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Vector-valued invariant means revisited

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Abstract

We show that a Banach space X is complemented in its ultraproducts if and only if for every amenable semigroup S the space of bounded X-valued functions defined on S admits (a) an invariant average; or (b) what we shall call "an admissible assignment". Condition (b) still provides an equivalence for quasi-Banach spaces, while condition (a) necessarily implies that the space is locally convex.

1. Introduction

Invariant means were definitely introduced in the study of functional equations by Székelyhidi in 1982 [12], although similar ideas appear in Pełczyński's dissertation [9]. Since then, the method of invariant means has been used by a number of authors for solving stability problems (see the recent book [6, Chapter 4] for an exposition; the papers [10,11] contain further information). Our purpose with this note is to give a simple approach to vector-valued invariants means which extends (and, we hope, clarifies) previous results by Gajda [3], Badora [1] and Ger [4]. To some extent, our results complement Zhang's illuminating monograph [14] (see also [15]). More precisely, we show that a Banach space is complemented in its ultraproducts (equivalently, in its bidual) if and only if for every amenable semi-

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group S the space $B(S,Y)$ of Y-valued bounded functions on S admits an invariant average; this is equivalent to admit what we call "an admissible assignment." For a quasi-Banach space Y, $B(S,Y)$ admits admissible assignments if and only if it is complemented in its ultraproducts. Finally, the possible equivalence between the existence of invariant averages and admissible assignments is settled with the proof that $B(S,Y)$ can only admit invariant averages when Y is locally convex.

2. Vector-valued invariant means

Let S be a (not necessarily commutative) semigroup and Y a Banach space. We denote by $B(S,Y)$ the Banach space of bounded functions $f: S \to Y$ normed by

$$\|f\|_\infty = \sup_{x\in S} \left\| f(x) \right\|_Y.$$

When $Y = \mathbb{K}$ is the ground field, we simply write $B(S)$. Given $f \in B(S,Y)$ and $z \in S$, the right translate of f by z is given by $f_z(x) = f(x+z)$. Left translates are defined in a similar way. We begin with the following:

Definition 1. A (right) invariant average for $B(S,Y)$ is a bounded linear operator $m: B(S,Y) \to Y$ satisfying:

- (Invariance) $m(f_z) = m(f)$ for all $f \in B(S,Y)$ and all $z \in G$.
- (Consistency) If $f(x) = f_0$ for every $x \in G$, then $m(f) = f_0$.

(Left invariant averages are defined in the obvious way.)

It is clear that the usual (scalar) invariant means are just invariant averages of norm 1 for $Y = \mathbb{R}$. A semigroup S is said to be (right) amenable if $B(S)$ admits a (right) invariant mean. It is well known that commutative semigroups are (two-sided) amenable [5, Theorem 17.5]. Our main positive result reads as follows.

Theorem 1. *Let S be an amenable semigroup and Y a Banach space. Suppose Y is complemented in its second dual by a projection π. Then $B(S,Y)$ admits an invariant average of norm at most $\|\pi\|$.*

Proof. What we shall see is that every (scalar, right) invariant mean on $B(S)$ "extends" in a natural way to an invariant average for $B(S,Y)$. So, let m be an invariant mean for $B(S)$ and $\pi: Y^{**} \to Y$ a bounded linear projection. Given $f \in B(S,Y)$, define $m^{**}(f) \in Y^{**}$ by

$$\left\langle m^{**}(f), y^* \right\rangle = m(y^* \circ f)$$

for $y^* \in Y^*$ and set

$$m_Y(f) = \pi\big(m^{**}(f)\big).$$

We claim that m_Y is an invariant average for $B(S, Y)$. That m_Y is linear and bounded by $\|\pi\|$ is obvious. To verify invariance, it clearly suffices to see that $m^{**}(f_z) = m^{**}(f)$. Fixing $y^* \in Y^*$ we have

$$\big\langle m^{**}(f_z), y^*\big\rangle = m(y^* \circ f_z) = m\big((y^* \circ f)_z\big) = m(y^* \circ f) = \big\langle m^{**}(f), y^*\big\rangle,$$

as desired. Finally, let us prove consistency. Suppose $f(x) = f_0$ for all $x \in S$. Then, for every $y^* \in Y^*$, one has $\langle m^{**}(f), y^*\rangle = m(y^* \circ f) = \langle y^*, f_0\rangle$ since $\langle y^*, f(x)\rangle = \langle y^*, f_0\rangle$ for all $x \in S$. Thus $m^{**}(f) = f_0$ and also $m_Y(f) = \pi(f_0) = f_0$. This completes the proof. □

Remark 1. The invariant averages of Theorem 1 have some additional properties. First, observe that $m^{**}(f)$ lies in the *weak-convex hull of $f(S)$ in Y^{**}. Thus, m^{**} is a (left) invariant *weak-mean in the sense of [14]. Hence, if $f(S)$ lies in a weakly compact subset of Y (which always occurs when Y is reflexive), then $m_Y(f)$ belongs to the norm convex hull of the range of f. (Compare to [3].)

Suppose $\|\pi\| = 1$ (for instance, if $Y = \ell_1$ or $L_1(0, 1)$, and, of course, if Y has the binary intersection property: a Banach space has the binary intersection property if and only if it is complemented in any superspace by a norm-one projection). Then $\|m_Y\| = 1$ and so $m_Y(f)$ lies in the ball of radius $\|f\|_\infty$ centered at the origin. (Compare to [4].)

