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Cauchy problem for the multi-dimensional Boussinesq type equation

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Abstract

The paper studies the existence and non-existence of global weak solutions to the Cauchy problem for the multi-dimensional Boussinesq type equation $u_{tt} - \Delta u + \Delta^2 u = \Delta\sigma(u)$. It proves that the Cauchy problem admits a global weak solution under the assumptions that $\sigma \in C(\mathbf{R})$, $\sigma(s)$ is of polynomial growth order, say p (> 1), either $0 \leqslant \sigma(s)s \leqslant \beta \int_0^s \sigma(\tau)\,d\tau$, $s \in \mathbf{R}$, where $\beta > 0$ is a constant, or the initial data belong to a potential well. And the weak solution is regularized and the strong solution is unique when the space dimension $N = 1$. In contrast, any weak solution of the Cauchy problem blows up in finite time under certain conditions. And two examples are shown.

Keywords: Cauchy problem; Boussinesq type equation; Global solutions; Blowup of solutions

1. Introduction

In this paper, we are concerned with the existence and non-existence of global weak solutions to the Cauchy problem for the multi-dimensional Boussinesq type equation

$$u_{tt} - \Delta u + \mu\Delta^2 u = \Delta\sigma(u) \quad \text{in } \mathbf{R}^N \times (0, \infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbf{R}^N, \tag{1.2}$$

where $\mu > 0$ is a constant and $\sigma(s)$ is a nonlinear function specified later.

When the space dimension $N = 1$, Eq. (1.1) becomes

$$u_{tt} - u_{xx} + \mu u_{xxxx} = \sigma(u)_{xx}, \tag{1.3}$$

and Eq. (1.3) is a generalization of the well-known Boussinesq equation

$$u_{tt} - u_{xx} + \mu u_{xxxx} = a\left(u^2\right)_{xx}, \tag{1.4}$$

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where $\mu > 0$ denotes the dispersive parameter depending on the compression and rigidity characteristics of the materials, $a = const \in \mathbf{R}$ is a constant coefficient controlling nonlinearity, u is the vertical deflection, and the quadratic nonlinearity accounts for the curvature of the bending beam. Equation (1.4) was derived by Boussinesq [4] in 1872 and is called the "good" Boussinesq equation because of its linear stability, it models small oscillations of nonlinear beams (see [24]), and it governs the oscillation of the nonlinear strings and the two-dimensional irrotational flows of a inviscid liquid in a uniform rectangular channel as well (see [2,3,14]).

Equation (1.4) with $\mu < 0$ is known as the "bad" Boussinesq equation because of its linear instability, it describes the propagation of long surface waves on shallow water, and it also arises in a large range of physical phenomena including the propagation of ion-sound waves in a uniform isotropic plasma and nonlinear lattice waves (see [17]). For the "bad" Boussinesq equation and its generalizations, there have been many researches from various points of view (cf. [1,6,7,10,13,26–28]).

With regard to the Cauchy problem of the "good" Boussinesq type equation, Bona and Sachs [3] established the global existence of smooth solutions under the assumptions that: "$1 < p < 5$, $(p-1)/4 < c^2 < 1$, where $c > 0$ is the wave speed, $\sigma(u)$ being either u^p with p integer or $|u|^{p-1}u$ with p real and initial data lie relatively close to stable solitary waves." Tsutsumi and Matahashi [20] proved the local and global well posedness by means of transforming the Cauchy problem of Eq. (1.3) into the system of nonlinear Schrödinger equations. Linares [14,15] improved the results in [20] and further discussed the asymptotic behavior of the solutions provided that $\sigma(u) = |u|^{p-1}u$ with $p > 1$ and the initial data are small. Sachs [19] proved that for a large set of initial data, the Cauchy problem of Eq. (1.4), with $\mu = a = 1$, has no smooth solution for all time, which was claimed by Kalantarov and Ladyzhenskaya [11]. Liu [16] studied the instability of solitary waves for Eq. (1.3). And some related works can be seen in [9,18]. Recently, Chen and Yang [5] proved the existence and uniqueness of global solution to the IBVP of Eq. (1.3) under the assumptions that "$\sigma \in C^3(\mathbf{R})$, $\sigma'(s)$ is bounded below, $\sigma''(0) = 0$ and $\sigma'''(s)$ satisfies local Lipschitz condition," and some blowup results were also given.

But to the authors' best knowledge, there are very few works on the multi-dimensional Boussinesq type equation. Recently, taking into account the role of viscosity in real process, Varlamov [21–25] studied in detail the existence, uniqueness and long-time asymptotics of solutions to the Cauchy problem and the IBVP of the damped Boussinesq equation

$$u_{tt} - 2b\Delta u_t - \Delta u + \alpha\Delta^2 u = \beta\Delta\left(u^2\right) \tag{1.5}$$

in the cases of space dimension $N = 1, 2, 3$, respectively. When $N = 1$, the long-time asymptotic expansion of the classical solution was computed for a Cauchy problem with periodic small initial data (see [22]), and the blowup of solutions (even for small initial data) was also studied (see [23]). When $N = 2$, Eq. (1.5) models the small nonlinear oscillations of elastic membranes, and the author constructed the global solution by using eigenfunction expansion method and computed the longtime asymptotics on the basis of obtained the serious representation for the IBVP of Eq. (1.5) in both a disk and a ball, respectively (see [24,25]).

When the space dimension $N = 2$, Akmel [1] established the global existence of small solutions for the "bad" Boussinesq equation provided that the potential $v(x)$ verifying $\hat{v}(\xi) \in C_0^\infty(\mathbf{R}^2)$, and he showed further that the small solution decays to zero when the nonlinear term possesses an appropriate lower bound.

But for general space dimension, does Cauchy problem (1.1), (1.2) still admit a global solution? In the case of space dimension $N = 1$, does the corresponding Cauchy problem still admit a global solution provided that the conditions in [3] are not valid (say $p \geqslant 5$)? Can the theoretically qualitative condition: "smallness of initial data" in [14,15] be replaced by a quantitative one? All these questions are still open.

Various versions of the Boussinesq equation discussed in literature all possess one obvious characteristic: they are perturbations of the linear wave equation that takes into account the effects of small nonlinearity and dispersion. From the physical points of view, in real process, dispersion plays an important spreading role for the energy gather arising from nonlinearity, and the interaction of it with the nonlinearity accompanies accumulation, balance and dissipation of the energy in the configurations. And hence, it is interesting to consider the interaction between the two factors. In fact, despite the status of Boussinesq type equations as the first model for nonlinear dispersive wave propagation, the mathematical theory for such equations is not nearly so complete as is the case for the Korteweg–de Vries-type equations (see [3,24]).

In the present paper, first, by using a Galerkin approximation scheme combined with a limiting process of the solutions of a series of periodic boundary value problems (PBVP), which is different from those used in [1,21–25], the authors establish the existence of global solutions to Cauchy problem (1.1), (1.2) under the assumptions that $\sigma \in C(\mathbf{R})$, $\sigma(s)$ is of polynomial growth order, say p (> 1), either $0 \leqslant \sigma(s)s \leqslant \beta \int_0^s \sigma(\tau)\,d\tau$, $s \in \mathbf{R}$, where $\beta > 0$ is a constant, or the initial data belong to a potential well W_0.

Second, the above-mentioned weak solution is regularized and the strong solution is proved to be unique when the space dimension $N = 1$. And where the restrictions for both p, i.e. "$1 < p < 5$ and $(p-1)/4 < c^2 < 1$" and the initial data, i.e. "the initial data lie relatively close to stable solitary waves" (see [3]) are removed.

Third, any weak solution of the Cauchy problem blows up in finite time under certain conditions.

The plan of the paper is as follows. The main results and some notations are stated in Section 2. The global existence of weak solutions to the PBVP and Cauchy problem (1.1), (1.2) is discussed in Sections 3 and 4, respectively. In Section 5, the weak solutions are regularized and so the strong solution is proved to be unique in the case of space dimension $N = 1$. And in Section 6, a blowup theorem is proved and two examples are shown.

2. Statement of main results

We first introduce the following abbreviations:

$$\begin{aligned} &\Omega = (-L, L)^N, \quad Q_T = \Omega \times (0, T), \quad L_p = L_p(\Omega), \quad W^{m,p} = W^{m,p}(\Omega),\\ &C_0^\infty = C_0^\infty(\Omega), \quad H^m = W^{m,2}, \quad \|\cdot\|_p = \|\cdot\|_{L_p}, \quad \|\cdot\|_{k,p} = \|\cdot\|_{W^{k,p}},\\ &\|\cdot\|_{H^k} = \|\cdot\|_{H^k(\Omega)}, \quad \|\cdot\| = \|\cdot\|_{L_2}, \quad \dot H^s(\mathbf{R}^N) = (-\Delta)^{-\frac{s}{2}} L_2(\mathbf{R}^N). \end{aligned} \tag{2.1}$$

We denote the operators $G = (-\Delta)^{-1}$, $G^{1/2} = (-\Delta)^{-1/2}$, $p' = \frac{p}{p-1}$, with $p \in (1, \infty)$, and we denote the measure of Ω by $|\Omega|$. And the notation $(\cdot,\cdot)$ for the L_2-inner product will also be used for the notation of duality paring between dual spaces.

Without loss of generality, we take $\mu = 1$ in Eqs. (1.1) and (1.3).

Applying G to both sides of Eq. (1.1), we have

$$Gu_{tt} + u - \Delta u + \sigma(u) = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty). \tag{2.2}$$

Definition. The function $u \in L_\infty([0, T]; H^1(\mathbf{R}^N))$, with $G^{1/2}u_t \in L_\infty([0, T]; L_2(\mathbf{R}^N))$, is called a weak solution of Cauchy problem (1.1), (1.2), if for any $\chi \in H^1(\mathbf{R}^N)$, $\operatorname{supp}\chi$ is a bounded set in $\mathbf{R}^N$, and a.e. $t \in [0, T]$,

$$\big(G^{1/2}u_{tt}, G^{1/2}\chi\big) + (u, \chi) + (\nabla u, \nabla \chi) + \big(\sigma(u), \chi\big) = 0, \tag{2.3}$$

$$u(\cdot, 0) = u_0 \quad \text{in } H^1\big(\mathbf{R}^N\big), \qquad G^{1/2}u_t(\cdot, 0) = G^{1/2}u_1 \quad \text{in } L_2\big(\mathbf{R}^N\big). \tag{2.4}$$

Let $v(x, t) = e^{-\lambda t}u(x, t)$, where $\lambda > 0$ is a constant. Then problem (1.1), (1.2) is equivalent to the problem

$$G\big(v_{tt} + 2\lambda v_t + \lambda^2 v\big) + v - \Delta v + \tilde\sigma(t, v) = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty), \tag{2.5}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \mathbf{R}^N, \tag{2.6}$$

where $v_0(x) = u_0(x)$, $v_1(x) = u_1(x) - \lambda u_0(x)$, $\tilde\sigma(t, v) = e^{-\lambda t}\sigma(e^{\lambda t}v)$.

