An arithmetical view to first-order logic

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A B S T R A C T
A value space is a topological algebra B equipped with a non-empty family of continuous quantifiers Q : B∗ → B. We will describe first-order logic on the basis of B. Operations of B are used as connectives and its relations are used to define statements. We prove under some normality conditions on the value space that any theory in the new setting can be represented by a classical first-order theory.

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1. Introduction

The Boolean notion of truth has always been central to mathematical reasoning. First-order logic (FOL) is based on this notion of truth. For example, by φ → ψ the following is usually meant: if φ is true then so is ψ. It is however natural to read this statement as: the truth value of ψ is not less than that of φ. In particular, the modus ponens rule may be written as

\[
\frac{\phi \leq \psi \quad \psi \leq \theta}{\psi \leq \theta}.
\]

This expression is very close to cancelation rules in arithmetic first introduced by the Persian mathematician Khwārazmi [5]:

\[
\begin{align*}
\phi & \leq \psi \\
\phi + \theta & \leq \psi + \theta & \text{(Al-Jabr)} \\
\phi + \theta & \leq \psi + \theta \\
\phi & \leq \psi & \text{(Al-Mughābala)}.
\end{align*}
\]

Historically, these rules were first considered as proof rules in arithmetic. Although the family of Boolean logics and particularly FOL have proved to be the most fruitful forms of logic, mathematicians usually change their components in order to fit their needs. However, it is interesting to see how the cancelation rules can be used in a different context.
to obtain expressive power for stating new mathematical notions. Metric logic [1] and probability logic [4] are important examples of logics obtained in this way. Recently, Ben-Yaacov and Poizat [2] proposed a generalization of first-order logic on the basis of the logical operations \( \land, \lor \) and \( \exists \). They of course recover the expressive power of FOL by injecting negation in the language as a possibility. This generalization has the advantage of accepting non-injective homomorphisms.

In this paper we give a rather arithmetical presentation of FOL by mainly replacing the notion of truth by the notion of value and using any fixed collection of connectives and quantifiers as logical operations. This framework is sufficiently general and has the possibility of accepting non-injective homomorphisms. However, to reduce complication, we take equations \( \phi = r \) as instances of statements forcing homomorphisms to be injective. To have a comparison with FOL, let us recall the situation there. FOL is a two-valued logic whose logical tools are all taken from \( \mathcal{B} = \{0, 1\} \) and its Boolean structure. More precisely, every Boolean operation is used as a connective and sup (or \( \exists \)) as a quantifier. Usually, the unit interval is regarded as an immediate generalization of \( \{0, 1\} \) as a value space. Metric space model theory (see [11]) uses any compact interval \([0, K]\) as a value space. This newly developed branch of model theory is convenient for studying metric spaces equipped with a family of uniformly continuous relations and operations. Metric model theory is itself a refinement of the general continuous model theory developed in [3] where any compact Hausdorff space is used as a value space. In this paper we mostly follow the lines of [3] in a rather discrete form, i.e. we allow discrete values but continuous logical operations. In other words, for each formula we assume a finite number of values while we perform continuous operations (e.g. addition) on the set of formulas. Recall that the Boolean value space \( \{0, 1\} \) may be regarded as a finite field. So, it is natural to replace it by any commutative ring, using ring operations as connectives and \( \Sigma, \Pi \) as quantifiers. Such value spaces are topologically discrete. In fact, any topological algebra equipped with a family of continuous quantifiers may be used as a value space. Given a value space \( \mathcal{B} \), we develop the basics of model theory with values in \( \mathcal{B} \). Then we show that when the value space enjoys some normality conditions, the corresponding logic is just a deformation of the classical first-order logic. The main goal of the paper is to provide a flexible framework for first-order logic. Many examples of theories are more naturally expressed in this way and we believe that the change of viewpoint leads to a better understanding of them. Below, after giving examples of value spaces, to reduce complications, we first consider the case where the value space is finite. Then we continue with the general case.

2. Value spaces

A topological algebra is a topological space \( \mathcal{B} \) equipped with a non-empty family \( \alpha, \beta, \ldots \) of continuous operations and a family \( =, \pi, \rho, \ldots \) of closed relations on it. A typical example of topological algebras is \((\mathbb{R}, +, \times, \leq)\). We always assume that \( \mathcal{B} \) contains two distinguished elements \( 0 \neq 1 \). Let \( \mathcal{B} \) be a topological algebra and \( \mathcal{B}^* \) the family of all its non-empty finite subsets.

**Definition 2.1.** A (unary) semi-quantifier on \( \mathcal{B} \) is any function \( \circ : \mathcal{B}^* \to \mathcal{B} \). If moreover \( \circ(r) = r \) for any \( r \in \mathcal{B} \), we call it a quantifier. A semi-quantifier \( \circ \) is localized at \( r_0 \) if for each \( A, \circ(A) = r_0 \) if and only if \( r_0 \in A \).

We usually say \( \circ \) is localized to mean it is localized at \( 0 \). We further need the semi-quantifiers to be continuous. For this purpose, we regard \( \circ \) as a sequence of functions \( \circ^n : \mathcal{B}^n \to \mathcal{B} \) defined by \( \circ^n(r_1, \ldots, r_n) = \circ\{r_1, \ldots, r_n\} \). So, the following properties hold:

- \( \circ^n(r_1, \ldots, r_n) \) is symmetric;
- \( \circ^n = \circ^n+1 \) where \( n \) is odd.

Conversely, any sequence of functions \( \circ^n : \mathcal{B}^n \to \mathcal{B} \) satisfying these properties defines a semi-quantifier. Then \( \circ \) is said to be continuous if every \( \circ^n \) is continuous where \( \mathcal{B}^n \) is endowed with the product topology. If \( \mathcal{B} \) is a pure algebra, any semi-quantifier is continuous.

**Definition 2.2.** A value space is a Hausdorff topological algebra \( \mathcal{B} \) equipped with a non-empty collection of continuous quantifiers (note that \( 0, 1 \in \mathcal{B} \)).

Any value space \( \mathcal{B} \) may be regarded as a first-order structure in a language containing symbols for its operations, relations and elements. We use sanserif letters \( u, v, \ldots \) as variables (or rather meta-variables) in that language and \( r, s, \ldots \) for elements of \( \mathcal{B} \). Also, by a meta-term we mean a term \( \tau(v_1, \ldots, v_k) \) in that language. Note that if \( \circ \) is a semi-quantifier of \( \mathcal{B} \), then for any meta-term \( \tau(v) \),

\[ \circ \circ \tau(r_1, \ldots, r_k) = \circ\{\tau(r_1), \ldots, \tau(r_k)\} \]

is a semi-quantifier. Similarly, \( \tau \circ, \circ \tau, \circ\circ \) and similar compositions are all semi-quantifiers on \( \mathcal{B} \) (e.g. \( \forall = \circ \exists \circ \)). By a semi-quantifier on \( \mathcal{B} \) we mean any one obtained in this way using its quantifiers.

**Definition 2.3.** A value space \( \mathcal{B} \) is conjunctive (resp. disjunctive) if for any compact \( X \subseteq \mathcal{B} \) and \( (r_1, \ldots, r_k) \in X^k \), there is a meta-term \( \tau(v_1, \ldots, v_k) \) such that for any \( (r'_1, \ldots, r'_k) \in X^k \), one has that \( \tau(r'_1, \ldots, r'_k) = 0 \) iff \( r'_1 = r_1, \ldots, r'_k = r_k \) (resp. \( r'_1 = r_1 \) or \( \ldots \) or \( r'_k = r_k \)).

