end{aligned}$$

for all admissible couples (p,q), $(\tilde{p},\tilde{q})$ introduced in Definition 6.1, for all $\sigma \geqslant \frac{n+1}{2}(\frac{1}{2}-\frac{1}{q})$, $\tilde{\sigma} \geqslant \frac{n+1}{2}(\frac{1}{2}-\frac{1}{\tilde{q}})$, and for a positive T to be determined later. According to the nonlinear assumption (39), we estimate the inhomogeneous term as follows:

$$\big\|F(v)\big\|_{L^{\tilde{p}'}([-T,T]; H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})} \lesssim \big\||v|^\gamma\big\|_{L^{\tilde{p}'}([-T,T]; H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})}.$$

Assuming $\sigma + \tilde{\sigma} - 1 \leqslant n(\frac{1}{\tilde{q}'} - \frac{1}{\tilde{q}_1'}) \leqslant 0$, we deduce from Sobolev's embedding (Proposition 3.1) that

$$\begin{aligned} &\|u\|_{L^\infty([-T,T]; H^{\sigma-\frac{1}{2},\frac{1}{2}})} + \|\partial_t u\|_{L^\infty([-T,T]; H^{\sigma-\frac{1}{2},-\frac{1}{2}})} + \|u\|_{L^p([-T,T]; L^q)} \\ &\quad \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \|v\|^\gamma_{L^{\tilde{p}'\gamma}([-T,T]; L^{\tilde{q}_1'\gamma})}. \end{aligned}$$

In order to remain within the same function space, we require that $q = \tilde{q}_1'\gamma$. After applying Hölder's inequality in time, we obtain

$$\begin{aligned} &\|u\|_{L^\infty([-T,T]; H^{\sigma-\frac{1}{2},\frac{1}{2}})} + \|\partial_t u\|_{L^\infty([-T,T]; H^{\sigma-\frac{1}{2},-\frac{1}{2}})} + \|u\|_{L^p([-T,T]; L^q)} \\ &\quad \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + T^\lambda \|v\|^\gamma_{L^p([-T,T]; L^q)}. \end{aligned} \tag{41}$$

Here we have assumed $p > \tilde{p}'\gamma$ and set $\lambda = \frac{1}{\tilde{p}'} - \frac{\gamma}{p} > 0$. It remains for us to check that the following conditions can be fulfilled simultaneously:

$$\begin{cases} \text{(i)} & p > \tilde{p}'\gamma, \\ \text{(ii)} & 0 < \frac{1}{\tilde{q}'} \leqslant \frac{\gamma}{q} < 1, \\ \text{(iii)} & \frac{n-1}{2} - \frac{n+1}{2}\left(\frac{1}{q} + \frac{1}{\tilde{q}}\right) \leqslant n\left(\frac{1}{\tilde{q}'} - \frac{\gamma}{q}\right), \\ \text{(iv)} & \frac{2}{p} + \frac{n-1}{q} \geqslant \frac{n-1}{2}, \\ \text{(v)} & \frac{2}{\tilde{p}} + \frac{n-1}{\tilde{q}} \geqslant \frac{n-1}{2}, \\ \text{(vi)} & \left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right] \times \left(\frac{n-3}{2(n-1)}, \frac{1}{2}\right), \\ \text{(vii)} & \left(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}\right) \in \left(0, \frac{1}{2}\right] \times \left(\frac{n-3}{2(n-1)}, \frac{1}{2}\right). \end{cases} \tag{42}$$

Suppose indeed that there exist indices p, q, $\tilde{p}$, $\tilde{q}$ satisfying all conditions in (42). Then (41) shows that Φ maps X into itself, where X denotes the Banach space

$$X = \left\{u \;\middle|\; u \in C\left([-T, T]; H^{\sigma-\frac{1}{2},\frac{1}{2}}\left(\mathbb{H}^n\right)\right) \cap L^p\left([-T, T]; L^q\left(\mathbb{H}^n\right)\right), \partial_t u \in C\left([-T, T]; H^{\sigma-\frac{1}{2},-\frac{1}{2}}\left(\mathbb{H}^n\right)\right)\right\},$$

equipped with the norm

$$\|u\|_X = \|u\|_{L^\infty([-T,T];H^{\sigma-\frac{1}{2},\frac{1}{2}})} + \|\partial_t u\|_{L^\infty([-T,T];H^{\sigma-\frac{1}{2},-\frac{1}{2}})} + \|u\|_{L^p([-T,T];L^q)}.$$

Moreover we shall show that Φ is a contraction on the ball

$$X_M = \left\{u \in X \;\middle|\; \|u\|_X \leqslant M\right\},$$

provided the time $T > 0$ is sufficiently small and the radius $M > 0$ is sufficiently large. Let $v, \tilde{v} \in X$ and $u = \Phi(v)$, $\tilde{u} = \Phi(\tilde{v})$. By arguing as above and using Hölder's inequality, we have

$$\begin{aligned} \|u - \tilde{u}\|_X &\leqslant C\left\|F(v) - F(\tilde{v})\right\|_{L^{\tilde{p}'}([-T,T];H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})} \\ &\leqslant C\left\|\left\{|v|^{\gamma-1} + |\tilde{v}|^{\gamma-1}\right\}|v - \tilde{v}|\right\|_{L^{\tilde{p}'}([-T,T];L^{\tilde{q}'_1})} \\ &\leqslant CT^\lambda\left\{\|v\|^{\gamma-1}_{L^p([-T,T];L^q)} + \|\tilde{v}\|^{\gamma-1}_{L^p([-T,T];L^q)}\right\}\|v - \tilde{v}\|_{L^p([-T,T];L^q)} \\ &\leqslant CT^\lambda\left\{\|v\|^{\gamma-1}_X + \|\tilde{v}\|^{\gamma-1}_X\right\}\|v - \tilde{v}\|_X. \end{aligned} \tag{43}$$

If $\|v\|_X \leqslant M$ and $\|\tilde{v}\|_X \leqslant M$, then (41) yields on the one hand

$$\|u\|_X \leqslant C\left\{\|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + T^\lambda M^\gamma\right\}$$

and

$$\|\tilde{u}\|_X \leqslant C\left\{\|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + T^\lambda M^\gamma\right\},$$

while (43) yields on the other hand

$$\|u-\tilde{u}\|_X \leqslant 2CT^{\lambda}M^{\gamma-1}\|v-\tilde{v}\|_X.$$

Thus, if we choose $M>0$ so large that $\frac{M}{2} \geqslant C\{\|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}}\}$ and $T>0$ so small that $CT^{\lambda}M^{\gamma} \leqslant \frac{M}{2}$ and $2CT^{\lambda}M^{\gamma-1} \leqslant \frac{1}{2}$, then

$$\|u\|_X \leqslant M, \qquad \|\tilde{u}\|_X \leqslant M \quad \text{and} \quad \|u-\tilde{u}\|_X \leqslant \frac{1}{2}\|v-\tilde{v}\|_X$$

if $v, \tilde{v} \in X_M$ and $u=\Phi(v)$, $\tilde{u}=\Phi(\tilde{v})$. Hence the map Φ is a contraction on the complete metric space X_M and the fixed point theorem allows us to conclude.

Let us eventually prove the existence of couples (p,q) and $(\tilde{p},\tilde{q})$ satisfying all conditions in (42). Condition (42)(iii) amounts to

$$\frac{2n\gamma-n-1}{q} + \frac{n-1}{\tilde{q}} \leqslant n+1 \quad \text{i.e.} \quad \frac{1}{\tilde{q}} \leqslant \frac{n+1}{n-1} - \frac{2n\gamma-n-1}{n-1}\frac{1}{q}. \tag{44}$$

By combining (44) with (42)(ii) and (42)(vi), we deduce that

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}.$$

This implies that $\gamma \leqslant \tilde{\gamma}_\infty = \frac{n^2+2n-7}{(n+1)(n-3)} = 1 + \frac{4(n-1)}{(n+1)(n-3)}$. By combining (44) with (42)(vii), we obtain

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{\tilde{q}} \leqslant \min\left\{\frac{1}{2}, \frac{n+1}{n-1} - \frac{2n\gamma-n-1}{n-1}\frac{1}{q}\right\}, \qquad \frac{1}{\tilde{q}} \neq \frac{1}{2}.$$

By combining (44) with (42)(vii), we also obtain $\frac{1}{q} \leqslant \frac{n+5}{2(2n\gamma-n-1)}$. In summary, the conditions on q reduce to

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \min\left\{\frac{1}{2}, \frac{1}{\gamma}, \frac{2}{(\gamma-1)(n+1)}, \frac{n+5}{2(2n\gamma-n-1)}\right\}, \qquad \frac{1}{q} \neq \frac{1}{2}, \frac{1}{\gamma},$$

or case by case to

- $1<\gamma \leqslant \gamma_1$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} < \frac{1}{2}$,
- $\gamma_1 < \gamma \leqslant \gamma_2$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{n+5}{2(2n\gamma-n-1)}$,
- $\gamma_2 < \gamma \leqslant \tilde{\gamma}_\infty$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}$.