For any $u \in H_0^1(\Omega)$, let

$$\tilde u(x) = \begin{cases} u(x), & x \in \Omega,\\ 0, & x \in \mathbf{R}^N - \Omega, \end{cases}$$

and define

$$Gu = G\tilde u, \qquad G^{1/2}u = G^{1/2}\tilde u. \tag{2.7}$$

Then $Gu \in \dot H^3(\mathbf{R}^N)$ and $G^{1/2}u \in \dot H^2(\mathbf{R}^N)$, Gu and $G^{1/2}u$ are meaningful.

We first consider the PBVP of Eq. (2.5)

$$v|_{\partial\Omega} = 0, \quad v(x, t) = v(x + 2Le_i, t), \quad x \in \mathbf{R}^N,\ t > 0, \tag{2.8}$$

$$v(x, 0) = \varphi(x), \quad v_t(x, 0) = \psi(x), \quad x \in \mathbf{R}^N, \tag{2.9}$$

where $x + 2Le_i = (x_1, \dots, x_{i-1}, x_i + 2L, x_{i+1}, \dots, x_N)$, $L > 0$ is a real number, $\varphi \in H_0^1(\Omega)$, $\psi \in L_2(\Omega)$, with $\varphi|_{\partial\Omega} = \psi|_{\partial\Omega} = 0$, $\varphi(x) = \varphi(x + 2Le_i)$, $\psi(x) = \psi(x + 2Le_i)$, $x \in \mathbf{R}^N$, $i = 1, \dots, N$.

Definition. The function $v \in L_\infty([0,T]; H_0^1)$, with $G^{1/2}v_t \in L_\infty([0,T]; L_2)$, is called a weak solution of problem (2.5), (2.8), (2.9), if for any $\chi \in H_0^1$, and a.e. $t \in [0,T]$,

$$\big(G^{1/2}v_{tt}, G^{1/2}\chi\big) + 2\lambda\big(G^{1/2}v_t, G^{1/2}\chi\big) + \lambda^2\big(G^{1/2}v, G^{1/2}\chi\big) + (v,\chi) + (\nabla v, \nabla\chi) + \big(\tilde\sigma(t,v),\chi\big) = 0, \quad (2.10)$$

$$v(\cdot,0) = \varphi \quad \text{in } H^1, \qquad G^{1/2}v_t(\cdot,0) = G^{1/2}\psi \quad \text{in } L_2. \quad (2.11)$$

Define the potential wells with parameter t,

$$W_t = \big\{v(t) \in H_0^1 \mid I\big(v(t)\big) = \|v(t)\|_{H^1}^2 - e^{\lambda(p-1)t}\|v(t)\|_{p+1}^{p+1} > 0\big\} \cup \{0\}, \quad (2.12)$$

$$\tilde W_t = \big\{v(t) \in H^1\big(\mathbf{R}^N\big) \mid I\big(v(t)\big) = \|v(t)\|_{H^1(\mathbf{R}^N)}^2 - e^{\lambda(p-1)t}\|v(t)\|_{L_{p+1}(\mathbf{R}^N)}^{p+1} > 0\big\} \cup \{0\}, \quad t \in [0,T]. \quad (2.13)$$

For later purposes we introduce the functional J defined by

$$J\big(v(t)\big) := \frac{1}{2}\|v(t)\|_{H^1}^2 - \frac{b}{p+1}e^{\lambda(p-1)t}\|v(t)\|_{p+1}^{p+1}, \quad t \in [0,T], \quad (2.14)$$

for suitable $v(t)$. Obviously,

$$J\big(v(t)\big) = \frac{1}{2}I\big(v(t)\big) + b_1 e^{\lambda(p-1)t}\|v(t)\|_{p+1}^{p+1} = \frac{b}{p+1}I\big(v(t)\big) + b_1\|v(t)\|_{H^1}^2 \quad (2.15)$$

for all such $v(t)$, where and in the sequel $b_1 = \frac{p+1-2b}{2(p+1)}$.

Lemma 2.1. *For any $T > 0$ and each $t \in [0,T]$, W_t is a neighborhood of 0 in H_0^1 provided $1 < p \leqslant \frac{N+2}{(N-2)^+}$ ($p < +\infty$), where $a^+ = \max\{a, 0\}$.*

Proof. By the Gagliardo–Nirenberg inequality,

$$\begin{aligned} e^{\lambda(p-1)t}\|v(t)\|_{p+1}^{p+1} &\leqslant C_* e^{\lambda(p-1)t}\|v(t)\|^{(p+1)(1-\theta)}\|\nabla v(t)\|^{(p+1)\theta} \\ &\leqslant C_* e^{\lambda(p-1)t}\|v(t)\|_{H^1}^{p-1}\|v(t)\|_{H^1}^2 \\ &< \|v(t)\|_{H^1}^2, \quad t \in [0,T], \end{aligned} \quad (2.16)$$

as long as $\|v(t)\|_{H^1} < (1/C_*)^{1/(p-1)}e^{-\lambda T}$, $t \in [0,T]$, where $\theta = \frac{N(p-1)}{2(p+1)}$ ($\leqslant 1$). Lemma 2.1 is proved. □

Remark 1. From (2.16) we know that $H_0^1 \hookrightarrow L_{p+1}$, with the embedding constant C_* independent of Ω.

Now we state the main results of the paper.

Theorem 2.1. *Assume that*

(H_1) $\sigma \in C(\mathbf{R})$, $|\sigma(s)| \leqslant b|s|^p$, $s \in \mathbf{R}$, *with* $1 < p \leqslant \frac{N+2}{(N-2)^+}$ ($p < +\infty$).
(H_2) $G^{1/2}\psi \in L_2$, $\varphi \in H_0^1$, *and one of the following conditions holds*:
(i)

$$0 \leqslant \sigma(s)s \leqslant \beta\int_0^s \sigma(\tau)\,d\tau, \quad s \in \mathbf{R}, \quad (2.17)$$

where $\beta > 0$ is a constant.
(ii) $b < \frac{p+1}{2}$, $\varphi \in W_0$, *i.e.* $\varphi = 0$ *or*

$$I(\varphi) = \|\varphi\|_{H^1}^2 - \|\varphi\|_{p+1}^{p+1} > 0, \quad (2.18)$$

and

$$E^*(0) < \frac{b_1}{2e} C_*^{\frac{-2}{p+1}}, \tag{2.19}$$

where $b_1 = \frac{p+1-2b}{2(p+1)}$ (>0), C_* *as shown in* (2.16) *and*

$$E^*(0) = \frac{1}{2}\left(\left\|G^{1/2}\psi\right\|^2 + \lambda^2\left\|G^{1/2}\varphi\right\|^2 + \|\varphi\|_{H^1}^2\right) + \int_\Omega \int_0^\varphi \sigma(s)\,ds\,dx + \lambda\left(G^{1/2}\psi, G^{1/2}\varphi\right) + \lambda^2\left\|G^{1/2}\varphi\right\|^2, \tag{2.20}$$

with $\lambda = 2/(3T)$.

Then PBVP (2.5), (2.8), (2.9) *admits a weak solution on* $[0, T]$.

Theorem 2.2. *In addition to assumption* (H_1), *we assume that*

(H_3) $G^{1/2}u_1 \in L_2(\mathbf{R}^N)$, $u_0 \in H^1(\mathbf{R}^N)$, *and one of the following conditions holds*:
- (i) *inequality* (2.17).
- (ii) $b < \frac{p+1}{2}$, $u_0 \in \tilde{W}_0$, *i.e.* $u_0 = 0$ *or*

$$I(u_0) = \|u_0\|_{H^1(\mathbf{R}^N)}^2 - \|u_0\|_{L_{p+1}(\mathbf{R}^N)}^{p+1} > 0, \tag{2.21}$$

and

$$\tilde{E}^*(0) < \frac{b_1}{2e} C_*^{\frac{-2}{p+1}}, \tag{2.22}$$

where C_* *as shown in* (2.16), *and*

$$\tilde{E}^*(0) = \frac{1}{2}\left(\left\|G^{1/2}(u_1 - \lambda u_0)\right\|_{L_2(\mathbf{R}^N)}^2 + \lambda^2\left\|G^{1/2}u_0\right\|_{L_2(\mathbf{R}^N)}^2 + \|u_0\|_{H^1(\mathbf{R}^N)}^2\right) + \int_{\mathbf{R}^N} \int_0^{u_0} \sigma(s)\,ds\,dx + \lambda\left(G^{1/2}u_1, G^{1/2}u_0\right), \tag{2.23}$$

with $\lambda = 2/(3T)$.

Then Cauchy problem (1.1), (1.2) *admits a weak solution on* $[0, T]$.

Remark 2. Since for $T > 1$,

$$\tilde{E}^*(0) \leqslant 2\left(\left\|G^{1/2}u_1\right\|_{L_2(\mathbf{R}^N)}^2 + \left\|G^{1/2}u_0\right\|_{L_2(\mathbf{R}^N)}^2\right) + \|u_0\|_{H^1(\mathbf{R}^N)}^2 + \int_{\mathbf{R}^N} \int_0^{u_0} \sigma(s)\,ds\,dx \equiv \bar{E}^*(0),$$

(2.22) holds true for all $T > 1$ as long as $\bar{E}^*(0) < b_1 C_*^{\frac{-2}{p+1}}/2e$.

In the case of space dimension $N = 1$, the weak solutions can be regularized.