There are many continuous quantifiers in nature. Below we recall some value spaces and quantifiers on them. It is easy to see that any finite disjunctive value space is conjunctive.
1. Pointed quantifier on a (pure) universal algebra $B$: Let $\varnothing[r] = r$ for any $r$ and, if $A$ has more than one element, $\varnothing(A) = 0$ if $0 \in A$ and $= 1$ otherwise. Then $\varnothing$ is a localized quantifier.

2. Let $(B, +, \cdot, 0)$ be a commutative ring (with the discrete topology). Then $\Sigma$ and $\Pi$ define quantifiers on $B$. If addition is divisible, $\frac{\sum_i r_i}{n}$ defines the ‘mean’ quantifier on $B$. Any integral domain equipped with $\Pi$ is localized. In $\mathbb{Z}_2$ we have $\Pi = \forall$ and $\Sigma = \exists$. Also, $(\mathbb{Z}_4, \Pi)$ is localized, but not conjunctive (see below). If $B$ is possibly a non-commutative ordered ring, the ordered product $\Pi$ is again a quantifier. The following map is a quantifier on $\mathbb{R}^+$ for any $p > 0$:

$$\varnothing_p : \{r_1, \ldots, r_n\} \mapsto \left(\frac{p}{r_1 + \cdots + r_n}\right)^p.$$  

Note that $\varnothing_p$ is not continuous (e.g. at $(1, 1)$) with respect to the Euclidean topology.

3. Let $(\mathbb{A}, +, \cdot, 0, 1)$ be a discretely ordered commutative ring which is also a unique factorization domain (e.g. $\mathbb{Z}$, $\mathbb{Z}[x]$). Let $\{r_1, \ldots, r_n\}$ denote the least positive common multiple of $r_1, \ldots, r_n$ (we take it to be zero only if some $r_i = 0$). Assume $sg$ is the sign function, i.e. $sg(r)$ is zero if $r = 0$ and the sign of $r$ otherwise. Then

$$\text{lcm} : \{r_1, \ldots, r_n\} \mapsto [r_1, \ldots, r_n] \cdot \text{sg}(r_1 \cdot \cdots \cdot r_n)$$

is a localized quantifier.

4. Any lattice $(A, \wedge, \vee, 0, 1)$ equipped with $\varnothing = \inf$ and $\varnothing' = \sup$ is a value space. Note that $B = \{\{0, 1\}, \wedge, \vee, \exists\}$ is non-conjunctive, even though $r = 1 \wedge s = 1$ is equivalent to $r \wedge s = 1$. It is however localized (here at 1); see [2]. If $B$ is a Banach lattice or an ordered ring with the order topology, then ‘inf’ and ‘sup’ are continuous quantifiers.

5. For any points $r_1, \ldots, r_k$ in the complex plane let $\varnothing[r_1, \ldots, r_k]$ be the centroid (center of gravity) of the convex polygon generated by $r_1, \ldots, r_k$. Then $\varnothing$ is continuous with respect to the Euclidean topology. It is however not localized.

6. Let $\varnothing$ and $\varnothing'$ be quantifiers on $B$ and set

$$\varnothing' \circ \varnothing(A) = \varnothing'[(\varnothing(B) : \emptyset \neq B \subseteq A)].$$

Then $\varnothing' \circ \varnothing$ is a quantifier. If $\varnothing$ and $\varnothing'$ are continuous then so is $\varnothing' \circ \varnothing$.

Any integral domain is disjunctive. A finite commutative ring $B$ is conjunctive if and only if there is a polynomial $\tau(u, v)$ with coefficients in $B$ having exactly one root. This is due to the presence of subtraction and the fact that we only need to characterize pairs of points. In particular, any finite field $B$ is conjunctive: If $m = |B| - 1$, then $(u^m - 1)(v^m - 1) = 1$ has a unique solution. In fact:

**Lemma 2.4.** (i) Any integral domain (with discrete topology) is conjunctive.

(ii) $\mathbb{Z}_4$ is not conjunctive.

(iii) $\mathbb{Z}_5$ is conjunctive but not disjunctive.

**Proof.** (i) Let $X = \{r_1, \ldots, r_m\}$ be a finite subset of $B$. Then, if $(u, v) \in X^2$, one has $(u, v) \neq (r_1, r_2)$ if and only if

$$\tau(u, v) = \prod_{i \neq 1}(u - r_i) \prod_{j \neq 2}(v - r_j) = 0.$$  

So the range of $\tau$ on $X^2$ has the form $\{0, r\}$ where $r \neq 0$. Now, $(u, v) = (r_1, r_2)$ if and only if $\tau(u, v) = r$. A similar argument works for the general case.

(ii) Assume $(0, 0)$ is the unique solution of $\tau(u, v) = 0$. Then $\tau$ has the form

$$\tau(u, v) = ru + sv + \text{terms of degree at least 2}.$$  

Putting $u, v = 0, 2$, we see that $2r, 2s, 2r + 2s$ are all non-zero. This is impossible.

(iii) It is easy to check that $u^2 - v^2 + uv = 0$ has a unique solution in $\mathbb{Z}_6$. On the other hand, assume the only solutions of $\tau(u, v) = 0$ are those $(r, s)$ where at least one of $r, s$ is zero. Let $k, l, m$ be respectively the sum of the coefficients of terms containing $u, v$ and $uv$. Then putting $(0, 1), (1, 0)$ and $(3, 4)$ in the equation we obtain $k = l = 0$ and $3k + 4l \neq 0$ which is impossible. □

**Definition 2.5.** A value space $B$ is normal if it is conjunctive and has a localized semi-quantifier.

So, the value space of example 3 and also any ordered ring equipped with the order topology, absolute value and quantifier $\text{Min}[r_1, \ldots, r_n]$ is normal. Similarly, any integral domain equipped with the discrete topology and $\Pi$ is normal.
3. Satisfaction

Let $B$ be a value space. Operations, relations and elements of $B$ are respectively called connectives, canons and values. We usually assume that equality is a canon, although we may sometimes imagine the contrary. We will use connectives, canons and quantifiers as logical symbols and formulate first-order logic with truth values in $B$. As usual, we dispose of an infinite list $x, y, \ldots$ of individual variables.

Let $L$ be an ordinary first-order language containing a family of function, relation and constant symbols. The equality symbol $e$ is always considered as a relation symbol. One must be careful of the distinction between $e$ and $=$ (we may use $a = b$ for $e(a, b) = 1$). A relation on a non-empty set $M$ is any function $R : M^n \to B$ with finite range (this rules out most familiar examples from the continuous model theory). In particular, the characteristic function of the diagonal is a relation. An $L$-structure (or model) is the same as the ordinary $L$-structure except that interpretations of relation symbols take (a finite number of) values in $B$. So, taking $B = (\mathbb{Z}_2, \prec)$ we get the usual first-order structures. Terms are defined as usual. Formulas are defined as follows: (i) Any value $r$ is an atomic formula. (ii) If $R \subseteq L$ is an $n$-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is an atomic formula. (iii) If $a$ is a $k$-ary connective and $\phi_1, \ldots, \phi_n$ are formulas then $a(\phi_1, \ldots, \phi_n)$ is a formula. (iv) If $M$ is a $\sigma$-structure, $\sigma$ is a quantifier and $x$ is a variable then $\sigma \phi$ is a formula.

