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Discrete Mathematics 281 (2004) 137–148
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Maximum sizes of graphs with given domination parameters

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Received 28 January 2002; received in revised form 1 July 2003; accepted 16 July 2003

Abstract

We find the maximum number of edges for a graph of given order and value of parameter for several domination parameters. In particular, we consider the total domination and independent domination numbers.

Keywords: Extremal graph; Size; Domination; Edges; Bipartite

1. Introduction

A set S of vertices is a dominating set of a graph G if every vertex not in S is adjacent to a vertex in S; it is a total dominating set if every vertex is adjacent to a vertex in S (so that a total dominating set exists only for graphs without isolates). A set is an independent dominating set if it is both dominating and independent (no edge joins two members). The domination number $\gamma(G)$, total domination number $\gamma_t(G)$, and independent domination number $i(G)$, are the minimum cardinalities of a dominating, total dominating, and independent dominating set of G, respectively. For much about these, consult Ref. [6].

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doi:10.1016/j.disc.2003.07.010

Vizing [10] determined the maximum number of edges in a graph with a given domination number. For a graph with order n and domination number k, he showed that the maximum size is $\lceil (n-k+2)(n-k)/2 \rceil$ provided $k \geqslant 2$. However, we are more interested in what the extremal graphs look like. So we state his result as follows. By an edge cover we mean a set of edges such that every vertex is incident with at least one of the edges.

Theorem 1 (Vizing [10]). *Consider graphs of order n with domination number* $k \geqslant 2$. *Then the maximum size is uniquely attained by the graph constructed as follows: take the complete graph on* $n-k+2$ *vertices and remove a minimum edge cover and then add* $k-2$ *isolated vertices.*

A rendition of the proof is given in [6]. Other people have generalised the result, see for example [5,7–9]. If we restrict our consideration to bipartite graphs, then the following can be read out of results from Ferneyhough et al. [4]. By the almost balanced complete bipartite graph on n vertices we mean $K(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$.

Theorem 2 (Ferneyhough et al. [4]). *Consider bipartite graphs of order n with domination number* $k \geqslant 2$. *Then the maximum size is attained by the graph constructed as follows: take an almost balanced complete bipartite graph on* $n-k+2$ *vertices and add* $k-2$ *isolated vertices.*

Here, we consider the problem of the maximum size for the independent and total domination numbers, both for general graphs and when restricted to bipartite graphs. The question for other domination parameters was looked at in [1,3,9], inter alia.

What we find most interesting is the behaviour of the extremal graphs (that is, those with the maximum number of edges). For example, consider graphs with total domination number k where k is even. Then, we show that an extremal graph is the complete graph together with some isolated edges. On the other hand, if one restricts consideration to only the bipartite graphs, then an extremal graph is a balanced complete bipartite graph with an edge cover removed, together with some isolated edges. In contrast, for ordinary domination the edge cover is removed in the extremal graph for general graphs.

We need the following notation. For a graph G, the set of vertices is V and the set of edges is E. We use $N(v)$ for the neighbourhood of a vertex v; $N[v]$ for $\{v\} \cup N(v)$; $N(S)$ for the neighbourhood $\bigcup_{v \in S} N(v)$ of a set S; and for a set X of vertices $N_X(v) = N(v) \cap X$. The maximum degree is denoted by Δ. Further, $q(G)$ is the number of edges in G, and $q(A,B)$ is the number of edges with one end in A and one end in B.