Let us turn to the indices p and $\tilde{p}$. According to (42), we have

$$\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right) \leqslant \frac{1}{p} \leqslant \frac{1}{2}$$

and

$$\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{q}}\right) \leqslant \frac{1}{\tilde{p}} \leqslant \min\left\{\frac{1}{2}, 1-\frac{\gamma}{p}\right\}, \qquad \frac{1}{\tilde{p}} \neq 1-\frac{\gamma}{p}.$$

By taking into account the previous conditions on q, we end up with the following conditions on p and $\tilde{p}$:

$$\begin{cases} \text{(i)} \quad \dfrac{n-1}{2}\left(\dfrac{1}{2}-\dfrac{1}{q}\right) \leqslant \dfrac{1}{p} \leqslant \min\left\{\dfrac{1}{2}, \dfrac{5-n}{4\gamma}+\dfrac{n-1}{2\gamma\tilde{q}}\right\}, \qquad \dfrac{1}{p} \neq \dfrac{5-n}{4\gamma}+\dfrac{n-1}{2\gamma\tilde{q}}, \\ \text{(ii)} \quad \dfrac{n-1}{2}\left(\dfrac{1}{2}-\dfrac{1}{\tilde{q}}\right) \leqslant \dfrac{1}{\tilde{p}} \leqslant \min\left\{\dfrac{1}{2}, 1-\dfrac{\gamma}{p}\right\}, \qquad \dfrac{1}{\tilde{p}} \neq 1-\dfrac{\gamma}{p}. \end{cases} \tag{45}$$

There exist indices p and $\tilde{p}$ which satisfy (45) provided that $\frac{1}{\tilde{q}} > \frac{\gamma}{2} + \frac{n-5}{2(n-1)} - \frac{\gamma}{q}$. We thus have to find $\tilde{q}$ such that

$$\max\left\{\frac{n-3}{2(n-1)}, \frac{\gamma}{2}+\frac{n-5}{2(n-1)}-\frac{\gamma}{q}\right\} \leqslant \frac{1}{\tilde{q}} \leqslant \min\left\{\frac{1}{2}, \frac{n+1}{n-1}-\frac{2n\gamma-n-1}{(n-1)q}\right\}, \tag{46}$$

with $\frac{1}{\tilde{q}} \neq \frac{1}{2}, \frac{\gamma}{2}+\frac{n-5}{2(n-1)}-\frac{\gamma}{q}$. This implies that q has to satisfy the following conditions:

$$\begin{aligned} &\max\left\{\frac{n-3}{2(n-1)}, \frac{1}{2}-\frac{2}{\gamma(n-1)}\right\} \\ &\quad \leqslant \frac{1}{q} \leqslant \min\left\{\frac{1}{2}, \frac{1}{\gamma}, \frac{2}{(\gamma-1)(n+1)}, \frac{n+5}{2(2n\gamma-n-1)}, \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}\right\}, \end{aligned} \tag{47}$$

with $\frac{1}{q} \neq \frac{1}{2}-\frac{2}{\gamma(n-1)}, \frac{1}{2}, \frac{1}{\gamma}, \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}$. The fact that $\frac{n-3}{2(n-1)} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}$ easily implies that $\gamma < \gamma_4 < \tilde{\gamma}_\infty$. The fact that $\frac{1}{2}-\frac{2}{\gamma(n-1)} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}$ implies that $\gamma < \gamma_3$. In summary, here are the final conditions on q, depending on γ and possibly on the dimension n:

(A) $1 < \gamma \leqslant \gamma_1 = 1+\frac{3}{n}$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} < \frac{1}{2}$.
(B) $\gamma_1 < \gamma \leqslant \gamma_2 = \frac{(n+1)^2}{n^2-2n+5}$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{n+5}{2(2n\gamma-n-1)}$.
(C) $\gamma_2 < \gamma < \gamma_{\text{conf}}$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}$ when $n \geqslant 5$.
When $n = 4$, we distinguish two subcases:
- $\gamma_2 < \gamma \leqslant 2$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}$,
- $2 < \gamma < \gamma_{\text{conf}}$ and $\frac{1}{2}-\frac{2}{\gamma(n-1)} < \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}$.

(D) When $n \geqslant 6$, we distinguish two subcases:
- $\gamma_{\text{conf}} \leqslant \gamma \leqslant 2$ and $\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}$,
- $2 < \gamma < \gamma_4$ and $\frac{1}{2}-\frac{2}{\gamma(n-1)} < \frac{1}{q} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}$.

When $n = 5$, we replace γ_4 by γ_3.
When $n = 4$, $\gamma_{\text{conf}} \leqslant \gamma < \gamma_3$ and $\frac{1}{2}-\frac{2}{\gamma(n-1)} < \frac{1}{q} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}$.

Let us now examine these cases separately.

Case (A). In this case, we choose successively q such that

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} < \frac{1}{2},$$

$\tilde{q}$ satisfying (46), and p, $\tilde{p}$ satisfying (45). Thus, when $1 < \gamma \leqslant \gamma_1$ and $\sigma > 0$, there exists always an admissible couple (p,q) such that all conditions (42) are satisfied and $\sigma \geqslant \frac{(n+1)}{2}(\frac{1}{2}-\frac{1}{q})$.

Case (B). In this case, we choose successively q such that

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{n+5}{2(2n\gamma-n-1)},$$

$\tilde{q}$ satisfying (46), and p, $\tilde{p}$ satisfying (45), and a correspondent $\tilde{q}$ which satisfies (46). Thus, when $\gamma_1 < \gamma \leqslant \gamma_2$ and $\sigma \geqslant \frac{n+1}{4} - \frac{(n+1)(n+5)}{4(2n\gamma-n-1)}$, there exists always an admissible couple (p,q) such that all conditions (42) are satisfied and $\sigma \geqslant \frac{(n+1)}{2}(\frac{1}{2}-\frac{1}{q})$.

Case (C). Assume first that $n \geqslant 5$. We choose successively q such that

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}, \tag{48}$$

$\tilde{q}$ satisfying (46), and p, $\tilde{p}$ satisfying (45).

Assume next that $n = 4$. If $\gamma_2 < \gamma \leqslant 2$, we choose q according to (48). If $2 < \gamma < \gamma_{\text{conf}}$, we replace (48) by

$$\frac{1}{2} - \frac{2}{\gamma(n-1)} < \frac{1}{q} \leqslant \frac{2}{(\gamma-1)(n+1)}.$$

In both cases, we can choose afterwards $\tilde{q}$, p, $\tilde{p}$ satisfying (46) and (45).

In summary, when $\gamma_2 < \gamma < \gamma_{\text{conf}}$ and $\sigma \geqslant \frac{n+1}{4} - \frac{1}{\gamma-1}$, there exists always an admissible couple (p,q) such that all conditions (42) are satisfied and $\sigma \geqslant \frac{(n+1)}{2}(\frac{1}{2}-\frac{1}{q})$.

Case (D). Assume first that $n \geqslant 6$. If $\gamma_{\text{conf}} \leqslant \gamma \leqslant 2$, we choose successively q such that

$$\frac{n-3}{2(n-1)} \leqslant \frac{1}{q} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}, \tag{49}$$

$\tilde{q}$ satisfying (46), and p, $\tilde{p}$ satisfying (45). If $2 < \gamma < \gamma_4$, (49) is replaced by

$$\frac{1}{2} - \frac{2}{\gamma(n-1)} < \frac{1}{q} < \frac{n+7-\gamma(n-1)}{2(\gamma-1)(n+1)}. \tag{50}$$

Assume next that $n = 5$. We choose again q according to (49) if $\gamma_{\text{conf}} \leqslant \gamma \leqslant 2$ and according to (50) if $2 < \gamma < \gamma_3$. In both cases, we can choose afterwards $\tilde{q}$, p, $\tilde{p}$ satisfying (46) and (45).

Assume eventually that $n = 4$. Then we choose q according to (49) and $\tilde{q}$, p, $\tilde{p}$ satisfying (46) and (45).