Free variables are defined as usual. A sentence is a formula without free variables. If $\phi(x)$ is a formula and $\bar{a} \in M$, the value of $\phi(\bar{a})$ in $M$, denoted as $\phi^M(\bar{a})$, is defined inductively. If $\phi$ is atomic, $\phi^M(\bar{a})$ is already defined. For the connective and quantifier cases we set:

- $\alpha(\phi_1, \ldots, \phi_n)(\bar{a}) = \alpha(\phi_1^M(\bar{a}), \ldots, \phi_n^M(\bar{a}));$
- $\neg (\sigma \phi(x, y))(\bar{a}) = \neg \sigma(\phi^M(\bar{a}, b) : b \in M)$.

Note that if $\phi(x)$ is not free in $\phi$, then $\sigma \phi^M(\bar{a}) = \phi^M(\bar{a})$ because $\sigma$ is the identity on $B^1$. Furthermore, $\phi^M$ always has a finite range and, if $\sigma$ is a sentence, $\sigma^M$ is uniquely determined. It is also clear that $\sigma x y$ (which may be abbreviated $\sigma xy$) and $\sigma xy$ have different meanings. However, one easily shows that:

**Lemma 3.1.** If $\sigma$ is a localized semi-quantifier, then $\sigma(\neg \phi(x)) = 0$ and only if there is $\bar{a} \in M$ such that $\phi^M(\bar{a}) = 0$. In particular, $\sigma x y = 0$ and $\sigma x y = 0$ are equivalent.

A statement is an expression of the form

$$\pi(\phi_1, \ldots, \phi_n)$$

where $\pi$ is any k-ary canon. If every $\phi_i$ is atomic, it is called an atomic statement. In particular, every equation $\phi(x) = \psi(x)$ is an atomic statement. Statements are denoted by $\delta, \epsilon, \delta(x)$ etc. A statement without free variables is called a closed statement. A statement $\pi(\phi_1, \ldots, \phi_n)$ is satisfied by $\bar{a}$ in $M$, denoted as $M \models \pi(\phi_1(\bar{a}), \ldots, \phi_n(\bar{a}))$, if $\pi(\phi_1^M(\bar{a}), \ldots, \phi_n^M(\bar{a}))$ holds. Any satisfiable set of closed statements is called a theory. The set of all closed statements satisfied in $M$ is called the theory of $M$. Two structures $M$ and $N$ are elementarily equivalent, denoted as $M \equiv N$, if they have the same theories. Since we use equality as a canon, this means that $\sigma^M = \sigma^N$ for any sentence $\sigma$. Also, $M \models T$ and $T \models \delta$ have their obvious definitions. A theory is complete if it is maximally satisfiable. If $T$ is complete, then for each $\sigma$ there is a unique value $r$ such that $T \models \sigma = r$.

**Definition 3.2.** An injective function $f : M \to N$ is an embedding if for any atomic formula $\phi(x)$ and any $\bar{a} \in M$, $\phi^M(\bar{a}) = \phi^N(f(\bar{a}))$. Likewise, $f$ is an elementary embedding if for any formula $\phi(x)$ and $\bar{a} \in M$, $\phi^M(\bar{a}) = \phi^N(f(\bar{a}))$. A surjective elementary embedding is called an isomorphism. Also, submodel $M \subseteq N$ and elementary submodel $M \preceq N$ are defined in the obvious way.

Note that every surjective embedding is an isomorphism.

**Proposition 3.3** (Tarski–Vaught Test). Let $M \subseteq N$ and assume that for any formula $\phi(x, y)$, $\bar{a} \in M$ and $r \in B$, if there is $b \in N$ such that $\phi^N(\bar{a}, b) = r$, then there is $c \in M$ such that $\phi^N(\bar{a}, c) = r$. Then $M \preceq N$. If $B$ is an integral domain or $\mathbb{Z}_3$, equipped with $\Pi$, this is a necessary condition. Also, this is a necessary condition for any normal value space.

**Proof.** We prove by induction on the complexity of formulas that for any $\phi(\bar{a})$ and $\bar{a} \in M$, $\phi^M(\bar{a}) = \phi^N(\bar{a})$. This is obviously true for atomic formulas. The connective cases are trivial. Let $\sigma$ be a quantifier and $\sigma \phi^N(\bar{a}, x) = r$. Let $\{t_1, \ldots, t_k\}$ be the range of $\phi^N(\bar{a}, x)$. By the assumption and the induction hypothesis, this is the range of $\sigma \phi^M(\bar{a}, x)$ too. Therefore, $\sigma \phi^M(\bar{a}, x) = r$.

If $B$ is an integral domain or $\mathbb{Z}_3$, consider the statement $\Pi x(\phi(\bar{a}, x) - r) = 0$ and note that in $\mathbb{Z}_3$, $rs = 0$ if and only if $r = s = 2$ or one of them is zero. The normal case is similar, as $\phi = r$ can be replaced by some $\psi = 0$ and that $\sigma$ is localized. $\square$

4. Chains and compactness

Let $M_0 \subseteq M_1 \subseteq \cdots$ be a chain of L-structures. By definition, if $R$ is relation symbol, $R^{M_i}$ has a finite range for each $i$. So, it is clear that the union of a chain may not be a structure. More generally, interesting results cannot be obtained without imposing further conditions on L-structures or on the value space. Below, we first assume that the value space is finite and prove the fundamental theorem. Then we consider the general case.
4.1. Finite value spaces

Assume \( B \) is a finite value space. It clear that the union of a chain \( M_0 \subseteq \cdots \subseteq M_i \subseteq \cdots, i \in I \), of \( L \)-structures is an \( L \)-structure. Furthermore:

**Proposition 4.1 (Elementary Chain Theorem).** Let

\[
M_0 \leq \cdots \leq M_i \leq \cdots \quad i \in I
\]

be an elementary chain of \( L \)-structures. Then there is an \( L \)-structure \( M = \bigcup_{i \in I} M_i \) such that \( M_i \leq M \) for any \( i \).

**Proof.** Let \( M = \bigcup_{i \in I} M_i \) defined above. We show by induction that for any \( \phi, i \) and \( \bar{a} \in M_i, \phi^M(\bar{a}) = \phi^M(\bar{a}) \). Let us consider the non-trivial case \( \bar{x} \phi(\bar{a}, x) \). Let \( \{r_1, \ldots, r_k\} \) be the range of \( \phi^M(\bar{a}, x) \). Then for some \( j, \phi^M_i(\bar{a}, x) \) takes all these values. So,

\[
\bigvee x \phi^M_i(\bar{a}, x) = \bigvee x \phi^M_i(\bar{a}, x) = \bigvee x \phi^M_i(\bar{a}, x).
\]

Now we state the fundamental theorem of ultraproducts. First we recall a basic fact from general topology. Let \( X \) be a topological space and \( (u_i)_{i \in I} \) an indexed family of elements of \( X \). If \( D \) is a filter on \( I \) then \( x \) is the \( D \)-limit of \( (u_i)_{i \in I} \) if for each neighborhood \( U \) of \( x \) the set \( \{i : u_i \in U\} \) belongs to \( D \). In this case one writes \( \lim_D u_i = u \). Note that if \( I \) is discrete, this means that \( u_i = u \) for \( D \)-almost all \( i \). It is well-known that is compact Hausdorff if and only if for every indexed family \( (u_i)_{i \in I} \) of elements of \( X \) and every ultrafilter \( D \) on \( I \), \( \lim_D u_i \) exists and is unique. Furthermore, if \( X, Y \) are compact Hausdorff and \( f : X \to Y \) is continuous, then for any indexed family \( (u_i)_{i \in I} \) in \( X \) one has that

\[
\lim_D f = \lim_D f(u_i).
\]

Assume \( I \) is an index set and \( D \) an ultrafilter over \( I \). Let \( L \) be a language and \( \{M_i\}_{i \in I} \) an indexed family of \( L \)-structures. Let \( M = \prod_D M_i \) be the usual set theoretic ultraproduct of the \( L \)-structure whose elements are denoted by \( [a_i] \), etc. We define an \( L \)-structure on \( M \) setting:

- \( c^M = [c^M] \);
- \( F^M([a_i^1], \ldots, [a_i^n]) = [F^M(a_i^1, \ldots, a_i^n)] \);
- \( R^M([a_i^1], \ldots, [a_i^n]) = \lim_D R^M(a_i^1, \ldots, a_i^n) \).

