Tensegrity deployment using infinitesimal mechanisms

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Abstract

Kinematic properties of tensegrity structures reveal that an ideal way of motion is by using their infinitesimal mechanisms. For example in motions along infinitesimal mechanisms there is no energy loss due to linearly kinetic tendon damping. Consequently, a deployment strategy which exploits these mechanisms and uses the structure’s nonlinear equations of motion is developed. Desired paths that are tangent to the directions determined by infinitesimal mechanisms are constructed and robust nonlinear feedback control is used for accurate tracking of these paths. Examples demonstrate the feasibility of this approach and further analysis reveals connections between the power and energy dissipated via damping, infinitesimal mechanisms, speed of the motion, and deployment time.

1. Introduction

Classical tensegrity structures (Fig. 1) are assemblies of flexible elements, called tendons, and disjoint bars (Snelson, 1996). This combination gives tensegrity structures a fascinating form, with the disconnected bars apparently floating in a network of tendons. The tensioned tendons appear to give integrity to the structure, hence the acronym tensegrity (Sadao, 1996). Readers interested in this concept’s evolution and extensions that include connected bars and other rigid bodies may consult Juan and Tur (2008), Skelton and DeOliveira (2009) or Sultan (2009a). In this article the structures of interest are defined via key properties and modeling assumptions.

The key defining property of tensegrity structures, identified by early tensegrity researchers (Calladine, 1978; Pellegrino and Calladine, 1986), is that they can achieve equilibrium configurations under zero external actions (i.e. forces or torques) and with all tendons in tension. This property is called prestressability and these equilibrium configurations, prestressable configurations (see Tibert and Pellegrino (2003a) for a review of methods to find such configurations). An immediate consequence of prestressability is that the structure is statically indeterminate at any prestressable configuration, i.e. the equilibrium equations have multiple solutions for the internal forces.

Another key property of classical tensegrities is that they have kinematically indeterminate prestressable configurations with internal infinitesimal mechanisms. A configuration is kinematically indeterminate if infinitesimal displacements are possible with no changes in the lengths of the structural members (Calladine, 1978). Such displacements are called infinitesimal mechanisms. The adjective “internal” is sometimes used to emphasize the fact that tensegrity infinitesimal mechanisms are intrinsic to the structure and not due to effects such as rigid body motions, which involve large displacements with no changes in the lengths of the structure’s members (see Pellegrino and Calladine (1986) for details on this topic). Note that, in general, i.e. not limiting the discussion to tensegrity, mechanisms lead to changes in the structural member lengths that are at least of second order in terms of displacements and are classified according to this order, culminating with finite mechanisms, which result in zero changes in the structural member lengths for large displacements, thus being similar to rigid body motions in this respect (the interested reader may refer to Pellegrino and Calladine (1986), Calladine and Pellegrino (1991) or Vassart et al. (2000) and references therein).

The existence of mechanisms is a major advantage for structures which require change of configuration (e.g. morphing structures, robots, deployable structures, etc.). Indeed, mechanisms enable configuration changes without modifications in the internal member lengths. For infinitesimal mechanisms this is of course valid for infinitesimal displacements while for finite mechanisms it is valid even for large displacements. A structure with mechanisms has increased “mobility” compared to structures without mechanisms, making it more amenable to dynamic applications which involve configuration changes. Clearly, this is true for any structure with mechanisms, including articulated assemblies composed only of bars. In structures with tendons and mechanisms, the mechanisms provide another advantage for dynamic applications. Specifically for tensegrity, the energy dissipated via linearly
kinetic tendon damping is zero in motions that occur along infinitesimal mechanisms. It can also be shown, using a simple approximation, that the variation in the potential elastic energy of a tensegrity structure when displacements along infinitesimal mechanisms occur is small. More generally, i.e., not limiting the discussion to tensegrity and infinitesimal displacements, if a structure with tendons has finite mechanisms the energy dissipated via linearly kinetic tendon damping and the potential elastic energy variation are both zero for large displacements along these mechanisms because tendon lengths do not change (of course the above rationale assumes that tendon rest-lengths are constant). The previous paragraph outlined major advantages mechanisms provide for dynamic applications, especially for structures with tendons like tensegrity, emphasizing that motions along mechanisms are desired. However, kinematic analysis, which is used to identify mechanisms, is basically a geometric study and by itself cannot address the questions if such motions are feasible and how they can be achieved. For this purpose, the dynamic equations of motion must be employed. Furthermore, for large displacement applications such as deployment, nonlinear dynamics equations are required.

Nonlinear ordinary differential equations were used to model tensegrity's dynamics in a deployment strategy in Sultan and Skelton (2003). In that work mechanisms were not exploited. Instead, the system was controlled using tendons to maintain the state space trajectory of the deployment process close to an equilibrium manifold. The evolution of the structure was quasi-static, facilitating satisfaction of structural integrity and collision avoidance constraints. Sultan et al. (2002) also developed a non quasi-static reconfiguration procedure which exploits the mathematical structure of the nonlinear equations of motion and symmetrical tensegrity configurations. Tendon control and, in some cases, external torque control applied to a rigid element of the structure, was used to achieve symmetrical motions. Working on other tensegrity deployment problems, Tibert and Pellegrino (2002, 2003b) disputed tendon control claiming that it is technologically complicated and proposed deployment using foldable/telecopeable struts. A disadvantage of this strategy is that the structure has slack tendons until fully deployed. Fest et al. (2004) studied the potential of telescopic struts in the shape control of a tensegrity structure assuming quasi-static evolution. Motro et al. (2006) proposed deployable tensegrity rings that can be assembled in pedestrian bridges. Smalli and Motro (2007) investigated folding of tensegrity systems by creating finite mechanisms. Finite mechanisms have also been exploited in Rhode-Barbarigos et al. (2012) in a study of ring modules (see also Rhode-Barbarigos et al. (2010)) for deployable footbridges, where the structure is deployed assuming sufficient damping and quasi-static evolution. A key idea in using finite mechanisms in tensegrity deployment is to “activate” these mechanisms, for example by changing the lengths of telescopic struts or tendons. The main disadvantage associated with this procedure is that instabilities are introduced when finite mechanisms are created. These issues are amply described in Motro (2003) Chapter 6.

As emphasized in the above, many successful deployment methods are quasi-static. The structure's generalized velocities and accelerations are very small and the state space trajectory of the deployment process is maintained close to an equilibrium set. Quasi-static strategies are very effective when damping is large because it naturally facilitates small accelerations and velocities. This explains the success of quasi-static deployment procedures in the presence of considerable damping. However, for many applications one would actually like to reduce damping because of its detrimental effects. Damping is a thermodynamically irreversible process which may result in large energy dissipation and non-desirable thermal effects. On one hand, it is well known that these effects are particularly damaging for tendons composed of certain materials such as elastomers. On another hand, such materials may actually be required, especially in deployment applications. This is so because deployment requires large geometry changes that may easily translate into the requirement that tendons tolerate large strain variations, as it will be revealed by examples included in this article. The requirement for large strains is fulfilled by tendons made of elastomers. Therefore, developing deployment strategies in which the energy dissipated via tendon damping is small is important. Also, quasi-static deployment strategies are inherently slow because they require small velocities and accelerations that usually result in long deployment times. This can be reduced by solving a constrained optimization problem aimed at minimizing the deployment time, which is not an easy task (see Sultan and Skelton (2003) for such an example).