2. Total domination

We will need the following bound of Cockayne et al. [2] on the total domination number:

Proposition 3 (Cockayne et al. [2]). *For a connected graph G without isolated vertices of order $n \geqslant 3$ it holds that $\gamma_t(G) \leqslant 2n/3$.*

For $1 \leqslant k \leqslant n$ define $q(n,k)$ as follows:

$$q(n,k) := \begin{cases} \binom{n-k+2}{2} + \frac{k}{2} - 1 & \text{if } k \text{ is even,} \\ \binom{n-k+1}{2} + \frac{k}{2} + \frac{1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

We will need the following properties of $q(n,k)$:

Lemma 4. (a) *If $1 \leqslant a \leqslant c$ and $1 \leqslant b \leqslant d$ then $q(c+d, a+b) \geqslant q(c,a) + q(d,b)$, unless $a = c$ and a is odd, or $b = d$ and b is odd.*

(b) *If $2 \leqslant k \leqslant n-1$ then $q(n-1,k) < q(n,k) < q(n,k-1)$.*

(c) *If $1 \leqslant k \leqslant n$ and $0 \leqslant \ell \leqslant k-1$ then $q(n-\ell, k-\ell) \leqslant q(n,k-1)$.*

(d) *If $3 \leqslant k \leqslant n-2$ and k is odd then $q(n-1,k) + n - k = q(n,k) = q(k-1,k-2) + (n-k)(n-k+1)/2$.*

Proof. Arithmetic. □

Theorem 5. *Let G be a graph (without isolates) with order n and with total domination number $k \geqslant 2$. Then $q(G) \leqslant q(n,k)$, and this bound is sharp.*

Proof. Suppose the bound is false for some n and k. Then let G be a counter-example with minimum order.

Then G is connected, since by part (a) of the above lemma $q(n,k) \geqslant q(m,i) + q(n-m, k-i)$. (Note that a graph of odd order cannot have total domination number equal to its order.) In particular, $\Delta \geqslant 2$. The bound is clearly true for $k = 2$ (as $q(n,2) = \binom{n}{2}$). So $k \geqslant 3$. By Proposition 3, $k \leqslant 2n/3$.

Let u be a vertex of maximum degree Δ in G. Let $A = \{v_1, v_2, \ldots, v_\Delta\}$ be the set of neighbours of u and let $B = V - (A \cup \{u\})$. Since $k \geqslant 3$, B is nonempty.

Claim 1. $|N_B(v_i)| \leqslant n - \Delta - k + 1$ *for all* $i \in \{1, \ldots, \Delta\}$.

Let $S_i = \{u, v_i\} \cup (B - N_B(v_i))$. If the graph $G[S_i]$ induced by S_i contains no isolated vertex, then S_i is a total dominating set of G. If $G[S_i]$ contains isolated vertices, these vertices must be in B; replacement of each such vertex by a neighbour in $A \cup N_B(v_i)$ yields a total dominating set of G with at most $|S_i|$ vertices. That is, $\gamma_t(G) \leqslant |S_i|$. Hence

$$k \leqslant |S_i| = n - \Delta + 1 - |N_B(v_i)|,$$

which proves the claim. □

Claim 2. $\Delta \leqslant n - k$.

By the above claim, $n-\Delta-k+1 \geqslant 0$, so $\Delta \leqslant n-k+1$. Assume $\Delta = n-k+1$. Then $|N_B(v_i)| = 0$ for all $i \in \{1,\ldots,\Delta\}$. So there is no edge between A and B, which contradicts the fact that G is connected. □

Claim 3. $q(G[B]) \leqslant q(n-\Delta-1, k-2)$.

Let I_B be the set of isolates in $G[B]$ with $|I_B| = l$, and let S be a minimum total dominating set of $G[B-I_B]$. If $l=0$, then $T_1 = S \cup \{u, v_1\}$ is a total dominating set of G, and so $|T_1| \geqslant k$. Then $\gamma_t(G[B]) \geqslant k-2$ and by the above lemma, $q(G[B]) \leqslant q(n-\Delta-1, k-2)$.

If $l \geqslant 1$, then choose vertex v_i that has a neighbour in I_B, say x, and let $T_2 = S \cup (I_B - \{x\}) \cup \{u, v_i\}$. The set T_2 dominates G. If any vertex w in $G[T_2]$ is isolated, then it is a vertex of I_B, and one can replace w in T_2 by a neighbour in A. Thus, T_2 can be modified without increasing its cardinality to be a total dominating set of G, and so $|T_2| \geqslant k$. Then $\gamma_t(G[B-I_B]) \geqslant k-1-l$.