Clearly, \( M \) is an \( L \)-structure. Furthermore:

**Theorem 4.2 (Fundamental Theorem).** For any formula \( \phi(\bar{x}) \) and any tuple \( ([a_i^1], \ldots, [a_i^n]) \)

\[
\phi^M([a_i^1], \ldots, [a_i^n]) = \lim_D \phi^M(a_i^1, \ldots, a_i^n).
\]

**Proof.** By induction on the complexity of formulas. By definition, the claim holds for atomic formulas. The connective cases are obvious. Assume the claim holds for \( \phi(\bar{x}, y) \) and \( \bar{\phi} \) is a quantifier. For simplicity assume \( \bar{x} = \emptyset \). There are values \( r_1, \ldots, r^k \) such that for \( D \)-almost all \( i \), the range of \( \phi^M_i(y) \) consists of \( \{r_1, \ldots, r^k\} \). By the induction hypothesis, \( \phi^M(y) \) can take the values \( r_1, \ldots, r^n \) and these are the only possible ones. Therefore,

\[
(\bar{\phi} \psi(y))^M = \lim_D (\bar{\phi} \psi(y))^M = \bar{\phi}[\{r_1, \ldots, r^k\}].
\]

Using the fundamental theorem one can prove the compactness theorem (see below) and use it to deduce the upward Löwenheim–Skolem theorem. One can also apply Tarski’s test to prove the downward one. The elementary diagram of \( M \), denoted by \( ediam(M) \), is the set of all closed equations with parameters in \( M \) which are satisfied in \( M \). Clearly \( M \preceq N \) is equivalent to \( N \models ediam(M) \). Using this, one shows that every model has elementary extensions of arbitrarily large cardinalities. All these results and many other ones are valid in the more general case where the value space is infinite, except that we must put uniform bounds on the ranges of relations. In the next subsection we consider the general case.

4.2. Infinite value spaces

In this subsection and also in the rest of the paper, we assume that the value space is arbitrary, probably having a non-trivial topology. Obviously, some price should be paid in order to be able to obtain interesting results. For this purpose, we impose some limitations on the interpretations of relation symbols. More precisely, we index a bound on each relation symbol of the language and demand that, whenever interpreted, its range be limited by that bound.

**Definition 4.3.** A relational bound for a language \( L \) is a function \( b \) which associates with each relation symbol \( R \in L \) a pair \( b(R) = (n_R, B_R) \) where \( n_R \geq 1 \) is a natural number and \( B_R \) is a compact subset of \( B \). If \( B_R \) is finite (e.g. when \( B \) is discrete), we let \( n_R = |B_R| \), so that practically \( b(R) = B_R \). In particular, \( b(e) = \{0, 1\} \). A model \( M \) is called a \( b \)-model if for any relation symbol \( R \in L \), \( R^M \) takes at most \( n_R \) values from \( B_R \). A theory is \( b \)-satisfiable if it has a \( b \)-model. Likewise, \( \models b \) and related \( b \)-indexed notions are defined.
The assignment \( b \) can be uniformly extended to all formulas by setting:

\[
- B_{\alpha(\phi_1, \ldots, \phi_k)} = \alpha(B_{\phi_1}, \ldots, B_{\phi_k}) \quad \text{and} \quad n_{\alpha(\phi_1, \ldots, \phi_k)} = n_{\phi_1} \cdots n_{\phi_k};
- B_{\exists\phi} = \exists^{n_{\phi}}(B_{\phi}) \quad \text{and} \quad n_{\exists\phi} = 2^{n_{\phi}} - 1.
\]

It is easy to see that in any \( b \)-model \( M \), for any formula \( \phi \), \( \phi^M \) takes at most \( n_{\phi} \) values in the compact set \( B_{\phi} \).

As before, assume \( I \) to be an index set and \( D \) an ultrafilter over \( I \). Let \( \{M_i\}_{i \in I} \) be an indexed family of \( b \)-models. Then \( M = \prod_D M_i \) is defined as before and it is easily seen to be a \( b \)-model. Furthermore:

**Theorem 4.4 (Fundamental Theorem).** For any formula \( \phi(x) \) and any tuple \( ([a^n_1], \ldots, [a^n_k]) \)

\[
\phi^M([a^n_1], \ldots, [a^n_k]) = \lim_{\varnothing} \phi^M(a^n_1, \ldots, a^n_k).
\]

**Proof.** By induction on the complexity of formulas. By definition, the claim holds for atomic formulas. The connective cases are obvious. Assume that the claim holds for \( \phi(x) \) and \( \circ \) is a quantifier. For simplicity assume \( x = \varnothing \). There is an \( n \) such that for \( D \)-almost all \( i \), \( \phi^M(y) \) takes exactly \( n \) values, say \( r_i^1, \ldots, r_i^k \in B_{\phi} \). Then, \( \phi^M(y) \) can take the values \( \lim_D r_i^1, \ldots, \lim_D r_i^k \).

However, by the induction hypothesis, these are the only values that it can take. So, by the continuity of \( \circ^n \),

\[
(\circ y \phi(y))^M = \circ^n(\phi^M([a^n_1]), \ldots, \phi^M([a^n_k]))
\]

\[
= \circ^n(\lim_D \phi^M(a^n_1), \ldots, \lim_D \phi^M(a^n_k))
\]

\[
= \lim_{\varnothing} \circ^n(\phi^M(a^n_1), \ldots, \phi^M(a^n_k)) = \lim_{\varnothing} (\circ y \phi(y))^M.
\]

**Corollary 4.5.** For any model \( M \) the diagonal embedding \( \varnothing : M \to \prod_D M \) is an elementary embedding.

**Theorem 4.6 (Compactness Theorem).** Let \( \Gamma \) be a collection of statements. If any finite subset of \( \Gamma \) has a \( b \)-model, then \( \Gamma \) has a \( b \)-model.

**Proof.** Let \( I \) be the family of all finite subsets of \( \Gamma \). For any \( i \in I \), let \( M_i \) be a \( b \)-model such that \( M_i \models i \) and set \( i = \{ j \in I : i \subseteq j \} \).

By finite \( b \)-satisfiability, there is an ultrafilter \( D \) over \( I \) which contains every \( i \). Set \( M = \prod_D M_i \) and assume that \( \pi(\sigma_1, \ldots, \sigma_k) \) is a statement in \( \Gamma \). Then, for any \( i \) containing \( \pi(\sigma_1, \ldots, \sigma_k) \), we have \( \pi(\sigma^M_1, \ldots, \sigma^M_k) \).

Now, since \( \pi \) is a closed we can have

\[
\pi(\lim_D \sigma^M_1, \ldots, \lim_D \sigma^M_k).
\]

So by the fundamental theorem, \( \pi(\sigma^M_1, \ldots, \sigma^M_k) \).

