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Pseudo almost periodic solutions for equation with piecewise constant argument

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ABSTRACT

By using the roughness theory of exponential dichotomies and the contraction mapping, some sufficient conditions are obtained for the existence and uniqueness of pseudo almost periodic solution of the above differential equation with piecewise constant argument

$$\frac{dx}{dt} = A(t)x(t) + \sum_{j=0}^{r} A_j(t)x\big([t-j]\big) + g\big(t, x(t), x\big([t]\big), \ldots, x\big([t-r]\big)\big).$$

1. Exponential dichotomy and pseudo almost periodic sequences

1.1. Introduction

Differential equations with piecewise constant argument arise in an attempt to the theory of functional differential equations with continuous argument to differential equations with discontinuous arguments and have applications in certain biomedical models (e.g., see [2]) and impulsive and loaded equations of control theory. The strong interest in these equations is the fact that they describe hybrid dynamic systems (a continuous and discrete combination) and, therefore, combine properties of both differential and difference equations. In this work, we investigate the existence of pseudo almost periodic solution to pseudo almost periodic difference equation. First we need some general results on exponential dichotomy of linear difference equations, and properties of almost periodic functions that are not continuous.

The theory of exponential dichotomy has played a central role in the study of ordinary differential equations and diffeomorphisms for finite dimensional dynamic systems. This theory, which addresses the issue of strong transversality in dynamic systems, originated in the pioneering works of Lyapunov (1892) and Poincaré (1890). During the last few years one finds an ever growing use of exponential dichotomies to study the dynamic structures of various partial differential delay equations.

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The existence problem of periodic and almost periodic solutions has been one of the most attractive topics in the qualitative theory of ordinary or functional differential equations for its significance in the physic sciences.

Let P_T be the set of periodic functions with period T. Then the linear differential equation of first order

$$\frac{dx}{dt} = ax(t) + f(t), \tag{1}$$

has a unique T-periodic solution if $a \neq 0$.

In [8], Yuan and Hong proved the existence of periodic solutions for the following differential equation with piecewise constant argument (abbreviated in EPCA)

$$\frac{dx}{dt} = ax(t) + \sum_{i=-N}^{N} a_i x\big([t+i]\big) + f(t), \quad N \geqslant 1 \tag{2}$$

where the period T is a rational number and [.] denotes the greatest integer function.

In [5], it has been shown that EPCA has a quasi periodic solution and no periodic solution when the period is irrational. The appearance of quasi periodic rather than periodic solutions is due to piecewise constant argument. This new phenomenon illustrates a crucial difference between ODE and EPCA.

Eq. (2) describes model of disease dynamic, see Busenberg and Cooke [2]. An application of EPCA is the stabilization of the hybrid control system with feedback delays, see Cooke and Weiner [3], Shah and Weiner [7].

In [1] Ait Dads et al. consider the problem of existence of pseudo almost periodic solutions to the following second order neutral delay differential equation

$$\frac{d^2x}{dt^2}\big[x(t) + px(t-1)\big] = qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t), \tag{3}$$

where p, q are given constants and $f : \mathbb{R} \to \mathbb{R}$ is a given pseudo almost periodic function.

This work deals with the following EPCA

$$\frac{dx}{dt} = A(t)x(t) + \sum_{j=0}^{r} A_j(t)x\big([t-j]\big) + g\big(t, x(t), x\big([t]\big), \ldots, x\big([t-r]\big)\big). \tag{4}$$

In this work we present some results concerning pseudo almost periodic solutions of some difference equations using the theory of exponential dichotomy.

Dichotomy theories for difference equations are applied to obtain stability criteria for a class of discrete reaction–diffusion equations.

The rest of the paper is organized as follows. In next section, some definitions and preliminary results are introduced. We show some interesting properties about discontinuous pseudo almost periodic functions. Section 3 is devoted to establish some criteria for the existence and uniqueness of pseudo almost periodic solutions for some difference equation. Section 4 is devoted to the problem of existence and uniqueness of pseudo almost periodic solution for Eq. (10).

This work is motivated by the work of R. Yuan [9]. But Looking to the work of Yuan, we note that, in this one there are a number of inaccuracies in the text concerning the almost periodicity of the function $g(t, \varphi_1(t), \varphi_1([t]), \ldots, \varphi_1([t]-r))$. We begin by introducing some Banach spaces in which the equation will be well posed.

2. Some preliminary results on exponential dichotomy and discontinuous almost periodic functions

2.1. Exponential dichotomy in discrete case

Definition 1. Let $(A(n))_{n\in\mathbb{Z}}$ be a sequence in $GL_p(\mathbb{R})$ (the group of invertible matrix). Consider the following linear difference equation:

$$x_{n+1} = A(n)x_n. \tag{5}$$

We say that $Y(n)$ is a fundamental matrix of (5) if

$$Y(0) = I \quad \text{and} \quad \forall n \in \mathbb{Z}, \quad Y(n+1) = A(n)Y(n).$$

We say that (5) has an exponential dichotomy on $\mathbb{Z}$ with parameters (P, K, α) if there exist nonnegative numbers K and α and a projection matrix P $(P^2 = P)$ such that

$$\big|Y(m)PY^{-1}(n)\big| \leqslant Ke^{-\alpha(m-n)}, \quad m \geqslant n,$$
$$\big|Y(m)QY^{-1}(n)\big| \leqslant Ke^{-\alpha(n-m)}, \quad n \geqslant m,$$

where $Q = I - P$.

Remark 2. We can make explicit the fundamental matrix $Y(n)$ by

$$Y(n) = \begin{cases} A(n-1)\dots A(1)A(0) & \text{if } n > 0, \\ I & \text{if } n = 0, \\ A^{-1}(n)A^{-1}(n+1)\dots A^{-1}(-1) & \text{if } n < 0. \end{cases}$$

Definition 3. A sequence $(x_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{R}^N$ is called almost periodic if: $\forall\varepsilon > 0$,

$$T(x,\varepsilon) := \{\tau \in \mathbb{Z};\ \forall n \in \mathbb{Z},\ \|x_{n+\tau} - x_n\| < \varepsilon\}$$

is relatively dense.

The set of such sequences is denoted by $AP(\mathbb{Z},\mathbb{R}^N)$.

If $N = 1$, we use the notation $AP(\mathbb{Z})$.

$(B, \|.\|_\infty)$ denotes the space of bounded real sequences.