This article directly addresses the last two issues. A fast deployment procedure, specifically focused on achieving small energy dissipation via tendon damping, is developed. Because in motions along infinitesimal mechanisms the energy dissipated via linearly kinetic tendon damping is zero, a natural solution is to use these mechanisms for deployment. For this purpose, desired paths that are tangent to the directions determined by infinitesimal mechanisms are created. The requirement of quasi-static motion is eliminated and the desired paths are not constrained to be close to an equilibrium set. The amplitude of the structure's motion is also not restricted to small variations around equilibria, therefore nonlinear ordinary differential equations are used to describe the structure's dynamics. Furthermore, robust nonlinear feedback controllers are designed to guarantee that the state space trajectories of the deployment process, called actual paths, track the desired paths in the presence of uncertainties. These controllers use only torques and eventually forces applied to the bars, which are technologically easy to implement. Because the actual paths follow closely trajectories that are tangent to infinitesimal mechanisms it is expected that the energy dissipated via tendon damping is small. Examples reveal the feasibility of the procedure on a tensegrity simplex as well as on a much more complex tensegrity tower. Correlations between the power dissipated via damping, the speed of the motion, and the infinitesimal mechanisms, as well as the influence of the deployment time on the energy dissipated via damping are analyzed. Issues related to material selection, structural integrity, robustness of the design are also amply discussed.

2. Mathematical modeling, prestressability, mechanisms

2.1. Modeling assumptions

The bars are stiff in comparison with the tendons and the mass of each bar is large relative to the mass of each tendon. Therefore,
the bars are modeled as rigid elements and the tendons as massless elastic elements. The only damping forces are due to the tendons and are linearly kinetic, i.e. the magnitude of the damping force in tendon $j$, $D_j$, is

$$D_j = d_j |\dot{q}_j|$$

(1)

where $d_j > 0$ is the damping coefficient of the tendon and $\dot{q}_j$ is the time derivative of the tendon length. Tendons exert elastic and damping forces on the nodes they are attached to only when they are in tension, otherwise they are slack and exert no force. External forces and torques may be applied to the bars but any other effects (e.g., gravity) are ignored.

### 2.2. Nonlinear equations of motion

Derivation of the equations of motion is a straightforward application of Lagrange equations and has been presented elsewhere (e.g., Sultan (2009a) provides details). Here a summary is given. The tensegrity system is holonomic so a set of independent generalized coordinates (IGC) can be selected to describe its configuration with respect to an inertial reference frame. These IGC can be, for example, coordinates associated with the rigid bars. Let $q \in \mathbb{R}^N$ be the vector of IGC, where $N$ is the number of IGC required to describe the configuration of the system and $\mathbb{R}^N$ the set of $N$ dimensional real vectors. Then the equations of motion are

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + A(q)D^T(q)\dot{q} + A(q)\psi(q) = H(q)F$$

(2)

where $M(q)$ is the mass matrix and the elements of matrix $C(q, \dot{q})$ are

$$C_{ij} = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ik}}{\partial q_j} \right) \dot{q}_k.$$  

(3)

Matrix $A(q)$, of size $N \times E$ where $E$ is the number of tendons, depends only on geometry, its elements being given by $A_{ij} = \partial l_j / \partial q_i$. Here $l_j$ is the length of tendon $j$, which is a function of $q$. Note that "$E" is used for the number of tendons to emphasize the key assumption that they are elastic. Vectors $A(q)D^T(q)\dot{q}$ and $A(q)\psi(q)$ represent the tendon damping and elastic effects, respectively (here $D$ is a diagonal matrix, $D = \text{Diag}[d_j]$, and $\psi(q)$ is the vector of tendon tensions). The term $H(q)F$ is the vector of generalized forces due to external actions (i.e. forces, torques) applied to the bars, where $F$ is the vector representing these actions (see Sultan (2009a) for details; note that a negative sign is used to express the vector of generalized forces due to external actions in Sultan (2009a) for consistency with the rest of the equations, i.e. matrix $H(q)$ is the negative of the one used here).

The structure of these equations is typical of many mechanical systems. Particular to tensegrity systems are the properties discussed next.

### 2.3. Prestressability and infinitesimal mechanisms

A prestressable configuration is an equilibrium achieved under no external actions (i.e. $F = 0$) and with all tendons in tension. From (2) the prestressability conditions are

$$A_q T_0 = 0, \quad T_0 > 0, \quad j = 1, \ldots, E$$

(4)

where $A_q = A(q_0)$, $T_0 = T(q_0)$, and $q_0$ is a prestressable configuration. The necessary condition for (4) to have solutions it that the kernel of $A_q$ is nonzero ($\text{rank}(A_q) < E$). If $T_0$ is an orthonormal basis for this kernel then

$$T_0 = P S P^T = [P_1 \quad P_2 \quad \ldots \quad P_1]$$

(5)

where $P_1, \ldots, P_1$ are real scalars called pretension coefficients and $S$ is the number of pretension states. If a set of pretension coefficients exists such that $T_0 > 0$ for all $j = 1, \ldots, E$, then $q_0$ is a prestressable configuration. Static indeterminacy is obvious from (4) and (5): once such a set of pretension coefficients has been found, other selections for these coefficients are possible which result in positive tendon tensions. Finding prestressable configurations is a geometry problem, because matrix $A_q = A(q_0)$ depends only on geometry. Material properties are necessary to compute the rest-lengths required to achieve a prestressable configuration. For example if the tendons are linearly elastic, $T_j = k_j l_j/r_j - 1$ for $l_j > r_j$ and the rest-lengths are easily computed using

$$r_j = \frac{k_j l_j}{T_0 P_j + k_j}, \quad j = 1, \ldots, E$$

(6)

where $k_j = S_j E_j$, with $S_j$ the cross section area, $E_j$ Young’s modulus of elasticity for tendon $j$, $l_j = l_j(q_0)$, and $T_0 P_j$ the $j$th row of $T_0$.

If the kernel of $A_q$ is nonzero ($\text{rank}(A_q) < N$) the structure is kinematically indeterminate at $q_0$. Indeed, let $d_q = [d_{q_1} \quad \ldots \quad d_{q_E}]^T$ be an infinitesimal displacement in the kernel of $A_q$, i.e. $A_q d_q = 0$. Then, in the first approximation, tendon lengths do not change due to this displacement:

$$A_q d_q = 0 \iff d_{q_j} A_q = 0 \iff d_l = \sum_{i=1}^{N} \frac{\partial l_j}{\partial q_i} d_{q_i} = 0, \quad \forall j = 1, \ldots, E.$$  

(7)

Here $d_l$ is the variation of the length of tendon $j$ due to $d_q$. Bars are rigid, so structural members do not change their lengths due to $d_q$. Of course, this analysis shows only that $d_q$ is an infinitesimal mechanism, which is sufficient for the work presented in this article. Note also the notation "$d_q" to emphasize the infinitesimal character of the mechanism.

Infinitesimal mechanisms depend only on geometry, however in combination with inertial, stiffness, and damping characteristics they influence the dynamical properties (e.g., stability) of the structure. Importantly, infinitesimal mechanisms are crucial in the next “energetic” analysis.

### 2.4. Energetic analysis of infinitesimal mechanisms

For infinitesimal mechanism displacement, $d_l = 0$ (see (7)) so $l_j = \lim_{d_q \rightarrow 0} d_{q_j} / dt = 0$ for all $j = 1, \ldots, E$. From (1) it follows that all damping forces are zero, i.e. $D_j = 0, \quad j = 1, \ldots, E$, so no energy dissipation via linearly kinetic tendon damping occurs in motions along an infinitesimal mechanism, $d_q$.

