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Semicontinuous Nonstationary Stochastic Games

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1. INTRODUCTION

In this paper, we study a zero-sum discrete-time nonstationary stochastic game model with dependent-on-time metric state spaces, separable metric action spaces of players, and semicontinuous admissible action mappings. The transition law in our model is a sequence of weakly continuous transition probabilities associating with every n-stage history of the game a probability distribution of the $(n+1)$th state, and the payoff is a bounded-below lower semicontinuous function defined on the space of all histories of the game. We prove, under certain additional compactness conditions, that such a game has a value, the minimizer has an optimal strategy while, for each $\varepsilon > 0$, the maximizer has an ε-optimal strategy. We also discuss a question of approximation of an infinite horizon game by means of some finite horizon subgames.

The game introduced here constitutes a natural generalization of the nonstationary stochastic game model developed in an earlier paper of the author [16] where the state and action spaces are assumed to be countable sets. It also includes the so-called continuous Markov games studied among others by Maitra and Parthasarathy [13, 14], Parthasarathy [20], Kumar and Shiau [12], and Rieder [23]. For further bibliographic notes and some comments on the model and the results obtained we refer the reader to Sections 3 and 4.

The organization of this paper is as follows. Section 2 gives some basic facts concerning transition probabilities and multifunctions. The stochastic game model is described in Section 3 and the main results are stated in Section 4. In Section 5 we establish a minimax selection theorem which is crucial in our developement. Finally, Section 6 presents the proofs of the main results.

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2. Preliminaries

Let N denote the set of positive integers, R the set of real numbers, and R^+ the set of real numbers augmented by the point $+\infty$. Let X be a metric space. We write $M(X)$ for the set of all bounded-below Borel measurable functions $w: X \to R^+$, and $\mathbf{C}(X)$ for the set of all lower semicontinuous functions in $M(X)$. By $C(X)$ we denote the set of all bounded continuous functions in $M(X)$. Let $\mathscr{B}_X$ be the σ-algebra of Borel subsets of X and let P_X be the space of all probability measures on $\mathscr{B}_X$ endowed with the weak topology (cf. [19] or [3, Chap. 7]). For any $x \in X$, we denote by p_x the probability measure on $\mathscr{B}_X$ which assigns unit point mass to x, i.e., $p_x(B) = 1$ if and only if $x \in B$. It is known that the mapping $\delta: X \to P_X$ defined by $\delta(x) = p_x$ is a homeomorphism [3, p. 130]. Let H and X be metric spaces. By a Borel measurable transition probability from H to X we mean a function $q: H \to P_X$ such that, for each $B \in \mathscr{B}_X$, $q(B \mid \cdot)$ is a Borel measurable function on H. (Here $q(B \mid \cdot)$ means $q(\cdot)(B)$.) It is known that every Borel measurable function q from H to P_X (endowed with the Borel σ-algebra) is a transition probability, and if X is also separable, then $q: H \to P_X$ is a transition probability if and only if q is Borel measurable (cf. [21, Lemma 6.1; 5, Theorem 2.1; 3, Proposition 7.25]). If q is continuous (with respect to the weak topology in P_X), then q is said to be a continuous transition probability from H to X.

Let $A: H \to \mathscr{B}_X$ be a (set-valued) mapping. For any $S \subset X$, we define

$$A^{-1}(S) = \{h \in H: A(h) \cap S \neq \varnothing\}.$$

If $A^{-1}(S)$ is closed (open) in H for each closed (open) subset S of X, then A is said to be upper (lower) semicontinuous. A mapping $A: H \to \mathscr{B}_X$ is called continuous if it is both lower and upper semicontinuous. In the sequel we shall need the following facts.

Lemma 2.1. *Let A be a lower semicontinuous mapping from a metric space H to the nonempty complete subsets of a separable metric space X. Then there exists a sequence of Borel measurable functions $f_n: H \to P_X$, $n \in N$, such that the set $\{f_n(h): n \in N\}$ is dense in $P_{A(h)}$ for each $h \in H$.*

Proof. First, we note that A is weakly measurable in the sense of Himmelberg [8]. Using [8, Theorem 5.6], we infer that there exists a sequence $\{a_m\}$ of Borel measurable functions $a_m: H \to X$ such that $\{a_m(h)\}$ is dense in $A(h)$ for each $h \in H$. Let Q be the set of all sequences $(\lambda_1, \lambda_2, ...)$ of non-negative rational numbers such that all but finitely many λ_m's are 0 and $\sum_n \lambda_n = 1$. Clearly, Q may be represented as a denumerable sequence, say $\{\lambda^n\}$. For each $\lambda^n = (\lambda_1^n, \lambda_2^n, ...) \in Q$, let $f_n: H \to P_X$ be defined by $f_n(h) = \sum_m \lambda_m^n p_{a_m(h)}$, $h \in H$. By the Borel measurability of a_m's and [3, Corollary 7.21.1], each f_n is Borel measurable. From [19, Theorem 6.3], we now conclude that, for each $h \in H$, the set $\{f_n(h): n \in N\}$ is dense in $P_{A(h)}$.