Hence, since $|B| = n-\Delta-1$, and by the above lemma,

$$\begin{aligned} q(G[B]) = q(G[B-I_B]) &\leqslant q(n-\Delta-1-l, k-1-l) \\ &\leqslant q(n-\Delta-1, k-2). \end{aligned} \tag{1}$$

It can be checked that the inequality still holds for $k=3$ if we extend the definition of $q(n,k)$ to allow $k=1$. □

We now use Claim 1 to bound the number of edges in G.

$$\begin{aligned} 2q(G) &= \deg(u) + \sum_{i=1}^{\Delta} \deg(v_i) + q(A,B) + 2q(G[B]) \\ &\leqslant (\Delta^2+\Delta) + \sum_{i=1}^{\Delta} |N_B(v_i)| + 2q(G[B]) \\ &\leqslant \Delta^2 + \Delta + \Delta(n-\Delta-k+1) + 2q(G[B]) \\ &= \Delta(n-k+2) + 2q(G[B]). \end{aligned} \tag{2}$$

Inequality (2) and Claim 3 together yield

$$2q(G) \leqslant \Delta(n-k+2) + 2q(n-\Delta-1, k-2). \tag{3}$$

Assume now that k is even. The right-hand side in inequality (3) is a parabola as a function of Δ, and since the second derivative is positive, it is maximised at an extremum viz., $\Delta = 2$ or $\Delta = n-k$. Some arithmetic shows that this extremum is $2q(n,k)$ (the value for $\Delta = 2$), unless $k \geqslant n-1$, which is impossible. Hence we have a contradiction.

So k is odd.

Suppose first that $\Delta \leqslant n-k-1$. Again the right-hand side of inequality (3) is maximised at extremum of $\Delta = n-k-1$ or $\Delta = 2$. Arithmetic shows these have the same value, viz. $2q(n,k)+2$. That is,

$$2q(G) \leqslant 2q(n,k)+2.$$

Since G is a counter-example, $q(G) > q(n,k)$. Thus the above inequality holds with equality.

Therefore, inequality (3) and so inequality (2) holds with equality, which implies in turn that Claim 1 holds with equality. That is, $|N_B(v_i)| = n-\Delta-k+1$ for all i. In particular, this is true for $i=1$. Hence, by the proof of Claim 1, the number of vertices needed to dominate the set $B_1 = B - N_B(v_1)$ equals $|B_1|$. Hence $G[B_1]$ contains no vertex of degree greater than 1 and no vertex in $N_B(v_1)$ has more than one neighbour in B_1. In particular, $q(G[B_1]) \leqslant |B_1|/2$ and $q(B_1, N_B(v_1)) \leqslant |N_B(v_1)|$. Therefore,

$$\begin{aligned} 2q(G[B]) &\leqslant (|B_1| + 2|N_B(v_1)|) + 2\binom{|N_B(v_1)|}{2} \\ &= k-2+(n-\Delta-k+1)(n-\Delta-k+2). \end{aligned}$$

In conjunction with inequality (2), we obtain

$$2q(G) \leqslant \Delta(n-k+2)+k-2+(n-\Delta-k+1)(n-\Delta-k+2). \tag{4}$$

The right-hand side of inequality (4) is maximised if $\Delta = n-k-1$ or $\Delta = 2$. Some arithmetic shows that these give the same value viz. $2q(n,k)+1$. Thus

$$2q(G) \leqslant 2q(n,k)+1.$$

This implies that $q(G) \leqslant q(n,k)$ by integrality, a contradiction.

Therefore, $\Delta = n-k$.