So, finite \( b \)-satisfiability implies \( b \)-satisfiability. Upward, downward (taking into account the number of logical operations in this case), elementary chain and many other basic results can be proved when restricted to \( b \)-models. For example for the upward one we have:

**Proposition 4.7.** Every \( b \)-model has elementary extensions of arbitrarily large cardinals. Every theory \( T \) with infinite models (i.e. the \( b \)-model for some bound \( b \)) has models of arbitrarily large cardinalities.

Also, Vaught’s test for completeness is easily proved:

**Corollary 4.8.** Assume that the number of logical operations of \( B \) is countable. Let \( T \) be a countable theory which is \( \kappa \)-categorical for some \( \kappa \geq \aleph_0 \). Then \( T \) is complete.

In some situations, finite \( b \)-satisfiability can be replaced by “countable satisfiability”. Recall that a topological space is \( \sigma \)-compact if it is a countable union of compact sets.

**Proposition 4.9.** Assume that \( L \) has a finite number of relation symbols and \( B \) is \( \sigma \)-compact. If every countable subset of \( \Gamma \) has a model, then \( \Gamma \) has a model.

**Proof.** For simplicity, in this paragraph by an \( n \)-model we mean a model \( M \) for which \( M^k \) takes at most \( n \) values for any \( R \in L \). Let \( B_0 \subseteq B_1 \subseteq \cdots \) be a sequence of compact sets which cover \( B \). We claim that there are \( n \) and \( k \) such that any finite subset of \( I^n \) has an \( n \)-model whose relations take values in \( B_k \). Assume not. Then, for each \( n, k \) there is a finite subset \( I^n_k \subseteq I^n \) such that for any \( n \)-model of \( I^n_k \), some relation takes values outside \( B_k \). Let \( M \) be a model of \( \bigcup_{i \in I^n_k} I^n_{ik} \). Then there are \( n, k \) such that \( M \) is an \( n \)-model and all its relations take values in \( B_k \). But, this is a contradiction since \( I^n_{ik} \) has no \( n \)-model all of whose relations take values in \( B_k \). Now we use the compactness theorem.

More generally, assume that \( B \) is covered by a number \( \kappa \) of compact subsets \( (\kappa \geq \aleph_0) \) and the number of relation symbols of the language is \( \lambda \). If every subset of \( I^n \) of cardinality \( \leq \kappa^\lambda \) has a model, then \( I^n \) has a model.

The usual quantifiers \( \exists, \forall \) and Boolean operations \( \land, \lor \) in the two-valued logic can be easily simulated in various value spaces. For example, if \( B \) is an ordered ring equipped with the absolute value operation and the summation quantifier \( \Sigma \), the expression \( \forall x (\phi(x) = 0) \) can be stated as \( \Sigma_x |\phi(x)| = 0 \). In \( \mathbb{Z} \), \( |\psi| \leq |\psi \phi| \) states that \( \phi = 0 \) implies \( \psi = 0 \). Also, \( \phi(1 - \phi) = 0 \) states that the range of \( \phi \) is at most \( 0, 1 \).
Example 4.10. A coloring for a graph $G$ is a partition of $G$ such that no adjacent vertices lie in the same partition. An easy consequence of the classical compactness theorem is that if every countable subgraph of $G$ is colorable with a finite number of colors then so is $G$. Let us prove this in the framework of $B = \mathbb{Z}$. Let $G$ be an uncountable graph. Let $L = \{R, F\}$ where $R$ is a binary and $F$ is a unary function symbol. The theory of graphs is then stated as follows:

- $R(x, y) = 0$ or $1$;
- $R(x, x) = 0$;
- $R(x, y) = R(y, x)$.

Let $\Gamma$ be the union of $\text{diag}(G)$ (consisting of all atomic equalities with parameters in $G$ satisfied in it) with the axioms of graphs and the statement

$$\prod_{i \in \omega} |1 - R(x, y)| + |F(x) - F(y)| \geq 1$$

which says that if $x, y$ are adjacent, then $F(x) \neq F(y)$. Then, $\Gamma$ is countably and hence totally satisfiable. If $H \models \Gamma$ we have $G \subseteq H$ and $F^{H}$ is a finite coloring of $H$. Therefore, $F^{H} |_{c}$ is a finite coloring of $G$.

5. Axiomatizability

In this section we study the impact of $\mathbb{H}$ on axiomatization theorems. Let $\mathbb{B}$ be a relational bound for $L$. By Mod$_{\mathbb{B}}(T)$ we mean the class of all $b$-models of $T$. A class $K$ of $\mathbb{B}$-models in the language $L$ is a $b$-elementary class if there is a collection $\Gamma$ of statements such that $\text{Mod}_{\mathbb{B}}(\Gamma) = K$. Axiomatizability theorems can be proved for various value spaces. Below we consider two non-discrete and discrete cases containing respectively $(\mathbb{R}, +, \times, |, \text{sup})$ and commutative rings equipped with $\mathbb{H}$ and the discrete topology.

**Proposition 5.1 (Axiomatizability).** Assume $(\mathbb{B}, +, \cdot, |, \leq)$ is an ordered ring equipped with the order topology and any non-empty set of quantifiers. Let $\mathbb{B}$ be an arbitrary relational bound for $L$. Then, a class $K$ of $\mathbb{B}$-models is $\mathbb{B}$-elementary iff it is closed under elementary equivalence and ultraproduct.

**Proof.** We prove the non-trivial part. Assume that $\mathbb{K}$ is closed under elementary equivalence and ultraproduct. Let $\mathbb{B}$ be the set of all inequalities satisfied in every member of $\mathbb{K}$. Every member of $\mathbb{K}$ is a $\mathbb{B}$-model of $\Gamma$. Conversely, assume $M \models_{\mathbb{B}} \Gamma$. We show that $M \in \mathbb{K}$. Let $I$ be the collection of all finite sets $i$ of equalities $\sigma = r$ such that $M \models i$. For each $i \in I$ let $\hat{i} = \{j \in I : i \subseteq j\}$. The collection of sets of the form $\hat{i}$ is contained in a non-principal ultrafilter $\mathcal{D}$ over $I$. First, we prove that every $i \in I$ is satisfiable in some member of $\mathbb{K}$. We may assume $i$ consists only of a single equality $\sigma = 0$. Note that for any $\epsilon > 0$, there must exist an $N_{i} \in \mathcal{K}$ such that $|\sigma^{N_{i}}| < \epsilon$, otherwise since $M \models |\sigma| \geq \epsilon$ for some $\epsilon > 0$ which is impossible. If $\mathbb{B}^{>0}$ has a least element, we deduce immediately that there is an $N \in \mathcal{K}$ with $\sigma^{N} = 0$. Suppose $\mathbb{B}^{>0}$ has no least element. Let $\kappa$ be the least (regular) cardinal number anti-order isomorphic to a cofinal subset $\kappa \subseteq \mathbb{B}^{>0}$. Let $\mathcal{D}_{\kappa}$ be a non-principal ultrafilter over $\kappa$ containing any initial segment $\{i \in \kappa : i \leq \iota_{0}\}$. Then for each $\iota_{0} \in \kappa$ we dispose of a $b$-model $N_{\iota_{0}}$ such that $|\sigma^{N_{\iota_{0}}}| < \iota_{0}$. So, for each $\iota_{0} \in \kappa$ we have $\prod_{\iota_{0}} N_{\iota_{0}} \models |\sigma| < \iota_{0}$ which means that $\prod_{\mathcal{D}_{\kappa}} N_{\iota} \models \sigma = 0$. Therefore, $\sigma = 0$ is always satisfiable in $\mathcal{K}$. Now, for each $i \in I$ take a model $M_{i} \in \mathcal{K}$ such that $M_{i} \models i$. Then, $\prod_{\mathcal{D}} M_{i}$ belongs to $\mathcal{K}$ and is elementarily equivalent to $M$. □

A similar result holds for any discrete value space. The proof is left to the reader.