In summary, when $\gamma_{\text{conf}} \leqslant \gamma < \gamma_\infty$ and $\sigma > \frac{n}{2} - \frac{2}{\gamma-1}$, there exists always an admissible couple (p,q) such that all conditions (42) are satisfied and $\sigma \geqslant \frac{n+1}{2}(\frac{1}{2}-\frac{1}{q})$.

This concludes the proof of Theorem 7.2. □

Remark 7.3. Notice that, in dimension $n = 3$, the Strichartz estimates are available in the triangle T_3 without the endpoint (see Remark 6.2). By arguing as above, we prove that the NLW (38) is locally well-posed in $H^{\sigma-\frac{1}{2},\frac{1}{2}} \times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ if (see Fig. 6)

- $1 < \gamma \leqslant \gamma_1 = 2$ and $\sigma > 0$;
- $2 < \gamma < \gamma_{\text{conf}} = 3$ and $\sigma \geqslant C_2(\gamma) = 1 - \frac{1}{\gamma-1}$;
- $3 \leqslant \gamma < \gamma_3 = \frac{11+\sqrt{73}}{6}$ and $\sigma > C_3(\gamma) = \frac{3}{2} - \frac{2}{\gamma-1}$.

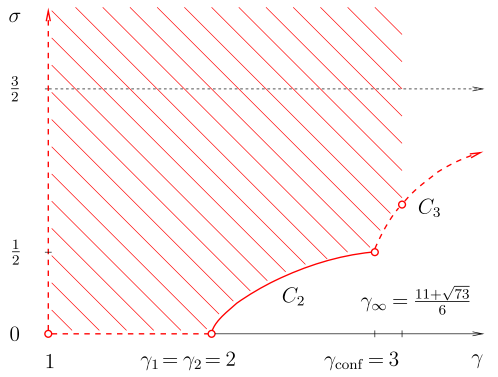

Fig. 6. Regularity in dimension $n = 3$.

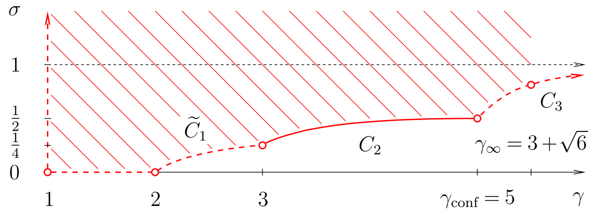

Fig. 7. Regularity in dimension $n = 2$.

Remark 7.4. In dimension $n = 2$, the Strichartz estimates are available in the region T_2 (see Remark 6.5). By following again the same line of the above proof, we obtain that the NLW (38) is locally well-posed in $H^{\sigma-\frac{1}{2},\frac{1}{2}} \times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ if (see Fig. 7)

- $1 < \gamma \leqslant 2$ and $\sigma > 0$;
- $2 \leqslant \gamma \leqslant 3$ and $\sigma > \widetilde{C}_1(\gamma) = \frac{3}{4} - \frac{3}{2}\frac{1}{\gamma}$;
- $3 < \gamma < \gamma_{conf} = 5$ and $\sigma \geqslant C_2(\gamma) = \frac{3}{4} - \frac{1}{\gamma-1}$;
- $5 \leqslant \gamma < \gamma_3 = 3 + \sqrt{6}$ and $\sigma > C_3(\gamma) = 1 - \frac{2}{\gamma-1}$.

Appendix A

In this appendix we collect some lemmata in Fourier analysis on $\mathbb{R}$ which are used for the kernel analysis in Section 4 and in Appendix C.

Lemma A.1. *Let a be a compactly supported homogeneous symbol on $\mathbb{R}$ of order $d > -1$. In other words, a is a smooth function on $\mathbb{R}^*$, whose support is bounded in $\mathbb{R}$ and which has the following behavior at the origin*:

$$\sup_{\lambda \in \mathbb{R}^*} |\lambda|^{\ell-d} \left|\partial_\lambda^\ell a(\lambda)\right| < +\infty \quad \forall \ell \in \mathbb{N}.$$

Then its Fourier transform

$$k(x) = \int_0^{+\infty} d\lambda\, a(\lambda) e^{i\lambda x}$$

is a smooth function on $\mathbb{R}$, *with the following behavior at infinity*:

$$k(x) = \mathrm{O}\big(|x|^{-d-1}\big) \quad \textit{as } |x| \to \infty.$$

More precisely, let N *be the smallest integer* $> d+1$. *Then* $\exists C \geqslant 0$, $\forall x \in \mathbb{R}^*$,

$$\big|k(x)\big| \leqslant C|x|^{-d-1} \sum_{\ell=0}^{N} \sup_{\lambda\in\mathbb{R}^*} \big(1+|\lambda|\big)^{\ell-d} \big|\partial_\lambda^\ell a(\lambda)\big|.$$

Proof. Let us split up

$$a(\lambda) = \sum_{j=-\infty}^{+\infty} \chi\big(2^{-j}\lambda\big) a(\lambda)$$

and $k = \sum_{j=-\infty}^{+\infty} k_j$ accordingly, using a homogeneous dyadic partition of unity

$$1 = \sum_{j=-\infty}^{+\infty} \chi\big(2^{-j}\cdot\big)$$

on $(0,\infty)$. Notice that a_j hence k_j vanishes for j large, since a is compactly supported. By the Leibniz formula, we obtain, for every $\ell \in \mathbb{N}$,

$$\begin{aligned} |x|^\ell \big|k_j(x)\big| &\leqslant \int_{|\lambda|\asymp 2^j} d\lambda\, \big|\partial_\lambda^\ell\big\{\chi\big(2^{-j}\lambda\big)a(\lambda)\big\}\big| \\ &\lesssim \sum_{k=0}^{\ell} 2^{-kj} \int_{|\lambda|\asymp 2^j} d\lambda\, |\lambda|^{d-\ell+k} \lesssim 2^{j(1+d-\ell)}. \end{aligned}$$

Let $N \in \mathbb{N}^*$ such that $N > d+1$. Then

$$\begin{aligned} \big|k(x)\big| &\leqslant \sum_{2^j \leqslant |x|^{-1}} \big|k_j(x)\big| + \sum_{2^j \geqslant |x|^{-1}} \big|k_j(x)\big| \\ &\lesssim \sum_{2^j \leqslant |x|^{-1}} 2^{j(1+d)} + |x|^{-N} \sum_{2^j \geqslant |x|^{-1}} 2^{j(1+d-N)} \lesssim |x|^{-d-1}. \qquad \square \end{aligned}$$

Lemma A.2. *Let* a *be an inhomogeneous symbol on* $\mathbb{R}$ *of order* $d \in \mathbb{R}$. *In other words,* a *is a smooth function on* $\mathbb{R}$ *such that*

$$\sup_{\lambda\in\mathbb{R}} \big(1+|\lambda|\big)^{\ell-d} \big|\partial_\lambda^\ell a(\lambda)\big| < +\infty \quad \forall \ell \in \mathbb{N}.$$

Then its Fourier transform

$$k(x) = \int_{-\infty}^{+\infty} d\lambda\, a(\lambda) e^{i\lambda x}$$

is a smooth function on $\mathbb{R}^*$, *which has the following asymptotic behaviors*:

(i) *At infinity*, $k(x)=\mathrm{O}(|x|^{-\infty})$. *More precisely, for every* $N>d+1$, *there exists* $C_N\geqslant 0$ *such that, for every* $x\in\mathbb{R}^*$,

$$\bigl|k(x)\bigr|\leqslant C_N|x|^{-N}\sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^{N-d}\bigl|\partial_\lambda^N a(\lambda)\bigr|.$$

(ii) *At the origin*,

$$k(x)=\begin{cases}\mathrm{O}(1) & \text{if } d<-1,\\ \mathrm{O}(\log\frac{1}{|x|}) & \text{if } d=-1,\\ \mathrm{O}(|x|^{-d-1}) & \text{if } d>-1.\end{cases}$$

More precisely:

- *If* $d<-1$, *then there exists* $C\geqslant 0$ *such that, for every* $x\in\mathbb{R}$,

$$\bigl|k(x)\bigr|\leqslant C\sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^{-d}\bigl|a(\lambda)\bigr|.$$

- *If* $d=-1$, *then there exists* $C\geqslant 0$ *such that, for every* $0<|x|<\frac{1}{2}$,

$$\bigl|k(x)\bigr|\leqslant C\log\frac{1}{|x|}\Bigl\{\sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)\bigl|a(\lambda)\bigr|+\sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^2\bigl|a'(\lambda)\bigr|\Bigr\}.$$

- *If* $d>-1$, *let* N *be the smallest integer* $>d+1$. *Then there exists* $C\geqslant 0$ *such that, for every* $0<|x|<1$,

$$\bigl|k(x)\bigr|\leqslant C|x|^{-d-1}\sum_{\ell=0}^{N}\sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^{\ell-d}\bigl|\partial_\lambda^\ell a(\lambda)\bigr|.$$

(iii) *Similar estimates hold for the derivatives*

$$\partial_x^\ell k(x)=\int_{-\infty}^{+\infty}d\lambda\,(i\lambda)^\ell a(\lambda)e^{i\lambda x}$$

which correspond to symbols $a_\ell(\lambda)=(i\lambda)^\ell a(\lambda)$ *of order* $d+\ell$.