The potential elastic energy of the structure is due only to the tendons. The variation of this energy when the structure deforms along an infinitesimal mechanism can be estimated using the tangent stiffness matrix at a prestressable configuration, $q_0$. This matrix, labeled $K_0$, can be written as (see Sultan, 2013 for details) $K_0 = \partial A(T)/\partial q_1 \cdots \partial A(T)/\partial q_N |_0$ (8)

where "$0" indicates that the matrix on the right is evaluated at $q_0$. Using $A_q = \partial l_j / \partial q_i$ and the relationship between tendon elastic forces and their lengths written in generic form, i.e. $T_j = T_j(l)$ where $T_j(l)$ denotes a differentiable function, (8) becomes

$$K_0 = A_q G_0 A_q^T + \left[ \frac{\partial A_1 / \partial q_1 \cdots \partial A_N / \partial q_N }{\text{Diag}[T_0]} \right]$$

(9)

where $K_0$ is the material stiffness matrix, $K_0$ the geometric stiffness matrix, $G_0 = \text{Diag}([\partial l_j / \partial q_i]) > 0$ is the matrix of axial stiffnesses, assumed strictly positive here, and $\text{Diag}[T_0]$ is a block-diagonal matrix of size $NE \times N$ which has the column vector $T_0$ on the main diagonal. Using (5), (9) becomes

...
\[ K_s = \sum_{i=1}^{n} \frac{A}{\partial \theta_i} \] where \( K_i = \left[ \frac{\partial A_i}{\partial \theta_1} \ldots \frac{\partial A_i}{\partial \theta_n} \right] \).

Since \( G_0 > 0 \), the material stiffness matrix, \( K_m = A_0 G_0 A_0^T \), is positive semi-definite and its directions of positive semi-definiteness are the directions of the infinitesimal mechanisms. More details on these derivations, as well as an alternative formulation for the stiffness matrix decomposition which uses tendon force density coefficients, can be found in Sultan (2013).

The potential elastic energy variation along an infinitesimal mechanism can be approximated as

\[ dW_{el} = 0.5d^T K_d dq = 0.5s^T \left( A_0 G_0 A_0^T + \sum_{i=1}^{n} \frac{A}{\partial \theta_i} \right) dq \]

so the only contribution to this energy’s variation is due to the geometric stiffness and \( \lim_{\rho \to 0} dW_{el} = 0 \). Practically, infinitesimal mechanisms are directions along which the structure is very compliant (i.e. easy to deform) especially at low pretension. Clearly, the assumption of infinitesimal mechanism displacement is crucial in (11).

These “energetic” results are summarized in Lemma 1 below.

**Lemma 1.** If the structure deforms along an infinitesimal mechanism the energy dissipated via linearly kinetic tendon damping is zero while the variation in the potential elastic energy is only due to the geometric stiffness matrix and is linear in pretension.

**Observation:** The key assumption which leads to zero energy dissipation via tendon damping in the above is that the tendons are affected by linearly kinetic damping forces. This assumption is acceptable for a vast array of materials (e.g., from metals to many elastomers operating in their range of linear behavior). Also note that the specific order of the mechanism was not necessary in Lemma 1, only its infinitesimal character was used. Therefore these energetic properties are only guaranteed locally.

### 3. Deployment using infinitesimal mechanisms

#### 3.1. Motion control strategy

Lemma 1 indicates that from the point of view of energy loss due to damping an ideal way of motion is along infinitesimal mechanisms because the energy dissipated via tendon damping is zero. As already emphasized, this property is only guaranteed locally due to the assumption of infinitesimal displacement so the structure’s motion should be only locally connected to infinitesimal mechanism directions. To ensure such motions the following strategy is used. First, a path that is tangent to a set of infinitesimal mechanism directions is constructed in the configuration space of the structure (i.e. the N-dimensional space of the generalized coordinates). This will be referred to as the desired path. The state space trajectory of the structure’s motion, further referred to as the actual (or deployment) path, must track the desired path. To achieve this goal, feedback control is used. It is important to remark that there is no need to maintain the actual path close to an equilibrium manifold or to enforce quasi-static evolution, so fast deployment is possible. Preliminary attempts at tensegropy motion control along these lines were presented in a conference (Sultan, 2009b). Here complete investigations are performed that include comprehensive studies in the context of tensegropy folding and unfolding.

The first step in designing this control strategy, i.e. finding infinitesimal mechanisms, has already been explained. In the following, the desired path’s construction and tracking controller design are described.

#### 3.2. Desired path construction

Let \( w_j \in \mathbb{R}^N, j = 1, \ldots, M \), be a set of configurations with infinitesimal mechanisms. For each \( w_j \in \mathbb{R}^N \), a direction parallel to an infinitesimal mechanism, \( v_j \), is selected. The problem of building the desired path is then reduced to constructing a curve that passes through \( w_j \in \mathbb{R}^N, j = 1, \ldots, M \), and is tangent to the corresponding \( v_j \in \mathbb{R}^N \) at each of these points. A simple solution to this problem is given by the following Lemma.

**Lemma 2.** Let \(( \tau_j, w_j, v_j) \), \( j \in \mathbb{R}, w_j \in \mathbb{R}^N, v_j \in \mathbb{R}^N \), be a prescribed sequence with times \( \tau_j \) strictly increasing. The piecewise cubic polynomial, with respect to time, given by

\[
q_d(t) = \frac{1}{6} c_j(t^3 - t_0^3) + \frac{1}{2} a_j(t^2 - t_0^2) - \frac{1}{2} G_j^2 + \alpha(t - t_j) + w_j, \quad t_j \leq t \leq t_{j+1},
\]

for \( j = 1, \ldots, M - 1 \) where

\[
c_j = -12(w_{j+1} - w_j) + 6(v_{j+1} + v_j)(t_{j+1} - t_j),
\]

\[
a_j = \frac{v_{j+1} - v_j}{t_{j+1} - t_j} + \frac{t_{j+1} + t_j}{t_{j+1} - t_j} \left( 6(w_{j+1} - w_j) - 3(t_{j+1} + v_j)(t_{j+1} - t_j) \right)
\]

passes through \( w_j \) and is tangent to \( v_j \) at time \( t_j \), for all \( j = 1, \ldots, M \).

This result can be easily proved using the generic formula for cubic polynomials for adjacent intervals and imposing the appropriate conditions on the values at the end points, i.e. \( q_d(t_j) = w_j, q_d(t_j) = v_j \) for \( j = 1, \ldots, M \). Algebraic manipulations then lead to (12)-(14). Alternatively, one can simply verify these formulas by substituting \( t = t_j \) into the formulas for \( q_d(t) \) in (12) and for \( q_d(t) \) (also see Sultan et al. (2007) for more details and other applications of this result).

#### 3.3. Robust nonlinear feedback control

Once a desired path has been constructed, active control can be applied to guarantee accurate tracking of this path. For the examples presented next, a nonlinear feedback controller has been selected. This controller has advantages such as guaranteed robustness properties, simplicity because it avoids online computation of the regressor matrix, and previous success in tracking fast and large amplitude motions (see Ziehe and Corless (1997) and Sultan et al. (2000)).