Claim 4. *If there is a pair of vertices x and y such that* $(N(x)-\{y\}) \subseteq (N(y)-\{x\})$, *then* $G-y$ *has isolated vertices.*

Suppose there exists such a pair and $G-y$ has no isolates. Then $G-y$ has a total dominating set and any total dominating set of $G-y$ is one for G also. So $\gamma_t(G-y) \geqslant \gamma_t(G)$. Thus, by Claim 2 and the above lemma,

$$q(G) = q(G-y) + \deg(y) \leqslant q(n-1,k) + n-k = q(n,k),$$

a contradiction. □

Claim 5. *$G-u$ has no isolates.*

Suppose v_1 is isolated in $G-u$; that is, $\deg(v_1)=1$. If we return to the derivation of inequality (2), it follows that now $2q(G) \leqslant (n-k)(n-k+1)+1+2q(G[B])$, and so by the above lemma

$$2q(G) \leqslant (n-k)(n-k+1)+1+2q(k-1,k-2) = 2q(n,k)+1,$$

a contradiction, since $q(n,k)$ is integral. □

So $G-u$ has no isolates. This means by Claim 4 that the neighbourhood of any v_i cannot be contained in A. Combined with Claim 1, it follows that *every vertex in A* *has exactly one neighbour in* B. In particular,

$$q(A,B) = |A| = \Delta. \tag{5}$$

A vertex in B cannot be adjacent only to vertices of A, since then its neighbourhood would be a subset of $N(u)$, contradicting Claims 4 and 5. So every vertex of B has a neighbour in B. That is, $G[B]$ has no isolates. Note that $|B| = k - 1$.

We claim that $\gamma_t(G[B]) \geqslant |B| - 1$. For otherwise, one can add u and v_1 to a minimum total dominating set of $G[B]$ to obtain a total dominating set of G of cardinality at most $k - 1$, a contradiction. By Proposition 3, this means that at most one component of $G[B]$ is not K_2, and if so, that component must have three vertices. But since $|B|$ is even, $G[B]$ is the union of K_2's. In particular, $q(G[B]) = |B|/2$.

Suppose vertex $x_1 \in B$ is an end-vertex in G. Then let x_2 be its neighbour in B. Since G is connected, x_2 is adjacent to a vertex of A, say v_1. Since $G - v_1$ has no isolates, this contradicts Claim 4 with $x = x_1$ and $y = v_1$. Thus, every vertex in B has a neighbour in A.

Recall that every vertex in A has a unique neighbour in B. Let x_1 and x_2 be any two vertices of A with different neighbours in B. Extend to a set S by choosing for each remaining vertex of B any neighbour in A. So $|S| = n - \Delta - 1$. Since $|S| < k$, the set S is not a total dominating set of G. Thus, there is a vertex w not totally dominated by S (that is, $w \notin N(S)$); necessarily $w \in A$. If x_1 and x_2 are adjacent then w is distinct from them.

If we now consider the complement graph, this means that either x_1 and x_2 are adjacent in the complement, or they have a common neighbour $w \in A$ in the complement. That is, x_1 and x_2 are connected in $\bar{G}[A]$. Since this holds true for any pair with different neighbours in B, it follows that $\bar{G}[A]$ is connected. Thus there are at least $|A| - 1$ edges missing from $G[A]$. Hence,

$$q(G[A]) \leqslant \binom{\Delta}{2} - (\Delta - 1). \tag{6}$$

From inequalities (1), (5) and (6), and the value of Δ, it follows that

$$\begin{aligned} q(G) &\leqslant \deg u + q(G[A]) + q(A,B) + q(G[B]) \\ &\leqslant \Delta + \binom{\Delta}{2} - (\Delta - 1) + \Delta + (k-1)/2 \\ &= q(n,k), \end{aligned}$$

as some arithmetic shows. But this is a contradiction.

In order to see that the bound is sharp, consider the graph $G(n,k)$ defined as follows. If k is even, let $G(n,k)$ be the union of K_{n-k+2} and $(k-2)/2$ copies of K_2. If k is odd, let $G(n,k)$ be the graph obtained from $G(n-2,k-1)$ by subdividing one edge of the component isomorphic to K_{n-k+1} twice. In both cases, the graph $G(n,k)$ has order n, total domination number k and size $q(n,k)$. □

3. Total domination number in bipartite graphs

We are only able to establish a sharp result for k even.