Proof. (i) Since k is the Fourier transform of a, then $x^N k(x)$ is the Fourier transform of $(i\partial_\lambda)^N a(\lambda)$, which is $\mathrm{O}((1+|\lambda|)^{d-N})$, hence integrable when $N>d+1$.

(ii) If $d<-1$, we simply estimate:

$$\bigl|k(x)\bigr|\leqslant\int_{-\infty}^{+\infty}d\lambda\,\bigl|a(\lambda)\bigr|\leqslant\sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^{-d}\bigl|a(\lambda)\bigr|\int_{-\infty}^{+\infty}d\lambda\,\bigl(1+|\lambda|\bigr)^d.$$

If $d \geqslant -1$, we split up

$$k(x) = \underbrace{\int_{-\infty}^{+\infty} d\lambda\, \chi_0\big(|x|\lambda\big)a(\lambda)e^{i\lambda x}}_{k_0(x)} + \underbrace{\int_{-\infty}^{+\infty} d\lambda\, \chi_\infty\big(|x|\lambda\big)a(\lambda)e^{i\lambda x}}_{k_\infty(x)},$$

using smooth cut-off functions χ_0 and χ_∞ on $[0,+\infty)$ such that $1=\chi_0+\chi_\infty$, $\chi_0=1$ on $[0,1]$ and $\chi_\infty=1$ on $[2,+\infty)$. The first integral is estimated as above:

$$\begin{aligned}
\big|k_0(x)\big| &\leqslant \int_{|\lambda|\leqslant 2|x|^{-1}} d\lambda\, \big|a(\lambda)\big| \\
&\leqslant 2\sup_{\lambda\in\mathbb{R}}\big(1+|\lambda|\big)^{-d}\big|a(\lambda)\big| \int_0^{2|x|^{-1}} d\lambda\, (1+\lambda)^d \\
&\lesssim \sup_{\lambda\in\mathbb{R}}\big(1+|\lambda|\big)^{-d}\big|a(\lambda)\big| \begin{cases} \log\frac{1}{|x|} & \text{if } d=-1, \\ |x|^{-d-1} & \text{if } d>-1. \end{cases}
\end{aligned}$$

After N integrations by parts, the second integral becomes

$$k_\infty(x) = \left(\frac{i}{x}\right)^N \int_{-\infty}^{+\infty} d\lambda \left(\frac{\partial}{\partial\lambda}\right)^N \big\{\chi_\infty\big(|x|\lambda\big)a(\lambda)\big\}e^{i\lambda x}.$$

Hence

$$\begin{aligned}
\big|k_\infty(x)\big| &\lesssim |x|^{-N} \int_{|\lambda|\geqslant|x|^{-1}} d\lambda\, \big|\partial_\lambda^N a(\lambda)\big| + \sum_{0<\ell<N} |x|^{-\ell} \int_{|x|^{-1}\leqslant|\lambda|\leqslant 2|x|^{-1}} d\lambda\, \big|\partial_\lambda^\ell a(\lambda)\big| \\
&\lesssim |x|^{-d-1} \sum_{\ell=1}^{N-1} \sup_{\lambda\in\mathbb{R}}\big(1+|\lambda|\big)^{\ell-d}\big|a(\lambda)\big|.
\end{aligned}$$

This concludes the proof of (ii). The proof of (iii) is similar and we omit the details. $\square$

Lemma A.3. *Assume that*

$$a(\lambda) = \zeta\,\chi_\infty(\lambda)\lambda^{-m-1-i\zeta} + b(\lambda)$$

where $m\in\mathbb{N}$, $\zeta\in\mathbb{R}$, and b is a symbol of order $d<-m-1$. Then

$$\partial_x^m k(x) = \int_{-\infty}^{+\infty} d\lambda\, a(\lambda)(i\lambda)^m e^{i\lambda x}$$

is a bounded function at the origin. More precisely, there exists $C \geqslant 0$ such that, for every $0 < |x| < \frac{1}{2}$,

$$\left|\partial_x^m k(x)\right| \leqslant C\left\{1 + \zeta^2 + \sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^{-d}\left|b(\lambda)\right|\right\}.$$

Proof. Let us split up

$$\partial_x^m k(x) = \overbrace{i^m \int_2^{\frac{1}{|x|}} d\lambda\, \zeta \lambda^{-1-i\zeta} e^{i\lambda x}}^{k_1(x)} + \overbrace{i^m \zeta \int_{\frac{1}{|x|}}^{+\infty} d\lambda\, \lambda^{-1-i\zeta} e^{i\lambda x}}^{k_2(x)} + \underbrace{i^m \zeta \int_1^2 d\lambda\, \chi_\infty(\lambda) \lambda^{-1-i\zeta} e^{i\lambda x}}_{k_3(x)} + \underbrace{i^m \int_{-\infty}^{+\infty} d\lambda\, \lambda^m b(\lambda) e^{i\lambda x}}_{k_4(x)}.$$

The first two terms are estimated by integrations by parts. Specifically,

$$k_1(x) = i\lambda^{-i\zeta} e^{i\lambda x}\Big|_{\lambda=2}^{\lambda=\frac{1}{|x|}} + x \int_2^{\frac{1}{|x|}} d\lambda\, \lambda^{-i\zeta} e^{i\lambda x},$$

with $|\lambda^{-i\zeta} e^{i\lambda x}|_{\lambda=2}^{\lambda=\frac{1}{|x|}}| \leqslant 2$ and $|\int_2^{\frac{1}{|x|}} d\lambda\, \lambda^{-i\zeta} e^{i\lambda x}| \leqslant \frac{1}{|x|}$, while

$$k_2(x) = -\frac{i}{x}\lambda^{-1-i\zeta} e^{i\lambda x}\Big|_{\lambda=\frac{1}{|x|}}^{\lambda=+\infty} + \frac{\zeta - i}{x} \int_{\frac{1}{|x|}}^{+\infty} d\lambda\, \lambda^{-2-i\zeta} e^{i\lambda x},$$

with $|\lambda^{-1-i\zeta} e^{i\lambda x}|_{\lambda=\frac{1}{|x|}}^{+\infty}| \leqslant |x|$ and $|\int_{\frac{1}{|x|}}^{+\infty} d\lambda\, \lambda^{-2-i\zeta} e^{i\lambda x}| \leqslant |x|$. The last two terms are easy to estimate. Obviously $|k_3(x)| \leqslant 1$, while

$$\left|k_4(x)\right| \leqslant \sup_{\lambda\in\mathbb{R}}\bigl(1+|\lambda|\bigr)^{-d}\left|b(\lambda)\right| \underbrace{\int_{-\infty}^{+\infty} d\lambda\, \bigl(1+|\lambda|\bigr)^{m+d}}_{<+\infty}.$$

We conclude by summing up these four estimates. □

Appendix B

In this appendix we collect some properties of the Riesz distributions. We refer to [14, Ch. 1, §3 & Ch. 2, §2] or [20, Ch. III, §3.2] for more details. The Riesz distribution R_z^+ is defined by

$$\left\langle R_z^+, \varphi\right\rangle = \frac{1}{\Gamma(z)} \int_0^{+\infty} d\lambda\, \lambda^{z-1} \varphi(\lambda) \tag{51}$$

when $\operatorname{Re} z > 0$. It extends to a holomorphic family $\{R_z^+\}_{z\in\mathbb{C}}$ of tempered distributions on $\mathbb{R}$ which satisfy the following properties:

(i) $\lambda R_z^+ = z R_{z+1}^+ \ \forall z \in \mathbb{C}$,
(ii) $(\frac{d}{d\lambda}) R_z^+ = R_{z-1}^+ \ \forall z \in \mathbb{C}$,
(iii) $R_0^+ = \delta_0$ and more generally $R_{-m}^+ = (\frac{d}{d\lambda})^m \delta_0 \ \forall m \in \mathbb{N}$,
(iv) $R_{z+z'}^+ = R_z^+ * R_{z'}^+ \ \forall z, z' \in \mathbb{C}$.