Theorem 2.3. *Under the assumptions of Theorem* 2.2 (*with* $N = 1$), *if also* $\sigma \in C^4(\mathbf{R})$, $u_0 \in H^4(\mathbf{R})$ *and* $u_1 \in H^2(\mathbf{R})$, *then for any* $T > 0$, *Cauchy problem* (1.3), (1.2), *with* $N = 1$, *admits a unique strong solution*

$$u \in C\left([0, T]; H^4(\mathbf{R})\right) \cap C^1\left([0, T]; H^2(\mathbf{R})\right) \cap C^2\left([0, T]; L_2(\mathbf{R})\right).$$

Theorem 2.4. *Assume that*

(H_4) *there exists a constant* $\alpha > 0$ *such that*

$$\sigma(s)s \leqslant 2(2\alpha + 1)\int_0^s \sigma(\tau)\,d\tau, \quad s \in \mathbf{R}. \tag{2.24}$$

(H_5) $u_0 \in H^1(\mathbf{R}^N)$, $G^{1/2}u_1 \in L_2(\mathbf{R}^N)$ *such that either*

$$E(0) = \|G^{1/2}u_1\|^2_{L_2(\mathbf{R}^N)} + \|u_0\|^2_{H^1(\mathbf{R}^N)} + 2\int_{\mathbf{R}^N}\int_0^{u_0} \sigma(s)\,ds\,dx < 0 \tag{2.25}$$

or

$$0 \leqslant E(0) < \frac{(G^{1/2}u_0, G^{1/2}u_1)^2}{\|G^{1/2}u_0\|^2_{L_2(\mathbf{R}^N)}} \quad \text{with } \left(G^{1/2}u_0, G^{1/2}u_1\right) > 0. \tag{2.26}$$

Then any weak solution u of Cauchy problem (1.1), (1.2) *blows up in finite time* $\tilde{T}$, *i.e.*

$$\left\|G^{1/2}u(t)\right\|_{L_2(\mathbf{R}^N)} \to +\infty \quad \text{as } t \to \tilde{T}^-. \tag{2.27}$$

Remark 3. We set $\sigma(s) = a|s|^{p-1}s$ and $\sigma(s) = as^p$, with $a \neq 0$, $p > 1$, in Eq. (1.1), respectively. A simple verification shows that (2.24) is valid, with $\alpha = \frac{p-1}{4}$ (> 0), and then we have by Theorem 2.4 that any weak solution of the corresponding Cauchy problem blows up in finite time as long as the initial data satisfy condition (H_5).

3. Global existence of weak solutions to the PBVP

Lemma 3.1. *(See [8].) Let* $\Omega \subset \mathbf{R}^N$ *be a bounded domain, and* $\{w_j\}_{j=1}^\infty$ *a complete orthogonal system in* $L_2(\Omega)$. *Then for every* $\varepsilon > 0$ *there is a positive integer* N_ε *such that for all* $u \in W_0^{1,p}(\Omega)$,

$$\|u\| \leqslant \left[\sum_{j=1}^{N_\varepsilon}(u, w_j)^2\right]^{1/2} + \varepsilon\|u\|_{1,p},$$

where $2 \leqslant p < +\infty$.

Proof of Theorem 2.1. We look for approximate solutions v^n of problem (2.5), (2.8), (2.9) of the form

$$v^n(t) := \sum_{j=1}^n T_{jn}(t)w_j, \quad t \geqslant 0, \tag{3.1}$$

where $\{w_j\}_{j=1}^\infty$ is a Schauder basis in H_0^1 and at the same time an orthonormal basis in L_2, with $w_j(x) = w_j(x + 2Le_i)$, $x \in \mathbf{R}^N$, $i = 1, \ldots, N$, and the coefficients $\{T_{jn}\}_{j=1}^n$ satisfy $T_{jn}(t) = (v^n(t), w_j)$ with

$$\begin{aligned}&\left(G^{1/2}v^n_{tt}, G^{1/2}w_j\right) + 2\lambda\left(G^{1/2}v^n_t, G^{1/2}w_j\right) + \lambda^2\left(G^{1/2}v^n, G^{1/2}w_j\right)\\&\quad + \left(v^n, w_j\right) + \left(\nabla v^n, \nabla w_j\right) + \left(\tilde{\sigma}\left(t, v^n\right), w_j\right) = 0, \quad t > 0,\end{aligned} \tag{3.2}$$

$$v^n(0) = \varphi^n, \qquad G^{1/2}v^n_t(0) = G^{1/2}\psi^n, \tag{3.3}$$

for $j = 1, \ldots, n$, $n \in \mathbf{N}$, and where φ^n and ψ^n are in C_0^∞ such that

$$G^{1/2}\varphi^n \to G^{1/2}\varphi \quad \text{in } H^3, \qquad G^{1/2}\psi^n \to G^{1/2}\psi \quad \text{in } L_2 \text{ as } n \to \infty. \tag{3.4}$$

Substituting w_j in (3.2) by v^n_t and v^n, respectively, we have

$$\frac{d}{dt}\left[\frac{1}{2}\left(\left\|G^{1/2}v_t^n(t)\right\|^2+\lambda^2\left\|G^{1/2}v^n(t)\right\|^2+\left\|v^n(t)\right\|_{H^1}^2\right)+e^{-2\lambda t}\int_\Omega\int_0^{e^{\lambda t}v^n}\sigma(s)\,ds\,dx\right]$$
$$+2\lambda\left(\left\|G^{1/2}v_t^n(t)\right\|^2+e^{-2\lambda t}\int_\Omega\int_0^{e^{\lambda t}v^n}\sigma(s)\,ds\,dx\right)$$
$$=\lambda e^{-2\lambda t}\left(\sigma\left(e^{\lambda t}v^n\right),e^{\lambda t}v^n\right),\quad t>0, \tag{3.5}$$
$$\frac{d}{dt}\left[\left(G^{1/2}v_t^n,G^{1/2}v^n\right)+\lambda\left\|G^{1/2}v^n(t)\right\|^2\right]+\lambda^2\left\|G^{1/2}v^n(t)\right\|^2+\left\|v^n(t)\right\|_{H^1}^2+e^{-2\lambda t}\left(\sigma\left(e^{\lambda t}v^n\right),e^{\lambda t}v^n\right)$$
$$=\left\|G^{1/2}v_t^n(t)\right\|^2,\quad t>0. \tag{3.6}$$

The combination of (3.5) with (3.6) yields

$$\frac{d}{dt}E_n^*(t)+\bar{E}_n(t)=0,\quad t>0, \tag{3.7}$$

where

$$E_n^*(t)=\frac{1}{2}\left(\left\|G^{1/2}v_t^n(t)\right\|^2+3\lambda^2\left\|G^{1/2}v^n(t)\right\|^2+\left\|v^n(t)\right\|_{H^1}^2\right)+e^{-2\lambda t}\int_\Omega\int_0^{e^{\lambda t}v^n}\sigma(s)\,ds\,dx$$
$$+\lambda\left(G^{1/2}v_t^n,G^{1/2}v^n\right), \tag{3.8}$$

$$\bar{E}_n(t)=\lambda\left\|G^{1/2}v_t^n(t)\right\|^2+\lambda^3\left\|G^{1/2}v^n(t)\right\|^2+\lambda\left\|v^n(t)\right\|_{H^1}^2+2\lambda e^{-2\lambda t}\int_\Omega\int_0^{e^{\lambda t}v^n}\sigma(s)\,ds\,dx,\quad t\geqslant 0. \tag{3.9}$$

(1) If (H_2)(i) holds true, by noticing $(G^{1/2}v_t^n,G^{1/2}v^n)\leqslant\frac{\varepsilon}{2}\|G^{1/2}v^n\|^2+\frac{1}{2\varepsilon}\|G^{1/2}v_t^n\|^2$, and by taking $\varepsilon=2\lambda$ and $\varepsilon=\lambda/2$, respectively, we obtain

$$\frac{1}{4}\left\|G^{1/2}v_t^n(t)\right\|^2+\frac{1}{2}\lambda^2\left\|G^{1/2}v^n(t)\right\|^2+\frac{1}{2}\left\|v^n(t)\right\|_{H^1}^2+e^{-2\lambda t}\int_\Omega\int_0^{e^{\lambda t}v^n}\sigma(s)\,ds\,dx$$
$$\leqslant E_n^*(t)\leqslant\frac{2}{\lambda}\bar{E}_n(t),\quad t>0. \tag{3.10}$$

It follows from (H_1), Remark 1 and (3.4) that

$$\left|\int_\Omega\int_\varphi^{\varphi^n}\sigma(s)\,ds\,dx\right|\leqslant\left|\int_\Omega\sigma\left(\kappa^n\right)\left(\varphi^n-\varphi\right)dx\right|\leqslant b\left\|\kappa^n\right\|_{p+1}^p\left\|\varphi^n-\varphi\right\|_{p+1}\to 0$$

as $n\to\infty$, where $\kappa^n=\theta\varphi^n+(1-\theta)\varphi$, $0<\theta<1$. As a result,

$$E_n^*(0)\to E^*(0)\quad(>0)\text{ as }n\to\infty, \tag{3.11}$$

where $E^*(0)$ as shown in (2.20). Without loss of generality we assume that

$$E_n^*(0)<2E^*(0)\quad\text{for all }n. \tag{3.12}$$

The combination of (3.7) with (3.10) and (3.12) leads to

$$\frac{d}{dt}E_n^*(t)+\frac{\lambda}{2}E_n^*(t)\leqslant 0,$$
$$\frac{1}{4}\left\|G^{1/2}v_t^n(t)\right\|^2+\frac{1}{2}\lambda^2\left\|G^{1/2}v^n(t)\right\|^2+\frac{1}{2}\left\|v^n(t)\right\|_{H^1}^2+e^{-2\lambda t}\int_\Omega\int_0^{e^{\lambda t}v^n}\sigma(s)\,ds\,dx\leqslant 2E^*(0)e^{\frac{-\lambda t}{2}},\quad t>0. \tag{3.13}$$

By (H_2)(i) and (3.13),

$$\int_\Omega \left|\tilde\sigma\left(t, v^n\right)\right|^{(p+1)'} dx = e^{-(p+1)'\lambda t} \int_\Omega \left|\sigma\left(e^{\lambda t} v^n\right)\right|^{(p+1)'} dx$$
$$\leqslant b^{1/p} e^{-(p+1)'\lambda t} \int_\Omega \sigma\left(e^{\lambda t} v^n\right) e^{\lambda t} v^n \, dx$$
$$\leqslant \beta b^{1/p} e^{-(p+1)'\lambda t} \int_\Omega \int_0^{e^{\lambda t} v^n} \sigma(s)\, ds\, dx$$
$$\leqslant 2\beta b^{1/p} e^{\frac{(p-2)}{2p}\lambda t} E^*(0), \quad t > 0. \tag{3.14}$$

(2) If (H_2)(ii) holds true, by (3.4) and Remark 1,

$$I\left(\varphi^n\right) = \left\|\varphi^n\right\|_{H^1}^2 - \left\|\varphi^n\right\|_{p+1}^{p+1} \to I(\varphi) > 0 \quad \text{as } n \to \infty. \tag{3.15}$$

Without loss of generality we assume that $I(\varphi^n) > 0$, i.e. $\varphi^n \in W_0$ for all n. By the continuity of $I(v^n(t))$, there exists a right neighborhood $(0, \delta)$ such that $I(v^n(t)) > 0$, $t \in (0, \delta)$. We claim that for any $T > 0$, $v^n(t) \in W_t$, $t \in [0, T]$, i.e.