$PAP_0(\mathbb{Z})$ denotes the space of bounded real sequences $(x_n)_{n\in\mathbb{Z}}$ which satisfy

$$\lim_{N\to+\infty} \frac{1}{2N} \sum_{n=-N}^{N} |x_n| = 0.$$

We define the space of pseudo almost periodic sequences by: $PAP(\mathbb{Z}) = AP(\mathbb{Z}) + PAP_0(\mathbb{Z})$.

The space of pseudo periodic sequences by: $PP_\omega(\mathbb{Z}) = P_\omega(\mathbb{Z}) + PAP_0(\mathbb{Z})$, where $P_\omega(\mathbb{Z})$ denotes the space of ω-periodic real sequences.

Proposition 4. *(See [4].)*

(1) *Let $f \in AP(\mathbb{R},\mathbb{R}^N)$ and $\varepsilon > 0$, then $T(f,\varepsilon) \cap \mathbb{Z}$ is relatively dense.*
(2) *Let $N \in \mathbb{N}^*$, $x = (x_n)_{n\in\mathbb{Z}}$ a sequence with values in $\mathbb{R}^N$.*

Let us consider $\bar{x}$: $\mathbb{R} \to R^N$ which is defined by $\forall n \in \mathbb{Z}$, $\bar{x}(n) = x_n$ and $\bar{x}$ is affine in $[n, n+1]$. Then we have

$$T(\bar{x},\varepsilon) \cap \mathbb{Z} = T(x,\varepsilon) \quad \textit{and} \quad \bar{x} \in AP(\mathbb{R},\mathbb{R}^N) \quad \textit{if and only if} \quad x \in AP(\mathbb{Z},\mathbb{R}^N).$$

Theorem 5. *(See [8].) Consider the equation*

$$x_{n+1} = A(n)x_n + h_n, \tag{6}$$

where $(h_n)_n$ is a bounded sequence, $(A(n))_{n\in\mathbb{Z}}$ is a sequence in $GL_p(\mathbb{R})$. Let us assume that the homogeneous system associated to (6) *has an exponential dichotomy with parameters (P, K, α), then Eq.* (6) *has a unique bounded solution $(x_n)_{n\in\mathbb{Z}}$; furthermore, we have*:

$$\sup_{n\in\mathbb{Z}} |x_n| \leqslant K \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \sup_{n\in\mathbb{Z}} |h_n|.$$

Moreover, if we have $(A(n))_n \in AP(\mathbb{Z})$ (resp. $P_\omega(\mathbb{Z})$), and $(h_n)_n \in PAP(\mathbb{Z})$ (resp. $PP_w(\mathbb{Z})$) then $(x_n)_{n\in\mathbb{Z}} \in PAP(\mathbb{Z})$ (resp. $PP_w(\mathbb{Z})$).

2.2. Some results on discontinuous almost periodic functions

In the objective to study properties of solutions of differential equations with piecewise constant argument with almost periodic coefficients, we need to introduce some preliminary results on discontinuous almost periodic functions.

Let E be a Banach space, $B(E)$ denote the Banach space of bounded functions $x : \mathbb{R} \to E$ provided with the uniform norm topology $|x| = \sup_{t\in\mathbb{R}} |x(t)|$.

$$C_m(E) = \left\{ \begin{array}{l} x : \mathbb{R} \to E,\ \text{continuous on } \mathbb{R}\backslash\mathbb{Z},\ \text{such that } x \text{ has a finite limits} \\ \text{on the left and on the right of any point in } \mathbb{Z}. \end{array} \right\}$$

Let $\tau \in \mathbb{Z}$ and $f \in C_m(E)$, we define $f_\tau : t \to f(t+\tau)$. Then we have $f_\tau \in C_m(E)$.

Let us define

$$BC_m(E) = C_m(E) \cap B(E).$$

Then one can see that $BC_m(E)$ is closed in $B(E)$.

Definition 6. We call that $T(x,\varepsilon) \cap \mathbb{Z}$ is relatively dense in $\mathbb{R}$, if there exists a real number $L > 0$, such that for all $a \in \mathbb{R}$, $T(x,\varepsilon) \cap \mathbb{Z} \cap [a, a+L] \neq \varnothing$.

Remark 7. Without any loss of generality, the number L can be chosen as an integer number $L \in \mathbb{N}^*$, we may also replace $\forall a \in \mathbb{R}$, by $\forall a \in \mathbb{Z}$. Indeed if the intersection is not empty for $a \in \mathbb{Z}$, then $\forall a \in \mathbb{R}$, $T(x,\varepsilon) \cap \mathbb{Z} \cap [a, a+L+1] \neq \varnothing$. In what follows, we use that $T(x,\varepsilon) \cap \mathbb{Z}$ is relatively dense in $\mathbb{R}$ if and only if $\exists L \in \mathbb{N}^*$, $\forall a \in \mathbb{Z}$, $T(x,\varepsilon) \cap \mathbb{Z} \cap [a, a+L] \neq \varnothing$.

Let us define the space of almost periodic functions continuous on $\mathbb{R}\backslash\mathbb{Z}$ with values in E, which are discontinuous on $\mathbb{Z}$ and having a finite limits on the left and on the right of any point in $\mathbb{Z}$:

$$F_1(E) = \big\{x \in C_m(E),\ \text{such that } \forall \varepsilon > 0,\ T(x,\varepsilon) \cap \mathbb{Z} \text{ is relatively dense in } \mathbb{R}\big\}.$$

Lemma 8. *$F_1(E) = \{x \in BC_m(E), \forall \varepsilon > 0, T(x,\varepsilon) \cap \mathbb{Z}$ is relatively dense in $\mathbb{R}\}$ i.e. $F_1(E) \subseteq BC_m(E)$.*

Proof. Let $x \in C_m(E)$, such that $\forall \varepsilon > 0$, $T(x,\varepsilon) \cap \mathbb{Z}$ is relatively dense in $\mathbb{R}$. For $\varepsilon = 1$, there exist $L \in \mathbb{N}^*$, $\forall a \in \mathbb{Z}$, $\exists \tau_a \in [0, L] \cap \mathbb{Z}$ such that $\forall x \in \mathbb{R}$, $|f(x+a+\tau_a) - f(x)| < 1$, we chose $a = -E(x)$ where $E(x)$ is the integer part of x. Since f is bounded by some M on $[0, L+1]$ and we obtain that $\forall x \in \mathbb{R}$, $|f(x)| \leqslant M+1$. □

Lemma 9. *Let $f \in C_m(E)$ be such that for all number sequence $(t_n)_{n\geqslant 0} \in \mathbb{Z}^{\mathbb{N}}$ we can extract from $(f(\,.+t_n))_{n\geqslant 0}$ a subsequence which converges uniformly on $\mathbb{R}$, then f is bounded on $\mathbb{R}$.*