For $1 \leqslant k \leqslant n$ define

$$r(n,k) := ((n-k)(n-k+6)+2k)/4.$$

Theorem 6. *Let G be a bipartite graph with order n and total domination number k. Then $q(G) \leqslant r(n,k)$, and this bound is sharp for $k \geqslant 4$ and even.*

Proof. The proof is by induction on k. It is easily checked that any bipartite graph has at most $r(n,2)$ edges. So the bound is true for $k=2$ (even if not sharp).

So, let G be a bipartite graph with $\gamma_t(G)=k \geqslant 3$. We may assume that G is connected, since some arithmetic shows that $r(n,k) \geqslant r(m,i)+r(n-m,k-i)$.

Assume G has bipartition V_1, V_2. Choose $u \in V_1$ and $v \in V_2$ such that $uv \in E$ and $\deg u + \deg v$ is as large as possible. Say $\deg u = d_1$, $\deg v = d_2$ with $d_1 \geqslant d_2$. Since G is connected and a star has total domination number 2, $d_2 \geqslant 2$.

Let $A = N(v) - \{u\}$, $B = N(u) - \{v\}$, $C = V_1 - N(v)$ and $D = V_2 - N(u)$. Let $\alpha = |C \cup D| = n - d_1 - d_2$. Pick vertices $x \in A$ and $y \in B$ such that $|N_C(y)|$ and $|N_D(x)|$ are as large as possible. Then let $C^* = N_C(y)$ and $D^* = N_D(x)$.

Now, consider the set $S = \{u,v\} \cup C \cup D$. There is a total dominating set of cardinality (at most) $|S|$, for if $w \in S$ is isolated, then one can replace w by a neighbour in $A \cup B$. It follows that $|S| \geqslant k$, and so

$$\alpha \geqslant k-2. \tag{7}$$

Next, consider the set $T = \{u,v,x,y\} \cup (C - C^*) \cup (D - D^*)$. There is a total dominating set of cardinality (at most) $|T|$, for if $w \in T$ is isolated, then one can replace w by a neighbour in $(C^* \cup D^* \cup A \cup B) - \{x,y\}$. It follows that $|T| \geqslant k$, and so

$$|C^*| + |D^*| \leqslant \alpha + 4 - k. \tag{8}$$

Let H denote the subgraph induced by the nonisolated vertices of $G[C \cup D]$ and let L denote the isolates, $|L| = l$. Let U be a minimum total dominating set of H. For each vertex of L, pick a neighbour in $A \cup B$ and let L' denote the resultant set. Then $U \cup L' \cup \{u,v\}$ totally dominates G and thus $\gamma_t(H) = |U| \geqslant k-2-l$. Thus, by the inductive hypothesis,

$$q(G[C \cup D]) \leqslant r(\alpha - l, k-2-l) = r(\alpha, k-2) - l/2 \leqslant r(\alpha, k-2). \tag{9}$$

By the choice of u, each vertex in A has degree at most d_1. So, the sum of degrees of the vertices in $A \cup \{u\}$ is at most $d_1 d_2$. The same bound holds for the sum of degrees of vertices of $B \cup \{v\}$. The sum of degrees of the vertices in $C \cup D$ is at most $2q(G[C \cup D]) + q(C,B) + q(D,A)$. By the choice of x, a vertex in A has at most $|D^*|$ neighbours in D and so $q(D,A) \leqslant (d_2-1)|D^*|$. Similarly, $q(C,B) \leqslant (d_1-1)|C^*|$. Thus, using inequality (9),

$$2q(G) \leqslant 2d_1 d_2 + 2r(\alpha, k-2) + (d_1-1)|C^*| + (d_2-1)|D^*|.$$

Since $d_1 \geqslant d_2$ and by inequality (8), this expression is maximised as a function of $|C^*|$ and $|D^*|$ when $|C^*| = \alpha + 4 - k$ and $|D^*| = 0$. Thus,

$$2q(G) \leqslant 2d_1 d_2 + 2r(\alpha, k-2) + (d_1 - 1)(\alpha + 4 - k). \tag{10}$$

So,

$$4q(G) \leqslant E := 4d_1 d_2 + (\alpha - k + 2)(\alpha - k + 8) + 2(k-2) \\ + 2(d_1 - 1)(\alpha - k + 4).$$

If one replaces α by $n - d_1 - d_2$, one sees that E is a function of d_1 and d_2.