$$I\left(v^n(t)\right) > 0, \quad t \in [0, T]. \tag{3.16}$$

In fact, if there exists t_0: $0 < t_0 \leqslant T$ such that $v^n(t) \in W_t$, $t \in [0, t_0)$, and $v^n(t_0) \in \partial W_{t_0}$, i.e. $I(v^n(t_0)) = 0$. Then by (H_1), (H_2)(ii), (2.14) and (2.15),

$$\frac{1}{2}\left\|v^n(t)\right\|_{H^1}^2 + e^{-2\lambda t} \int_\Omega \int_0^{e^{\lambda t} v^n} \sigma(s)\, ds\, dx \geqslant J\left(v^n(t)\right) \geqslant \begin{cases} b_1 \|v^n(t)\|_{H^1}^2, \\ b_1 e^{\lambda(p-1)t} \|v^n(t)\|_{p+1}^{p+1}, & t \in [0, t_0]. \end{cases} \tag{3.17}$$

Hence (3.10) and (3.13) hold true on $[0, t_0]$. The combination of (3.17) with (3.13) yields

$$b_1 \left\|v^n(t)\right\|_{H^1}^2 \leqslant 2E^*(0) e^{\frac{-\lambda t}{2}}, \quad t \in [0, t_0]. \tag{3.18}$$

By taking $\lambda = 2/(3T)$, we get from (2.16) and (2.19) that

$$e^{\lambda(p-1)t} \left\|v^n(t)\right\|_{p+1}^{p+1} \leqslant C_* \left(\frac{2E^*(0)}{b_1}\right)^{\frac{p-1}{2}} e^{\frac{3}{4}\lambda(p-1)T} \left\|v^n(t)\right\|_{H^1}^2$$
$$\leqslant C_* \left(\frac{2eE^*(0)}{b_1}\right)^{\frac{p-1}{2}} \left\|v^n(t)\right\|_{H^1}^2 < \left\|v^n(t)\right\|_{H^1}^2, \quad t \in [0, t_0]. \tag{3.19}$$

In particular, $I(v^n(t_0)) > 0$, which violates the assumption. So (3.16) is valid. And hence (3.17)–(3.19) hold true on $[0, T]$.

The combination of (3.13) with (3.17) leads to

$$\frac{1}{4}\left\|G^{1/2} v_t^n(t)\right\|^2 + \frac{1}{2}\lambda^2 \left\|G^{1/2} v^n(t)\right\|^2 + \frac{b_1}{2}\left(\left\|v^n(t)\right\|_{H^1}^2 + e^{\lambda(p-1)t}\left\|v^n(t)\right\|_{p+1}^{p+1}\right) \leqslant 2E^*(0) e^{\frac{-\lambda t}{2}}, \quad t \in [0, T]. \tag{3.20}$$

By (H_2)(ii) and (3.20),

$$\int_\Omega \left|\tilde\sigma\left(t, v^n\right)\right|^{(p+1)'} dx \leqslant b^{(p+1)'} e^{-(p+1)'\lambda t} \int_\Omega \left|e^{\lambda t} v^n(t)\right|^{p+1} dx \leqslant \frac{4}{b_1} b^{(p+1)'} e^{\frac{(p-2)}{2p}\lambda t} E^*(0), \quad t \in [0, T], \tag{3.21}$$

where $\lambda = 2/(3T)$. Integrating (3.2) over $(0, t)$, we obtain

$$\big(G^{1/2}v_t^n(t),G^{1/2}w_j\big)+2\lambda\big(G^{1/2}v^n(t),G^{1/2}w_j\big)+\int_0^t\Big(\lambda^2\big(G^{1/2}v^n,G^{1/2}w_j\big)$$
$$+\big(v^n,w_j\big)+\big(\nabla v^n,\nabla w_j\big)+\big(\tilde\sigma\big(t,v^n\big),w_j\big)\Big)\,d\tau$$
$$=\big(G^{1/2}\psi^n,G^{1/2}w_j\big)+2\lambda\big(G^{1/2}\varphi^n,G^{1/2}w_j\big),\quad t>0, \tag{3.22}$$
$$v^n(0)=\varphi^n. \tag{3.23}$$

For any $T>0$, it follows from (3.13), (3.14) (if (H_2)(i) holds true) or (3.20), (3.21) (if (H_2)(ii) holds true) that the nonlinear terms appearing in (3.22) are uniformly bounded on $[0,T]$, so the solution of problem (3.22), (3.23) exists on $[0,T]$ for each n.

From (3.13), (3.14) (if (H_2)(i) holds true) or (3.20), (3.21) (if (H_2)(ii) holds true) we can extract a subsequence from $\{v^n\}$, still denoted by $\{v^n\}$, such that

$$G^{1/2}v^n\to G^{1/2}v\quad\text{in } L_\infty\big([0,T];L_2\big)\text{ weak}^*, \tag{3.24}$$
$$v^n\to v\quad\text{in } L_\infty\big([0,T];H_0^1\big)\text{ weak}^*, \tag{3.25}$$
$$G^{1/2}v_t^n\to G^{1/2}v_t\quad\text{in } L_\infty\big([0,T];L_2\big)\text{ weak}^*, \tag{3.26}$$
$$\tilde\sigma\big(t,v^n\big)\to\gamma\quad\text{in } L_\infty\big([0,T];L_{(p+1)'}\big)\text{ weak}^* \tag{3.27}$$

as $n\to\infty$. It follows from Lemma 3.1, (3.25), (3.13) (if (H_2)(i) holds true) or (3.20) (if (H_2)(ii) holds true) that for every $\varepsilon>0$, there exists a positive integer N_ε such that for all v^n, when $n\to\infty$,

$$\big\|v^n(t)-v(t)\big\|\leqslant\left[\sum_{j=1}^{N_\varepsilon}\big(v^n-v,w_j\big)^2\right]^{1/2}+\varepsilon\big\|v^n(t)-v(t)\big\|_{H^1}\leqslant CE^*(0)\varepsilon,\quad t\in[0,T]. \tag{3.28}$$

By the arbitrariness of ε,

$$v^n\to v\quad\text{in } L_\infty\big([0,T];L_2\big)\text{ and a.e. on } Q_T, \tag{3.29}$$

as $n\to\infty$. By (3.29) and the continuity of σ, for a.e. $t\in[0,T]$,

$$\tilde\sigma\big(t,v^n\big)\to\tilde\sigma(t,v),\quad\text{a.e. on }\Omega \tag{3.30}$$

as $n\to\infty$. It follows from the Egoroff theorem that for any $\delta>0$, there exists a measurable set $\Omega_\delta\subset\Omega$, $|\Omega_\delta|<\delta$ such that for a.e. $t\in[0,T]$, $\tilde\sigma(t,v^n)\to\tilde\sigma(t,v)$ uniformly on $\tilde\Omega=\Omega-\Omega_\delta$ as $n\to\infty$. Hence, for any p_1: $p_1>p$, by the embedding theorem, (3.14) (if (H_2)(i) holds true) or (3.21) (if (H_2)(ii) holds true), we have

$$\big\|\tilde\sigma\big(t,v^n\big)-\tilde\sigma(t,v)\big\|_{L_{(p_1+1)'}(\Omega_\delta)}\leqslant\big\|\tilde\sigma\big(t,v^n\big)-\tilde\sigma(t,v)\big\|_{L_{(p+1)'}(\Omega_\delta)}\delta^{\frac{p_1-p}{(p+1)(p_1+1)}}$$
$$\leqslant M(T)\delta^{\frac{p_1-p}{(p+1)(p_1+1)}},\quad t\in[0,T], \tag{3.31}$$

where and in the sequel $M(T)$ denotes positive constants depending only on T. Therefore, when $n\to\infty$,

$$\big\|\tilde\sigma\big(t,v^n\big)-\tilde\sigma(t,v)\big\|_{L_{(p_1+1)'}(\Omega)}\leqslant\big\|\tilde\sigma\big(t,v^n\big)-\tilde\sigma(t,v)\big\|_{L_{(p_1+1)'}(\tilde\Omega)}+\big\|\tilde\sigma\big(t,v^n\big)-\tilde\sigma(t,v)\big\|_{L_{(p_1+1)'}(\Omega_\delta)}$$
$$\leqslant M(T)\delta^{\frac{p_1-p}{(p+1)(p_1+1)}},\quad t\in[0,T]. \tag{3.32}$$

By the arbitrariness of δ,

$$\tilde\sigma\big(t,v^n\big)\to\tilde\sigma(t,v)\quad\text{in } L_\infty\big([0,T];L_{(p_1+1)'}\big)\text{ as } n\to\infty. \tag{3.33}$$

It follows from (3.33) and the uniqueness of weak* limit that $\gamma(t)=\tilde\sigma(t,v)$ in $L_{(p_1+1)'}$ for a.e. $t\in[0,T]$. Since C_0^∞ is dense in L_{p+1}, for any $\kappa\in L_{p+1}$ there exists a sequence $\{\kappa_n\}$, $\kappa_n\in C_0^\infty$, such that $\kappa_n\to\kappa$ in L_{p+1} as $n\to\infty$. Hence,

$$\left|\left(\tilde{\sigma}(t,v)-\gamma(t),\kappa_n-\kappa\right)\right|\leqslant\left\|\tilde{\sigma}(t,v)-\gamma(t)\right\|_{(p+1)'}\|\kappa_n-\kappa\|_{p+1}\to 0\quad\text{as } n\to\infty,\ t\in[0,T],\tag{3.34}$$

$$\left(\tilde{\sigma}(t,v)-\gamma(t),\kappa\right)=\lim_{n\to\infty}\left(\tilde{\sigma}(t,v)-\gamma(t),\kappa_n\right)=0,\tag{3.35}$$

$$\tilde{\sigma}(t,v)=\gamma(t)\quad\text{in } L_{(p+1)'}\text{ for a.e. } t\in[0,T].\tag{3.36}$$

For any $\chi\in H_0^1$, it follows from (2.7) that $G^{1/2}\chi\in H^2$. Hence, letting $n\to\infty$ in (3.22) and making use of (3.24)–(3.27), (3.36) and (3.4) we obtain, for any $\chi\in H_0^1$ and a.e. $t\in[0,T]$,

$$\begin{aligned}&\left(G^{1/2}v_t(t),G^{1/2}\chi\right)+2\lambda\left(G^{1/2}v(t),G^{1/2}\chi\right)\\&\quad+\int_0^t\left[\lambda^2\left(G^{1/2}v,G^{1/2}\chi\right)+(v,\chi)+(\nabla v,\nabla\chi)+\left(\tilde{\sigma}(\tau,v),\chi\right)\right]d\tau\\&=\left(G^{1/2}\psi,G^{1/2}\chi\right)+2\lambda\left(G^{1/2}\varphi,G^{1/2}\chi\right).\end{aligned}\tag{3.37}$$