Proof. Suppose that f is not bounded, then there exits a sequence number (t_n) such that $|f(t_n)| \to \infty$. Putting $t_n = k_n + r_n$ where k_n is in $\mathbb{Z}$ and $0 \leqslant r_n < 1$, (r_n) is bounded, then it converges in term of subsequence $r_n \to r$ and

$$\big|f(t+k_n) - g(t)\big| \to 0 \quad \text{uniformly on } \mathbb{R},$$

this implies that

$$\big|f(t+k_n+r_n) - g(t+r_n)\big| \to 0 \quad \text{uniformly in } \mathbb{R}.$$

By taking $t = 0$ in the last formula, we obtain that

$$\big|f(k_n+r_n) - g(r_n)\big| \to 0,$$

but $g(r_n) \to g(r^-)$ or $g(r^+)$ because $g \in C_m(E)$ which implies that $f(t_n) \to g(r^-)$ or $g(r^+)$, which is a contradiction. □

Remark 10. When we call that $g(r_n)$ tends to $g(r-)$ or $g(r+)$ this suppose that $r_n - r$ has a constant sign, if no, we may extract a subsequence of r_n which goes to $r-$ or $r+$ and the problem is solved.

In fact if there exists an infinite of numbers n such that r_n is greater than r, one can extract a subsequence which goes to $r+$, if no, we have a subsequence which goes to $r-$.

Lemma 11. *Let $f \in C_m(E)$, then one has $f \in F_1(E)$ if and only if for all real number sequence $(t_n)_{n\geqslant 0} \in \mathbb{Z}^{\mathbb{N}}$ we can pick up from $(f(\,.+t_n))_{n\geqslant 0}$ a subsequence which converges uniformly on $\mathbb{R}$.*

Proof. From Lemmas 8 and 9 we work on the space $BC_m(E)$, we have to prove that $f \in F_1(E)$ if and only if $\{f_\tau, \tau \in \mathbb{Z}\}$ is a relatively compact part of $BC_m(E)$. Since $BC_m(E)$ is a complete space, it suffices to prove that $f \in F_1(E)$ if and only if $\{f_\tau, \tau \in \mathbb{Z}\}$ is precompact; mainly: for all $\varepsilon > 0$, there exist $\tau_1, \ldots, \tau_n \in \mathbb{Z}$ such that $\{f_\tau, \tau \in \mathbb{Z}\} \subset \bigcup_{i=1}^n B(f_{\tau_i}, \varepsilon)$.

Sufficient condition: Let $\varepsilon > 0$. Since $f \in F_1(E)$, then there exists $L \in \mathbb{N}^*$ such that $\forall a \in \mathbb{Z}$, $T(f,\varepsilon) \cap \mathbb{Z} \cap [a, a+L]$ is not empty, identically to $\forall a \in \mathbb{Z}$, $\exists \tau \in \mathbb{Z} \cap [a, a+L]$ such that $|f - f_\tau| < \varepsilon$, or yet, $\forall a \in \mathbb{Z}$, $\exists \tau \in \mathbb{Z} \cap [0, L]$ such that $|f - f_{a+\tau}| < \varepsilon$. By remarking that $|f - f_{a+\tau}| = |f_{-a} - f_\tau|$ one obtains: $\forall a' \in \mathbb{Z}$, $\exists \tau' \in \mathbb{Z} \cap [0, L]$ such that $|f_{a'} - f_{\tau'}| < \varepsilon$. If we put $\mathbb{Z} \cap [0, L] = \{\tau_1, \ldots, \tau_n\}$ one has the desired result.

Necessary condition: Let $\varepsilon > 0$. Then there exist $\tau_1, \ldots, \tau_n \in \mathbb{Z}$ such that $\{f_\tau, \tau \in \mathbb{Z}\} \subset \bigcup_{i=1}^n B(f_{\tau_i}, \varepsilon)$. Let $\ell \in \mathbb{N}$ such that $\forall i \in \{1, \ldots, n\}$, $\ell + \tau_i > 0$. If we put $L = \max_{1\leqslant i\leqslant n} \ell + \tau_i$ one has $L \in \mathbb{N}^*$. For $a \in \mathbb{Z}$, if we take $\tau = -\ell - a$, we obtain $\exists i \in \{1, \ldots, n\}$ such that $|f_{-\ell-a} - f_{\tau_i}| < \varepsilon$, mainly $|f - f_{a+\ell+\tau_i}| < \varepsilon$ and since $a + \ell + \tau_i \in [a, a+L] \cap \mathbb{Z}$, then $T(f,\varepsilon) \cap \mathbb{Z} \cap [a, a+L]$ is not empty. □

Lemma 12. *Let G and H be two Banach spaces, if $x \in F_1(G)$ and $y \in F_1(H)$ then $(x, y) \in F_1(G \times H)$.*

Proof. Let $(t_n)_{n\geqslant 0} \in \mathbb{Z}^{\mathbb{N}}$. Since $x \in F_1(G)$, then there exists a nondecreasing function φ such that $(x(\,.+t_{\varphi(n)}))_{n\geqslant 0}$ converges uniformly, on the other hand $y \in F_1(H)$ and $(t_{\varphi(n)})_{n\geqslant 0} \in \mathbb{Z}^{\mathbb{N}}$ then there exists a nondecreasing function ψ such that $(y(\,.+t_{\varphi\circ\psi(n)}))_{n\geqslant 0}$ converges uniformly, then if we put $\theta = \varphi \circ \psi$, we obtain that $(x(\,.+t_{\theta(n)}), y(\,.+t_{\theta(n)}))$ converges uniformly, then from Lemma 11 one has $(x, y) \in F_1(G \times H)$. □

Corollary 13. *Let $x \in F_1(G)$, $y \in F_1(H)$. Take $\varepsilon > 0$, we have $T(x,\varepsilon) \cap T(y,\varepsilon) \cap \mathbb{Z}$ is relatively dense, in particular, one has that if $x \in F_1(\mathbb{R}^N)$, then $\forall N \geqslant 1$, $\forall y \in AP(\mathbb{R}, \mathbb{R}^N)$ and $\forall \varepsilon > 0$, $T(x,\varepsilon) \cap T(y,\varepsilon) \cap \mathbb{Z}$ is relatively dense.*

Proof. From Lemma 12 one has $(x, y) \in F_1(G \times H)$, then $T((x, y), \varepsilon) \cap \mathbb{Z}$ is relatively dense, mainly $T(x, \varepsilon) \cap T(y, \varepsilon) \cap \mathbb{Z}$ is relatively dense. For the rest of the proof, we use the fact $AP(\mathbb{R}, \mathbb{R}^N) \subset F_1(\mathbb{R}^N)$. $\square$