LEMMA 2.2. [11, *Lemma* 3.1]. *Let B be an upper semicontinuous mapping from a metric space H to the nonempty compact subset of a separable metric space Y. Then there exists a Borel measurable function $g: H \to P_Y$ such that $g(h) \in P_{B(h)}$ for each $h \in H$.*

Proof. By [6, Theorem 1], there exists a Borel measurable selection of B, that is, a Borel measurable function $b: H \to Y$ such that $b(h) \in B(h)$ for each $h \in H$. Defining $g: H \to P_Y$ by $g(h) = p_{b(h)}$ and using [3, Corollary 7.21.1], we complete the proof.

3. THE STOCHASTIC GAME MODEL

A *zero-sum discrete-time nonstationary stochastic game* G which we consider is defined by a sequence of objects $\{S_n, X_n, Y_n, A_n, B_n, q_n, u; n \in N\}$ having the following meaning:

(i) S_n is a metric space, endowed with the Borel σ-algebra $\mathscr{B}_{S_n}$, the *state space* at stage n.

(ii) X_n and Y_n are separable metric spaces, endowed with their Borel σ-algebras $\mathscr{B}_{X_n}$ and $\mathscr{B}_{Y_n}$, the *action spaces* of players I and II, respectively, at stage n.

Let $H_1 = S_1$, $H_n = S_1 \times X_1 \times Y_1 \times \cdots \times S_n$, and $H_\infty = S_1 \times X_1 \times Y_1 \times S_2 \times X_2 \times Y_2 \times \cdots$. Then H_n is the set of histories up to stage $n \in N$, and H_∞ is the set of all histories of the game. We assume that the sets H_n and H_∞ are given the product topologies and the product σ-algebras.

(iii) A_n (B_n) is a mapping from H_n to the nonempty complete (compact) subsets of X_n (Y_n). We assume that A_n (B_n) is lower (upper) semicontinuous for each $n \in N$. The set $A_n(h_n)$ $(B_n(h_n))$ represents the *set of admissible actions* for player I (II) under the history $h_n \in H_n$.

(iv) q_n is a continuous function from $H_n \times X_n \times Y_n$ endowed with the product topology to $P_{S_{n+1}}$ equipped with the weak topology. In other words, q_n is a (weakly) continuous transition probability from $H_n \times X_n \times Y_n$ to S_{n+1}. The sequence $\{q_n\}$ constitutes the *transition law* of the game. For given a history h_n and actions x_n and y_n chosen by the players at stage n, $q_n(\cdot \mid h_n, x_n, y_n)$ is the conditional distribution of the state at stage $n+1$.

(v) $u \in \mathbf{C}(H_\infty)$ is the *payoff function* for player I.

The game is played as follows. The players I and II observe the initial state $s_1 \in S_1$ and choose simultaneously actions $x_1 \in A_1(s_1)$ and $y_1 \in B_1(s_1)$, respectively. Then the result (x_1, y_1) is announced to both of them and the game moves to a new state $s_2 \in S_2$ according to the probability distribution $q_1(\cdot \mid s_1, x_1, y_1)$, upon which I chooses $x_2 \in A_2(h_2)$ while II chooses $y_2 \in B_2(h_2)$, $(h_2 = (s_1, x_1, y_1, s_2))$, and so on. The result of this infinite sequence of moves is a point $h = (s_1, x_1, y_1, s_2, x_2, y_2, \ldots) \in H_\infty$ and II pays I the amount $u(h)$.

Let $\mathscr{F}_n$ $(n \in N)$ be the set of all Borel measurable transition probabilities $f_n \colon H_n \to P_{X_n}$ such that $f_n(h_n) \in P_{A_n(h_n)}$ for all $h_n \in H_n$. The set $\mathscr{F}_n$ is called the set of *feasible controls* of player I at stage n. Similarly, we define the set $\mathscr{G}_n$ of *feasible controls* of player II at stage n. Under our assumption (iii) we know from Lemmas 2.1 and 2.2 that $\mathscr{F}_n$ and $\mathscr{G}_n$ are nonempty for every $n \in N$.

A (Borel measurable) *strategy* for player I (II) is a sequence $f = \{f_n\}$ $(g = \{g_n\})$ where $f_n \in \mathscr{F}_n$ $(g_n \in \mathscr{G}_n)$ for each $n \in N$. We denote by $\mathscr{F}$ $(\mathscr{G})$ the set of all strategies for player I (II).

Let $f = \{f_n\} \in \mathscr{F}$ and $g = \{g_n\} \in \mathscr{G}$. For each $n \in N$, we define a mapping $Q_{f_n g_n} \colon M(H_{n+1}) \to M(H_n)$ by

$$(Q_{f_n g_n} w)(h_n) = \iiint w(h_n, x_n, y_n, s_{n+1}) \times q_n(ds_{n+1} \mid h_n, x_n, y_n) f_n(dx_n \mid h_n) g_n(dy_n \mid h_n),$$

where $w \in M(H_{n+1})$ and $h_n \in H_n$.