Differentiating (3.37) we get that for any $\chi\in H_0^1$ and a.e. $t\in[0,T]$, (2.10) is valid. By (3.24), (3.26) and (3.4),

$$\begin{aligned}&\left(G^{1/2}v^n,w_j\right)\to\left(G^{1/2}v,w_j\right)\quad\text{in } H^1[0,T]\text{ and in } C[0,T],\\&G^{1/2}v(0)=G^{1/2}\varphi\quad\text{in } L_2.\end{aligned}\tag{3.38}$$

Since $\varphi\in H_0^1$, applying the operators $G^{-1/2}=(-\Delta)^{1/2}$ and $G^{-1}=-\Delta$ to both sides of (3.38), we get

$$v(0)=\varphi\quad\text{in } H^1.\tag{3.39}$$

Letting $t=0$ in (3.37) and using (3.38) and the density of H_0^1 in L_2, we have

$$Gv_t(0)=G\psi\quad\text{in } L_2.\tag{3.40}$$

Applying $G^{-1/2}=(-\Delta)^{1/2}$ to both sides of (3.40) leads to

$$G^{1/2}v_t(0)=G^{1/2}\psi.\tag{3.41}$$

The combination of (2.10) with (3.39) and (3.41) indicates that v is a weak solution of PBVP (2.5), (2.8), (2.9). Theorem 2.1 is proved. □

4. Global weak solutions to the Cauchy problem

Proof of Theorem 2.2. We take a sequence $\{L_s\}$, where L_s (>1) are real numbers, and $L_s\to+\infty$ as $s\to\infty$. Let $\Omega_s=(-L_s,L_s)^N$. For each s we construct the potential wells with parameter t

$$W_t^s=\left\{v(t)\in H_0^1(\Omega_s)\;\middle|\;I_s\left(v(t)\right)=\left\|v(t)\right\|^2_{H^1(\Omega_s)}-e^{\lambda(p-1)t}\left\|v(t)\right\|^{p+1}_{L_{p+1}(\Omega_s)}>0\right\}\cup\{0\},\tag{4.1}$$

and the periodic functions $\varphi_s\in H_0^1(\Omega_s)$, $\psi_s\in L_2(\Omega_s)$, with $\varphi_s|_{\partial\Omega_s}=\psi_s|_{\partial\Omega_s}=0$, and

(i) $\varphi_s(x)=\varphi_s(x+2L_se_i)$, $\psi_s(x)=\psi_s(x+2L_se_i)$, $x\in\mathbf{R}^N$, $i=1,\dots,N$.
(ii) $\varphi_s(x)=v_0(x)$, $\psi_s(x)=v_1(x)$ on $\Omega_s^*=(-L_s+1,L_s-1)^N$, and

$$\begin{aligned}&\left\|G^{1/2}\varphi_s\right\|_{L_2(\Omega_s)}\leqslant\left\|G^{1/2}v_0\right\|_{L_2(\mathbf{R}^N)},\qquad\|\varphi_s\|_{H^1(\Omega_s)}\leqslant\|v_0\|_{H^1(\mathbf{R}^N)},\\&\|\varphi_s\|_{L_{p+1}(\Omega_s)}\leqslant\|v_0\|_{L_{p+1}(\mathbf{R}^N)},\qquad\left\|G^{1/2}\psi_s\right\|_{L_2(\Omega_s)}\leqslant\left\|G^{1/2}v_1\right\|_{L_2(\mathbf{R}^N)}.\end{aligned}\tag{4.2}$$

Let

$$\tilde{\varphi}_s(x)=\begin{cases}\varphi_s(x), & x\in\Omega_s,\\ 0, & x\in\mathbf{R}^N-\Omega_s,\end{cases}\qquad\tilde{\psi}_s(x)=\begin{cases}\psi_s(x), & x\in\Omega_s,\\ 0, & x\in\mathbf{R}^N-\Omega_s.\end{cases}\tag{4.3}$$

By (4.2),

$$\left\|G^{1/2}\tilde{\varphi}_s-G^{1/2}v_0\right\|_{L_2(\mathbf{R}^N)}\leqslant 2\left\|G^{1/2}v_0\right\|_{L_2(\mathbf{R}^N)}.\tag{4.4}$$

Hence,

$$\left\|G^{1/2}\tilde{\varphi}_s - G^{1/2}v_0\right\|_{L_2(\mathbf{R}^N)} = \left\|G^{1/2}\tilde{\varphi}_s - G^{1/2}v_0\right\|_{L_2(\mathbf{R}^N-\Omega_s^*)} \to 0 \tag{4.5}$$

as $s \to \infty$. Similarly,

$$\tilde{\varphi}_s \to v_0 \quad \text{in } H^1(\mathbf{R}^N) \cap L_{p+1}(\mathbf{R}^N), \qquad G^{1/2}\tilde{\psi}_s \to G^{1/2}v_1 \quad \text{in } L_2(\mathbf{R}^N) \tag{4.6}$$

as $s \to \infty$. It follows from (4.6) that

$$I_s(\varphi_s) = \|\varphi_s\|^2_{H^1(\Omega_s)} - \|\varphi_s\|^{p+1}_{L_{p+1}(\Omega_s)} \to I(v_0) = \|v_0\|^2_{H^1(\mathbf{R}^N)} - \|v_0\|^{p+1}_{L_{p+1}(\mathbf{R}^N)} \tag{4.7}$$

as $s \to \infty$. By the integral mean value theorem, the Hölder inequality and (4.6),

$$\begin{aligned}\left|\int_{\Omega_s}\int_0^{\varphi_s}\sigma(\tau)\,d\tau\,dx - \int_{\mathbf{R}^N}\int_0^{v_0}\sigma(\tau)\,d\tau\,dx\right| &\leqslant \left\|\sigma(\xi_s)\right\|_{L_{(p+1)'}(\mathbf{R}^N)}\|\tilde{\varphi}_s - v_0\|_{L_{p+1}(\mathbf{R}^N)} \\ &\leqslant C\|\xi_s\|^p_{L_{p+1}(\mathbf{R}^N)}\|\tilde{\varphi}_s - v_0\|_{L_{p+1}(\mathbf{R}^N)} \to 0\end{aligned} \tag{4.8}$$

as $s \to \infty$, where $\xi_s = v_0 + \theta_s\tilde{\varphi}_s$, $0 < \theta_s < 1$. By (4.5), (4.6) and (4.8),

$$E_s^*(0) \to \tilde{E}^*(0) \quad \text{as } s \to \infty, \tag{4.9}$$

where $E_s^*(0)$ as shown in (2.20) (substituting φ, ψ and Ω there by φ_s, ψ_s and Ω_s, respectively), and $\tilde{E}^*(0)$ as shown in (2.23), where the facts: $v_0 = u_0$ and $v_1 = u_1 - \lambda u_0$ have been used. By (2.21), (2.22), (4.7) and (4.9), without loss of generality we assume that

$$I(\varphi_s) > 0, \quad E_s^*(0) < \min\left\{\frac{b_1}{2e}C_*^{\frac{-2}{p-1}}, 2\tilde{E}^*(0)\right\} \quad \text{for all } s. \tag{4.10}$$

Repeating the arguments of Theorem 2.1, for each s we get a weak solution v^s of corresponding PBVP (2.5), (2.8), (2.9) (substituting L, Ω, φ and ψ there by L_s, Ω_s, φ_s and ψ_s, respectively), and v^s is a weak* limit of a sequence $\{v^{n_s}\}$, where v^{n_s} are the solutions of problem (3.2), (3.3) (substituting Ω, v^n, φ^n and ψ^n there by Ω_s, v^{n_s}, φ^{n_s} and ψ^{n_s}, respectively). And inequalities (3.13), (3.14) (if (H_2)(i) holds true) or (3.20), (3.21) (if (H_2)(ii) holds true) are valid for v^{n_s} (substituting v^n and Ω there by v^{n_s} and Ω_s, respectively).

(1) If (H_2)(i) holds true, then $F(e^{\lambda t}v^{n_s}) = \int_0^{e^{\lambda t}v^{n_s}}\sigma(\tau)\,d\tau \geqslant 0$ are measurable functions defined on Ω_s for a.e. $t \in [0,T]$. Since $v^{n_s} \to v^s$ a.e. on Ω_s for a.e. $t \in [0,T]$ as $n_s \to \infty$ (see (3.29)), and by the continuity of $F(e^{\lambda t}v^{n_s})$,

$$F(e^{\lambda t}v^{n_s}) \to F(e^{\lambda t}v^s) \quad \text{a.e. on } \Omega_s \text{ as } n_s \to \infty,$$

i.e. there exists a set $\tilde{\Omega}_s \subset \Omega_s$, $|\tilde{\Omega}_s| = 0$, such that

$$F(e^{\lambda t}v^{n_s}) \to F(e^{\lambda t}v^s) \quad \text{on } \Omega_s - \tilde{\Omega}_s \text{ as } n_s \to \infty.$$

Then, by the Fatou theorem, for a.e. $t \in [0,T]$,

$$\int_{\Omega_s} F(e^{\lambda t}v^s)\,dx = \int_{\Omega_s-\tilde{\Omega}_s} F(e^{\lambda t}v^s)\,dx \leqslant \liminf_{n_s\to\infty}\int_{\Omega_s-\tilde{\Omega}_s} F(e^{\lambda t}v^{n_s})\,dx = \liminf_{n_s\to\infty}\int_{\Omega_s} F(e^{\lambda t}v^{n_s})\,dx. \tag{4.11}$$

From the sequential lower semi-continuity of the norm and (4.11) we conclude that inequalities (3.13) and (3.14) are still valid for v^s (substituting v^n, Ω and $E^*(0)$ there by v^s, Ω_s and $\tilde{E}^*(0)$, respectively).

(2) If (H_2)(ii) holds true, from the sequential lower semi-continuity of the norm we deduce that inequalities (3.20) and (3.21) are still valid for v^s (substituting v^n, Ω and $E^*(0)$ there by v^s, Ω_s and $\tilde{E}^*(0)$, respectively).

Hence, by the same arguments used in the proof of Theorem 2.1 we can extract a subsequence from $\{v^s\}$, still denoted by $\{v^s\}$, such that, for any $\Omega_L = (-L,L)^N$ and $T > 0$,

$$\begin{aligned}&G^{1/2}v^s \to G^{1/2}v \quad \text{in } L_\infty([0,T]; L_2(\Omega_L)) \text{ weak}^*,\\ &v^s \to v \quad \text{in } L_\infty([0,T]; H_0^1(\Omega_L)) \text{ weak}^*,\\ &G^{1/2}v_t^s \to G^{1/2}v_t \quad \text{in } L_\infty([0,T]; L_2(\Omega_L)) \text{ weak}^*,\\ &\tilde{\sigma}(t,v^s) \to \tilde{\sigma}(t,v) \quad \text{in } L_\infty([0,T]; L_{(p+1)'}(\Omega_L)) \text{ weak}^*\end{aligned} \tag{4.12}$$

as $s \to \infty$. And by the same method used above we see that inequalities (3.13) and (3.20) still hold true for the limit function v (substituting v^n, Ω and $E^*(0)$ there by v, Ω_L and $\tilde{E}^*(0)$, respectively).