3. Mean results

Let $D_j(n)$, $j = 0, \dots, r$, be a sequence in $M_q(\mathbb{C})$ with $D_r(n)$ invertible, and consider the difference equation in $\mathbb{C}^q$,

$$x_{n+1} = \sum_{j=0}^{r} D_j(n) x_{n-j} + h_n, \quad n \in \mathbb{Z}, \tag{7}$$

such that $(h_n)_n$ is bounded. We start by studying the following homogeneous equation

$$x_{n+1} = \sum_{j=0}^{r} D_j(n) x_{n-j}. \tag{8}$$

By putting

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-r} \end{pmatrix}$$

and

$$D(n) = \begin{pmatrix} D_0(n) & D_1(n) & \cdots & \cdots & D_r(n) \\ I_q & 0 & \cdots & \cdots & 0 \\ 0 & I_q & \ddots & . & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_q & 0 \end{pmatrix},$$

Eq. (8) takes the form

$$X_{n+1} = D(n) X_n. \tag{9}$$

Remark 14. Denote if $D_r(n)$ is invertible then $D(n)$ is also.

One has the following theorem which is an extension of the result proved by R. Yuan [9].

Theorem 15. *If $S := \sup_{n\in\mathbb{Z}} \sum_{j=0}^{r} |D_j(n)| < 1$, then the system (9) has an exponential dichotomy on $\mathbb{Z}$ with parameters (P, M, α) such that*

$$P = Id, \qquad M = S^{-\frac{r}{r+1}}, \qquad \alpha = \frac{1}{r+1} \ln\left(\frac{1}{S}\right).$$

Proof. Let $Y(n)$ be the fundamental matrix of system (9), we must prove that $\forall n, p \in \mathbb{Z}$, with $n \geqslant p$,

$$\left|Y(n) Y^{-1}(p)\right| \leqslant M e^{-\alpha(n-p)},$$

or $\forall Z \in \mathbb{C}^{q(r+1)}$,

$$\left|Y(n) Z\right| \leqslant M e^{-\alpha(n-p)} \left|Y(p) Z\right|.$$

For all $k \in \mathbb{Z}$, putting $Z_k = Y(k) Z$ and $K = e^{-\alpha}$, it suffices to prove that for $n \geqslant p$,

$$|Z_n| \leqslant M K^{n-p} |Z_p|.$$

One has for all i,

$$Z_i = \begin{pmatrix} \mathfrak{z}_i \\ \mathfrak{z}_{i-1} \\ \vdots \\ \mathfrak{z}_{i-r} \end{pmatrix},$$

$Z_{i+1} = D(i) Z_i$, then Z_i is of the form $\mathfrak{z}_{i+1} = \sum_{j=0}^{r} D_j(i) \mathfrak{z}_{i-j}$, here $|Z_i|$ denotes $\max(|\mathfrak{z}_i|, \dots, |\mathfrak{z}_{i-r}|)$.

Let us prove by recurrence on i that

$$\forall i \geqslant p-r, \quad |\mathfrak{z}_i| \leqslant |Z_p| K^{i-p}.$$

The recurrence hypothesis: if $p-r \leqslant i \leqslant p$ one has $|\mathfrak{z}_i| \leqslant |Z_p| \leqslant |Z_p| K^{i-p}$. Assume that $|\mathfrak{z}_k| \leqslant |Z_p| K^{k-p}$, $k = i-r, \ldots, i$ and prove that $|\mathfrak{z}_{i+1}| \leqslant |Z_p| K^{i+1-p}$. In fact, one has

$$|\mathfrak{z}_{i+1}| \leqslant \sum_{j=0}^{r} \big|D_j(i)\big| K^{i-j-p} |Z_p| \leqslant K^{i-r-p} \sum_{j=0}^{r} \big|D_j(i)\big| Z_p| \leqslant K^{i-r-p} S |Z_p| \leqslant K^{i-r-p} K^{r+1} |Z_p| \leqslant K^{i+1-p} |Z_p|.$$

Finally, $\forall i \geqslant p-r$, $|\mathfrak{z}_i| \leqslant |Z_p| K^{i-p}$. Let now $i \geqslant p$, so one has

$$|Z_i| = \max\big(|\mathfrak{z}_i|, \ldots, |\mathfrak{z}_{i-r}|\big) \leqslant |Z_p| \max\big(K^{i-p}, \ldots, K^{i-r-p}\big) \leqslant |Z_p| K^{i-r-p} \leqslant M |Z_p| K^{i-p}. \qquad \square$$

Corollary 16. *If $S := \sup_{n\in\mathbb{Z}} \sum_{j=0}^{r} |D_j(n)| < 1$, then Eq. (7) has a unique bounded solution $(x_n)_n$. Moreover, if for $j = 0, \ldots, r$, $(D_j(n))_n$ is almost periodic, and $(h_n)_n \in PAP(\mathbb{Z}, \mathbb{R}^q)$, then $(x_n)_n \in PAP(\mathbb{Z}, \mathbb{R}^q)$.*

Proof. It is an immediate consequence of Theorems 5 and 15. $\square$

3.1. Pseudo almost periodic solution of equation with piecewise constant argument

We start the section with some definitions and notations with respect to the pseudo almost periodicity.

Definition 17.

$$E_0 = \big\{\varphi \in PAP_0\big(\mathbb{R}, \mathbb{R}^q\big) \text{ such that } \big(\varphi(n)\big)_{n\in\mathbb{Z}} \in PAP_0\big(\mathbb{Z}, \mathbb{R}^q\big)\big\},$$
$$E = AP\big(\mathbb{R}, \mathbb{R}^q\big) \oplus E_0.$$

For $\omega > 0$,

$$E^\omega = P_\omega\big(\mathbb{R}, \mathbb{R}^q\big) \oplus E_0.$$

$C_m = \{f : \mathbb{R} \to \mathbb{R}^q$ continuous on $\mathbb{R}\backslash\mathbb{Z}$, which have finite limits at left and right of any point in $\mathbb{Z}\}$.

$$BC_m = \{f \in C_m, \text{ and } f \text{ is bounded}\},$$
$$F_0 = \left\{f \in BC_m, \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \big|f(s)\big|\,ds = 0\right\},$$
$$F_1 = \big\{f \in BC_m, \ \forall\varepsilon > 0, \ T(f, \varepsilon) \cap \mathbb{Z} \text{ is relatively dense}\big\},$$
$$F_{1,\omega} = \big\{f \in BC_m, \ \forall t \in \mathbb{R}, \ f(t+\omega) = f(t)\big\},$$
$$F = F_1 + F_0 \quad \text{and} \quad F^\omega = F_{1,\omega} + F_0.$$