According to the theorem of Ionescu Tulcea (cf. [15, p. 162] or [3, Proposition 7.28]), for each pair $f = \{f_n\}$, $g = \{g_n\}$ of strategies there exists a unique conditional probability $P_{fg}(\cdot \mid s_1)$ on $X_1 \times Y_1 \times S_2 \times X_2 \times Y_2 \times \cdots$, endowed with the product σ-algebra, given the initial state s_1 such that, for every $w \in M(H_{n+1})$, we have

$$\int w(s_1, h) \, P_{fg}(dh \mid s_1) = (Q_{f_1 g_1} \cdots Q_{f_n g_n} w)(s_1), \qquad s_1 \in S_1. \tag{3.1}$$

Thus, each pair $(f, g) \in \mathscr{F} \times \mathscr{G}$ defines an *expected payoff* to player I in the game G at an initial state $s_1 \in S_1$ to be

$$E(u, f, g)(s_1) = \int u(s_1. h) \, P_{fg}(dh \mid s_1).$$

From (v) and the Ionescu Tulcea theorem [15], it follows that $E(u, f, g)$ is a bounded below Borel measurable function of the initial state.

In the sequel we shall make use of the following result of Kertz and Schäl (cf. [26, p. 209; 27, p. 361]).

LEMMA 3.1. *A function* $u: H_\infty \to R^+$ *belongs to* $\mathbf{C}(H_\infty)$ *if and only if there exist functions* $u_n \in C(H_n)$, $n \in N$, *such that* $u_n \leqslant u_{n+1}$ *for each* $n \in N$, *and* $u = \lim_n u_n$.

Of course, it is assumed above that every function u_n on H_n is also a function on H_∞ depending on the first $3n-2$ coordinates only.

Now, from Lemma 3.1, the monotone convergence theorem and (3.1), we infer that

$$E(u, f, g) = \lim_n E(u_{n+1}, f, g) = \lim_n Q_{f_1 g_1} \cdots Q_{f_n g_n} u_{n+1}, \tag{3.2}$$

for each $f = \{f_n\} \in \mathscr{F}$ and $g = \{g_n\} \in \mathscr{G}$.

Define, for each $s_1 \in S_1$,

$$L(G)(s_1) = \sup_{f \in \mathscr{F}} \inf_{g \in \mathscr{G}} E(u, f, g)(s_1)$$

and

$$U(G)(s_1) = \inf_{g \in \mathscr{G}} \sup_{f \in \mathscr{F}} E(u, f, g)(s_1).$$

Then $L(G)$ ($U(G)$) is called the *lower* (*upper*) *value function* of the game G. It is alway true that $L(G) \leqslant U(G)$. If $L(G) = U(G)$, this common function is called the *value function* of the game G and will be denoted by $V(G)$. Suppose that the value function exists and define $D = \{s_1 \in S_1 : V(G)(s_1) < +\infty\}$. Let $\varepsilon > 0$ be given.

A strategy $\bar{f} \in \mathscr{F}$ is called *ε-optimal* for player I if

$$\inf_{g \in \mathscr{G}} E(u, \bar{f}, g)(s_1) + \varepsilon \geqslant V(G)(s_1) \qquad \text{for all } s_1 \in D,$$

and

$$\inf_{g \in \mathscr{G}} E(u, \bar{f}, g)(s_1) \geqslant 1/\varepsilon \qquad \text{for all } s_1 \in S_1 - D.$$

A strategy $\bar{g} \in \mathscr{G}$ is called *optimal* for player II if

$$\sup_{f \in \mathscr{F}} E(u, f, \bar{g})(s_1) \leqslant V(G)(s_1) \qquad \text{for all } s_1 \in S_1.$$

The aim of this paper is to prove that the game G has a value function, player II has an optimal strategy while player I has an ε-optimal strategy for each $\varepsilon > 0$.

To simplify the derivations in the sequel, we employ the following operator terminology. For each $n \in N$ and $w \in M(H_{n+1})$, define the functions $L_n w$ and $U_n w$ on H_n by

$$(L_n w)(h_n) = \sup_{f_n \in \mathscr{F}_n} \inf_{g_n \in \mathscr{G}_n} (Q_{f_n g_n} w)(h_n)$$

and

$$(U_n w)(h_n) = \inf_{g_n \in \mathscr{G}_n} \sup_{f_n \in \mathscr{F}_n} (Q_{f_n g_n} w)(h_n),$$

where $h_n \in H_n$. If $L_n w = U_n w$ for some $w \in M(H_{n+1})$, then this common function will be denoted by $V_n w$.

Remark 3.1. The game model introduced here constitutes a generalization of the so-called *continuous* (*discounted* and *positive*) *Markov game* model described below. Assume that $S_n = S$, $X_n = X$, and $Y_n = Y$, for each $n \in N$, i.e., the state and action spaces are independent of time. Let $q_n = q$ for some continuous transition probability q from $S \times X \times Y$ to S, that is, $q_n(\cdot \mid h_n, x_n, y_n) = q(\cdot \mid s_n, x_n, y_n)$ for every $h_n = (s_1, x_1, y_1, \ldots, s_n) \in H_n$, $x_n \in X_n$, $y_n \in Y_n$, $n \in N$. Such a transition law is called *stationary*. Assume further that the payoff is accumulated over stages with a discount factor $\beta \in [0, 1]$, so that for each history $h = (s_1, x_1, y_1, \ldots) \in H_\infty$,

$$u(h) = \sum_{n=1}^{\infty} \beta^{n-1} r(s_n, x_n, y_n),$$

where r is a bounded continuous and nonnegative (payoff per stage) function on $S \times X \times Y$. A Markov game is called discounted (positive) when $\beta < 1$ ($\beta = 1$).