For a concise description of the controller, consider the generic system

\[ M(q, \mu) \ddot{q} + C(q, \dot{q}, \mu) \dot{q} + G(q, q, \mu) = u \]

where \( \mu \) represents uncertainties in the system. For example, parameters that are not known precisely or are likely to exhibit variations due to operational and environmental conditions (e.g., temperature, radiation, etc.) can be elements of \( \mu \). If \( q_d(t) \) is the desired path and the control law is

\[
u = q_d - N \psi, \quad \eta = \dot{\eta} + N \dot{\psi}, \quad \dot{\eta} = q - q_d
\]

with \( U, \Lambda, \varepsilon, \beta_{1-3} \), selected such that
U > γβ1A. A > γI. 0 < ε ≤ (√γ)² λmin(U)β0/β1. 0 < βdI ≤ M(q, µ) ≤ β1I.

\[ ||C(q, q, µ)|| ≤ β2||q||, \quad ||G(q, q, µ)|| ≤ β3, \]

then the tracking error, \( \bar{q}(t) = q(t) - q_d(t) \), satisfies (see Zenieh and Corless (1997))

\[ ||\bar{q}(t)|| ≤ \left( e_1||\bar{q}(t_0)|| + e_2||\bar{q}(t_0)|| \right) e^{-γ(t-t_0)} + r. \]

In these formulas, \( γ \) is the rate of convergence and \( r \) the error.

Also, \( ||\bullet|| \) represents the standard Euclidean norm for vectors and the maximum singular value norm for matrices (i.e. in (17), \( ||C(q, q, µ)|| \) is the maximum singular value of matrix \( C(q, q, µ) \)).

Theorem that inequalities (17) contain several scalars, i.e. \( β_{0,3} \), which can be determined analytically only for very simple systems.

In general, numerical procedures based on discretization of the region where the system trajectories are expected to lie can be used (see Sultan et al. (2000)). Heuristic determination of \( β_{0,3} \), which is also an alternative to discretization for very complex, large dimensional systems

where numerical computation of these constants is prohibitive. Note also from (16) that time derivatives of the desired path, \( q_d(t) \), up to the second order are required (due to the \( v \) term in \( µ \)).

If the desired path is of class \( C^2 \) almost everywhere with respect to time, having only a finite number of isolated discontinuities of the first kind (i.e. left and right limits exist), for numerical implementation at the points of discontinuity the average value of the left and right limits can be used. Numerous numerical experiments indicated that this approach works very well.

To apply this result, the equations of motion (2) are easily cast as in (15) by identifying terms:

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + A(q)D\dot{q}\dot{q} + A(q)T(q) = H(q)F. \]

Note that for simplicity and consistency with (2) the uncertainty term, \( µ \), has not been explicitly written in (20). The most important uncertain parameters are tendon material properties (damping coefficients, Young moduli) and their influence on the robustness of the design will be extensively investigated in the examples.

In the remainder of this article, the deployment procedure described in the above is applied to the folding and unfolding of some tensegrity structures. For simplicity the word “deployment” is used to denote folding and unfolding processes.

4. Tensegrity simplex example

4.1. Tensegrity simplex description

The first example used to verify the feasibility of this deployment approach is a tensegrity structure of low complexity called a tensegrity simplex. This structure is composed of three “vertical” tendons, \( A_1B_1 \), three “top” tendons”, \( B_iB_i \), and three identical bars of length \( l, A_iB_i \), attached via frictionless rotational joints to a fixed equilateral triangle, \( A_1A_2A_3 \), of side length \( b \) (see Fig. 2, where tendons are represented by thin lines and bars by thick lines). A dextral inertial reference frame \( \{b_1, b_2, b_3\} \) with origin at the centroid of \( A_1A_2A_3 \), \( b_1 \) parallel to \( A_1A_2 \), and \( b_2 \) perpendicular onto \( A_1A_2A_3 \) is introduced. The rotational degree of freedom around each bar’s longitudinal axis of symmetry, \( A_iB_i \), is ignored and the vector of generalized coordinates is \( q = [\delta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_2]^T \)

where \( \delta_1 \) is the angle between \( A_1B_1 \) and \( b_3 \) and \( \alpha_2 \) is the angle between \( A_iB_i \)’s projection onto \( A_1A_2A_3 \) and \( b_i \). The bars are assumed rigid while the tendons are massless, linearly elastic and affected by linearly kinetic damping. Each bar is acted upon by external torques, which represent the control vector, \( u \), in (15) or (20).

A particularly interesting class of prestressable configurations is the set of symmetrical prestressable configurations (see Fig. 3). Symmetrical configurations are described by

\[ q_0 = [\delta, \alpha, \alpha + 4\pi/3, \alpha + 2\pi/3]^T \]

where \( \alpha \in [0, 2\pi) \) is the angle between \( A_1B_1 \)’s projection onto \( A_1A_2A_3 \) and \( b_3 \) and \( \delta \in (0, \pi/2) \) the angle between \( A_iB_i \), \( i = 1, 2, 3 \), and \( b_3 \). In the following, the deployment strategy described in the previous section will be exemplified on the unfolding of the simplex between two symmetrical prestressable configurations.

In Sultan (2013) the prestressability conditions (4) have been solved and the infinitesimal mechanisms found for these configurations, leading to the following results.

![Fig. 2. Tensegrity simplex.](image)

![Fig. 3. Top view of a symmetrical configuration.](image)
Lemma 3. Symmetrical prestressable configurations exist if and only if \( \alpha \in (0, \pi/6) \), \( b < l\sqrt{3} \) and
\[
\sin \delta = \frac{b}{2v\sqrt{3}\sin \alpha}. \tag{22}
\]
At each symmetrical prestressable configuration there is one pretension state (so in (3), \( S = 1 \) and \( P = P_1 \)) and one infinitesimal mechanism given by
\[
dq = d\alpha [\tan \alpha \tan \delta - 1 \tan \alpha \tan \delta - 1 \tan \alpha \tan \delta - 1]^T, \quad dq \neq 0. \tag{23}
\]
Lemma 4. At any symmetrical prestressable configuration the tangent stiffness matrix is positive definite. Moreover, all of the symmetrical prestressable configurations are exponentially stable. These properties are valid regardless of the positive values of pretension and material properties consistent with the modeling assumptions.

Furthermore, the tendon tensions at a symmetrical prestressable configuration can be expressed analytically as follows:
\[
T_0 = PT_n, \quad T_n = \frac{1}{\sqrt{3}\sqrt{T_i^2 V^2 + B^2}} [\tilde{T}_V V \tilde{T}_V V \tilde{T}_V V B B B]^T, \quad P > 0. \tag{24}
\]
where \( \tilde{T}_V = (\cot \alpha - \sqrt{3}/2) \) and \( V \) and \( B \) are the lengths of the vertical and top tendons at such a configuration, expressed as
\[
V = (l^2 + b^2 - 2eb \sin \alpha \sin \delta)^{0.5}, \quad B = (3l^2 \sin \delta^2 + b^2 - 3eb \sin \delta \cos \alpha - \sqrt{3}eb \sin \delta \sin \alpha)^{0.5} \tag{25}
\]
respectively, with \( \sin \delta \) given by (22). The order of tendons in \( T_n \) is \( A_1 B_3, A_2 B_3, A_3 B_1, B_3 B_1, B_2 B_3, B_3 B_1 \). Of course at each symmetrical prestressable configuration the corresponding tendon rest-lengths required to achieve equilibrium must be computed using (6).

The compressive force in any bar, obtained from nodal equilibrium conditions, is
\[
C = \frac{L P T_V}{\sqrt{3}(T_i^2 V^2 + B^2)} \tag{26}
\]
It is crucial to remark that positive definiteness of the tangent stiffness matrix means that the structure is stiff at that configuration. Thus, another key property of classical tensegrity structures, identified as early as 1978 (see Calladine (1978)), namely that the infinitesimals are stiffened by pretension, is verified by Lemma 4.