Remark 18. The following example proves the fact that $\varphi \in PAP_0(\mathbb{R}, \mathbb{R})$, but $(\varphi(n))_{n\in\mathbb{Z}} \notin PAP_0(\mathbb{Z}, \mathbb{R})$:

$$\varphi(t) = \begin{cases} 1 - 2n^2|t-n| & \text{if } |t-n| \leqslant \frac{1}{2n^2}, \ n \in \mathbb{N}^*, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 19. Now, let us consider the following differential equation with piecewise constant argument

$$\frac{dx}{dt} = A(t)x(t) + \sum_{j=0}^{r} A_j(t) x\big([t-j]\big) + g\big(t, x(t), x\big([t]\big), \ldots, x\big([t-r]\big)\big), \tag{10}$$

where $[.]$ denotes the greatest integer function, $A, A_j : \mathbb{R} \to M_q(\mathbb{R})$ are almost periodic, $g : \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q \times \cdots \times \mathbb{R}^q \to \mathbb{R}^q$ is pseudo almost periodic.

Moreover, we assume that there exists η such that

$$\big|g(t, x_0, x_1, \ldots, x_{r+1}) - g(t, y_0, y_1, \ldots, y_{r+1})\big| \leqslant \eta \sum_{j=0}^{r+1} |x_j - y_j|, \quad t \in \mathbb{R}, \ x_j, y_j \in \mathbb{R}^q.$$

Remark 20. This last inequality is also satisfied by the almost periodic component of g.

We will start by studying the following equation

$$\frac{dx}{dt} = A(t)x(t) + \sum_{j=0}^{r} A_j(t)x\big([t-j]\big) + f(t). \tag{11}$$

By the variation constant formula, one has for $t \in [n, n+1[$:

$$x(t) = X(t)X^{-1}(n)x(n) + \int_n^t X(t)X^{-1}(s)\left[\sum_{j=0}^{r} A_j(s)x(n-j) + f(s)\right] ds, \tag{12}$$

where $X(t)$ denotes the fundamental matrix associated to the equation

$$\frac{dx}{dt} = A(t)x(t).$$

By the continuity of solutions, we get

$$x(n+1) = X(n+1)X^{-1}(n)x(n) + \int_n^{n+1} X(n+1)X^{-1}(s)\left[\sum_{j=0}^{r} A_j(s)x(n-j) + f(s)\right] ds,$$

which is the equation of the form:

$$x_{n+1} = \sum_{j=0}^{r} D_j(n)x_{n-j} + h_n,$$

where

$$D_0(n) = X(n+1)X^{-1}(n) + \int_n^{n+1} X(n+1)X^{-1}(s)A_0(s)\,ds,$$

$$D_j(n) = \int_n^{n+1} X(n+1)X^{-1}(s)A_j(s)\,ds, \quad j = 1, \dots, r,$$

$$h_n = \int_n^{n+1} X(n+1)X^{-1}(s)f(s)\,ds.$$

In the sequel, we assume that $\forall n \in \mathbb{Z}$, $D_r(n)$ is invertible.

Lemma 21. *(See [6].) Assume that $\forall t \in \mathbb{R}$, $|A(t)| \leqslant M$, then*

(i) *there exists $K_0 > 0$, such that*

$$\big|X(t)X^{-1}(s)\big| \leqslant K_0, \quad \text{for } 0 \leqslant t-s \leqslant 1;$$

(ii) *moreover, if $\tau \in T(A, \varepsilon)$, then*

$$\big|X(t+\tau)X^{-1}(s+\tau) - X(t)X^{-1}(s)\big| \leqslant K_0\varepsilon e^M \quad \text{for } 0 \leqslant t-s \leqslant 1.$$

Lemma 22.

(1) *If $f \in F_1$, then $(h_n)_n$ is an almost periodic sequence and the sequences $(D_j(n))_n$ for $0 \leqslant j \leqslant r$ are also almost periodic.*
(2) *If $f \in F_0$, then $(h_n)_n \in PAP_0(\mathbb{Z}, \mathbb{R}^q)$.*

Proof. (1) According to Lemma 21, for $\tau \in T(A, \varepsilon) \cap \mathbb{Z}$, one has

$$\big|X(n+1+\tau)X^{-1}(n+\tau) - X(n+1)X^{-1}(n)\big| \leqslant \varepsilon K_0 e^M,$$

it results that $(X(n+1)X^{-1}(n))_n$ is almost periodic. To end the proof, it suffices to prove that if $p \in F_1$, then the sequence

$$s_n := \int_n^{n+1} X(n+1)X^{-1}(s)p(s)\,ds,$$

is also almost periodic. One has, for $\tau \in T(p,\varepsilon) \cap T(A,\varepsilon) \cap \mathbb{Z}$,

$$\begin{aligned} s_{n+\tau} - s_n &= \int_n^{n+1} \bigl[X(n+1+\tau)X^{-1}(s+\tau)p(s+\tau) - X(n+1)X^{-1}(s)p(s)\bigr]\,ds \\ &= \int_n^{n+1} \bigl[\bigl(X(n+1+\tau)X^{-1}(s+\tau) - X(n+1)X^{-1}(s)\bigr)p(s+\tau) + X(n+1)X^{-1}(s)\bigl(p(s+\tau) - p(s)\bigr)\bigr]\,ds, \end{aligned}$$

then from Lemma 21,

$$|s_{n+\tau} - s_n| \leqslant K_0\varepsilon\bigl[e^M\|p\|_\infty + 1\bigr],$$

thus, $(s_n)_n$ is also almost periodic.