Claim 6. *The expression $E(d_1, d_2)$ has a maximum of $(n-k+3)^2 + 2k - 8$ attained uniquely by $d_1 = (n-k+3)/2$ and $d_2 = (n-k+1)/2$.*

By calculus and some arithmetic, there is only one point where both partial derivatives are zero. But further calculus shows that this point is a saddle point. So the function $E(d_1, d_2)$ achieves a maximum on the boundary of the region.

The region is bounded by the lines $\mathscr{L}_1 : d_2 = 0$, $\mathscr{L}_2 : d_2 = d_1$ and $\mathscr{L}_3 : \alpha = k - 2$. Some calculations show that the maximum value of E on $\mathscr{L}_1$ is achieved uniquely at $d_1 = d_2 = 0$; that is, on its intersection with $\mathscr{L}_2$. More calculations show that the maximum value of E on $\mathscr{L}_2$ is achieved uniquely at the point it intersects $\mathscr{L}_3$. Finally, some calculations show that E is maximised on $\mathscr{L}_3$ uniquely at $d_1 = (n-k+3)/2$ and $d_2 = (n-k+1)/2$. There it has the value $E^* = (n-k+3)^2 + 2k - 8$. This proves the claim. □

However, it turns out that this maximum is unattainable in the actual graph. For, this unique extremum has $d_1 > d_2$. Thus attaining E^* requires, by the optimisation on $|C^*|$ and $|D^*|$ that produced inequality (10), that $|C^*| = \alpha + 4 - k \geqslant 2$ and $|D^*| = 0$. It also requires that the sum of degrees of the vertices in $A \cup \{u\}$ be $d_1 d_2$. But these two facts mean that every vertex in $A \cup \{u\}$ is adjacent to all of $B \cup \{v\}$; which means that $C^* = \emptyset$, a contradiction. Hence $4q(G) \leqslant E^* - 1$, whence the desired bound.

To show that this bound is best possible, let $H(x)$ denote the balanced or almost balanced complete bipartite graph on x vertices with a minimum edge cover removed. Then take $H(n-k+4) \cup [(k-4)/2]K_2$. This has order n, total domination number k and $\lfloor r(n,k) \rfloor$ edges. □

Conjecture 7. Consider bipartite graphs with order n and total domination number k for $k \geqslant 3$ odd. Then the maximum size is $\lfloor s(n,k) \rfloor$ where

$$s(n,k) := ((n-k)(n-k+4) + 2k - 2)/4.$$

The conjectured extremal graph is $G(n-k+3) \cup [(k-3)/2]K_2$ where $G(x)$ denotes the complete bipartite graph $K(\lceil x/2 - 1 \rceil, \lfloor x/2 + 1 \rfloor)$ with a maximum matching removed.

It is not hard to show that the conjecture is true for $k = 3$. For, one cannot have in each partite set a vertex adjacent to all of the other partite set, since then the total domination number is 2. So the number of edges missing (from the graph being complete bipartite) is at least the cardinality of the smaller partite set. But if the two sets are equal and a matching is missing, the graph has total domination number 4.

4. Independent domination number

Since $i(G)$ is at most the order minus the maximum degree, it is easy to show that if n is a multiple of k then the complete multipartite graphs with k vertices in each partite set are extremal graphs. The next theorem extends this to the case where n is not a multiple of k.

For $1 \leqslant k \leqslant n$ define $t(n,k)$ by

$$t(n,k) := \binom{n}{2} - \frac{1}{2}n(k-1) - \frac{1}{2}r(k-r),$$

where $n = sk + r$ and $0 \leqslant r < k$.