From the arbitrariness of L and the bounded-ness of the norm of v (see (3.13) and (3.20)) we know that the limiting function

$$v \in L_\infty\big([0,T]; H^1\big(\mathbf{R}^N\big)\big), \quad \text{with } G^{1/2}v_t \in L_\infty\big([0,T]; L_2\big(\mathbf{R}^N\big)\big). \tag{4.13}$$

For any $\chi \in H^1(\mathbf{R}^N)$, supp χ is a bounded set in $\mathbf{R}^N$, there must be an $L > 0$ such that supp $\chi \subset \Omega_L$ and $\chi \in H_0^1(\Omega_L)$. Substituting v^n, w_j, φ^n and ψ^n in (3.22) by v^s, χ, φ_s and ψ_s, respectively, and letting $s \to \infty$, we obtain

$$\begin{aligned}&\big(G^{1/2}v_t(t), G^{1/2}\chi\big) + 2\lambda\big(G^{1/2}v(t), G^{1/2}\chi\big)\\&\quad + \int_0^t \big(\lambda^2\big(G^{1/2}v(\tau), G^{1/2}\chi\big) + \big(v(\tau),\chi\big) + \big(\nabla v(\tau), \nabla\chi\big) + \big(\tilde{\sigma}\big(\tau, v(\tau)\big),\chi\big)\big)\,d\tau\\&= \big(G^{1/2}v_1, G^{1/2}\chi\big) + 2\lambda\big(G^{1/2}v_0, G^{1/2}\chi\big), \quad t \in [0,T].\end{aligned} \tag{4.14}$$

By (4.12), for any $\chi \in C_0^\infty(\mathbf{R}^N)$,

$$\big(G^{1/2}v^s, \chi\big) \to \big(G^{1/2}v, \chi\big) \quad \text{in } H^1[0,T] \text{ and in } C[0,T]. \tag{4.15}$$

Then by (4.5),

$$\big(G^{1/2}v(0), \chi\big) = \big(G^{1/2}v_0, \chi\big). \tag{4.16}$$

Since $C_0^\infty(\mathbf{R}^N)$ is dense in $L_2(\mathbf{R}^N)$, from the continuity of the inner-product in L_2 we get that (4.16) holds true for arbitrary $\chi \in L_2(\mathbf{R}^N)$, and hence

$$G^{1/2}v(0) = G^{1/2}v_0 \quad \text{in } L_2\big(\mathbf{R}^N\big). \tag{4.17}$$

Since $G^{1/2}v_0 \in H^2(\mathbf{R}^N)$, applying $G^{-1/2}$ and G^{-1} to both sides of (4.17), we have

$$v(0) = v_0 \quad \text{in } H^1\big(\mathbf{R}^N\big). \tag{4.18}$$

For any $\chi \in C_0^\infty(\mathbf{R}^N)$, letting $t = 0$ in (4.14), exploiting (4.17) and using the same method as above, we get

$$\big(Gv_t(0), \chi\big) = (Gv_1, \chi), \quad Gv_t(0) = Gv_1 \quad \text{in } L_2\big(\mathbf{R}^N\big). \tag{4.19}$$

Since $G^{1/2}v_1 \in L_2(\mathbf{R}^N)$, applying $G^{-1/2}$ to both sides of (4.19), we obtain

$$G^{1/2}v_t(0) = G^{1/2}v_1 \quad \text{in } L_2\big(\mathbf{R}^N\big). \tag{4.20}$$

Differentiating (4.14) and combining the result with (4.18) and (4.20) we see that v is a weak solution of Cauchy problem (2.5), (2.6) on $[0,T]$. Therefore, $u = e^{\lambda t}v$ is a weak solution of Cauchy problem (1.1), (1.2) on $[0,T]$. Theorem 2.2 is proved. □

5. The case in one dimension

Lemma 5.1. *(See [29].) Assume that $G(z_1,\ldots,z_h)$ is a k-times continuously differentiable function with respect to variables $z_1,\ldots,z_h$ and $z_i \in L_\infty([0,T]; H^k(\Omega))$ $(i = 1,\ldots,h)$. Then*

$$\left\|\frac{\partial^k}{\partial x^k}G\big(z_1(\cdot,t),\ldots,z_h(\cdot,t)\big)\right\|^2 \leqslant C(\bar{M},k,h)\sum_{i=1}^h \big\|z_i(t)\big\|_{H^k}^2$$

where $\bar{M} = \max_{1\leqslant i\leqslant h}\max_{(x,t)\in\bar{Q}_T}|z_i(x,t)|$, $C(\bar{M},k,h)$ is a positive constant depending only on $\bar{M}$, k and h.

In this section, we still use the notations in (2.1), with $\Omega = (-L, L)$. Moreover, $\|\cdot\|_C = \|\cdot\|_{C(\Omega)}$.

Proof of Theorem 2.3. We first consider the PBVP of Eq. (1.3)

$$u|_{\partial\Omega}=0,\quad u(x,t)=u(x+2L,t),\quad x\in\mathbf{R},\ t>0, \tag{5.1}$$

$$u(x,0)=\xi(x),\quad u_t(x,0)=\eta(x),\quad x\in\mathbf{R}, \tag{5.2}$$

where $\xi\in H^4\cap H_0^1$, $\eta\in H^2\cap H_0^1$, with $\xi(x)=\xi(x+2L)$, $\eta(x)=\eta(x+2L)$, $x\in\mathbf{R}$. And it follows from (2.9) (with $N=1$) that $\xi(x)=\varphi(x)$, $\eta(x)=\psi(x)+\lambda\varphi(x)$.

We still start with the approximate solutions of PBVP (1.3), (5.1), (5.2) of the form

$$u^n(t):=\sum_{j=1}^{n}\tilde T_{jn}(t)w_j,\quad t\geqslant 0, \tag{5.3}$$

where $\{w_j\}_{j=1}^\infty$ is a Schauder basis in $H^4\cap H_0^1$ and at the same time an orthonormal basis in L_2, with $w_j(x)=w_j(x+2L)$, $x\in\mathbf{R}$, and the coefficients $\{\tilde T_{jn}\}_{j=1}^n$ satisfy $\tilde T_{jn}(t)=(u^n(t),w_j)$ with

$$\big(u_{tt}^n,w_j\big)-\big(u_{xx}^n,w_j\big)+\big(u_{x^4}^n,w_j\big)=\big(\sigma\big(u^n\big)_{xx},w_j\big),\quad t>0, \tag{5.4}$$

$$u^n(0)=\xi^n,\qquad u_t^n(0)=\eta^n, \tag{5.5}$$

for $j=1,\dots,n$, $n\in\mathbf{N}$, where and in the sequel $u_{x^k}=\frac{\partial^k u}{\partial x^k}$, $\xi^n,\eta^n\in C_0^\infty$ and

$$\xi^n\to u_0\quad\text{in }H^4,\qquad \eta^n\to u_1\quad\text{in }H^2\quad\text{as }n\to\infty. \tag{5.6}$$

By (3.13) (if (H_2)(i) holds true) or (3.20) (if (H_2)(ii) holds true), and the Gagliardo–Nirenberg inequality,

$$\begin{aligned}&\big\|u^n(t)\big\|_{H^1}=e^{\lambda t}\big\|v^n(t)\big\|_{H^1}\leqslant M(T),\\&\big\|u^n(t)\big\|_C\leqslant C\big\|u^n(t)\big\|^{\frac12}\big\|u_x^n(t)\big\|^{\frac12}\leqslant M(T),\quad t\in[0,T].\end{aligned} \tag{5.7}$$

Substituting w_j in (5.4) by $u_t^n+u_{x^4t}^n$, integrating by parts and exploiting Lemma 5.1, (5.7) and the Gronwall inequality, we get

$$\begin{aligned}&\frac12\frac{d}{dt}\big(\big\|u_t^n(t)\big\|^2+\big\|u_{x^2t}^n(t)\big\|^2+\big\|u^n(t)\big\|_{H^4}^2\big)\\&\quad=\big(\sigma\big(u^n\big)_{xx},u_t^n\big)+\big(\sigma\big(u^n\big)_{x^4},u_{x^2t}^n\big)+\big(u^n,u_t^n\big)\leqslant M(T)\big(\big\|u_t^n(t)\big\|^2+\big\|u_{x^2t}^n(t)\big\|^2+\big\|u^n(t)\big\|_{H^4}^2\big),\\&\big\|u_t^n(t)\big\|_{H^2}^2+\big\|u^n(t)\big\|_{H^4}^2\leqslant\big(\big\|\eta^n\big\|_{H^2}^2+\big\|\xi^n\big\|_{H^4}^2\big)e^{2M(T)T}\leqslant M(T),\quad t\in[0,T].\end{aligned} \tag{5.8}$$

Substituting w_j in (5.4) by u_{tt}^n and using the Hölder inequality, Lemma 5.1 and (5.8), we obtain

$$\big\|u_{tt}^n(t)\big\|\leqslant\big\|u_{x^2}^n(t)\big\|+\big\|u_{x^4}^n(t)\big\|+\|\sigma\big(u^n(t)\big)_{xx}\|\leqslant M(T),\quad t\in[0,T]. \tag{5.9}$$