The theory of zero-sum Markov games started with the fundamental paper of Shapley [30], in which the state and action spaces are assumed to be finite sets. Maitra and Parthasarathy have first studied Markov games with uncountable state and action spaces [13, 14]. They have assumed that S, X and Y are compact metric spaces. For further generalizations of Shapley's work allowing the state space in a Markov game to be a (standard) Borel (or even a metric) space we refer the reader to [4, 12, 17, 20, 23] and the references therein.

Remark 3.2. The game developed here is inspired by a nonstationary stochastic control system introduced by Hinderer [10] and subsequently studied by Schäl [25–28] and Kertz and Nachman [11]. Especially, this paper owes much to Schäl's work [26, 27].

Remark 3.3. A special case of the game introduced here has already been studied in [16] under the assumption that the state and action spaces are countable sets. Some different nonstationary stochastic games have been studied by Sengupta [29] and Schäl [28]. In the model of Sengupta [29] the state space is a compact metric space, the action spaces are finite, the payoff is lower semicontinuous on H_∞, but the transition law is stationary. Schäl has considered in [28] a game with Borel state and action spaces in which, at every stage $n \in N$, player I (similarly, player II) knows the sequence of states $(s_1, s_2, ..., s_n)$ occurring up to this stage and his own previous choices only.

Remark 3.4. In a subsequent paper [18], we investigate an alternative framework for zero-sum discrete-time nonstationary stochastic games involving universally measurable strategies. We assume there that S_n, X_n and Y_n are Borel spaces and the payoff u depends semicontinuously on the actions of one player only. It is also assumed that each transition probability q_n is continuous in the actions of one player only, but with respect to the strong topology in $P_{S_{n+1}}$.

4. Main Results

In the first place we shall consider the so-called *finite horizon games* in which the payoffs are decided in a finite number of stages.

Let $u_{m+1} \in \mathbf{C}(H_{m+1})$ be a bounded function. For each $n \leqslant m$, we denote by G_n^m a game which has the payoff function $u = u_{m+1}$ and proceeds from an arbitrary history $h_n \in H_n$ till stage m. (The games G_n^m, $m \geqslant n$, will play the crucial role in the analysis of the infinite horizon game G, cf. Sect. 6.) A strategy for player I (II) in such a game is simply a sequence $f = (f_n, ..., f_m)$ $(g = (g_n, ..., g_m))$, where $f_k \in \mathscr{F}_k$ $(g_k \in \mathscr{G}_k)$, $k = n, ..., m$. Let f and g be arbitrary strategies for players I and II, respectively, in the game G_n^m. Then the expected payoff to player I corresponding to f and g at a partial history $h_n \in H_n$ is given by

$$E(u_{m+1}, f, g)(h_n) = (Q_{f_n g_n} \cdots Q_{f_m g_m} u_{m+1})(h_n).$$

Of course, the value functions $L(G_n^m)$, $U(G_n^m)$, $V(G_n^m)$ and the optimal strategies of both players in the game G_n^m are defined just as in the game G.

We are now prepared to state the main results of this paper.

THEOREM 4.1. *The game G_n^m has a value function $V(G_n^m)$, player* II *has an optimal strategy, and for each $\varepsilon > 0$, player* I *has an ε-optimal strategy. Moreover, $V(G_n^m) \in \mathbf{C}(H_n)$ and*

$$V(G_n^m) = V_n \cdots V_m u_{m+1}.$$

Let $\{u_m\}$ be the sequence of functions from Lemma 3.1. For each $m \in N$, let $G^m = G_1^m$ be an m-stage game corresponding to the payoff u_{m+1}.

We now turn to the game G from Section 3. We have the following result.

THEOREM 4.2. *The game G has a value function $V(G)$, player* II *has an optimal strategy, and for each $\varepsilon > 0$, player* I *has an ε-optimal strategy. Moreover, $V(G) \in \mathbf{C}(S_1)$ and*

$$V(G) = \lim_m V(G^m).$$

COROLLARY 4.1. *Assume* (i), (ii), *and* (iv). *If in addition the admissible action mappings A_n and B_n ($n \in N$) are continuous and compact-valued while the payoff u is bounded and continuous on H_∞, then player* I *has an optimal strategy too.*

Remark 4.1. Suppose u is lower semicontinuous on H_∞. Then player I need not have optimal strategies even when his action spaces are finite. Often such a situation takes place in the positive Markov games [12, Example 1]. Therefore we do not make any compactness assumptions in Theorem 4.2 regarding the sets of admissible actions for player I.