(2) From Lemma 21,

$$|h_n| \leqslant K_0 \int_n^{n+1} \bigl|f(s)\bigr|\,ds, \tag{13}$$

since $f \in F_0$, gives that $(\int_n^{n+1} |f(s)|\,ds)_n \in PAP_0(\mathbb{Z},\mathbb{R})$, so $(h_n)_n \in PAP_0(\mathbb{Z},\mathbb{R}^q)$. $\square$

Theorem 23. *Assume that the system* (9) *has an exponential dichotomy. Then, for $f \in BC_m$, Eq.* (11) *has a unique bounded solution. Furthermore, there exists a positive constant c which is independent from f such that*

$$\|x\|_\infty \leqslant c\|f\|_\infty.$$

Moreover, if $f \in F$ (resp. $f \in F^\omega$, A and $(A_j)_{0\leqslant j\leqslant r}$ are ω-periodic where $\omega \in \mathbb{N}^*$), then the unique bounded solution of Eq. (11) *is in E (resp. E^ω).*

Proof. (1) *Case where $f \in BC_m$*. One has from (13) $|h_n| \leqslant K_0\|f\|_\infty$, then $(h_n)_n$ is bounded, and thanks to Theorem 5, Eq. (7) has a unique bounded solution $(x_n)_n$. (12) gives existence and uniqueness of x. On the other hand, there exists a constant c_1 which depends only on A and from the A_i such that

$$\sup_{n\in\mathbb{Z}} |x_n| \leqslant c_1 \sup_{n\in\mathbb{Z}} |h_n| \leqslant c_1K_0\|f\|_\infty.$$

For $t \in [n, n+1[$, one has from (12):

$$\begin{aligned} \bigl|x(t)\bigr| &\leqslant \bigl|X(t)X^{-1}(n)\bigr|\bigl|x(n)\bigr| + \int_n^t \bigl|X(t)X^{-1}(s)\bigr|\left(\sum_{j=0}^r \bigl|A_j(s)\bigr|\bigl|x(n-j)\bigr| + \bigl|f(s)\bigr|\right)ds \\ &\leqslant K_0\bigl|x(n)\bigr| + \int_n^t K_0\left(\sum_{j=0}^r \bigl|A_j(s)\bigr|\bigl|x(n-j)\bigr| + \bigl|f(s)\bigr|\right)ds \\ &\leqslant c_1K_0^2\|f\|_\infty + K_0\left(\sum_{j=0}^r \|A_j\|_\infty c_1K_0\|f\|_\infty + \|f\|_\infty\right). \end{aligned}$$

We see that x is bounded and

$$\sup_{t\in\mathbb{R}}\bigl|x(t)\bigr| \leqslant c\|f\|_\infty,$$

with

$$c = c_1K_0^2 + K_0\left(\sum_{j=0}^r \|A_j\|_\infty c_1K_0 + 1\right),$$

independent of f.

(2) *Case where* $f \in F_0$. According to Lemma 22 $(h_n)_n \in PAP_0(\mathbb{Z})$, Theorem 5 gives that $(x_n)_n \in PAP_0(\mathbb{Z})$.
On the other hand, (12) implies:

$$\left|x(t)\right| \leqslant K_0\left|x(n)\right| + K_0\left(\sum_{j=0}^{r} \|A_j\|_\infty \left|x(n-j)\right| + \int_n^{n+1} \left|f(s)\right| ds\right),$$

then

$$\int_n^{n+1} \left|x(t)\right| dt \leqslant K_0\left|x(n)\right| + K_0\left(\sum_{j=0}^{r} \|A_j\|_\infty \left|x(n-j)\right| + \int_n^{n+1} \left|f(s)\right| ds\right),$$

it follows that

$$\left(\int_n^{n+1} \left|x(t)\right| dt\right)_n \in PAP_0(\mathbb{Z}),$$

then $x \in PAP_0(\mathbb{R})$. Finally, $x \in E_0$.

(3) *Case where* $f \in F_1$. Lemma 22 gives that $(h_n)_n \in AP(\mathbb{Z})$, Theorem 5 implies that $(x_n)_n \in AP(\mathbb{Z})$, and as $f \in F_1$, then $T(f,\varepsilon) \cap T((x_n)_n,\varepsilon) \cap T(A,\varepsilon)$ is relatively dense. Let $\tau \in T(f,\varepsilon) \cap T((x_n)_n,\varepsilon)$.

We have for $n \in \mathbb{Z}$ and $t \in [n, n+1[$, $t+\tau \in [n+\tau, n+\tau+1[$. Then, from (12), we obtain

$$\begin{aligned} x(t+\tau) &= X(t+\tau)X^{-1}(n+\tau)x(n+\tau) + \int_{n+\tau}^{t+\tau} X(t+\tau)X^{-1}(s)\left(\sum_{j=0}^{r} A_j(s)x(n+\tau-j) + f(s)\right) ds \\ &= X(t+\tau)X^{-1}(n+\tau)x(n+\tau) + \int_n^t X(t+\tau)X^{-1}(s+\tau)\left[\sum_{j=0}^{r} A_j(s+\tau)x(n+\tau-j) + f(s+\tau)\right] ds, \end{aligned}$$

and

$$\left|x(t+\tau) - x(t)\right| \leqslant I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \left|X(t+\tau)X^{-1}(n+\tau)x(n+\tau) - X(t)X^{-1}(n)x(n)\right|, \\ I_2 &= \sum_{j=0}^{r} \int_n^t \left|\left(X(t+\tau)X^{-1}(s+\tau)\big(A_j(s+\tau)x(n+\tau-j)\big) - X(t)X^{-1}(s)\big(A_j(s)x(n-j)\big)\right)\right| ds, \\ I_3 &= \int_n^t \left|\left(X(t+\tau)X^{-1}(s+\tau)f(s+\tau) - X(t)X^{-1}(s)f(s)\right)\right| ds. \end{aligned}$$

One has from Lemma 21,

$$\begin{aligned} I_1 &\leqslant \left|\left(X(t+\tau)X^{-1}(n+\tau) - X(t)X^{-1}(n)\right)x(n+\tau)\right| + \left|X(t)X^{-1}(n)\big(x(n+\tau) - x(n)\big)\right| \\ &\leqslant K_0\varepsilon e^M \sup_{n\in\mathbb{Z}} |x_n| + K_0\varepsilon \\ &\leqslant \left(e^M c_1 K_0 \|f\|_\infty + 1\right)\varepsilon K_0, \\ I_2 &\leqslant \sum_{j=0}^{r} \int_n^t \lambda_j(s)\, ds, \end{aligned}$$

where

$$\begin{aligned} \lambda_j(s) &= \left|\left(X(t+\tau)X^{-1}(s+\tau)\big(A_j(s+\tau)x(n+\tau-j)\big) - X(t)X^{-1}(s)\big(A_j(s)x(n-j)\big)\right)\right| \\ &\leqslant \left|\left(X(t+\tau)X^{-1}(s+\tau) - X(t)X^{-1}(s)\right)\big(A_j(s+\tau)x(n+\tau-j)\big)\right| \\ &\quad + \left|X(t)X^{-1}(s)\big(A_j(s+\tau)x(n+\tau-j) - A_j(s)x(n-j)\big)\right| \end{aligned}$$

$$\begin{aligned}
&\leqslant K_0\varepsilon e^M \|A_j\|_\infty \sup_{n\in\mathbb{Z}} |x_n| + K_0\big|A_j(s+\tau)x(n+\tau-j) - A_j(s)x(n-j)\big| \\
&\leqslant K_0\varepsilon e^M \|A_j\|_\infty \sup_{n\in\mathbb{Z}} |x_n| + K_0\big|\big(A_j(s+\tau)-A_j(s)\big)x(n+\tau-j)\big| + \big|A_j(s)\big(x(n+\tau-j)-x(n-j)\big)\big| \\
&\leqslant \big(K_0^2 e^M \|A_j\|_\infty c_1\|f\|_\infty + K_0^2 c_1\|f\|_\infty + K_0\|A_j\|_\infty\big)\varepsilon.
\end{aligned}$$