Let $w^n=u^n-u^{n-1}$, then $w^n(t)$ satisfy

$$\big(w_{tt}^n,w_j\big)-\big(w_{xx}^n,w_j\big)+\big(w_{x^4}^n,w_j\big)=\big(\sigma\big(u^n\big)_{xx}-\sigma\big(u^{n-1}\big)_{xx},w_j\big),\quad t>0, \tag{5.10}$$

$$w^n(0)=\xi^n-\xi^{n-1},\qquad w_t^n(0)=\eta^n-\eta^{n-1}, \tag{5.11}$$

for $j=1,\dots,n$, $n\in\mathbf{N}$. Substituting w_j in (5.10) by $w^n+w_{x^4t}^n$ and exploiting the Lagrange mean value theorem, (5.8), Lemma 5.1 and the Gronwall inequality, we arrive at

$$\begin{aligned}&\frac12\frac{d}{dt}\big(\big\|w_t^n(t)\big\|^2+\big\|w_{x^2t}^n(t)\big\|^2+\big\|w^n(t)\big\|_{H^4}^2\big)\\&\quad=\big(\big(\sigma\big(u^n\big)-\sigma\big(u^{n-1}\big)\big)_{xx},w_t^n\big)+\big(\big(\sigma\big(u^n\big)-\sigma\big(u^{n-1}\big)\big)_{x^4},w_{x^2t}^n\big)+\big(w^n,w_t^n\big)\\&\quad\leqslant M(T)\big(\big\|w_t^n(t)\big\|^2+\big\|w_{x^2t}^n(t)\big\|^2+\big\|w^n(t)\big\|_{H^4}^2\big),\quad t\in[0,T],\end{aligned} \tag{5.12}$$

$$\big\|w_t^n(t)\big\|_{H^2}^2+\big\|w^n(t)\big\|_{H^4}^2\leqslant\big(\big\|\eta^n-\eta^{n-1}\big\|_{H^2}^2+\big\|\xi^n-\xi^{n-1}\big\|_{H^4}^2\big)e^{2M(T)T}\to 0 \tag{5.13}$$

uniformly on $[0,T]$ as $n\to\infty$. Substituting w_j in (5.10) by w_{tt}^n and exploiting Lemma 5.1 and (5.13), we have

$$\begin{aligned}\|w^n_{tt}(t)\| &\leqslant \|w^n_{x^2}(t)\| + \|w^n_{x^4}(t)\| + \|(\sigma(u^n(t)) - \sigma(u^{n-1}(t)))_{xx}\| \\ &\leqslant M(T)\|w^n(t)\|_{H^4} \to 0 \end{aligned} \tag{5.14}$$

uniformly on $[0, T]$ as $n \to \infty$. Then,

$$\|u^n_t(t) - u^m_t(t)\|^2_{H^2} + \|u^n(t) - u^m(t)\|^2_{H^4} + \|u^n_{tt}(t) - u^m_{tt}(t)\|^2 \to 0 \tag{5.15}$$

uniformly on $[0, T]$ as $m, n \to \infty$, i.e. $\{u^n\}$ is a Cauchy sequence in $C([0, T]; H^4) \cap C^1([0, T]; H^2) \cap C^2([0, T]; L_2)$. Hence

$$u^n \to u \quad \text{in } C([0, T]; H^4) \cap C^1([0, T]; H^2) \cap C^2([0, T]; L_2) \tag{5.16}$$

as $n \to \infty$. Letting $n \to \infty$ in (5.4) and (5.5) we get that u is a strong solution of PBVP (1.3), (5.1), (5.2).

Now, we restrict our attention to Cauchy problem (1.3), (1.2), with $N = 1$.

We still take a sequence $\{L_s\}$ as usual and construct the periodic functions $\xi_s(x) = \varphi_s(x)$, $\eta_s(x) = \psi_s(x) + \lambda\varphi_s(x)$, where $\varphi_s \in H^4(\Omega_s) \cap H^1_0(\Omega_s)$, $\psi_s \in H^2(\Omega_s) \cap H^1_0(\Omega_s)$, and φ_s and ψ_s satisfy conditions (i) and (ii) in the proof of Theorem 2.2, with $N = 1$ and $\Omega_s = (-L_s, L_s)$. Obviously, $\xi_s|_{\partial\Omega_s} = \eta_s|_{\partial\Omega_s} = 0$, $\xi_s(x) = \xi_s(x + 2L)$, $\eta_s(x) = \eta_s(x + 2L)$, $x \in \mathbf{R}$, and $\xi_s(x) = u_0(x)$, $\eta_s(x) = u_1(x)$ in $\Omega^*_s = (-L_s + 1, L_s - 1)$. Moreover, let

$$\|\xi_s\|_{H^4(\Omega_s)} \leqslant \|u_0\|_{H^4(\mathbf{R})}, \qquad \|\eta_s\|_{H^2(\Omega_s)} \leqslant \|u_1\|_{H^2(\mathbf{R})}, \tag{5.17}$$

and $\tilde{\xi}_s(x) = \tilde{\varphi}_s(x)$, $\tilde{\eta}_s(x) = \tilde{\psi}_s(x) + \lambda\tilde{\varphi}_s(x)$, where $\tilde{\varphi}_s(x)$, $\tilde{\psi}_s(x)$ as shown in (4.3), with $N = 1$. As a result, (4.5)–(4.10) hold true. Similarly,

$$\tilde{\xi}_s \to u_0 \quad \text{in } H^4(\mathbf{R}), \qquad \tilde{\eta}_s \to u_1 \quad \text{in } H^2(\mathbf{R}) \quad \text{as } s \to \infty. \tag{5.18}$$

As shown in the proof of Theorem 2.2, for each s we get a strong solution $u^s \in C([0, T]; H^4(\Omega_s)) \cap C^1([0, T]; H^2(\Omega_s)) \cap C^2([0, T]; L_2(\Omega_s))$ of corresponding problem (1.3), (5.1), (5.2) (substituting ξ, η and L there by ξ_s, η_s and L_s, respectively), and u^s is a weak* limit of a sequence $\{u^{n_s}\}$, where u^{n_s} are the solutions of problem (5.4), (5.5) (substituting u^n, ξ^n, η^n, ξ and η there by u^{n_s}, ξ^{n_s}, η^{n_s}, ξ_s and η_s, respectively). From the sequential lower semi-continuity of the norm we know that inequalities (5.7)–(5.9) still hold true for u^s (substituting u^n, ξ^n and η^n there by u^s, ξ_s and η_s, respectively). And inequalities (5.13), (5.14) still hold true for $w^s = u^s - u^{s-1}$ (substituting w^n, ξ^n and η^n there by w^s, ξ_s and η_s, respectively). Then $\{u^s\}$ is a Cauchy sequence in $C([0, T]; H^4(\Omega_L)) \cap C^1([0, T]; H^2(\Omega_L)) \cap C^2([0, T]; L_2(\Omega_L))$ and

$$u^s \to u \quad \text{in } C([0, T]; H^4(\Omega_L)) \cap C^1([0, T]; H^2(\Omega_L)) \cap C^2([0, T]; L_2(\Omega_L)) \tag{5.19}$$

as $s \to \infty$. And inequalities (5.7)–(5.9) are still valid for the limit function u. From the arbitrariness of L and the bounded-ness of the norm of u (see (5.8)–(5.9)) we know that the limiting function

$$u \in C([0, T]; H^4(\mathbf{R})) \cap C^1([0, T]; H^2(\mathbf{R})) \cap C^2([0, T]; L_2(\mathbf{R})), \tag{5.20}$$

and u is a strong solution of Cauchy problem (1.3), (1.2), with $N = 1$.

By using the standard method we easily get the uniqueness of the strong solution, so we omit the process. Theorem 2.3 is proved. □

6. Blowup of solutions

Lemma 6.1. *(See [12].) Assume that* $\phi \in C^2$, $\phi(t) \geqslant 0$ *satisfies*

$$\phi(t)\phi''(t) - (1 + \delta)\phi'^2(t) \geqslant 0$$

for certain real number $\delta > 0$, $\phi(0) > 0$ *and* $\phi'(0) > 0$. *Then there exists a real number* $\tilde{T}$: $0 < \tilde{T} \leqslant \frac{\phi(0)}{\delta\phi'(0)}$ *such that*

$$\phi(t) \to \infty \quad \text{as } t \to \tilde{T}^-.$$

Proof of Theorem 2.4. Taking L_2-inner product of Eq. (2.2) with u and u_t, respectively, we get

$$\big(G^{1/2}u_{tt}, G^{1/2}u\big) + \big\|u(t)\big\|^2 + \big\|\nabla u(t)\big\|^2 + \int_{\mathbf{R}^N} \sigma(u)u\,dx = 0, \quad t > 0, \tag{6.1}$$

$$E'(t) = 0, \quad E(t) = E(0), \quad t > 0, \tag{6.2}$$

where

$$E(t) = \big\|G^{1/2}u_t\big\|^2 + \big\|u(t)\big\|^2 + \big\|\nabla u(t)\big\|^2 + 2\int_{\mathbf{R}^N}\int_0^u \sigma(s)\,ds\,dx. \tag{6.3}$$

1. If (2.25) holds true, we set

$$\phi(t) = \big\|G^{1/2}u(t)\big\|^2 + \beta(t+t_0)^2. \tag{6.4}$$

Obviously,

$$\phi'(t) = 2\big(G^{1/2}u, G^{1/2}u_t\big) + 2\beta(t+t_0), \tag{6.5}$$

$$\phi'^2(t) \leqslant 4\phi(t)\big(\big\|G^{1/2}u_t(t)\big\|^2 + \beta\big), \tag{6.6}$$

$$\begin{aligned}\phi''(t) = 2\bigg[&2\alpha\big(\big\|u(t)\big\|^2 + \big\|\nabla u(t)\big\|^2\big) + 2(2\alpha+1)\int_{\mathbf{R}^N}\int_0^u \sigma(s)\,ds\,dx - \int_{\mathbf{R}^N}\sigma(u)u\,dx - (2\alpha+1)\big(E(0)+\beta\big)\bigg]\\ &+ 4(1+\alpha)\big(\big\|G^{1/2}u_t(t)\big\|^2 + \beta\big), \quad t > 0,\end{aligned} \tag{6.7}$$

where (6.1)–(6.3) have been used. Then

$$\phi(t)\phi''(t) - (1+\alpha)\phi'^2(t) \geqslant \phi(t)Q(t), \quad t \geqslant 0, \tag{6.8}$$

where

$$Q(t) = 2\bigg[2\alpha\big(\big\|u(t)\big\|^2 + \big\|\nabla u(t)\big\|^2\big) + 2(2\alpha+1)\int_{\mathbf{R}^N}\int_0^u \sigma(s)\,ds\,dx - \int_{\mathbf{R}^N}\sigma(u)u\,dx - (2\alpha+1)\big(E(0)+\beta\big)\bigg]. \tag{6.9}$$

By taking $\beta = -E(0)$ (> 0), we get from (2.24) and (6.9) that $Q(t) \geqslant 0$. And hence

$$\phi(t)\phi''(t) - (1+\alpha)\phi'^2(t) \geqslant 0, \quad t \geqslant 0. \tag{6.10}$$

Obviously $\phi(0) = \|G^{1/2}u_0\|^2 + \beta t_0^2 > 0$. By taking t_0 large enough such that $\phi'(0) = 2(G^{1/2}u_0, G^{1/2}u_1) + 2\beta t_0 > 0$, we deduce from Lemma 6.1 that there must be $\tilde{T} \leqslant \phi(0)/\alpha\phi'(0)$ such that $\phi(t) \to +\infty$ as $t \to \tilde{T}^-$, i.e.

$$\big\|G^{1/2}u(t)\big\|^2 \to +\infty \quad \text{as } t \to \tilde{T}^-. \tag{6.11}$$

2. If (2.26) holds true, we set

$$\phi(t) = \big\|G^{1/2}u(t)\big\|^2, \quad t \geqslant 0. \tag{6.12}$$

As a result, (6.5)–(6.9) still hold true, with $\beta = 0$. And it follows from (6.8) and (6.9) that

$$\phi(t)\phi''(t) - (1+\alpha)\phi'^2(t) \geqslant -2(2\alpha+1)E(0)\phi(t), \quad t \geqslant 0. \tag{6.13}$$

(1) If $E(0) = 0$, it follows from the facts: $\phi(0) = \|G^{1/2}u_0\|^2 > 0$, $\phi'(0) = 2(G^{1/2}u_0, G^{1/2}u_1) > 0$ and Lemma 6.1 that (6.11) holds true.