**Proposition 5.2 (Axiomatizability).** Assume $\mathbb{B}$ is conjunctive and disjunctive, and every $\mathbb{B}_{R}$ is finite (e.g. a Boolean algebra). Then, a class $\mathbb{K}$ of $\mathbb{B}$-models in the language $L$ is a $\mathbb{B}$-elementary class if and only if it is closed under elementary equivalence and ultraproduct.

It is easy to show that in general if both $\mathbb{K}$ and $\text{Mod}_{\mathbb{B}}(\emptyset) - \mathbb{K}$ are $\mathbb{B}$-elementary, then $\mathbb{K}$ is finitely $\mathbb{B}$-axiomatizable.

**Lemma 5.3.** Assume every canon of $\mathbb{B}$ is open (i.e. $\mathbb{B}$ is discrete as equality is a canon). Let $(\Gamma, \Delta)$ be a pair of theories with the property that for each finite $(\Gamma_{0}, \Delta_{0}) \subseteq (\Gamma, \Delta)$ there is a $\mathbb{B}$-model $M$ such that $M \models \Gamma_{0}$ and $M \not\models \delta$ for any $\delta \in \Delta_{0}$. Then there is a $\mathbb{B}$-model $M \models \Gamma$ such that $M \not\models \delta$ for any $\delta \in \Delta$.

**Proof.** Let

$$T = \Gamma \cup \{\pi^{c}(\sigma_{1}, \ldots, \sigma_{k}) : \pi(\sigma_{1}, \ldots, \sigma_{k}) \in \Delta\}$$

and $\mathbb{B}^{T}$ be the expansion of $\mathbb{B}$ by new canons $\pi^{c}$ for any canon $\pi$ of $\mathbb{B}$. Then $T$ is a finitely $\mathbb{B}$-satisfiable theory with respect to $\mathbb{B}$. □

**Corollary 5.4.** If every canon is open and $\Gamma \models_{\mathbb{B}} \delta$, then there is a finite $\Gamma_{0} \subseteq \Gamma$ such that $\Gamma_{0} \models_{\mathbb{B}} \delta$.

**Example 5.5.** Let $\mathbb{B} = \mathbb{R}$ and $L = \{R, c\}$ where $R$ is a unary relation symbol with $\mathbb{B}_{R} = \{0, 1\}$ and $c$ is a constant symbol. Then

$$\Gamma = \left\{R(c) \leq \frac{1}{n} : n \geq 1 \right\} \models_{\mathbb{B}} R(c) = 0.$$

However, for any finite subset $\Gamma_{f} \subseteq \Gamma$ we have $\Gamma_{f} \not\models_{\mathbb{B}} R(c) = 0$. As a consequence, $R(c) > 0$ is not an expressible statement.
Corollary 5.6. Assume $B$ is disjunctive and every canon is open. Let $T$ be a $b$-satisfiable theory and $\Delta$ a set of statements which is closed under disjunction. Then the following are equivalent:

(i) $T$ has a set of $b$-axioms $T_0$ such that $T_0 \subseteq \Delta$, i.e. $\text{Mod}_b(T_0) = \text{Mod}_b(T)$.

(ii) If $M, N$ are $b$-models, $M \models T$ and every statement in $\Delta$ which is satisfied in $M$ is satisfied in $N$, then $N$ is a model of $T$.

Proof. Assume that (ii) holds and $T$ is $b$-closed i.e. $T \models \psi$ implies $\psi \in T$. Let $T_0 = T \cap \Delta$. We have only to show that $T_0 \models T$. Let $N$ be a $b$-model of $T_0$ and

$$\Gamma = \{ \psi \in \Delta : N \not\models \psi \}.$$

Then, $\Gamma$ is closed under disjunction. Note that $T \not\models \psi$ for any $\psi \in \Gamma$. So, by the fundamental theorem there is a $b$-model $M \models T$ which is not a model of any statement in $\Gamma$. So, every $\Delta$-statement which is satisfied in $M$ is satisfied in $N$. Hence, $N$ is a model of $T$. □

6. Definable types

In classical logic $[\bar{a} | M \models \phi(\bar{x})]$ is called a definable set. Definable sets form a Boolean algebra or even a cylindrical algebra if we also take into account quantifiers. If $B$ is an arbitrary value space, a definable set is a set of the form $[\bar{a} | \phi(\bar{a}) = r]$ or more generally of the form $[\bar{a} | \pi(\phi_1(\bar{a}), \ldots, \phi_k(\bar{a}))]$. So, for example if $B$ is a ring, we deal with the ring of definable relations instead of the Boolean algebra of definable sets. In fact, if $B$ is an integral domain, this algebra corresponds to the Boolean algebra of definable sets. We will explore this in the next section. If $B$ is arbitrary and $T$ is a theory, the family of formulas with respect to $T$-equivalence forms a topological algebra of the same type as $B$, the topology being defined by basic open sets of the form

$$[\phi(\bar{x}) : \text{Im}(\phi) \subseteq U].$$

Now assume $L$ is a language, $b$ a relational bound for $L$ and $T$ a $b$-satisfiable $L$-theory. A partial $n$-type of $T$ is any set $\Gamma(x)$ of statements such that $T \cup \Gamma(x)$ is $b$-satisfiable. Maximal partial types are called types. If $p(x)$ is a type, for each formula $\phi(x)$ there is a unique $r \in B$ such that $p \models \phi = r$. We denote this $r$ by $\phi$. Clearly, every type is uniquely determined by its equations. The collection of all $n$-types of $T$ is denoted by $S_n(T)$. Let $U$ be a subset of $B$ and $\phi(x)$ a formula. Set

$$[\phi, U] = \{ p \in S_n(T) : \phi \in U \}.$$

We take sets of the form $[\phi, U]$, where $U$ is open, as basic open subsets of $S_n(T)$. Since $B$ is Hausdorff, $S_n(T)$ is always Hausdorff.

Lemma 6.1. Suppose $B$ is conjunctive, locally compact and first countable. Then, basic open sets form a basis for the topology of $S_n(T)$.

Proof. Assume $p \in [\phi, U] \cap [\psi, V]$. Assume $\phi = r_0, \psi = s_0$. Let $\tau(r, s)$ be a meta-term for which the unique solution of $\tau(u, v) = 0$ on $B \times B$ is $(r_0, s_0)$. Let $(W_n)_n$ be a basis of open sets with compact closures for 0. We claim that there is a neighborhood $W$ of 0 such that

$$\tau^{-1}(W) \cap (B \times B) \subseteq U \times V.$$

Assume not. Then for each $n$, there is an $(r_n, s_n)$ in $\tau^{-1}(W_n) \cap (B \times B)$ not belonging to $U \times V$. But, if $(u_0, v_0) \in B \times B$ is a cluster point of this sequence, we must have both $\tau(u_0, v_0) = 0$ and $\tau(u_0, v_0) \not\in U \times V$. This is a contradiction. Now, we have $p \in [\tau(\phi, \psi), W] \subseteq [\phi, U] \cap [\psi, V]$. □

Proposition 6.2. Assume basic open sets form a basis for the topology of $S_n(T)$. Then $S_n(T)$ is compact.