It results that

$$I_2 \leqslant K\varepsilon$$

where

$$K = \sum_{j=0}^{r} K_0^2 e^M \|A_j\|_\infty c_1\|f\|_\infty + K_0^2 c_1\|f\|_\infty + K_0\|A_j\|_\infty,$$

$$\begin{aligned}
I_3 &\leqslant \int_n^t \big|\big(X(t+\tau)X^{-1}(s+\tau) - X(t)X^{-1}(s)\big)f(s+\tau)\big|\,ds + \int_n^t \big|X(t)X^{-1}(s)\big(f(s+\tau)-f(s)\big)\big|\,ds \\
&\leqslant \big(e^M\|f\|_\infty + 1\big)K_0\varepsilon,
\end{aligned}$$

finally $x \in AP(\mathbb{R})$.

By superposition principle of solutions for the linear system we deduce the proof of the case where $f \in F$.

When $f \in F_\omega$, the proof is a simple consequence of existence and uniqueness. □

Proposition 24. *Let φ be a continuous and bounded function defined on $\mathbb{R}$ and $f_\varphi : t \mapsto g(t,\varphi(t),\varphi([t]),\dots,\varphi([t-r]))$; then*

(i) $f_\varphi \in BC_m$;
(ii) $f_\varphi \in F$ *if* $\varphi \in E$.

Proof. (i) is clear.

(ii) Let $\varphi \in E$, $\varphi = \varphi_1 + \varphi_0$, $\varphi_1 \in AP(\mathbb{R},\mathbb{R}^q)$ and $\varphi_0 \in E_0$. Let us denote

$$f_1(t) = g\big(t,\varphi_1(t),\varphi_1\big([t]\big),\dots,\varphi_1\big([t]-r\big)\big) \quad \text{and} \quad f_0 = f_\varphi - f_1.$$

Let us prove that $f_1 \in F_1$.

One has $f_1 \in BC_m$ and $W := \overline{\varphi_1(\mathbb{R})}^{r+2}$ is a compact set in $(\mathbb{R}^q)^{r+2}$. Denote

$$T(g_1,W,\varepsilon) = \big\{\tau \in \mathbb{R};\ \forall t \in \mathbb{R},\ \forall z \in W,\ \big|g_1(t+\tau,z) - g_1(t,z)\big| \leqslant \varepsilon\big\},$$

where g_1 is the almost periodic component of g.

Now, let $\varepsilon > 0$, prove that $T(f_1,\varepsilon)\cap\mathbb{Z}$ is relatively dense. We know that $T(g_1,W,\varepsilon)\cap T(\varphi_1,\varepsilon)\cap\mathbb{Z}$ is relatively dense, choosing τ in the last set, and it follows that:

$$\begin{aligned}
\big|f_1(t+\tau) - f_1(t)\big| \leqslant{}& \big|g_1\big(t+\tau,\varphi_1(t+\tau),\varphi_1\big([t]+\tau\big),\dots,\varphi_1\big([t]+\tau-r\big)\big) \\
&- g_1\big(t,\varphi_1(t+\tau),\varphi_1\big([t]+\tau\big),\dots,\varphi_1\big([t]+\tau-r\big)\big)\big| \\
&+ \big|g_1\big(t,\varphi_1(t+\tau),\varphi_1\big([t]+\tau\big),\dots,\varphi_1\big([t]+\tau-r\big)\big) \\
&- g_1\big(t,\varphi_1(t),\varphi_1\big([t]\big),\dots,\varphi_1\big([t]-r\big)\big)\big| \\
\leqslant{}& \varepsilon + (r+2)\eta\varepsilon.
\end{aligned}$$

Thus $T(f_1,\varepsilon)$ is relatively dense.

Finally $f_1 \in F_1$. Now, let us prove that $f_0 \in F_0$. One has $f_0 = f_\varphi - f_1$ and since we have that f_φ, f_1 are in BC_m, then $f_0 \in BC_m$. On the other hand,

$$\begin{aligned}
f_0(t) &= g\big(t,\varphi(t),\varphi\big([t]\big),\dots,\varphi\big([t]-r\big)\big) - g_1\big(t,\varphi_1(t),\varphi_1\big([t]\big),\dots,\varphi_1\big([t]-r\big)\big) \\
&= \underbrace{g\big(t,\varphi(t),\varphi\big([t]\big),\dots,\varphi\big([t]-r\big)\big) - g\big(t,\varphi_1(t),\varphi_1\big([t]\big),\dots,\varphi_1\big([t]-r\big)\big)}_{\psi(t)} \\
&\quad + \underbrace{g_0\big(t,\varphi_1(t),\varphi_1\big([t]\big),\dots,\varphi_1\big([t]-r\big)\big)}_{\gamma(t)}.
\end{aligned}$$

One has

$$\begin{aligned}
&\left|\psi(t)\right| \leqslant \eta\left(\left|\varphi_0(t)\right| + \left|\varphi_0\left([t]\right)\right| + \cdots + \left|\varphi_0\left([t]-r\right)\right|\right),\\
&\int_n^{n+1} \left|\psi(t)\right| dt \leqslant \eta\left(\int_n^{n+1} \left|\varphi_0(t)\right| dt + \left|\varphi_0(n)\right| + \cdots + \left|\varphi_0(n-r)\right|\right),
\end{aligned}$$

and

$$\left(\int_n^{n+1} \left|\varphi_0(t)\right| dt\right)_n \in PAP_0(\mathbb{Z}).$$

Then

$$\left(\int_n^{n+1} \left|\psi(t)\right| dt\right)_n \in PAP_0(\mathbb{Z}),$$

it follows that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left|\psi(t)\right| dt = 0.$$

It remains to verify that $\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |\gamma(t)|\, dt = 0$.