(2) If $E(0) > 0$, we set

$$P(t) = \phi^{-\alpha}(t), \quad t \geqslant 0. \tag{6.14}$$

Obviously,

$$P'(t) = -\alpha\phi^{-\alpha-1}(t)\phi'(t),$$
$$P''(t) = -\alpha\phi^{-\alpha-2}(t)\big(\phi(t)\phi''(t) - (1+\alpha)\phi'^2(t)\big) \leqslant 2\alpha(2\alpha+1)E(0)\phi^{-\alpha-1}(t), \quad t \geqslant 0. \tag{6.15}$$

By assumption, $P'(0) = -\alpha\phi^{-\alpha-1}(0)\phi'(0) < 0$. We claim

$$P'(t) < 0, \quad t > 0. \tag{6.16}$$

In fact, if there exists $t_0 > 0$ such that $P'(t) < 0$, $t \in [0, t_0)$ and $P'(t_0) = 0$, multiplying (6.15) by $2P'(t)$ we obtain

$$\frac{d}{dt}P'^2(t) \geqslant 4\alpha^2 E(0)\big(\phi^{-2\alpha-1}(t)\big)', \tag{6.17}$$
$$P'^2(t) \geqslant 4\alpha^2 E(0)\phi^{-2\alpha-1}(t) + P'^2(0) - 4\alpha^2 E(0)\phi^{-2\alpha-1}(0), \tag{6.18}$$
$$P'(t) \leqslant -\big(P'^2(0) - 4\alpha^2 E(0)\phi^{-2\alpha-1}(0)\big)^{1/2} < 0, \quad t \in [0, t_0]. \tag{6.19}$$

In particular, $P'(t_0) < 0$, which violates the assumption. So (6.16) is valid.

Therefore, (6.19) holds true for $t > 0$. Integrating (6.19) over $(0, t)$, we have

$$P(t) \leqslant P(0) - \big(P'^2(0) - 4\alpha^2 E(0)\phi^{-2\alpha-1}(0)\big)^{1/2} t, \quad t > 0. \tag{6.20}$$

It follows from (6.20) that there exists a number $\tilde{T}$: $0 < \tilde{T} \leqslant P(0)/(P'^2(0) - 4\alpha^2 E(0)\phi^{-2\alpha-1}(0))^{1/2}$ such that $P(\tilde{T}) = 0$, i.e. $\phi(t) \to +\infty$ as $t \to \tilde{T}^-$. Theorem 2.4 is proved. □

Example 1. We set $\sigma(s) = a|s|^{p-1}s$ in Eq. (1.1), where $a \neq 0$, $p > 1$ are real numbers.

1. When $a > 0$, $1 < p \leqslant \frac{N+2}{(N-2)^+}$ ($p < +\infty$), obviously, (2.17) holds true, with $\beta = p + 1$. So Cauchy problem (1.1), (1.2) admits a weak solution on $[0, T]$ (for any $T > 0$) as long as $G^{1/2}u_1 \in L_2(\mathbf{R}^N)$, $u_0 \in H^1(\mathbf{R}^N)$ (see Theorem 2.2).
2. When $a < 0$, $|a| < \frac{p+1}{2}$, $1 < p \leqslant \frac{N+2}{(N-2)^+}$ ($p < +\infty$), Cauchy problem (1.1), (1.2) admits a weak solution on $[0, T]$ (for any $T > 0$) as long as $u_0 \in \tilde{W}_0$ (see (2.21)), $G^{1/2}u_1 \in L_2(\mathbf{R}^N)$ such that (2.22) holds true (see Theorem 2.2).
3. When the space dimension $N = 1$ and $p > 4$, obviously $\sigma \in C^4(\mathbf{R})$. Then the weak solution of Cauchy problem (1.3), (1.2) in both cases 1 and 2 can be regularized, the strong solution u is unique and
$$u \in C\big([0, T]; H^4(\mathbf{R})\big) \cap C^1\big([0, T]; H^2(\mathbf{R})\big) \cap C\big([0, T]; L_2(\mathbf{R})\big) \tag{6.21}$$
as long as $u_0 \in H^4(\mathbf{R})$, $u_1 \in H^2(\mathbf{R})$ (see Theorem 2.3).
4. A simple verification shows that (2.24) is valid, with $\alpha = \frac{p-1}{4}$ (> 0). As a result, any weak solution of Cauchy problem (1.1), (1.2) blows up in finite time, i.e. (2.27) holds true as long as the initial data u_0 and u_1 satisfy condition (H_5) of Theorem 2.4 (see Theorem 2.4).

Example 2. We set $\sigma(s) = as^p$ in Eq. (1.1), where $a \neq 0$, $p > 1$ are real numbers.

1. When $a > 0$, p is an odd integer and $1 < p \leqslant \frac{N+2}{(N-2)^+}$ ($p < +\infty$), a simple computation shows that (2.17) holds true, with $\beta = p + 1$. So the same conclusion as in the case 1 of Example 1 is valid.
2. When $|a| < \frac{p+1}{2}$ and $1 < p \leqslant \frac{N+2}{(N-2)^+}$ ($p < +\infty$), we have the same conclusion as in the case 2 of Example 1.
3. When the space dimension $N = 1$, $p > 4$ or p is an integer, the same conclusion as in the case 3 of Example 1 holds true.
4. Obviously (2.24) holds, with $\alpha = \frac{p-1}{4}$ (> 0). And hence, we have the same conclusion as in the case 4 of Example 1.

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References

[1] D.G. Akmel, Global existence and decay for solutions to the "bad" Boussinesq equation in two space dimensions, Appl. Anal. 83 (2004) 17–36.
[2] J.L. Bona, R.A. Smith, A model for the two-way propagation of water waves in a channel, Math. Proc. Cambridge Philos. Soc. 79 (1976) 167–182.
[3] J.L. Bona, R.L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Comm. Math. Phys. 118 (1988) 15–29.
[4] J. Boussinesq, Theorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en comuniquant au liquide contene dans ce canal des vitesses sunsiblement parielles de la surface au fond, J. Math. Pures Appl. Ser. 2 (17) (1872) 55–108.
[5] G.-W. Chen, Z.-J. Yang, Existence and non-existence of global solutions for a class of non-linear wave equation, Math. Methods Appl. Sci. 23 (2000) 615–631.
[6] P.A. Clarkson, New exact solutions of the Boussinesq equations, European J. Appl. Math. 1 (1990) 279–300.
[7] P.A. Clarkson, M.D. Kruskal, New similarity reductions of the Boussinesq equation, J. Math. Phys. 30 (1989) 2201–2213.
[8] J. Clements, Existence theorems for a quasi-linear evolution equation, SIAM J. Appl. Math. 26 (1974) 745–752.
[9] J. Defrutos, T. Ortega, J.M. Sanz-Serna, Pseudo-spectral method for the "good" Boussinesq equation, Math. Comp. 57 (1991) 109–122.
[10] P. Deift, C. Tomei, E. Trubowitz, Inverse scattering and the Boussinesq equation, Comm. Pure Appl. Math. 35 (1982) 567–628.
[11] V.K. Kalantarov, O.A. Ladyzhenskaya, The occurrence of collapse for quasi-linear equations of parabolic and hyperbolic type, J. Sov. Math. 10 (1978) 53–70.
[12] H.A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$, Trans. Amer. Math. Soc. 192 (1974) 1–21.
[13] H.A. Levine, B.D. Sleeman, A note on the non-existence of global solutions of initial boundary value problems for the Boussinesq equation $u_{tt} = 3u_{xxxx} + u_{xx} - 12(u^2)_{xx}$, J. Math. Anal. Appl. 107 (1985) 206–210.
[14] F. Linares, Global existence of small solutions for a generalized Boussinesq equation, J. Differential Equations 106 (1993) 257–293.
[15] F. Linares, Asymptotic behavior of solutions of a generalized Boussinesq type equation, Nonlinear Anal. 25 (1995) 1147–1158.
[16] Y. Liu, Instability of solitary waves for generalized Boussinesq equation, J. Dynam. Differential Equations 5 (1993) 537–558.
[17] V.G. Makhankov, Dynamics of classical solutions, Phys. Rev. Lett. 35 (c) (1978) 1–128.
[18] A.K. Pani, H. Saranga, Finite element Galerkin method for the "good" Boussinesq equation, Nonlinear Anal. 29 (1997) 937–956.
[19] R.L. Sachs, On the blow-up of certain solutions of the "good" Boussinesq equation, Appl. Anal. 36 (1990) 145–152.
[20] M. Tsutsumi, T. Matahashi, On the Cauchy problem for the Boussinesq-type equation, Math. Japan 36 (1991) 371–379.
[21] V. Varlamov, Existence and uniqueness of a solution to the Cauchy problem for the damped Boussinesq equation, Math. Methods Appl. Sci. 19 (1996) 639–649.
[22] V. Varlamov, On spatially periodic solutions of the damped Boussinesq equation, Differential Integral Equations 10 (1997) 1197–1211.
[23] V. Varlamov, Long-time asymptotics of solutions of the second initial-boundary value problem for the damped Boussinesq equation, Abstr. Appl. Anal. 2 (1998) 281–289.
[24] V. Varlamov, Eigenfunction expansion method and the long-time asymptotics for the damped Boussinesq equation, Discrete Contin. Dyn. Syst. 7 (2001) 675–702.
[25] V. Varlamov, A. Balogh, Forced nonlinear oscillations of elastic membranes, Nonlinear Anal. Real World Appl. 7 (2006) 1005–1028.
[26] Z.-J. Yang, G.-W. Chen, Blowup of solutions for a class of generalized Boussinesq equations, Acta Math. Sci. 16 (1996) 31–40.
[27] Z.-J. Yang, On local existence of initial boundary value problems for the "bad" Boussinesq-type equation, Nonlinear Anal. 51 (2002) 1259–1271.
[28] Z.-J. Yang, X. Wang, Blowup of solutions for the "bad" Boussinesq-type equation, J. Math. Anal. Appl. 285 (2003) 282–293.
[29] Y.-L. Zhou, H.-Y. Fu, Nonlinear hyperbolic systems of higher order generalized Sine–Gordon type, Acta Math. Sinica 26 (1983) 234–249.