4.2. The unfolding problem

The simplex is initially in a symmetrical prestressable configuration characterized by a height of \( 0.5 \) m and \( \alpha = \alpha_0 = 5.6^\circ, \quad \delta = \delta_0 = 80.4^\circ \) (Fig. 4) and must be unfolded to reach a final symmetrical prestressable configuration of height \( 2.5 \) m and with \( \alpha = \pi/10^\circ, \quad \delta = \delta_0 = 33.65^\circ \) (Fig. 5). The deployment time, \( \tau \), is prescribed.

The structure’s parameters were selected as (in SI units)
\[
b = 1, \quad l = 3, \quad m = 1/3, \quad P = 100, \quad d_j = 0.1, \quad k_j = 50, \quad j = 1, 2, 3, \quad k_j = 10, \quad j = 4, 5, 6 \tag{27}
\]
where \( m \) is the mass of a bar. The transversal moment of inertia of each bar is, \( J = m l^2/12 = 1/4 \) kg m².

To construct the desired path, \( q_{0}(t), \quad M = 12 \) points were selected on the equilibrium path, equidistantly placed within the interval \( [\alpha_0, \alpha_f] \), i.e. \( M = 12 \) values for \( \alpha \) were generated as \( \alpha_0 = \alpha_0 + (\alpha_f - \alpha_0)(j - 1)/(M - 1), \quad j = 1, \ldots, M \). The corresponding vectors \( w_j \) were computed using formula (21) for \( q_{0} \) with \( \alpha = \alpha_0 \) and with \( \delta \) given by (22). Vectors \( v_j \) were obtained by normalizing to unit Euclidean norm the corresponding infinitesimal mechanisms given by (23), and times \( t_j \) were chosen as \( t_j = \tau(j - 1)/(M - 1) \). These quantities were used in (12)–(14) to generate the desired path.

For control design the following parameters were selected:
\[
\rho_0 = 0.1, \quad \beta_1 = 2, \quad \beta_2 = 2, \quad \beta_3 = 150, \quad Q = 211, \quad \Lambda = 111, \quad e = 0.945. \tag{28}
\]

Tendon rest-lengths were fixed to the values corresponding to the initial configuration (i.e. 0.96 m for vertical tendons and 1.64 m for top tendons) and were switched to the values corresponding to the final configuration (1.22 m for vertical tendons and 0.87 m for top tendons) at time \( t = \tau \), when the controls, \( u \), were also fixed to zero (see for example Sultan and Skelton (2003) for a discussion about how tendon rest-lengths can be modified in practice: e.g. tendons whose rest-lengths must increase are rolled out of a device whereas tendons whose rest-lengths must decrease are rolled out of a device). The values of the rest-lengths were computed using (6) with the normalized tensions, \( T_{0n} \), given by (24), and using (25) for the tendon lengths in a symmetrical prestressable configuration, \( l_{0n} = l_0/(q_{0n}) \), with \( \delta \) given by (22).

4.3. Simulation results

Eqs. (20) and (16) were used to simulate the closed loop behavior of the system. Fig. 6 shows projections onto the \( \alpha_{11}, \delta_{11} \) plane of
the directions of the infinitesimal mechanisms and of the desired, actual, and equilibrium paths for $\tau = 5$ s (similar behavior was observed for all of the other generalized coordinates). Clearly, the quality of tracking is very good: the actual path ideally follows the desired path. Remark also that the points represented in Fig. 6 on the actual path correspond to equal time intervals. In the vicinity of the inflexion points on the actual path, where the infinitesimal mechanisms are located, these points are spread apart, whereas in the vicinity of the turning points on the actual path these points are very close to each other. It is then clear that motion in the vicinity of infinitesimal mechanisms (i.e. at the inflexion points on the actual path) is much more rapid than motion far from these mechanisms (e.g., at the turning points on the actual path). On the other hand, Fig. 7, which depicts the distribution of power dissipated via tendon damping along the actual path, shows that power loss due to damping is minimal when the actual path follows closely the infinitesimal mechanism directions despite of the rapid motion there, and much larger at the turning points, regardless of the slow motion there. This is a very important observation, which shows that the geometry of the motion has a dominant effect on the energy dissipated via linearly elastic tendon damping compared to the speed of the motion. Thus, for small energy dissipation via tendon damping it is more important to focus on controlling the motion such that geometry is properly exploited by using infinitesimal mechanisms, than to focus on achieving slow motion. An explanatory note may be required to understand Fig. 7: in this Figure the curve in the $x_{11}$, $\delta_{11}$ plane represents the projection of the actual path on this plane, whereas the out of plane curve represents the values of the dissipated power at the corresponding points on the actual path (effectively the curve in the $x_{11}$, $\delta_{11}$ plane is the projection of the out of plane curve on this plane). When the out of plane curve comes in contact with the $x_{11}$, $\delta_{11}$ plane the dissipated power is zero. This happens precisely at the inflexion points on the actual path, where the actual path is practically tangent to the infinitesimal mechanisms due to the exceptional tracking of the desired path (the best points to visualize this are towards high values of $x_{11}$, i.e. the end point for $x_{11} = 80.4^\circ$ or the point corresponding to $x_{11} = 67^\circ$).

Note that the tendons are always in tension throughout the motion (representative tendon tension time histories are depicted in Fig. 8), sufficient clearance between bars is maintained so collisions are avoided (the minimum distance between bars is 0.19 m), and the controls vary within acceptable limits (results are not reproduced here for brevity).

Fig. 9 shows snapshots of the deployment process at equal time intervals (for simplicity the connections to the fixed ground via rotational joints were not represented). The structure's shape variation is more impressive in the initial phase due to larger variations in the generalized coordinate values in that phase of the motion. Fig. 10, which shows time histories of two generalized coordinates, clearly indicates the size of these variations. For example $\delta_{11}$, which is directly related to the “height” of the structure that is easy to visualize in Fig. 9, varies substantially in the initial phase, when “Time” (or $t$ in all of the relevant equations) is small, and varies very little when “Time” approaches $\tau = 5$, thus explaining Fig. 9.

Fig. 10 also shows that the structure settles down to the final equilibrium configuration very quickly. This is explained by the quality of tracking and the exponential stability of this configuration. At $t = \tau$ all generalized coordinate values are practically equal to those of the final equilibrium configuration and all generalized velocities are very close to zero. These, combined with the exponential stability of the final equilibrium configuration, result in immediate convergence to the desired final equilibrium configuration, once the tendon rest-lengths are switched to the values corresponding to this configuration.

4.4. Further evaluation and discussion

4.4.1. Energy computation

The energy dissipated via tendon damping during the feedback controlled deployment process is $W_{\text{diss}} = \int_0^T q^T A(q) \dot{q}^T A(q) \dot{q} dt = 0.44$, while the mechanical work of the controls is $W_c = \int_0^T q^T \dot{q} dt = 58.20$ (simple numerical integration has been used
for these computations). This last value is easily explained as follows. The initial potential elastic energy of the structure, at \( t = 0 \), is
\[
W_{\text{el}} = \sum_{j=1}^{6} k_j (l_j - r_j)^2 / 2r_j = 150.37 \text{ J.}
\]
Here \( l_j \) and \( r_j \) denote the \( j \)th tendon length and rest-length at the initial equilibrium configuration, respectively. At the end of the feedback controlled deployment process, i.e. at \( t = \tau \), right before the tendon rest-lengths are switched to the values corresponding to the desired final equilibrium configuration and the controls are set to zero, the potential elastic energy is 208.85 J. The kinetic energy is due to the motion of the bars and at the same instant is very small, \( W_k = 0.16 \) J. Since this energy and the energy dissipated via damping are so small (i.e. \( W_d = 0.16 \) J and \( W_{\text{diss}} = 0.44 \) J), the work of the controls is practically dictated by the variation in the potential elastic energy that must be achieved (simple energy balance verifies these calculations). If one wants to reduce the mechanical work required from the controls, reduction of this potential elastic energy variation should be achieved. Other strategies could be pursued, like for example designing controllers and deployment paths that specifically reduce the control energy. These strategies may involve control procedures that are technologically different, such as rest-length control to maintain the variation in potential elastic at low levels even when tendon lengths exhibit large variations (see Sultan and Skelton (2003) for continuous rest-length control in tensegrity quasi-static deployment). Clearly this requires more controls and more complex procedures. Here only torques applied to bars are used and the rest-lengths are fixed during the controlled phase of the deployment, being switched only once, to the values corresponding to the desired final equilibrium. To what extent infinitesimal mechanisms can be effectively used to solve other type of problems may be a topic of future research.