According to the compactness of $W = \overline{\varphi_1(\mathbb{R})}^{r+2}$, g_1 is uniformly continuous on $\mathbb{R}\times W$.

Let $\varepsilon > 0$, $\exists\delta \in\,]0, \varepsilon[$ such that:

$$\forall z, z' \in W\colon\quad \left\|z - z'\right\| \leqslant \delta \quad\Rightarrow\quad \forall t \in \mathbb{R},\quad \left|g_1(t, z) - g_1\left(t, z'\right)\right| \leqslant \varepsilon,$$

and $\exists z_1, \ldots, z_m \in W$, such that $W \subset \bigcup_{i=1}^m B(z_i, \delta)$.

Let

$$\begin{aligned}
&h(t) = \left(\varphi_1(t), \varphi_1\left([t]\right), \ldots, \varphi_1\left([t]-r\right)\right),\\
&B_i = \left\{t \in \mathbb{R},\ h(t) \in B(z_i, \delta)\right\},\\
&E_1 = B_1, \qquad E_i = B_i \setminus \bigcup_{j=1}^{i-1} B_j,
\end{aligned}$$

for $2 \leqslant i \leqslant m$. One has $\mathbb{R} = \bigcup_{i=1}^m E_i$.

If $t \in E_i$, then

$$\begin{aligned}
\left|\gamma(t)\right| &\leqslant \left|g_0\left(t, h(t)\right) - g_0(t, z_i)\right| + \left|g_0(t, z_i)\right|\\
&\leqslant \left|g_1\left(t, h(t)\right) - g_1(t, z_i)\right| + \left|g\left(t, h(t)\right) - g(t, z_i)\right| + \left|g_0(t, z_i)\right|\\
&\leqslant \varepsilon + \eta\left|h(t) - z_i\right| + \left|g_0(t, z_i)\right|\\
&\leqslant (1+\eta)\varepsilon + \left|g_0(t, z_i)\right|,
\end{aligned}$$

so we have

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \left|\gamma(t)\right| dt &= \sum_{i=1}^m \frac{1}{2T}\int_{[-T,T]\cap E_i} \left|\gamma(t)\right| dt\\
&\leqslant (1+\eta)\varepsilon \sum_{i=1}^m \frac{mes([-T,T]\cap E_i)}{2T} + \sum_{i=1}^m \frac{1}{2T}\int_{[-T,T]\cap E_i} \left|g_0(t, z_i)\right| dt\\
&\leqslant (1+\eta)\varepsilon + \sum_{i=1}^m \frac{1}{2T}\int_{-T}^{T} \left|g_0(t, z_i)\right| dt,
\end{aligned}$$

which implies that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left|\gamma(t)\right| dt = 0.$$

Thus $f_0 = \psi + \gamma \in F_0$, finally $f_\varphi = f_1 + f_0 \in F$. $\square$

Theorem 25. *Let $g \in PAP(\mathbb{R} \times (\mathbb{R}^q)^{r+2}, \mathbb{R}^q)$ such that*

$$\exists\eta > 0, \quad \forall t \in \mathbb{R}, \quad \forall x, y \in (\mathbb{R}^q)^{r+2},$$

$$\left|g(t,x) - g(t,y)\right| \leqslant \eta|x-y|, \quad \text{where } |x-y| = \sum_{j=0}^{r+1} |x_j - y_j|.$$

Then:

(i) *There exists $\eta^* > 0$ such that for $0 < \eta < \eta^*$, Eq. (10) has a unique solution in $PAP(\mathbb{R}, \mathbb{R}^q)$.*
(ii) *Moreover, if $g \in PP_\omega(\mathbb{R} \times \mathbb{R}^q, \mathbb{R}^q)$ one has: For $\omega \in \mathbb{N}^*$, the solution is in $PP_\omega(\mathbb{R}, \mathbb{R}^q)$.*
(iii) *For $\omega = \frac{n_0}{m_0} \in \mathbb{Q}^{+*}$, the solution is in $PP_{n_0}(\mathbb{R}, \mathbb{R}^q)$.*

Proof. (i) Let $\varphi \in PAP(\mathbb{R})$, in particular φ is continuous and bounded on $\mathbb{R}$. Then thanks to Proposition 24, $f_\varphi \in BC_m$, and from Theorem 23, equation

$$\frac{dx}{dt} = A(t)x(t) + \sum_{j=0}^{r} A_j(t)x\big([t-j]\big) + f_\varphi(t)$$

has a unique bounded solution denoted by $T\varphi$.

We define the operator $T : PAP(\mathbb{R}, \mathbb{R}^q) \to C_b(\mathbb{R}, \mathbb{R}^q)$ (the space of continuous and bounded functions).

For $\varphi, \psi \in PAP(\mathbb{R})$, $T\varphi - T\psi$ is the unique bounded solution of

$$\frac{dx}{dt} = A(t)x(t) + \sum_{j=0}^{r} A_j(t)x\big([t-j]\big) + f_\varphi(t) - f_\psi(t),$$

then, from Theorem 23, there exists a constant c which depends only of A and A_i such that

$$\|T\varphi - T\psi\|_\infty \leqslant c\|f_\varphi - f_\psi\|_\infty.$$

Thus

$$\forall\varphi, \psi \in PAP\big(\mathbb{R}, \mathbb{R}^q\big), \quad \|T\varphi - T\psi\|_\infty \leqslant (r+2)c\eta\|\varphi - \psi\|_\infty. \tag{14}$$

Putting $\eta^* = \frac{1}{(r+2)c}$, for $0 < \eta < \eta^*$ one has $(r+2)c\eta < 1$.

On the other hand, follows to Proposition 24, if $\varphi \in E$, then $f_\varphi \in F$ and from Theorem 23, $T\varphi$ is in E, so $T(E) \subset E$, moreover E is a Banach space, then T has a unique fixed point x in E, and from (14) x is the unique fixed point of T, in $PAP(\mathbb{R}, \mathbb{R}^q)$. Finally, Eq. (10) admits a unique solution in $PAP(\mathbb{R}, \mathbb{R}^q)$.

(ii) Moreover for $g \in PP_\omega$, one has $t \to x(t+\omega)$ is also a solution, and by uniqueness $x \in PP_\omega(\mathbb{R}, \mathbb{R}^q)$.

(iii) If $\omega = \frac{n_0}{m_0} \in \mathbb{Q}^{+*}$, then $g \in PP_{n_0}$ and $n_0 \in \mathbb{N}^*$. Then from the previous comment $x \in PP_{n_0}(\mathbb{R}, \mathbb{R}^q)$. $\square$

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