Remark 4.2. Theorem 4.1 generalizes [16, Proposition 2.1] while Theorem 4.2 extends [16, Theorem 4.1] where the state and action spaces are assumed to be countable sets.

5. A MINIMAX SELECTION THEOREM

In this section we establish a minimax selection theorem which is crucial in our development. A related result was given by Rieder [23, Theorem 4.1] in the context of Markov games.

THEOREM 5.1. *Assume* (i)–(iv). *If $w_{n+1} \in \mathbf{C}(H_{n+1})$, then*

$$L_n w_{n+1} = U_n w_{n+1} = V_n w_{n+1}, \qquad \textit{and} \qquad V_n w_{n+1} \in \mathbf{C}(H_n).$$

Moreover, there exists $\bar{g}_n \in \mathscr{G}_n$ such that

$$V_n w_{n+1} = \sup_{f_n \in \mathscr{F}_n} Q_{f_n \bar{g}_n} w_{n+1}, \tag{5.1}$$

and if $w_{n+1} \in \mathbf{C}(H_{n+1})$ is bounded, then, for each $\varepsilon > 0$, there exists $\bar{f}_n \in \mathscr{F}_n$ such that

$$V_n w_{n+1} \leqslant \inf_{g_n \in \mathscr{G}_n} Q_{\bar{f}_n g_n} w_{n+1} + \varepsilon. \tag{5.2}$$

Before we prove the above result, let us state some auxiliary lemmas.

LEMMA 5.1. *Let H and T be metric spaces and let $w \in \mathbf{C}(H \times T)$. Then there exists a sequence of functions $w_n \in C(H \times T)$, $n \in N$, such that each w_n is continuous in $h \in H$, uniformly in $t \in T$, and $w_n \nearrow w$ as $n \to \infty$.*

Proof. Denote by d_1 and d_2 the metrics in H and T, respectively. Define

$$\varphi_n(h, t) = \inf_{(a,b) \in H \times T} [w(a, b) + n(d_1(a, h) + d_2(b, t))],$$

and

$$w_n(h, t) = \min\{\varphi_n(h, t), n\}, \qquad n \in N, (h, t) \in H \times T.$$

By the proof of the theorem of Baire [1, p. 390], $\varphi_n \nearrow w$ as $n \to \infty$. Hence $w_n \nearrow w$ as $n \to \infty$. Note that $w_n \in C(H \times T)$ for each $n \in N$. Moreover, for all $h_1, h_2 \in H$, $t \in T$, and $n \in N$, we have

$$|w_n(h_1, t) - w_n(h_2, t)| \leqslant n d_1(h_1, h_2),$$

which implies that each w_n is continuous in $h \in H$, uniformly in $t \in T$.

LEMMA 5.2. *Let H and T be metric spaces and let $w \in C(H \times T)$. Assume that w is continuous in $h \in H$, uniformly in $t \in T$, and define the function $\hat{w}: H \times P_T \to R$ by*

$$\hat{w}(h, p) = \int w(h, t)\, p(dt), \qquad (h, p) \in H \times P_T.$$

Then $\hat{w} \in C(H \times P_T)$.

Proof. The proof is straightforward.

The following lemma is an extension of [3, Proposition 7.31].

LEMMA 5.3. *Let H, T be metric spaces, $x \in \mathbf{C}(H \times T)$, and let $q: H \to P_T$ be a continuous transition probability from H to T. Define $\lambda: H \to R^+$ by*

$$\lambda(h) = \int w(h, t)\, q(dt \mid h), \qquad h \in H. \tag{5.3}$$

Then $\lambda \in \mathbf{C}(H)$.

Proof. Let $\{w_n\}$ be the sequence from Lemma 5.1. Define λ_n by (5.3) where w is replaced by w_n. It is easy to deduce from Lemma 5.2 that $\lambda_n \in C(H)$ for each $n \in N$. By the monotone convergence theorem, $\lambda = \sup_n \lambda_n$, which implies that $\lambda \in \mathbf{C}(H)$.

The following lemma extends [25, Lemma 3.4].

LEMMA 5.4. *Let H be a metric space, X, Y separable metric spaces, and let $w \in \mathbf{C}(H \times X \times Y)$. Define $\hat{w}: H \times P_X \times P_Y \to R^+$ by*

$$\hat{w}(h, p, r) = \iint w(h, x, y)\, p(dx)\, r(dy), \qquad (h, p, r) \in H \times P_X \times P_Y. \quad (5.4)$$

Then $\hat{w} \in \mathbf{C}(H \times P_X \times P_Y)$.

Proof. Put $T = X \times Y$ and take the sequence $\{w_n\}$ from Lemma 5.1. Let $\hat{w}_n$ be defined by (5.4) where w is replaced by w_n. By [3, Lemma 7.12] and Lemma 5.2, $\hat{w}_n \in C(H \times P_X \times P_Y)$ for each $n \in N$. From the monotone convergence theorem, we infer that $\hat{w} = \sup_n \hat{w}_n$. Thus, $\hat{w} \in \mathbf{C}(H \times P_X \times P_Y)$.