Hence

$$\langle R_z^+, \varphi\rangle = \left\langle \left(\frac{d}{d\lambda}\right)^m R_{z+m}^+, \varphi \right\rangle = \frac{(-1)^m}{\Gamma(z+m)} \int_0^{+\infty} d\lambda\, \lambda^{z+m-1} \left(\frac{d}{d\lambda}\right)^m \varphi(\lambda)$$

when $\operatorname{Re} z > -m$. The Riesz distribution $R_z^- = (R_z^+)^\vee$ is defined similarly. Their Fourier transforms are given by

(v) $\mathcal{F} R_z^\pm = e^{\pm i \frac{\pi}{2} z} (x \pm i0)^{-z} \ \forall z \in \mathbb{C}$,

where

$$\langle (x \pm i0)^z, \varphi \rangle = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} dx\, (x \pm i\varepsilon)^z \varphi(x)$$

when $\operatorname{Re} z > -1$ and

$$(x \pm i0)^z = \Gamma(z+1) \left\{ R_{z+1}^+ + e^{\pm i\pi z} R_{z+1}^- \right\}$$

in general (notice that there are actually no singularities in the last expression).

Appendix C

In this appendix we prove the local kernel estimates

$$\left|\widetilde{w}_t^\infty(r)\right| \lesssim \begin{cases} |t|^{-\frac{n-1}{2}} & \text{if } n \geqslant 3, \\ |t|^{-\frac{1}{2}} (1 - \log|t|) & \text{if } n = 2 \end{cases} \tag{52}$$

stated in Theorem 4.2(i)(a) under the assumptions $0 < |t| \leqslant 2$, $0 \leqslant r \leqslant 3$ and $\operatorname{Re} \sigma = \frac{n+1}{2}$. By symmetry, we may assume again that $t > 0$.

- *Case* 1: Assume that $r \leqslant \frac{t}{2}$.

By using the first integral representation of the spherical functions in (6), we obtain

$$\widetilde{w}_t^\infty(r) = \frac{e^{\sigma^2}}{\Gamma(-i \operatorname{Im} \sigma)} \int_K dk\, e^{-\rho \mathrm{H}(a_{-r}k)} \int_1^\infty d\lambda\, \chi_\infty(\lambda) a(\lambda) e^{i\lambda\{t - \mathrm{H}(a_{-r}k)\}}, \tag{53}$$

where

$$a(\lambda)=\left|\mathbf{c}(\lambda)\right|^{-2}\lambda^{-\tau}\left(\lambda^2+\tilde{\rho}^2\right)^{\frac{\tau}{2}-\frac{\sigma}{2}}.$$

According to Lemma A.2 in Appendix A, since $\chi_\infty a$ is a symbol of order $\frac{n-3}{2}$ and

$$\left|t-\mathrm{H}(a_r k)\right|\geqslant t-r\geqslant\frac{t}{2},$$

the inner integral in (53) is

$$\mathrm{O}\left(|\sigma|^N\left|t-\mathrm{H}(a_{-r}k)\right|^{-\frac{n-1}{2}}\right)=\mathrm{O}\left(|\sigma|^N t^{-\frac{n-1}{2}}\right),$$

where N is the smallest integer $>\frac{n-1}{2}$. Hence

$$\left|\widetilde{w}_t^{\infty}(r)\right|\lesssim t^{-\frac{n-1}{2}}.$$

- *Case* 2: Assume that $\frac{t}{2}<r<t$.

By using the third integral formula for spherical functions in (6), we are lead to estimate the expression

$$(\sinh r)^{2-n}\int_{-r}^{+r}du\,(\cosh r-\cosh u)^{\frac{n-3}{2}}\int_1^{\infty}d\lambda\,\chi_\infty(\lambda)a(\lambda)e^{i\lambda(t-u)}. \tag{54}$$

Let us expand

$$\left|\mathbf{c}(\lambda)\right|^{-2}\lambda^{-\tau}\left(\lambda^2+\tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}}=\text{const.}\,\lambda^{\frac{n-3}{2}-i\,\mathrm{Im}\,\sigma}+\mathrm{O}\left(|\sigma|\lambda^{\frac{n-5}{2}}\right),$$

as $\lambda\to+\infty$, and

$$a(\lambda)=\overbrace{\text{const.}\,\lambda^{\frac{n-3}{2}-i\,\mathrm{Im}\,\sigma}}^{\widetilde{a}(\lambda)}+b(\lambda)$$

accordingly. Since $\chi_\infty b$ is a symbol of order $\frac{n-5}{2}$, its contribution to (54) can be estimated by

$$|\sigma|^N(\sinh r)^{2-n}\int_{-r}^{+r}du\,(\cosh r-\cosh u)^{\frac{n-3}{2}}(t-u)^{-\frac{n-3}{2}}. \tag{55}$$

Here we have applied again Lemma A.2 and N is the smallest integer $>\frac{n-1}{2}$. By using

$$\begin{cases}\sinh r\asymp r,\\ \cosh r-\cosh u=2\sinh\dfrac{r-u}{2}\sinh\dfrac{r+u}{2}\asymp(r-u)(r+u),\\ t-u\geqslant r-u,\end{cases}$$

we end up with the estimate

$$|\sigma|^N r^{2-n} \int_{-r}^{+r} du\,(r+u)^{\frac{n-3}{2}} \lesssim |\sigma|^N r^{-\frac{n-3}{2}} \asymp |\sigma|^N t^{-\frac{n-3}{2}}.$$

Notice that the previous computations are valid in dimension $n>3$. In dimension $n=3$, the last estimate becomes $|\sigma|^2(1-\log t)$ while, in dimension $n=2$, (55) is replaced by

$$|\sigma| \int_{-r}^{+r} \frac{du}{\sqrt{\cosh r-\cosh u}} \asymp |\sigma| \int_{-r}^{+r} \frac{du}{\sqrt{r^2-u^2}} \asymp |\sigma|.$$

Similarly

$$\int_0^2 d\lambda\, \chi_0(\lambda)\widetilde{a}(\lambda) e^{i\lambda(t-u)}$$

yields a bounded contribution to (54). Let us eventually analyze the remaining contribution of

$$\int_0^{+\infty} d\lambda\, \lambda^{\frac{n-3}{2}-i\operatorname{Im}\sigma} e^{i\lambda(t-u)}, \tag{56}$$

which is a classical distribution. According to the properties of the Riesz distributions (51) in Appendix B, we have indeed

$$\int_0^{+\infty} d\lambda\, \lambda^{\frac{n-3}{2}-i\operatorname{Im}\sigma} e^{i\lambda(t-u)} = \Gamma\left(\frac{n-1}{2}-i\operatorname{Im}\sigma\right) e^{\frac{\pi}{2}\operatorname{Im}\sigma+i\frac{\pi}{4}(n-1)}(t-u)^{-\frac{n-1}{2}+i\operatorname{Im}\sigma}$$

and it remains for us to estimate the expression

$$\frac{\Gamma(\frac{n-1}{2}-i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)} (\sinh r)^{2-n} \int_{-r}^{+r} du\,(\cosh r-\cosh u)^{\frac{n-3}{2}}(t-u)^{-\frac{n-1}{2}+i\operatorname{Im}\sigma}. \tag{57}$$

In order to do so, we discuss separately the odd and even-dimensional cases.

- *Subcase* 2(a): Assume that $n=2m+1$ is odd.