Theorem 8. *Let G be a graph with order n and independent domination number k. Then $q(G) \leqslant t(n,k)$ and this bound is sharp.*

Proof. In a graph G with $i(G) = k$, every vertex is in a (maximal) independent set of cardinality at least k. We show that if every vertex is in an independent set of size k, then $q(G) \leqslant t(n,k)$. This is a consequence of the following result, which is probably known, applied to the complement of G.

Lemma 9. *Let G be a graph with order n such that every vertex is in a k-clique. Then the number of edges is at least $\frac{1}{2}n(k-1) + \frac{1}{2}r(k-r)$, where $n = sk + r$ and $0 \leqslant r < k$.*

Proof. Without loss of generality, we may assume that G is such a graph of minimum size. Every vertex in G has degree at least $k-1$; let A denote the set of vertices of G of degree exactly $k-1$. Since every vertex of G is in a k-clique, there is a collection $\mathscr{K}_1, \ldots, \mathscr{K}_m$ of distinct k-subsets of V each of which induces a clique in G and whose union is V. Define $A_i = A \cap \mathscr{K}_i$ with $a_i = |A_i|$. Note that A_i is the set of those vertices in $\mathscr{K}_i$ which are not contained in any other $\mathscr{K}_j$.

Consider any set $\mathscr{K}_i$ for $i \geqslant 2$. By the minimality of G, A_i is nonempty. Suppose $A_i \neq \mathscr{K}_i$. Consider the graph G' formed by replacing $\mathscr{K}_i$ by another set $\mathscr{K}'_i$ that contains A_i with $\mathscr{K}'_i - A_i$ part of $\mathscr{K}_1$. That is, delete the $a_i(k-a_i)$ edges between A_i and $\mathscr{K}_i - A_i$, and any edge in $G[\mathscr{K}_i - A_i]$ joining two vertices no longer in a common $\mathscr{K}_j$, and insert the $a_i(k-a_i)$ edges between A_i and $\mathscr{K}'_i - A_i$. So $q(G') \leqslant q(G)$. Further, G' still has the property that every vertex is in a k-clique.

Hence, if we number the vertices of $\mathscr{K}_1$ as $v_1, \ldots, v_k$, we may rearrange cliques such that for every $i \geqslant 2$ the clique $\mathscr{K}_i$ consists of A_i and the first $k - a_i$ vertices of $\mathscr{K}_1$. That is, for every $i \geqslant 2$, the set $\mathscr{K}_i$ is either disjoint from the others, or it overlaps the first $k - a_i$ vertices of $\mathscr{K}_1$ but is otherwise disjoint.

Suppose two sets $\mathscr{K}_i$ and $\mathscr{K}_j$ with $i, j \geqslant 2$ both intersect $\mathscr{K}_1$. Consider replacing $\mathscr{K}_i$ and $\mathscr{K}_j$ as follows. If $a_i + a_j \leqslant k$, then replace both with a single set $\mathscr{K}'_i$ which consists of $A_i \cup A_j$ and the first $k - a_i - a_j$ vertices of $\mathscr{K}_1$. If $a_i + a_j > k$ then replace both with a set $\mathscr{K}'_i$ which consists of k vertices of $A_i \cup A_j$ and a set $\mathscr{K}'_j$ which consists of the remaining vertices of $A_i \cup A_j$ together with the first $2k - a_i - a_j$ vertices of $\mathscr{K}_1$. Some arithmetic shows that this reduces the number of edges, a contradiction. (The reductions in the number of edges are $a_i a_j$ and $(k - a_i)(k - a_j)$, respectively.)