### 4.4.2. Tendon material and modeling assumptions verification

Tendon strains, computed as
\[
e_j = \left( \frac{l_j - r_j}{r_j} \right), \quad j = 1, \ldots, 6,
\]
nearby strains vary between 1.11 (static) and 1.7 (dynamic) for vertical tendons and 0.2 (dynamic) and 1.58 (static) for top tendons. Here “static” means that the value is achieved at an equilibrium (initial or final prestressable configuration) whereas “dynamic” means that it is attained during motion. For such large strains elastomers are recommended. Many elastomers resist much larger strains, while also exhibiting linearly elastic behavior, in agreement with the modeling assumptions made herein. For example Wang et al. (2002) report elastomers that are linearly elastic for strains between 0 and almost 4 while Sonnenschein et al. (2013) and Lee et al. (2009) report biocompatible elastomers that resist strains of 7 or larger. Therefore, there is a large range of elastomers that can be used to manufacture the tendons. Using an average density of 1000 kg/m\(^3\) and an average Young modulus of \( E_j = 1 \text{ MPa} \) (values that are in the typical range for these elastomers, see for example Table 6 in Sonnenschein et al. (2013)) and the values for \( k_j = S_j E_j \) given in (27), the maximum mass of each vertical tendon is 0.061 kg, achieved in the final prestressable configuration, and the maximum mass of each top tendon is 0.016 kg, achieved in the initial prestressable configuration. These values, obtained using the corresponding tendon rest-lengths for volume computation, are much smaller than the mass of each bar (\( m = 0.33 \text{ kg} \)), in agreement with the modeling assumption that tendon mass is...
negligible compared to bar mass. Moreover, other values can be used for material properties in the range that is typical for linearly elastic elastomers to yield even smaller tendon mass.

4.4.3. Bar loading and verification

A note is also made here regarding mechanical loads experienced by bars. In quasi-static deployment approaches the structure is always close to equilibria, so static values of forces and torques to design the structure’s members are used. Moreover, because reliable values for typical material properties that are used in the structural design process (e.g., elasticity moduli, strength limits, etc.) are determined in experimental conditions that are also quasi-static, such an approach is justified. When the structure experiences dynamic behavior that departs from quasi-static conditions, the simplest strategy to account for this difference is to use high values for the safety coefficients. For example in the particular situation exemplified herein, the maximum value for the static compressive force in bars (obtained from (26)) is \( C = 82.15 \, \text{N} \), achieved in the initial equilibrium configuration. If the bars are pipes of exterior radius \( R \), designing them against buckling gives the following formula for the minimum bar mass:

\[
m_{\min} = \pi \rho_b l \left( R^2 - \sqrt{R^4 - \frac{4l^2 c_b}{\pi^2 E_b}} \right)
\]

where \( \rho_b \), \( E_b \), and \( \gamma_b \) are bar density, Young modulus, and safety coefficient, respectively. For example for bars made of Titanium, with \( R = 0.025 \, \text{m} \), \( \rho_b = 4400 \, \text{kg/m}^3 \), \( E_b = 110 \, \text{GPa} \) and a large safety coefficient, \( \gamma_b = 5 \), (29) yields \( m_{\min} = 0.14 \, \text{kg} \) which is much smaller than the bar mass used in simulation (i.e. \( m = 0.33 \, \text{kg} \)).

4.4.4. Robustness analysis

Robustness of the design is also an important issue, especially when elastomers are used, because these materials usually exhibit variations in their properties. Of course, large safety coefficients, as discussed in the above, guarantee some robustness for structural design, but the robustness of the control design must also be verified. Therefore, a robustness study was carried out in which the Young moduli of all tendons were modified randomly by up to \( \pm 20\% \) with respect to their nominal values, \( E_j = 1 \, \text{MPa}, j = 1, \ldots, 6 \). The tendon damping coefficients were also randomly modified between 0 and 0.4. The corresponding closed loop behaviors were obtained, and that controls vary between acceptable limits were also verified. The energy dissipated via tendon damping was computed as the mechanical work of the damping forces,

\[
W_{\text{damp}} = \int_0^\tau q^T A(q) \text{D} A^T(q) q \, dq
\]

using simple numerical integration. Fig. 13 shows that this energy decreases as the deployment time increases, reaching a minimum for approximately \( \tau \approx 25 \, \text{s} \). For practical applications fast deployment is of interest so the deployment time should be kept low. Fig. 12 also shows that large deployment time leads to large amplitude motions, which may not be desired. If one wants to reduce the time for which minimum energy dissipation via damping is achieved, solving a multi-objective, constrained optimization problem may be attempted.

A final observation emphasizes the fact that the control actions (i.e. the elements of vector \( u \)) are external torques applied to the bars. Clearly, compared to other control actions proposed for tensegrity deployment such as tendon control or telescopic struts, these are easier to implement.

5. Tensegrity tower example

The tensegrity simplex is a simple structure, which was instrumental in presenting, with many details, the application of the
deployment approach. In this section, the feasibility and scalability of the approach is illustrated on a much more complex structure, a tensegrity tower.

5.1. Tensegrity tower description

A three stage SVDT tensegrity tower consists of 36 tendons and 9 identical bars of length $l$. The SVDT denomination is related to classes of tendons called “Saddle”, “Vertical”, “Diagonal”, and “Top” respectively (and for more details see Sultan and Skelton (2003)). The tower is depicted in Fig. 14 where thin lines represent tendons and thick lines represent bars. Three bars are attached via frictionless rotational joints at $A_i$, $i = 1, 2, 3$, to a fixed base equilateral triangle of side length $b$, labeled $A_1A_2A_3$. The dextral inertial reference frame, $(b_1, b_2, b_3)$, and the angles $x_l$ and $y_l$ for a generic bar, $A_lB_l$, are defined similarly with the ones for the simplex. Stages are composed of bars $A_iB_i$, $i = 1, 2, 3$. As in the simplex example, the tendons are assumed massless, linearly elastic, and affected by linearly kinetic damping, while the bars are rigid and each bar’s rotational degree of freedom around $A_{il}$ is ignored. The controls are torques applied to the bars of the first stage and torques and forces applied to the bars of the second and third stages.