After $m-1$ integrations by parts, (57) becomes

$$(-i\operatorname{Im}\sigma)(\sinh r)^{1-2m} \int_{-r}^{+r} du\,(t-u)^{-1+i\operatorname{Im}\sigma} \sum_{j=1}^{m-1} a_j(u)(\cosh r-\cosh u)^{m-j-1},$$

where $a_j(u)$ is a linear combination of monomials $(\sinh u)^{j'}(\cosh u)^{j''}$ with $j',j''\geqslant 0$, $j'+j''=j$ and $j'\geqslant 2j+1-m$. In particular $a_{m-1}(u)=(m-1)!(\sinh u)^{m-1}$. After one more integration by parts, we get

$$(m-1)!(\sinh r)^{-m}\big\{(t-r)^{i\operatorname{Im}\sigma}+(-1)^m(t+r)^{i\operatorname{Im}\sigma}\big\}$$
$$+(\sinh r)^{1-2m}\int_{-r}^{+r}du\,(t-u)^{i\operatorname{Im}\sigma}\sum_{j=1}^{m-1}\widetilde{a}_j(u)(\cosh r-\cosh u)^{m-j-1},$$

where $\widetilde{a}_j(u)=\mathrm{O}(r^{\max\{0,2j-m\}})$ and $\cosh r-\cosh u=2\sinh\frac{r-u}{2}\sinh\frac{r+u}{2}=\mathrm{O}(r^2)$, hence the last sum is $\mathrm{O}(r^{m-2})$ and the last integral is $\mathrm{O}(r^{m-1})$. Notice that these terms vanish when $m=1$. Thus (57) is $\mathrm{O}(r^{-m})=\mathrm{O}(t^{-\frac{n-1}{2}})$, when $n=2m+1$ is odd.

- *Subcase* 2(b): Assume that $n=2m$ is even.

After $m-1$ integrations by parts, (57) becomes this time

$$\frac{\Gamma(\frac{1}{2}-i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)}(\sinh r)^{2-2m}\int_{-r}^{+r}du\,(t-u)^{-\frac{1}{2}+i\operatorname{Im}\sigma}\sum_{j=1}^{m-1}a_j(u)(\cosh r-\cosh u)^{m-j-\frac{3}{2}},\tag{58}$$

where $a_{m-1}(u)=\frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}}(\sinh u)^{m-1}$ and the other $a_j(u)$ are as before. Since

$$\frac{\Gamma(\frac{1}{2}-i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)}=\mathrm{O}\big(|\sigma|^{\frac{1}{2}}\big),$$
$$a_j(u)(\cosh r-\cosh u)^{m-j-\frac{3}{2}}=\mathrm{O}\big(r^{m-2}\big)\quad\forall 1\leqslant j\leqslant m-2,$$
$$\int_{-r}^{+r}du\,(t-u)^{-\frac{1}{2}}\asymp\frac{r}{\sqrt{t}}\asymp\sqrt{t},$$

the $m-2$ first terms in (58) are $\mathrm{O}(|\sigma|^{\frac{1}{2}}t^{-\frac{n-1}{2}})$. Let us turn to the last term

$$\frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)}(\sinh r)^{2-2m}$$
$$\times\int_{-r}^{+r}du\,(t-u)^{-\frac{1}{2}+i\operatorname{Im}\sigma}(\sinh u)^{m-1}(\cosh r-\cosh u)^{-\frac{1}{2}},\tag{59}$$

which is obtained by taking $j=m-1$ in (58). Let us split the integral in (59) as follows:

$$\int_{-r}^{r}=\int_{-r}^{0}+\int_{0}^{2r-t}+\int_{2r-t}^{r}.\tag{60}$$

Notice that our current assumption $\frac{t}{2}<r<t$ implies that $0<2r-t<r$. Since

$$\cosh r-\cosh u=2\sinh\frac{r-u}{2}\sinh\frac{r+u}{2}\asymp(r-u)(r+u),$$

the contribution to (59) of the first integral in (60) can be estimated by

$$|\sigma|^{\frac{1}{2}} t^{-\frac{1}{2}} r^{\frac{1}{2}-m} \int_{-r}^{0} \frac{du}{\sqrt{r+u}} \asymp |\sigma|^{\frac{1}{2}} t^{-\frac{n-1}{2}}$$

and the contribution to (59) of the last integral in (60) by

$$|\sigma|^{\frac{1}{2}} (t-r)^{-\frac{1}{2}} r^{\frac{1}{2}-m} \int_{2r-t}^{r} \frac{du}{\sqrt{r-u}} \asymp |\sigma|^{\frac{1}{2}} t^{-\frac{n-1}{2}}.$$

We handle the remaining integral by performing the change of variables

$$v = \frac{t-r}{t-u} \quad \Longleftrightarrow \quad u = t - \frac{t-r}{v}$$

and by integrating by parts

$$\begin{aligned}
&\frac{\Gamma(\frac{1}{2} - i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)} \int_{0}^{2r-t} du\, (t-u)^{-\frac{1}{2}+i\operatorname{Im}\sigma} (r-u)^{-\frac{1}{2}} \overbrace{\left(\frac{\sinh\frac{r-u}{2}}{\frac{r-u}{2}}\right)^{-\frac{1}{2}} \left(\sinh\frac{r+u}{2}\right)^{-\frac{1}{2}} (\sinh u)^{m-1}}^{A(r,u)} \\
&= \frac{\Gamma(\frac{1}{2} - i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)} (t-r)^{i\operatorname{Im}\sigma} \int_{1-\frac{r}{t}}^{\frac{1}{2}} dv\, v^{-1-i\operatorname{Im}\sigma} (1-v)^{-\frac{1}{2}} A\left(r, t - \frac{t-r}{v}\right) \\
&= \frac{\Gamma(\frac{1}{2} - i\operatorname{Im}\sigma)}{\Gamma(1-i\operatorname{Im}\sigma)} (t-r)^{i\operatorname{Im}\sigma} v^{-i\operatorname{Im}\sigma} (1-v)^{-\frac{1}{2}} A\left(r, t - \frac{t-r}{v}\right)\Bigg|_{v=1-\frac{r}{t}}^{v=\frac{1}{2}} \\
&\quad - \frac{\Gamma(\frac{1}{2} - i\operatorname{Im}\sigma)}{2\Gamma(1-i\operatorname{Im}\sigma)} (t-r)^{i\operatorname{Im}\sigma} \int_{1-\frac{r}{t}}^{\frac{1}{2}} dv\, v^{-i\operatorname{Im}\sigma} (1-v)^{-\frac{3}{2}} A\left(r, t - \frac{t-r}{v}\right) \\
&\quad - \frac{\Gamma(\frac{1}{2} - i\operatorname{Im}\sigma)}{\Gamma(1-i\operatorname{Im}\sigma)} (t-r)^{1+i\operatorname{Im}\sigma} \int_{1-\frac{r}{t}}^{\frac{1}{2}} dv\, v^{-2-i\operatorname{Im}\sigma} (1-v)^{-\frac{1}{2}} \partial_2 A\left(r, t - \frac{t-r}{v}\right).
\end{aligned}$$

All resulting expressions are $\mathrm{O}(|\sigma|^{-\frac{1}{2}} t^{m-\frac{3}{2}})$, since

$$\frac{\Gamma(\frac{1}{2} - i\operatorname{Im}\sigma)}{\Gamma(1-i\operatorname{Im}\sigma)} = \mathrm{O}\big(|\sigma|^{-\frac{1}{2}}\big), \qquad A(r,u) = \mathrm{O}\big(t^{m-\frac{3}{2}}\big) \quad \text{and} \quad \partial_2 A(r,u) = \mathrm{O}\big(t^{m-\frac{5}{2}}\big).$$

Thus (59) and hence (58), (57) are $\mathrm{O}(|\sigma|^{\frac{1}{2}} t^{-\frac{n-1}{2}})$.

As a conclusion, we have obtained the following estimate in all dimensions $n \geqslant 2$:

$$\big|\widetilde{w}_t^{\infty}(r)\big| \lesssim t^{-\frac{n-1}{2}} \quad \text{when } \frac{t}{2} < r < t.$$

- *Case* 3: Assume that $r > t$.

In this case we estimate $\widetilde{w}_t(r)$ using the inverse Abel transform. More precisely, we apply the inversion formulae (13) and (14) to the Euclidean Fourier transform

$$\widetilde{g}_t^{\infty}(r) = \frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)} \int_1^{+\infty} d\lambda\, \chi_\infty(\lambda) \left|\mathbf{c}(\lambda)\right|^{-2} \lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} e^{it\lambda} \cos \lambda r.$$

- *Subcase* 3(a): Assume that $n = 2m + 1$ is odd.