Proof. Let $\{X_i\}_{i \in I}$ be a family of closed sets having a finite intersection property. Every closed set in $S_n(T)$ is an intersection of sets of the form $[\phi, F]$ where $F \subseteq B$ is closed. So, without loss of generality, we may assume that for each $i \in I, X_i = [\phi_i, F_i]$ where $F_i$ is closed. We may use each $F_i$ as a new (closed) canon for $B$. So, for each finite set $I_0 \subseteq I$, there is a type $p$ such that $\phi_i \in F_i$ for any $i \in I_0$. Since $p$ is realized in some model of $T$, $\{\phi_i \in F_i : i \in I_0\}$ is satisfiable in some model of $T$. So, by compactness, $\{\phi_i \in F_i : i \in I\}$ is satisfiable in some model $M \models T$, say by $\bar{a}$. Then, if $p(\bar{x}) = tp(\bar{a})$, for each $i$ we have $\phi_i \in F_i$. So, the family $\{X_i\}_{i \in I}$ has non-empty intersection. □

Note however that $S_n(T)$ is not necessarily totally disconnected. This is the case if every $B$ is finite. In the case where the value space is a ring (and in the presence of the equality canon), types correspond to ideals in the algebra of definable relations.
7. Representation in FOL

In this section we prove representation for languages equipped with a finitary relational bound. Representation is, in general, a method for representing arbitrary classes of structures by elementary classes. Here we rather use it as a method for representing arbitrary theories by classical theories. In particular, imposing some conditions on the value space, we obtain a model-complete theory which represents it. Below, we use notions of formula, model, satisfaction etc. in both classical and extended senses. Let $L$ be a language and $b$ a fixed relational bound such that for each $R \in L$, $B_R$ is finite. Let $T$ be a $b$-satisfiable theory. Then, each formula $\phi(\bar{x})$ when interpreted in a $b$-model of $T$ takes values in the finite set $B_\phi$. Let $L'$ be the language containing for each $L$-formula $\phi(\bar{x})$ with $|\bar{x}| = n \geq 1$ and $r \in B_\phi$ an $n$-ary relation symbol $R'_\phi$. Let $\Omega_b$ be the universal closure of the following classical $L'$-formulas, for any $\phi(\bar{x})$, $\alpha(\phi_1, \ldots, \phi_k)(\bar{x})$, $\lor \phi(\bar{x}, y)$ and $r \in B$, whenever the right-hand side is non-empty:

$$\forall \bar{x} \left( \bigvee_{r \in B_\phi} R'_\phi(\bar{x}) \right)$$

$$R'_{\lor \phi}(\bar{x}) \equiv \bigvee_{r \in B_\phi} \left( \bigwedge_{i} R'_\phi(\bar{x}) \right)$$

$$R'_{\lor \phi}(\bar{x}) \equiv \bigvee_{r \in B_\phi} \left[ \left( \bigwedge_{s \in A} \exists y R'_\phi(\bar{x}, y) \right) \land \left( \bigwedge_{s \in B_\phi \setminus A} \neg \exists y R'_\phi(\bar{x}, y) \right) \right].$$

Any $L$-sentence $\sigma$ may be regarded as a formula $\sigma(x)$ where $x$ is a dummy variable for it (we may also make use of 0-ary relation symbols as atomic sentences). Let

$$T' = \Omega_b \cup \{ \forall x R'_\phi(x) : T \models \sigma \equiv r \}.$$

Note that $T'$ is a $\forall \exists$-theory. Also, for each $M \models T$, let $M'$ be the classical $L'$-structure whose universe is $M$ and whose relations are the sets

$$(R'_\phi)^M = \{ \bar{a} : R'_\phi(\bar{a}) = r \}.$$  

So, for any $L$-formula $\phi(\bar{x})$, $r \in B_\phi$ and $\bar{a}$, we have $M \models \phi(\bar{a}) = r$ iff $M' \models R'_\phi(\bar{a})$.

**Lemma 7.1.** (i) For any $L$-structure $M$, $M$ is a $b$-model iff $M' \models \Omega_b$.

(ii) For any $b$-models $M, N, M \preceq N$ iff $M' \subseteq N'$.

(iii) For any $b$-model $M \models T$, if and only if $M' \models T'$.

(iv) For any $\bar{M} \models T'$ there is a model $M \models T$ such that $M' = \bar{M}$.

(v) For any equality $\sigma = r$, $T \models \sigma = r$ if and only if $T' \models \forall x R'_\phi(x)$.

(vi) If $T'$ is complete, then $T$ is $b$-complete (i.e. maximally $b$-satisfiable).

**Proof.** (ii) Obvious.

(iii) Obvious.

(iv) Assume $\bar{M} \models T'$. We define an $L$-structure $M$ on the base set of $\bar{M}$. For this purpose, for each atomic $L$-formula $\theta(\bar{x})$ and $\bar{a} \in M$ set $\theta^M(\bar{a}) = r$ iff $\bar{M} \models R'_\phi(\bar{a})$. It is clear that this turns $M$ into a $b$-model. Now, we show that $M' = \bar{M}$ by showing that for any atomic formula $\phi(\bar{x})$, $r \in B_\phi$ and $\bar{a}$, $M' \models R'_\phi(\bar{a})$ iff $\bar{M} \models R'_\phi(\bar{a})$. We do this by induction on the complexity of $\phi$. By definition, the claim holds for any atomic $\phi$. The connective and quantifier cases are also clear if we look at the sentences of $\Omega_b$. Also, note that by (iii) $M$ is a model of $T$.

(v) Immediate from (iii) and (iv).

(vi) Assume $M, N \models T$. For any $L$-sentence $\sigma$ there is an $r$ such that $M \models \sigma = r$. So, we have $M' \models \forall x R'_\phi(x)$. But, as $M' \equiv N'$, we must have $N' \models \forall x R'_\phi(x)$ which implies that $N \models \sigma = r$. This shows that $T$ is complete. □

**Corollary 7.2.** $(\text{Mod}_b(T), \preceq)$ and $(\text{Mod}(T'), \subseteq)$ are isomorphic as categories.

This shows that, at least as far as theories bounded by a ‘discrete’ relational bound are concerned, extended theories bring nothing mathematically new to model theory. This includes especially the case where $T$ is complete.

If we impose stronger conditions of the value space, we will be able to obtain a much better description of extended theories in terms of classical theories. Let us assume that the value space is normal. Then, any statement $\phi(\bar{x}, y) = r$ can be replaced by a statement $\phi'(\bar{x}, y) = 0$. With such notation let $\Omega'_b$ be the union of $\Omega_b$, and the universal closure of the following formulas:

$$\exists y R'_{\phi(\bar{x}, y)}(\bar{x}).$$
Also, let $T'' = T' \cup \Omega$. Note that Lemma 7.1 and Corollary 7.2 still hold with $\Omega$, replaced by $\Omega''$, and $T'$ replaced by $T''$. The following lemma is obvious.

**Lemma 7.3.** Assume $\mathbb{B}$ is normal, $L$ is a language and $M, N$ are any $L$-structures. Then, if $M \preceq N$ and $\phi^N(\bar{b}) = 0$ for some $\bar{b} \in N$, there is an $\bar{a} \in M$ such that $\phi^M(\bar{a}) = 0$.

**Proposition 7.4.** Assume $\mathbb{B}$ is normal and $\mathbb{b}$ is as before. Then, for any $L$-theory $T$, (i) $T''$ is model-complete. In fact, $T''$ has elimination of quantifiers. (ii) $T''$ is complete if and only if $T$ is $\mathbb{b}$-complete.