Hence, it follows that at most two sets intersect, and the remaining sets are each disjoint. It follows that the overlap between the two sets that intersect is precisely $k-r$, and thus the number of edges in G is at least $(s+1)\binom{k}{2} - \binom{k-r}{2}$. □

In order to see that the bound of Theorem 8 is sharp, consider the graph $G(n,k)$ defined as follows. Let $n = sk + r$ with $0 \leqslant r < k$. Start with a complete s-partite graph with k vertices in each of the first $s-1$ partite sets and $k+r$ vertices in the last partite set. Then in the last partite set, add edges to form a $K(r,r)$. (We might have $s = 1$, but the complete 1-partite graph is by definition empty.) For example, $G(sk,k)$ is the complete s-partite graph with k vertices in each partite set. It is easily shown that $G(n,k)$ has independent domination number k. □

We establish the corresponding result for bipartite graphs next. The nonmonotonicity is perhaps surprising: for fixed order n, the maximum number of edges decreases from $k = 2$ until approximately $k = n/2 - \sqrt{n/2}$, then increases until $k = n/2$, and then decreases again.

Theorem 10. *Consider bipartite graphs of order n and independent domination number $k \geqslant 2$. Then the maximum size is attained by*

(a) *either $K(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ minus $k-1$ edges incident with one vertex, or $K(k, n-k)$, if $k \leqslant n/2$; and*

(b) *$K(n-k, n-k) \cup (2k-n)K_1$, otherwise.*

Proof. (a) Consider a bipartite graph G of order n and independent domination number k. If G is complete bipartite, then the independent domination number is the size of the smaller partite set. So G must be $K(k, n-k)$. Otherwise, there are two nonadjacent vertices that are in opposite partite sets. At least $k-2$ vertices must be not dominated by that pair, and so at least $k-1$ edges must be missing from G. It remains to observe that the possible extremal graph has independent domination number k.

(b) Since either partite set is an independent dominating set in a bipartite graph without isolated vertices, there must be at least $2k-n$ isolated vertices. □

The extremal graphs are unique.

5. Other parameters: the domination chain

The analogous results for some related parameters are straight-forward. A set S is irredundant if for every vertex v in S there is a private neighbour p_v—a vertex in its closed neighbourhood which is not dominated by the rest of S.

The domination chain involves six parameters:

$$\mathrm{ir}(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \beta(G) \leqslant \Gamma(G) \leqslant \mathrm{IR}(G),$$

where the lower irredundance number $\mathrm{ir}(G)$ is the minimum cardinality of a maximal irredundant set; the independence number $\beta(G)$ is the maximum cardinality of an independent set; the upper domination number $\Gamma(G)$ is the maximum cardinality of a minimal dominating set; and the (upper) irredundance number $\mathrm{IR}(G)$ is the maximum cardinality of an irredundant set. For bipartite graphs the latter three parameters are equal.

Theorem 11. *Consider graphs of order n with value of parameter $k \geqslant 2$.*

(a) *For* ir, *the maximum size is attained by the graph constructed as follows: take the complete graph on $n-k+2$ vertices, then remove a minimum edge cover and then add $k-2$ isolated vertices.*

(b) *For β, Γ and* IR, *the maximum size is attained by the graph constructed as follows: take the complete graph and remove the edges of a k-clique.*

Proof. (a) Because of Vizing's result and the domination chain, it suffices to observe that the extremal graphs for Theorem 1 have $\gamma = \mathrm{ir}$.

(b) An independent set and a minimal dominating set are both irredundant. Every vertex in an irredundant set of cardinality k is nonadjacent to at least $k-1$ vertices. Hence the upper bound holds for irredundance. To conclude, it suffices to check that the stated extremal graph has parameters equal to k. □

Theorem 12. *Consider bipartite graphs of order n with value of parameter k.*

(a) *For* ir, *the maximum size is attained by the almost balanced complete bipartite graph on $n-k+2$ vertices together with $k-2$ isolated vertices.*

(b) *For $\beta = \Gamma = \mathrm{IR}$, the value $k \geqslant n/2$ and the maximum size is attained by the complete bipartite graph $K(k, n-k)$.*

Proof. (a) By Theorem 2 and the domination chain, it suffices to observe that the extremal graphs for that theorem have $\gamma = \mathrm{ir}$.

(b) Trivial. □

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