Then, up to a multiplicative constant,

$$\widetilde{w}_t^{\infty}(r) = \left(\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^m \widetilde{g}_t^{\infty}(r).$$

Let us expand

$$\Bigl(\overbrace{\frac{1}{\sinh r}}^{\frac{r}{\sinh r}\,\frac{1}{r}} \frac{\partial}{\partial r}\Bigr)^m = \sum_{\ell=1}^{m} \alpha_\ell^0(r) \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^\ell \tag{61}$$

and furthermore

$$\left(\frac{1}{r} \frac{\partial}{\partial r}\right)^\ell = \sum_{k=1}^{\ell} \beta_{\ell,k} r^{k-2\ell} \left(\frac{\partial}{\partial r}\right)^k. \tag{62}$$

The coefficients $\beta_{\ell,k}$ in (62) are constants, while the coefficients $\alpha_\ell^0(r)$ in (61) are smooth functions on $\mathbb{R}$, which are linear combinations of products

$$\left(\frac{r}{\sinh r}\right) \times \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^{\ell_2} \left(\frac{r}{\sinh r}\right) \times \cdots \times \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^{\ell_m} \left(\frac{r}{\sinh r}\right)$$

with $\ell_2 + \cdots + \ell_m = m - \ell$. Consider first

$$\frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)} \int_1^{\frac{6}{r}} d\lambda\, \chi_\infty(\lambda) \lambda^{-\tau} \left(\lambda^2 + \tilde{\rho}^2\right)^{\frac{\tau-\sigma}{2}} e^{it\lambda} \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^\ell \cos \lambda r. \tag{63}$$

Since $\chi_\infty(\lambda)\lambda^{-\tau}(\lambda^2 + \tilde{\rho}^2)^{\frac{\tau-\sigma}{2}} e^{it\lambda} = \mathrm{O}(\lambda^{-m-1})$ according to the assumption $\operatorname{Re}\sigma = m + 1$ and $(\frac{1}{r}\frac{\partial}{\partial r})^\ell \cos \lambda r = \mathrm{O}(\lambda^{2\ell})$ by Taylor's formula, the expression (63) is

$$\begin{cases} \mathrm{O}(1) & \text{if } 1 \leqslant \ell < \dfrac{m}{2}, \\ \mathrm{O}\left(\log \dfrac{1}{r}\right) & \text{if } \ell = \dfrac{m}{2}, \\ \mathrm{O}\left(r^{m-2\ell}\right) & \text{if } \dfrac{m}{2} < \ell \leqslant m, \end{cases}$$

hence $\mathrm{O}(r^{-m})$ in all cases. Consider next

$$\frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)}\int_{\frac{6}{r}}^{+\infty} d\lambda\,\lambda^{-\tau}\big(\lambda^2+\tilde{\rho}^2\big)^{\frac{\tau-\sigma}{2}} r^{k-2\ell}\left(\frac{\partial}{\partial r}\right)^k e^{i(t\pm r)\lambda}. \tag{64}$$

Since $(\frac{\partial}{\partial r})^k e^{i(t\pm r)\lambda} = (\pm i\lambda)^k e^{i(t\pm r)\lambda}$ and

$$\lambda^{-\tau}\big(\lambda^2+\tilde{\rho}^2\big)^{\frac{\tau-\sigma}{2}}(\pm i\lambda)^k e^{i(t\pm r)\lambda} = \mathrm{O}\big(\lambda^{k-m-1}\big),$$

the expression (64) is easily seen to be $\mathrm{O}(r^{m-2\ell})$ as long as $k<m$. For the remaining case, where $k=\ell=m$, let us expand

$$\lambda^{-\tau}\big(\lambda^2+\tilde{\rho}^2\big)^{\frac{\tau-\sigma}{2}}\lambda^m = \lambda^{-1-i\operatorname{Im}\sigma}\left(1+\frac{\tilde{\rho}^2}{\lambda^2}\right)^{\frac{\tau-\sigma}{2}} = \lambda^{-1-i\operatorname{Im}\sigma} + \mathrm{O}\big(|\sigma|\lambda^{-3}\big)$$

and split

$$\int_{\frac{6}{r}}^{+\infty} = \int_{\frac{6}{r}}^{\frac{6}{r}+\frac{1}{r\pm t}} + \int_{\frac{6}{r}+\frac{1}{r\pm t}}^{+\infty}$$

in (64). On the one hand, the resulting integrals

$$I_\pm = \frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)}\int_{\frac{6}{r}}^{\frac{6}{r}+\frac{1}{r\pm t}} d\lambda\,\lambda^{-1-i\operatorname{Im}\sigma} e^{i(t\pm r)\lambda} \tag{65}$$

and

$$II_\pm = \frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)}\int_{\frac{6}{r}+\frac{1}{r\pm t}}^{+\infty} d\lambda\,\lambda^{-1-i\operatorname{Im}\sigma} e^{i(t\pm r)\lambda} \tag{66}$$

are uniformly bounded. This is proved by integrations by parts:

$$\begin{aligned} I_\pm &= \frac{e^{\sigma^2}}{\Gamma(1-i\operatorname{Im}\sigma)} \overbrace{\lambda^{-i\operatorname{Im}\sigma} e^{i(t\pm r)\lambda}\Big|_{\lambda=\frac{6}{r}}^{\lambda=\frac{6}{r}+\frac{1}{r\pm t}}}^{\mathrm{O}(1)} \\ &\quad \mp i\frac{e^{\sigma^2}}{\Gamma(1-i\operatorname{Im}\sigma)}(r\pm t) \underbrace{\int_{\frac{1}{r}}^{\frac{1}{r}+\frac{1}{r\pm t}} d\lambda\,\lambda^{-i\operatorname{Im}\sigma} e^{i(t\pm r)\lambda}}_{\mathrm{O}(\frac{1}{r\pm t})} = \mathrm{O}(1), \end{aligned}$$

while

$$II_{\pm} = \mp i \frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)} \frac{1}{r \pm t} \overbrace{\lambda^{-1-i\operatorname{Im}\sigma} e^{i(t\pm r)\lambda} \Big|_{\lambda=\frac{6}{r}+\frac{1}{r\pm t}}^{\lambda=+\infty}}^{\mathrm{O}(r\pm t)}$$
$$\mp i \frac{e^{\sigma^2}(1+i\operatorname{Im}\sigma)}{\Gamma(-i\operatorname{Im}\sigma)} \frac{1}{r \pm t} \underbrace{\int_{\frac{6}{r}+\frac{1}{r\pm t}}^{+\infty} d\lambda\, \lambda^{-2-i\operatorname{Im}\sigma} e^{i(t\pm r)\lambda}}_{\mathrm{O}(r\pm t)} = \mathrm{O}(1).$$

Hence the contributions of (65) and (66) to (64) are $\mathrm{O}(r^{-m})$. On the other hand, the remainder's contribution to (64) is obviously $\mathrm{O}(r^{2-m})$. As a conclusion,

$$\left|\widetilde{w}_t^{\infty}(r)\right| \lesssim r^{-m} \lesssim t^{-\frac{n-1}{2}}$$

when $n = 2m+1$ is odd.

- *Subcase* 3(b): Assume that $n = 2m$ is even $\geqslant 4$.

Then, up to a multiplicative constant,

$$\widetilde{w}_t^{\infty}(r) = \frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)} \int_r^{+\infty} ds\, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^m \widetilde{g}_t^{\infty}(s). \tag{67}$$

Let us split

$$\int_r^{+\infty} = \int_r^{6} + \int_6^{+\infty}. \tag{68}$$

The following estimate is obtained by resuming the proof of Theorem 4.2(i)(b) in the odd-dimensional case:

$$\left|\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^m \widetilde{g}_t^{\infty}(s)\right| \lesssim e^{-ms} \quad \forall s \geqslant 6.$$

Since

$$\int_6^{+\infty} ds\, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} e^{-ms} \lesssim \int_0^{+\infty} \frac{du}{\sqrt{\sinh u}} < +\infty,$$

the contribution to (67) of the second integral in (68) is uniformly bounded. Thus we are left with the contribution of the first integral, which is a purely local estimate.

Lemma C.1. *Let m be an integer $\geqslant 2$ and let $\lambda \geqslant 1$, $r \leqslant 3$.*

(i) *Assume that $\lambda r \leqslant 6$. Then*

$$\theta(\lambda, r) = \int_r^6 ds \, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^m \cos \lambda s$$

is $O(\lambda^{2m-1-\varepsilon} r^{-\varepsilon})$, for every $\varepsilon > 0$.

(ii) *Assume that $\lambda r \geqslant 6$. Then*

$$\theta^{\pm}(\lambda, r) = \int_r^6 ds \, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^m e^{\pm i\lambda s}$$

has the following behavior:

$$\theta^{\pm}(\lambda, r) = c_{\pm} \lambda^{m-\frac{1}{2}} (\sinh r)^{\frac{1}{2}-m} e^{\pm i\lambda r} + O\big(\lambda^{m-1} r^{-m}\big)$$

where $c_{\pm}$ is a nonzero complex constant.