This system has $N = 36$ independent generalized coordinates, selected as the angles $\delta_j$, $x_j$, $y_j$, $i = 1, 2, 3$, $j = 1, 2, 3$, and the center of mass inertial Cartesian coordinates for the bars of the second and third stages, $x_{yj}, y_{yj}, z_{yj}, i = 1, 2, 3$, $j = 2, 3$. The vector of independent generalized coordinates is thus:

$$q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$$

where

$$q_1 = [x_{11} \ z_{11} \ \delta_{11} \ x_{12} \ \delta_{12} \ x_{13} \ \delta_{13}]^T,$$$$

q_2 = [x_{21} \ y_{21} \ \delta_{21} \ x_{22} \ \delta_{22} \ x_{23} \ \delta_{23}]^T,$$$$

q_3 = [x_{31} \ y_{31} \ \delta_{31} \ x_{32} \ \delta_{32} \ x_{33} \ \delta_{33}]^T, \quad j = 2, 3.$$

For this tower, symmetrical configurations (as shown in Fig. 15) are defined as follows. Triangles $A_{ij}A_{i+1j}B_{i+1j}$ and $B_{i}B_{i+1}B_{i+2}$ are equilateral triangles of side length $b$ and all angles $\delta_j$ are equal, $\delta_0 = \delta$. Bars are parallel as follows: $A_{1j}B_{1j} || A_{2j}B_{2j} || A_{3j}B_{3j} || A_{3j}B_{2j} || A_{3j}B_{1j} || A_{3j}B_{2j}$. The projections onto the plane $(b_1, b_2)$ of nodes $A_{ij}, B_{i}, B_{i+1}, B_{i+2}, A_{i+2j}, l = 1, 2$, form regular hexagons. Planes $A_{1j}A_{2j}A_{3j}$ and $A_{i+2j}A_{i+2j+1}A_{i+2j+2}, j = 1, 2$, are parallel, the distance between $A_{ij+1}A_{ij+1}A_{ij+2}$ and $B_{i}B_{i+1}B_{i+2}$ is the same for $j = 1, 2$, and it is called the overlap, $h$. The set of symmetrical configurations is easily parameterized using three parameters, $x_{1i} = x$, $\delta$, and $h$, resulting in:

$$x_{11} = x_{12} = x_{13} = x + \frac{4\pi}{3}, \quad x_{31} = x_{12} = x_{23} = x + \frac{2\pi}{3}, \quad x_{22} = x_{33} = x_{11} = x,$$

$$x_{12} = \frac{b}{2} + \frac{l}{2} \sin(\delta) \cos(x_{12}), \quad y_{12} = \frac{b\sqrt{3}}{6} - \frac{l}{2} \sin(\delta) \sin(x_{12}), \quad x_{22} = \frac{b}{2} - \frac{l}{2} \sin(\delta) \cos(x_{22}),$$

$$y_{22} = \frac{b\sqrt{3}}{6} - \frac{l}{2} \sin(\delta) \sin(x_{22}), \quad x_{32} = -\frac{b\sqrt{3}}{3} + \frac{l}{2} \sin(\delta) \cos(x_{32}), \quad y_{32} = -\frac{b\sqrt{3}}{3} + \frac{l}{2} \sin(\delta) \sin(x_{32}),$$

$$x_{13} = \frac{l}{2} \sin(\delta) \cos(x_{13}), \quad y_{13} = \frac{b\sqrt{3}}{3} + \frac{l}{2} \sin(\delta) \sin(x_{13}), \quad x_{23} = \frac{b}{2} + \frac{l}{2} \sin(\delta) \cos(x_{23}),$$

$$y_{23} = -\frac{b\sqrt{3}}{6} + \frac{l}{2} \sin(\delta) \sin(x_{23}), \quad x_{33} = -\frac{b}{2} + \frac{l}{2} \sin(\delta) \cos(x_{33}), \quad y_{33} = -\frac{b\sqrt{3}}{6} + \frac{l}{2} \sin(\delta) \sin(x_{33}),$$

$$z_{l2} = \frac{3l}{2} \cos(\delta) - h, \quad z_{l3} = \frac{5l}{2} \cos(\delta) - 2h, \quad i = 1, 2, 3.$$
This set was analyzed in Sultan (2013) for prestressability solutions (i.e. solutions to (4)) and for stiffness and stability properties of the corresponding prestressable symmetrical configurations. Here a subset of these configurations is used to generate infinitesimal mechanisms. Specifically, for \( a = 0 \) the prestressability conditions (4) for symmetrical prestressable configurations were numerically solved for \( l = 1 \, \text{m} \), \( b = 0.67 \, \text{m} \) and \( d = \frac{40}{C_{14}} \), \( \frac{80}{C_{14}} / C_{13} \). For each value \( d \in [40^\circ, 80^\circ] \) that was considered a unique solution for the overlap, \( h \), was obtained, generating a point on the equilibrium path. At each point on this path the equilibrium matrix, \( A_0 \), of size 36 x 36, has rank 35 so there is one pretension and one infinitesimal mechanism. Similarly with the tensegrity simplex example, the stiffness matrix is positive definite and each point on this path is an exponentially stable equilibrium regardless of the positive pretension and material properties consistent with the modeling assumptions (see Sultan (2013) for more details).

5.2. Tensegrity tower folding

The structure is in a “tall” initial symmetrical prestressable configuration, characterized by \( a = a_i = 0 \), \( d = d_i = 40^\circ \) and a height of 1.63 m, and must be folded into a final symmetrical prestressable configuration characterized by \( a = a_f = 0 \), \( d = d_f = 80^\circ \) and a height of 0.45 m (see Fig. 20). To apply the previous procedure, \( M = 11 \) points were selected on the equilibrium path described before, equidistantly placed between \( d_i \) and \( d_f \), i.e. \( M = 11 \) values for \( d \) were generated as \( d_j = d_i + (d_f - d_i)/(M - 1) \cdot j \), \( j = 1, \ldots, M \). Each vector \( w_j \), which is the corresponding generalized coordinate vector, was computed using the formula for \( q \) in (31) combined with (32), where \( a = 0 \), \( d = d_j \), and the overlap \( h \) was determined by numerically solving the prestressability conditions (4). The corresponding infinitesimal mechanisms were computed from the kernel of \( A_0^T \) and normalized.
to unit Euclidean norm to obtain the vectors $v_j$. Times $t_j$ were chosen as $t_j = \tau(j - 1)/(M - 1)$. Then the desired path was generated using (12)–(14). Other parameters of the structure were chosen as (SI units)

$$m = 1, \quad J = mL^2/12 = 1/12, \quad d_j = 0.1, \quad P = 82$$  \hspace{1cm} (33)

where $m$ and $J$ are the mass and central transversal moment of inertia of a bar, and $d_j$ and $P$ are damping and pretension coefficients. The pretension level was selected sufficiently large such that all tendons are in tension during motion. The values of $k_j = S_jE_j$ were selected equal to 50 N for all tendons except for nine vertical tendons ($A_{11}B_{21}, A_{21}B_{31}, A_{31}B_{11}, A_{12}B_{32}, A_{22}B_{12}, A_{32}B_{12}, A_{13}B_{21}, A_{23}B_{21}$, $A_{33}B_{31}$) for which they were set equal to 5 N. For control design the following parameters were chosen:

$$\beta_0 = 0.02, \quad \beta_1 = 1.1, \quad \beta_2 = 10, \quad \beta_3 = 400, \quad Q = 51I, \quad \Lambda = 11I, \quad \varepsilon = 0.5$$  \hspace{1cm} (34)

Similarly with the simplex example, the rest-lengths of the tendons in the initial and final prestressable configurations were computed using (6) and for the simulation results reported next, these rest-lengths were kept fixed, equal to their values in the initial prestressable configuration, until $t = \tau$ when they were switched to the values corresponding to the final prestressable configuration and the controls, $u$, were fixed to zero.