LEMMA 5.5. *Let $w_{n+1} \in \mathbf{C}(H_{n+1})$, $n \in N$. Assume* (iv) *and define $K_n: H_n \times P_{X_n} \times P_{Y_n} \to R^+$ by*

$$K_n(h_n, p, r) = \iiint w_{n+1}(h_n, x_n, y_n, s_{n+1})\, q_n(ds_{n+1} \mid h_n, x_n, y_n)\, p(dx_n)\, r(dy_n), \quad (5.5)$$

$(h_n, p, r) \in H_n \times P_{X_n} \times P_{Y_n}$. Then $K_n \in \mathbf{C}(H_n \times P_{X_n} \times P_{Y_n})$.

Proof. This follows directly from Lemmas 5.3 and 5.4.

Now we are ready to prove the theorem.

Proof of Theorem 5.1. The proof utilizes some arguments given by Rieder in the proof of Theorem 4.1 in [23]. We recall that the mapping A_n (B_n) is lower semicontinuous (upper semicontinuous and compact-valued). By [3, Proposition 7.22], for each $h_n \in H_n$, $P_{B_n(h_n)}$ is a compact subset of P_{Y_n}. From [9, Theorem 3], we know that the mapping $h_n \to P_{B_n(h_n)}$ is upper semicontinuous too.

Note that

$$\begin{aligned}(U_n w_{n+1})(h_n) &= \inf_{r \in P_{B_n(h_n)}} \sup_{p \in P_{A_n(h_n)}} K_n(h_n, p, r) \\ &= \inf_{r \in P_{B_n(h_n)}} \sup_{x \in A_n(h_n)} K_n(h_n, p_x, r), \end{aligned} \quad (5.6)$$

and

$$(L_n w_{n+1})(h_n) = \sup_{p \in P_{A_n(h_n)}} \inf_{r \in P_{B_n(h_n)}} K_n(h_n, p, r), \quad (5.7)$$

where $h_n \in H_n$ and K_n is the function (5.5).

From (5.6), (5.7), Lemma 5.5, and the Fan minimax theorem [7, Theorem 2], we get $L_n w_{n+1} = U_n w_{n+1} = V_n w_{n+1}$. The fact that $V_n w_{n+1} \in \mathbf{C}(H_n)$ follows now from (5.6), Lemma 5.5 and the theorems of

Berge [2, pp. 115, 116] while the existence of $\bar{g}_n \in \mathcal{G}_n$ satisfying (5.1) follows from [22, Theorem 4.9]. It remains to prove that there exists $\bar{f}_n \in \mathcal{F}_n$ satisfying (5.2). By Lemma 2.1, there exists a sequence $\{f_n^m\} \subset \mathcal{F}_n$ such that, for each $h_n \in H_n$, the set $\{f_n^m(h_n): m \in N\}$ is dense in $P_{A_n(h_n)}$. By Lemma 5.5 and the theorem of Berge [2, p. 115], the function $M_n: H_n \times P_{X_n} \to R$ defined by

$$M_n(h_n, p) = \inf_{r \varepsilon P_{B_n(h_n)}} K_n(h_n, p, r)$$

is lower semicontinuous in p, for each $h_n \in H_n$. Consequently, the equality (5.7) can be rewritten as

$$(V_n w_{n+1})(h_n) = (L_n w_{n+1})(h_n) = \sup_{m \in N} \inf_{r \varepsilon P_{B_n(h_n)}} M_n(h_n, f_n^m(h_n)), \qquad (5.8)$$

where $h_n \in H_n$.

Let $\varepsilon > 0$ be given. Let $\{E_m\}$ be a sequence of subsets of H_n defined by

$$E_1 = \{h_n \in H_n: (V_n w_{n+1})(h_n) \leqslant M_n(h_n, f_n^1(h_n)) + \varepsilon\},$$

and (5.9)

$$E_m = \{h_n \in H_n: (V_n w_{n+1})(h_n) \leqslant M_n(h_n, f_n^m(h_n)) + \varepsilon\} - \bigcup_{k=1}^{m-1} E_k, \quad \text{for } m \geqslant 2.$$

Clearly, $E_k \cap E_m = \varnothing$ for $k \neq m$, and by (5.8), $\bigcup_{m \in N} E_m = H_n$. Moreover, $E_m \in \mathcal{B}_{H_n}$ for each $m \in N$. Let $K = \{m_k \in N: E_{m_k} \neq \varnothing\}$. Define $D_k = E_{m_k}$, where $m_k \in K$. Clearly, $\{D_k\}$ is a measurable partition of H_n. Now, let us define $\bar{f}_n \in \mathcal{F}_n$ by

$$\bar{f}_n(h_n) = f_n^{m_k}(h_n) \qquad \text{when } h_n \in D_k.$$

Then from (5.9), it follows that $\bar{f}_n$ satisfies (5.2). Thus, the proof is complete.

6. Proofs of the Main Results

The proof of Theorem 4.1 is based on Theorem 5.1 and proceeds along similar lines as that of Proposition 2.1 in [16], where the state and action spaces are countable sets.