**Proof.** (i) We use Robinson’s test. Suppose $M' \subseteq N, \bar{b} \in M'$ and $N' \models \exists \bar{x} \theta(\bar{x}, \bar{b})$ where $\theta(\bar{x}, \bar{y})$ is a quantifier-free $L'$-formula. Without loss of generality assume $\theta(\bar{x}, \bar{y}) = \bigwedge_{i=1}^{k} R_{\phi_i}^{N}(\bar{x}, \bar{y})$. Then there is an $\bar{a}$ such that $\phi^N_i(\bar{a}, \bar{b}) = 0$ for any $i = 1, \ldots, k$. Since $\mathbb{B}$ is conjunctive, there is a meta-term $\tau(\bar{u}_1, \ldots, \bar{u}_k)$ such that for any $r_1 \in \mathbb{B}_{\phi_1}, \ldots, r_k \in \mathbb{B}_{\phi_k}, \tau(r_1, \ldots, r_k) = 0$ if $r_1 = 0, \ldots, r_k = 0$. So, we have

$$\tau(\phi_1, \ldots, \phi_k)^N(\bar{a}, \bar{b}) = 0.$$

Therefore, since $M \preceq N$, there is an $\bar{a}' \in M$ such that

$$\tau(\phi_1, \ldots, \phi_k)^M(\bar{a}', \bar{b}) = 0$$

from which we deduce $\phi^M_i(\bar{a}', \bar{b}) = 0$ for any $i = 1, \ldots, k$. Hence $M' = \exists \bar{x} \theta(\bar{x}, \bar{b})$. For elimination of quantifiers note that $\Omega''$ is equivalent to a universal theory. Also, any universal model-complete theory has elimination of quantifiers.

(ii) By Lemma 7.1 we have only to show that every complete theory $T$ satisfies the elementary joint embedding property. Let $M, N = T$. Let $\phi(\bar{x})$ be a formula, $r \in \mathbb{B}_{\phi}$ and $\bar{a} \in M$. Then there is a meta-term $\tau(\bar{u})$ such that $\phi^M(\bar{a}) = r$ if and only if $\tau(\phi^M(\bar{a})) = 0$. Since $\mathbb{B}$ is localized, there are a quantifier $\exists$ and a meta-term $\tau'(\bar{u})$ such that the last assertion is equivalent to $\bigwedge_{i=1}^{k} R_{\phi_i}^{N}(\bar{x}, \bar{y})$ = 0. So, since $M \equiv N$, this implies that $\exists \bar{x} \bigwedge_{i=1}^{k} \tau'(\phi^N_i(\bar{a}, \bar{b})) = 0$ which in turn implies that there is $\bar{b} \in N$ such that $\tau(\phi^N_i(\bar{a}, \bar{b})) = 0$. Hence, $\phi^N(\bar{b}) = r$. This shows that every sentence in $\text{edag}(M)$ is satisfiable in $N$. In fact, using conjunctivity, we can show that $\text{edag}(M) \cup \text{edag}(N)$ is finitely satisfiable in $N$. Hence, $M$ and $N$ can be elementarily embedded in a third $\mathbb{b}$-model. \qed

**Corollary 7.5.** Assume $\mathbb{B}$ is normal and the number of connectives and quantifiers is countable. Then for any countable language $L$ the following holds:

- For any finite $M, N, M \equiv N$ implies $M \simeq N$.
- The elementary amalgamation and elementary joint embedding properties hold.
- The Keisler–Shelah ultrapower theorem holds.
- No complete $L$-theory can have exactly two countable models.
- Assume $T$ is a complete $L$-theory which is categorical for some $\kappa \geq \aleph_1$. Then $T$ is categorical for every $\kappa \geq \aleph_1$.

There is a correspondence between types of $T$ and types of $T''$. Let $p(\bar{x})$ be a type of $T$. Set

$$p'(\bar{x}) = \{ R_{\phi}^{N}(\bar{x}) : p \vdash \phi(\bar{x}) = r \}.$$

Note that $p'(\bar{x})$ is always satisfiable in a model of $T''$. In fact, by quantifier elimination it is a complete type of $T''$. Moreover, $\phi(\bar{x}) = r \models T$ $p$ iff $R_{\phi}^{N}(\bar{x}) \models_{p'} p'$. In particular, $S_{0}(T)$ and $S_{n}(T'')$ are homeomorphic as topological spaces.

**Corollary 7.6.** Any non-principal type of $T$ is omitted in a countable model of $T$. If $T$ is complete, the Ryll-Nardzewski theorem holds.

There is a natural correspondence between other phenomena in $T$ and $T''$ such as saturation, primeness etc. Normality is however not the only case where we expect usual properties to appear. Below we mention without proof a proposition stating that elementary amalgamation holds for the case where a missing localized quantifier is the limit of a sequence of non-localized quantifiers.

**Proposition 7.7.** Let $(\mathbb{R}^{+}, +, \cdot, \times)$ be equipped with the discrete topology and quantifiers

$$\exists_{p} : \{ r_1, \ldots, r_k \} \mapsto (r_1^p + \cdots + r_k^p)^{\frac{1}{p}}$$

for any $p \geq 1$. Then the elementary amalgamation property holds in any language $L$. 


8. Discussion

As stated at the beginning of the paper, our aim was to give a rather arithmetical (or algebraic) presentation of first-order logic. Two typical examples of value spaces are $\mathbb{Z}$ and $\mathbb{R}$; hence the choice of the adjective ‘arithmetical’ in our title. It is natural to try to find an axiom system for such interesting cases as the ring of integers and to try to prove a completeness theorem. For example, if $\pi(u_1, \ldots, u_n)$ is an atomic meta-formula (a formula in the language of the value space) such that $\forall \bar{u} \pi(\bar{u})$ holds in $\mathbb{B}$, then $\pi(\phi_1, \ldots, \phi_n)$ should be considered as an axiom. Also, if $\pi_1(\bar{u}), \ldots, \pi_k(\bar{u})$, $\pi(\bar{u})$ are atomic meta-formulas and $\forall \bar{u} (\bigwedge_{i=1}^k \pi_i(\bar{u}) \rightarrow \pi(\bar{u}))$ holds in $\mathbb{B}$, then

$$
\pi_1(\phi_1, \ldots, \phi_n), \ldots, \pi_k(\phi_1, \ldots, \phi_n)
$$

should be considered as a logical rule. The “Al-Jabr” and “Al-Mughābala” rules mentioned in the introduction are instances of such rules. Since such axiom systems depend on the true meta-sentences, one may ask to what extent the usage of a particular value space $\mathbb{B}$ is essential. Is it possible to find the logic on the basis of the meta-theory of $\mathbb{B}$? For example, we may read the conditional expression $\phi \rightarrow \psi$ as $\text{val}(\phi) \leq \text{val}(\psi)$ while ignoring the exact truth values of $\phi$ and $\psi$. In this way, usually, proof rules in an axiom system correspond to the axioms of some meta-theory (i.e. the common theory of the intended value spaces). Clearly, such a meta-theory needs to satisfy some unifying conditions such as joint embedding and amalgamation.

Just as for FOL which has several generalizations, components of $\mathbb{B}$-logic may be changed in order to obtain other variants. For example, one option is to allow non-deterministic connectives or quantifiers. This situation occurs for example when the value space is a non-commutative ring. Then the quantifier $\Pi$ generates a collection of values instead of just one value. Another option is to expand the logic with an integration quantifier. For example, there are interesting algebraic structures which are equipped with a finitely additive measure, e.g. $\mathbb{Z}$ with the density measure which attributes the value $\frac{1}{n}$ to cosets of $n\mathbb{Z}$. Using an integration quantifier we may reach sufficient expressive power for studying such structures. In this way we obtain a probability logic less powerful than standard probability logic.

References