Proof. We first prove (i). Recall that

$$\left(\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^m (\cos \lambda s) = \begin{cases} O(\lambda^{2m}) & \text{if } \lambda s \leqslant 6, \\ O(\lambda^m s^{-m}) & \text{if } \lambda s \geqslant 6, \end{cases}$$

hence $(\frac{1}{\sinh s} \frac{\partial}{\partial s})^m (\cos \lambda s) = O(\lambda^{2m-1-\varepsilon} s^{-1-\varepsilon})$ in both cases. By combining this estimate with

$$\sinh s \asymp s, \quad \text{and} \quad \cosh s - \cosh r \asymp s^2 - r^2,$$

and by performing an elementary change of variables, we reach our conclusion:

$$\big|\theta(\lambda, r)\big| \lesssim \lambda^{2m-1-\varepsilon} \int_r^6 ds \, s^{-\varepsilon} \big(s^2 - r^2\big)^{-\frac{1}{2}} \leqslant \lambda^{2m-1-\varepsilon} r^{-\varepsilon} \underbrace{\int_1^{+\infty} ds \, s^{-\varepsilon} \big(s^2 - 1\big)^{-\frac{1}{2}}}_{<+\infty}.$$

We next prove (ii). Recall that

$$\left(\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^m \big(e^{\pm i\lambda s}\big) = \left(\frac{\pm i\lambda}{\sinh s}\right)^m e^{\pm i\lambda s} + O\big(\lambda^{m-1} s^{-m-1}\big).$$

The remainder's contribution to $\theta^{\pm}(\lambda, r)$ is estimated as above:

$$\int_r^6 ds \, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \lambda^{m-1} s^{-m-1} \lesssim \lambda^{m-1} \int_r^6 ds \, s^{-m} \big(s^2 - r^2\big)^{-\frac{1}{2}} \lesssim \lambda^{m-1} r^{-m}.$$

In order to handle the contribution of $(\frac{\pm i\lambda}{\sinh s})^m e^{\pm i\lambda s}$ to $\theta^{\pm}(\lambda,r)$, let us perform the change of variables $s=r(1+u)$, so that

$$\int_r^6 ds = r\int_0^{\frac{6}{r}-1} du,$$

and let us expand

$$\begin{aligned}
&\big(\overbrace{\cosh s-\cosh r}^{2\sinh\frac{s-r}{2}\sinh\frac{s+r}{2}}\big)^{-\frac12}(\sinh s)^{1-m}e^{\pm i\lambda s}\\
&\quad= r^{-m}e^{\pm i\lambda r}\underbrace{\left(\frac{\sinh\frac{ru}{2}}{\frac{ru}{2}}\right)^{-\frac12}\left(\frac{\sinh r(1+\frac{u}{2})}{r(1+\frac{u}{2})}\right)^{-\frac12}\left(\frac{\sinh r(1+u)}{r(1+u)}\right)^{1-m}}_{A(r,u)}\\
&\qquad\times\underbrace{u^{-\frac12}\left(1+\frac{u}{2}\right)^{-\frac12}(1+u)^{1-m}}_{B(u)}e^{\pm i\lambda ru}.
\end{aligned}$$

Notice that the expressions $A(r,u)$ and $B(u)$ can be expanded as follows:

$$A(r,u)=\left(\frac{\sinh r}{r}\right)^{\frac12-m}+\underbrace{\sum_{j=1}^{+\infty}A_j(r)(ru)^j}_{\widetilde{A}(r,u)}, \tag{69}$$

$$B(u)=u^{-\frac12}+\underbrace{\sum_{j=1}^{+\infty}B_j^0u^{j-\frac12}}_{\widetilde{B}(u)}\quad\text{for } u \text{ small}, \tag{70}$$

$$B(u)=\sqrt{2}u^{-m}+\sum_{j=1}^{+\infty}B_j^{\infty}u^{-j-m}\quad\text{for } u \text{ large}. \tag{71}$$

Using these behaviors and integrating by parts, we can estimate

$$\int_0^{\frac{6}{r}-1} du\, e^{\pm i\lambda ru}\widetilde{A}(r,u)B(u)=\frac{1}{\pm i\lambda r}e^{\pm i\lambda ru}\widetilde{A}(r,u)B(u)\Big|_{u=0}^{u=\frac{6}{r}-1}-\frac{1}{\pm i\lambda r}\int_0^{\frac{6}{r}-1} du\, e^{\pm i\lambda ru}\frac{\partial}{\partial u}\big\{\widetilde{A}(r,u)B(u)\big\}$$

by $\mathrm{O}(\frac{1}{\lambda r})$. The integrals

$$\int_0^1 du\, e^{\pm i\lambda ru}\widetilde{B}(u)\quad\text{and}\quad\int_1^{\frac{6}{r}-1} du\, e^{\pm i\lambda ru}B(u)$$

are estimated similarly. In summary, we showed that

$$\theta^{\pm}(\lambda,r)=(\pm i)^m\lambda^m(\sinh r)^{\frac{1}{2}-m}r^{\frac{1}{2}}e^{\pm i\lambda r}\int_0^1 du\, e^{\pm i\lambda r u}u^{-\frac{1}{2}}+\mathrm{O}\big(\lambda^{m-1}r^{-m}\big)$$

and we conclude by using the behavior of the elementary integral

$$\int_0^1 du\, e^{\pm i\lambda r u}u^{-\frac{1}{2}}=\lambda^{-\frac{1}{2}}r^{-\frac{1}{2}}\underbrace{\int_0^{+\infty} du\, e^{\pm iu}u^{-\frac{1}{2}}}_{\text{constant}}+\mathrm{O}\big(\lambda^{-1}r^{-1}\big). \qquad \square$$

From now on, the discussion of Subcase 3(b) is similar to Subcase 3(a). On the one hand, we deduce from Lemma C.1(i) that

$$\int_1^{\frac{6}{r}} d\lambda\, \chi_\infty(\lambda)\lambda^{-\tau}\big(\lambda^2+\tilde\rho^2\big)^{\frac{\tau-\sigma}{2}}e^{it\lambda}\theta(\lambda,r)=\mathrm{O}\big(r^{\frac{1}{2}-m}\big).$$

On the other hand, by expanding

$$\lambda^{-\tau}\big(\lambda^2+\tilde\rho^2\big)^{\frac{\tau-\sigma}{2}}=\lambda^{-\sigma}\left(1+\frac{\tilde\rho^2}{\lambda^2}\right)^{\frac{\tau-\sigma}{2}}=\lambda^{-m-\frac{1}{2}-i\operatorname{Im}\sigma}+\mathrm{O}\big(|\sigma|\lambda^{-m-\frac{5}{2}}\big)\quad \forall\lambda\geqslant 2$$

and $\theta^{\pm}(\lambda,r)$ according to Lemma C.1(ii), we have

$$\frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)}\int_{\frac{6}{r}}^{+\infty} d\lambda\, \chi_\infty(\lambda)\lambda^{-\tau}\big(\lambda^2+\tilde\rho^2\big)^{\frac{\tau-\sigma}{2}}e^{it\lambda}\theta^{\pm}(\lambda,r)=c_{\pm}(I_{\pm}+II_{\pm})(\sinh r)^{\frac{1}{2}-m}+\mathrm{O}\big(r^{\frac{1}{2}-m}\big),$$

where $I_{\pm}$ and $II_{\pm}$ denote the integrals (65) and (66), which are uniformly bounded and whose sum is equal to

$$\frac{e^{\sigma^2}}{\Gamma(-i\operatorname{Im}\sigma)}\int_{\frac{6}{r}}^{+\infty} d\lambda\, \lambda^{-1-i\operatorname{Im}\sigma}e^{i(t\pm r)\lambda}.$$

As a conclusion, we obtain again

$$\big|\widetilde{w}_t^{\infty}(r)\big|\lesssim r^{\frac{1}{2}-m}\lesssim t^{-\frac{n-1}{2}}.$$

Remark C.2. The analysis above still holds in dimension $n=2$, except for the first estimate in Lemma C.1, which is replaced by

$$\theta(\lambda,r)=\mathrm{O}\left(\lambda\log\frac{2}{r}\right).$$

As a result,

$$\left|\widetilde{w}_t^{\infty}(r)\right| \lesssim |t|^{-\frac{1}{2}}\big(1-\log|t|\big).$$

Remark C.3. In order to estimate the wave kernel for small time, we might have used the *Hadamard parametrix* [21, §17.4] instead of spherical analysis.

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