Proof of Theorem 4.1. The proof proceeds by induction. Let $\varepsilon > 0$ and $n \in N$ be given. If $m = n$, then the result follows immediately from Theorem 5.1. Fix $m \geqslant n$ and suppose the result holds for every game G_n^m.

Consider an arbitrary game G_n^{m+1} with a bounded payoff $u_{m+2} \in \mathbf{C}(H_{m+2})$. By Theorem 5.1,

$$L_{m+1} u_{m+2} = U_{m+1} u_{m+2} = V_{m+1} u_{m+2},$$

and there exist $\bar{f}_{m+1} \in \mathscr{F}_{m+1}$ and $\bar{g}_{m+1} \in \mathscr{G}_{m+1}$ such that

$$Q_{f_{m+1}\bar{g}_{m+1}} u_{m+2} \leqslant V_{m+1} u_{m+2} \leqslant Q_{\bar{f}_{m+1}g_{m+1}} u_{m+2} + \varepsilon/2, \tag{6.1}$$

for every $f_{m+1} \in \mathscr{F}_{m+1}$ and $g_{m+1} \in \mathscr{G}_{m+1}$.

Let G_n^m be the game with the payoff $u_{m+1} = V_{m+1} u_{m+2}$. Clearly, u_{m+1} is bounded and from Theorem 5.1, it follows that $u_{m+1} \in \mathbf{C}(H_{m+1})$. By our induction hypothesis the game G_n^m has a value function $V(G_n^m)$, $V(G_n^m) = V_n \cdots V_m u_{m+1}$, player II has an optimal strategy, say g', and player I has an $\varepsilon/2$-optimal strategy, say f'. Let $\bar{f} = (f', \bar{f}_{m+1})$ and $\bar{g} = (g', \bar{g}_{m+1})$. Then from our induction hypothesis and (6.1), it follows that

$$E(u_{m+2}, f, \bar{g}) \leqslant V(G_n^m) \leqslant E(u_{m+2}, \bar{f}, g) + \varepsilon,$$

for every strategies f and g for players I and II, respectively. Hence

$$U(G_n^{m+1}) \leqslant V(G_n^m) \leqslant L(G_n^{m+1}) + \varepsilon.$$

This implies that $V(G_n^{m+1})$ exists and $V(G_n^{m+1}) = V(G_n^m) = V_n \cdots V_m V_{m+1} u_{m+2}$. Moreover, $\bar{g}$ is an optimal strategy for player II in G_n^{m+1} while $\bar{f}$ is an ε-optimal strategy for player I in G_n^{m+1}. Thus, the result follows.

Throughout the sequel, let $\{u_m\}$ be the sequence from Lemma 3.1. Let G_n^m be the finite horizon game with the payoff function $u = u_{m+1}$. By Theorem 4.1, every game G_n^m has a value function $V(G_n^m)$, which belongs to $\mathbf{C}(H_n)$. The sequence $\{V(G_n^m)\}$ is nondecreasing, for each $n \in N$, because so is $\{u_m\}$. Therefore, for each $n \in N$, we can define

$$W_n = \lim_m V(G_n^m).$$

Clearly, $W_n \in \mathbf{C}(H_n)$, for each $n \in N$. Since $V(G_1^m) = V(G^m) = L(G^m) \leqslant L(G)$, for every $m \in N$, so we have

$$W_1 = \lim_m L(G^m) \leqslant L(G). \tag{6.2}$$

Now we state some auxiliary lemmas.

LEMMA 6.1. *For each $k<n$, we have $u_k \leqslant W_n$.*

Proof. Let $k<n$. We have $u_k(h_k) \leqslant u_{n+1}(h_k, h)$ for every $h_k \in H_k$, and $h \in X_k \times Y_k \times S_{k+1} \times X_{k+1} \times Y_{k+1} \times \cdots \times S_{n+1}$. Hence $u_k \leqslant V(G_n^n) \leqslant W_n$, which completes the proof.

LEMMA 6.2. *Let X be a set, Y a compact metric space, and $w_n: X \times Y \to R^+$, $n \in N$, a sequence of functions. Assume that $w_n \leqslant w_{n+1}$, and $w_n(x, \cdot) \in \mathbf{C}(Y)$ for each $x \in X$ and $n \in N$. Then*

$$\lim_n \inf_{y \in Y} \sup_{x \in X} w_n(x, y) = \inf_{y \in Y} \sup_{x \in X} \lim_n w_n(x, y).$$

Proof. This follows from the fact that the upper envelope of a family of lower semicontinuous functions is a lower semincontinuous function and from [24, Proposition 10.1].

LEMMA 6.3. *For each $n \in N$, we have $W_n = V_n W_{n+1}$.*

Proof. By Theorem 4.1, for every $m \geqslant n+1$, we have $V(G_n^m) = V_n V(G_{n+1}^m)$. Moreover, we know that $V(G_k^m) \in \mathbf{C}(H_k)$ for every $k \leqslant m$. From Lemma 5.5, the compactness of sets $P_{B_n(h_n)}$, $h_n \in H_n$, the monotone convergence theorem, and Lemma 6.2, it follows that

$$W_n = \lim_m V(G_n^m) = \lim_m V_n V(G_{n+1}^m) = V_n \lim_m V(G_{n+1}^m) = V_n W_{n+1},$$

which terminates the proof.