5.3. Simulation results

Eqs. (20) and (16) were used to simulate the closed loop system. Fig. 16 shows projections onto the $x_{11}, \delta_{11}$ plane of the infinitesimal mechanism directions and of the desired, actual, and equilibrium paths for $\tau = 5$ s, where the points represented in Fig. 16 on the actual path correspond to equal time intervals (all of the other generalized coordinates behave similarly). Fig. 17 shows the distribution of the power dissipated via tendon damping along the actual path. These figures reinforce conclusions reached in the tensegrity simplex example: tracking is excellent and the power dissipated via damping is practically zero when the actual path follows closely the infinitesimal mechanism directions, where motion is fast, and much larger at the turning points, where motion is slow. Also, like in the tensegrity simplex example, during the entire process sufficient clearance between bars was maintained (the minimum distance between bars was 0.11 m), all tendons were in tension,
and all controls displayed acceptable variations. Time histories for representative tendon tensions and for two generalized coordinates are given in Figs. 18 and 19, respectively. As in the tensegrity simplex case, the structure settles down to the final equilibrium configuration very quickly because of the exceptionally good tracking performance of the controller and the exponential stability of the final equilibrium configuration (see Fig. 19). At \( t = \tau \) all generalized coordinate values are almost equal to the values for the final equilibrium configuration and all generalized velocities are almost zero, so immediate convergence to the final desired equilibrium configuration is achieved. Lastly, Fig. 20 shows snapshots of the folding process, illustrating conclusions similar with the ones reached in the tensegrity simplex example: the variation in the structure’s shape is more impressive when larger variations in the values of the generalized coordinates occur (in this case, in the final phase of the motion).

The other conclusions reached when the simplex example was analyzed are also reinforced by this example. For example the tendons experience strains that are within the linear elasticity range of many elastomers. Specifically, the maximum tendon strain is static, equal to 3.06, achieved in the initial equilibrium configuration, whereas some elastomers reported, for example in Sonnenschein et al. (2013), easily exhibit much larger strains (of around 7). The mass of each tendon is also very small compared to the mass of each bar: for example, the maximum tendon mass is 0.053 kg (corresponding to the final equilibrium configuration), which is negligible compared to the bar mass (1 kg) used in the simulations. Also, similarly with the tensegrity simplex example, the maximum static compressive force in bars allows for large safety coefficients: for example for tubular bars made of Titanium with an exterior radius of 2.5 mm the necessary mass, computed using (29), is 0.025 kg for a safety coefficient equal to 5. This enables large dynamic loads during the folding process. Of course other bar materials can be used that result in larger \( m_{\text{min}} \), but for typical metallic materials this will still be well below the bar mass used in simulations.

A robustness study similar to the one performed in the tensegrity simplex example was carried out for the tower folding process. Specifically, for each tendon the Young modulus and damping coefficient were randomly perturbed as follows: the Young modulus was modified with up to 20% from the nominal value of 1 MPa and the damping coefficient was given a value between 0 and 0.4. These perturbations were independent from one tendon to another and applied simultaneously. Then the resulting closed loop system, obtained using these perturbed values for tendon material properties and the nominal controller was used to simulate the folding process for \( \tau = 5 \) s. Note that the nominal controller was designed for nominal (unperturbed) values of Young moduli and damping coefficients and was used to generate Figs. 16–20. The corresponding “aggregate” error, defined as in the simplex example, was computed. Fig. 21, which shows the distribution of this error for 200 test cases, including the nominal design for reference, confirms the strong robustness properties of the nominal controller with respect to variations in
tendon material properties. In Fig. 21 the term “perturbed designs” refers to the perturbed closed loop systems, while “nominal design” refers to the nominal closed loop system (i.e. corresponding to no perturbation in material properties). Also, during these simulations the conditions that all tendons are in tension and sufficient clearance between bars is maintained were satisfied.

The influence of the deployment time on the folding process was also evaluated like in the tensegrity simplex example. Fig. 22 shows relevant paths for different deployment times, indicating that the amplitude of the motion increases with the deployment time. This behavior is expected and similar with the behavior observed when unfolding of the tensegrity simplex was examined (Fig. 12). Fig. 23 shows the variation of the energy dissipated via tendon damping with the deployment time. Compared to the tensegrity simplex situation (Fig. 13) the minimum is more pronounced and achieved for a much smaller value (around $\tau \approx 15$ s), but the behavior is qualitatively similar.

5.4. Folding non-symmetrical configurations

In the previous examples the structures were deployed (i.e. unfolded or folded) between symmetrical configurations. There are well known advantages of using symmetries in structural analysis and in the investigation of mechanisms (see for example Fowler and Guest (2000), Guest and Fowler (2006), and the references therein). First, constraints that enforce symmetries (e.g., (21) for the tensegrity simplex and (32) for the tower) simplify the prestressability conditions (4) because many conditions become redundant. The number of conditions is drastically reduced, enabling analytical solutions (e.g., in the tensegrity simplex example) or very efficient numerical solutions (e.g., in the tensegrity tower example). Symmetries also facilitate mechanisms because in a configuration that is symmetric geometrical constraints that are otherwise independent become redundant, resulting in a less constrained structure with mechanisms.

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Fig. 25. Deployment sequence for the non-symmetrical tower folding process.
However, in some situations it may be necessary to involve non-symmetrical configurations. For example, due to gradual modification of tendon material properties, the initial symmetrical prestressable configuration of the tensegrity tower in Fig. 20 may evolve into a non-symmetrical configuration. Therefore, the folding problem in Section 5.2 will require folding the structure from a non-symmetrical prestressable configuration into the final symmetrical prestressable configuration depicted in Fig. 20. In the following it is shown that the proposed approach is easily applicable in such situations also.

Finding non-symmetrical prestressable configurations of tensegrities is a well studied field (see for example Bel Hadj Ali et al. (2011), Ebara and Kanno (2010), Koohestani and Guest (2013) and Zhang et al. (2006)). The configuration used in this example was found using a simple dynamic relaxation approach: the nonlinear equations of motion (2), in which the Young modulus of each tendon was randomly modified by up to 20% and F was set to 0, were numerically integrated using as initial conditions zero generalized velocities and the coordinate values of the initial symmetrical prestressable configuration in Fig. 20. The rest-lengths were fixed to the values corresponding to the initial symmetrical prestressable configuration in Fig. 20. Due to tendon damping, the numerical solution of (2) thus obtained settled down to a new non-symmetrical prestressable configuration corresponding to the modified tendon material properties. This configuration is exponentially stable. At this configuration the square equilibrium manifold. Therefore, fast and robust control of large motions is possible with small energy dissipation via linearly kinetic tendon damping.

Examples reveal the feasibility of the approach, first in the unfolding of a tensegrity simplex with 6 independent generalized coordinates, and then in the folding of a much more complex tensegrity tower with 36 independent generalized coordinates. Exceptionally good tracking is always achieved and further investigations reveal that the power dissipated via linearly kinetic tendon damping is minimal when the system’s trajectory follows closely the infinitesimal mechanism directions despite of the fact that motion is relatively fast there. On the other hand in motions away from these directions the dissipated power is much larger even though the motion is relatively slow there. Therefore, for small energy dissipation via linearly kinetic tendon damping, the geometry of the motion is more important that its speed. Analysis of tendon behavior reveals that material selection is not a major issue, especially in the light of the discovery of novel elastomers that obey the modeling assumptions made herein. Similarly, structural integrity of the structure members can be guaranteed via appropriate selection of safety coefficients and materials. Furthermore, the closed loop system is extremely robust with respect to modifications of material properties that are expected to occur when elastomers are used. Finally, the strategy is not limited to deployment between symmetrical prestressable configurations and it does not need an equilibrium path, being rather general.

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References