Proof of Theorem 4.2. We have already noted that $W_n \in \mathbf{C}(H_n)$, for each $n \in N$. By Theorem 5.1, for each $n \in N$, there exists $\bar{g}_n \in \mathscr{G}_n$ such that

$$V_n W_{n+1} = \sup_{f_n \in \mathscr{F}_n} Q_{f_n \bar{g}_n} W_{n+1}. \tag{6.3}$$

Let $U_{\bar{g}_n} W_{n+1}$ denote the right-hand side of (6.3). Let $f = \{f_n\} \in \mathscr{F}$ be arbitrary and let $k \in N$. Using Lemmas 6.3 and 6.1, we get

$$\begin{aligned} W_1 = V_1 \cdots V_n W_{n+1} = U_{\bar{g}_1} \cdots U_{\bar{g}_n} W_{n+1} &\geqslant Q_{f_1 \bar{g}_1} \cdots Q_{f_n \bar{g}_n} W_{n+1} \\ \geqslant Q_{f_1 \bar{g}_1} \cdots Q_{f_n \bar{g}_n} u_{k+1} = Q_{f_1 \bar{g}_1} \cdots Q_{f_k \bar{g}_k} u_{k+1}, &\qquad \text{where } n > k. \end{aligned}$$

This and (3.1) imply

$$W_1 \geqslant E(u_{k+1}, f, \bar{g}),$$

where $\bar{g}=\{\bar{g}_n\}$, f is an arbitrary strategy for player I, and $k \in N$. By (3.2), we get

$$W_1 \geqslant \lim_k E(u_{k+1}, f, \bar{g}) = E(u, f, \bar{g}),$$

for each $f \in \mathscr{F}$. Hence

$$W_1 \geqslant \sup_{f \in \mathscr{F}} E(u, f, \bar{g}) \geqslant U(G).$$

This and (6.2) imply that the game G has a value function $V(G)$, $V(G) = \lim_m V(G^m) = W_1 \in \mathbf{C}(S_1)$, and $\bar{g}$ is an optimal strategy for player II.

Let $\varepsilon > 0$ be given. We shall construct an ε-optimal strategy for player I. Recall that $D = \{s_1 \in S_1 : V(G)(s_1) < +\infty\}$. By Theorem 4.1, for each $m \in N$, player I has an $\varepsilon/2$-optimal strategy in the game G^m, say $f^m = (f_1^m, ..., f_m^m)$. Let $f = \{f_n\} \in \mathscr{F}$ be fixed. Define $\bar{f}^m \in \mathscr{F}$ by $\bar{f}^m = (f_1^m, ..., f_m^m, f_{m+1}, f_{m+2}, ...)$.

We have already shown that $V(G) = \lim_m V(G^m)$. Thus, in a standard way (cf. the proof of Theorem 5.1) we can find a measurable partition $\{D_k\}$ of D ($k \in K_1 \subset N$), a measurable partition $\{E_k\}$ of $S_1 - D$ ($k \in K_2 \subset N$), and subsets $\{G^{m_k}\}$ and $\{G^{m'_k}\}$ of $\{G^m\}$ such that

$$V(G^{m_k})(s_1) + \varepsilon/2 \geqslant V(G)(s_1) \qquad \text{when } s_1 \in D_k,$$

and

$$V(G^{m'_k})(s_1) \geqslant 1/\varepsilon + \varepsilon/2 \qquad \text{when } s_1 \in E_k.$$

Let $k \in K_1$ and $s_1 \in D_k$. Then we have

$$\begin{aligned} \varepsilon + \inf_{g \in \mathscr{G}} E(u, \bar{f}^{m_k}, g)(s_1) &\geqslant \varepsilon + \inf_{g \in \mathscr{G}} E(u_{m_k+1}, \bar{f}^{m_k}, g)(s_1) \\ &\geqslant \varepsilon/2 + V(G^{m_k})(s_1) \geqslant V(G)(s_1). \end{aligned} \tag{6.4}$$

Let $k \in K_2$ and $s_1 \in E_k$. Then we have

$$\begin{aligned} \varepsilon/2 + \inf_{g \in \mathscr{G}} E(u, \bar{f}^{m'_k}, g)(s_1) &\geqslant \varepsilon/2 + \inf_{g \in \mathscr{G}} E(u_{m'_k+1}, \bar{f}^{m'_k}, g)(s_1) \\ &\geqslant V(G^{m'_k})(s_1) \geqslant 1/\varepsilon + \varepsilon/2. \end{aligned}$$

Hence

$$\inf_{g \in \mathscr{G}} E(u, \bar{f}^{m'_k}, g)(s_1) \geqslant 1/\varepsilon. \tag{6.5}$$

Let $\bar{f}$ be a strategy for player I relying on using $\bar{f}^{m_k}(\bar{f}^{m'_k})$ when the initial state s_1 belongs to D_k (E_k). By (6.4) and (6.5), such a strategy $\bar{f}$ is ε-optimal for player I. Thus, the proof is complete.

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