Finally, assume that Y is a boundedly complete Banach lattice with strong unit e. Then there is a projection of Y^{**} onto Y of norm at most λ, where λ is the least number for which the order interval $[-\lambda \cdot e, \lambda \cdot e]$ contains the unit ball of Y. So, in this case we have $\|m_Y\| \leqslant \lambda$. (Compare to [4].)

We close the section showing that Theorem 1 is a sharp result.

Theorem 2. *Let Y a Banach space. Suppose that for every commutative semigroup S there is an invariant average m for $B(S, Y)$ with $\|m\| \leqslant K$. Then Y is complemented in its second dual by a projection of norm at most K.*

Proof. The proof is based on the "principle of local reflexivity" of Lindenstrauss and Rosenthal [8] which asserts that every Banach space is locally complemented in its bidual. Precisely, given $\varepsilon > 0$ and a subspace F of Y^{**} with $Y \subset F \subset Y^{**}$ and F/Y finite-dimensional, there exists a linear projection $P : F \to Y$ such that $\|P\| \leqslant 1 + \varepsilon$. (There are many proofs of the principle of local reflexivity in the literature; our favourite is Dean [2].)

Consider the set

$$S = \big\{(F, \varepsilon)\colon\ Y \subset F \subset Y^{**},\ \dim(F/Y) < \infty,\ 0 < \varepsilon \leqslant 1\big\}$$

endowed with the binary operation

$$(F,\varepsilon)(E,\delta) = \big(F+E, \min\{\varepsilon,\delta\}\big).$$

Clearly, S is a commutative semigroup with identity $(Y,1)$.

Now, for each $(F,\varepsilon)\in S$, take a projection $P_F^\varepsilon : F\to Y$ with $\|P_F^\varepsilon\|\leqslant 1+\varepsilon$ and define a mapping $\Phi: S\times Y^{**}\to Y$ as

$$\Phi(F,\varepsilon,x) = \begin{cases} P_F^\varepsilon(x) & \text{if } x\in F,\\ 0 & \text{otherwise.}\end{cases}$$

In this way, for each fixed $x\in Y^{**}$, we obtain a function $\Phi(\cdot,\cdot,x)\in B(S,Y)$ given by $\Phi(\cdot,\cdot,x)(F,\varepsilon)=\Phi(F,\varepsilon,x)$.

Let $m = m_{(F,\varepsilon)}(\cdot)$ be an invariant average for $B(S,Y)$ (here, the subscript indicates that m acts on functions of the variable (F,ε) and define a map $P: Y^{**}\to Y$ as

$$P(x) = m_{(F,\varepsilon)}\big(\Phi(F,\varepsilon,x)\big).$$

Clearly, $P(y)=y$ for all $y\in Y$, by consistency of m. That P is homogeneous is obvious. Let us show that P is additive. Fix $x,z\in Y^{**}$. Then

$$\begin{aligned}
P(x+z) &= m_{(F,\varepsilon)}\big(\Phi(F,\varepsilon,x+z)\big)\\
&= m_{(F,\varepsilon)}\big(\Phi\big(F+[x,z],\varepsilon,x+z\big)\big)\\
&= m_{(F,\varepsilon)}\big(P^\varepsilon_{F+[x,z]}(x,z)\big)\\
&= m_{(F,\varepsilon)}\big(P^\varepsilon_{F+[x,z]}(x)+P^\varepsilon_{F+[x,z]}(z)\big)\\
&= m_{(F,\varepsilon)}\big(P^\varepsilon_{F+[x,z]}(x)\big)+m_{(F,\varepsilon)}\big(P^\varepsilon_{F+[x,z]}(z)\big)\\
&= m_{(F,\varepsilon)}\big(\Phi\big(F+[x,z],\varepsilon,x\big)\big)+m_{(F,\varepsilon)}\big(\Phi\big(F+[x,z],\varepsilon,z\big)\big)\\
&= m_{(F,\varepsilon)}\big(\Phi(F,\varepsilon,x)\big)+m_{(F,\varepsilon)}\big(\Phi(F,\varepsilon,z)\big)\\
&= P(x)+P(z).
\end{aligned}$$

Thus, P is a linear projection of Y^{**} onto Y. It remains to show that $\|P\|\leqslant\|m\|$. Let $x\in Y^{**}$ and $\delta>0$ be fixed. We have

$$\begin{aligned}
\big\|P(x)\big\| &= \big\|m_{(F,\varepsilon)}\big(\Phi(F,\varepsilon,x)\big)\big\|\\
&= \big\|m_{(F,\varepsilon)}\big(\Phi\big(F+[x],\min\{\varepsilon,\delta\},x\big)\big)\big\|\\
&\leqslant \|m\|\big\|P^{\min\{\varepsilon,\delta\}}_{F+[x]}\big\|\|x\| \leqslant (1+\delta)\|m\|\|x\|,
\end{aligned}$$

and since δ was arbitrary we conclude that $\|P\|\leqslant\|m\|$, which ends the proof. □

Observe that the semigroup used in the proof of Theorem 2 is a directed set (in fact, a lattice). The following definition isolates the relevant property of $B(S,Y)$.

Definition 2. Let $(S,\leqslant)$ be a directed set. An admissible assignment in $B(S,Y)$ is a bounded linear operator $a: B(S,Y)\to Y$ such that $a(f)=f_0$ if $f(x)=f_0$ eventually.

We say that $f(x) = f_0$ eventually if there is some $y \in S$ such that $f(x) = f_0$ for every $x \geqslant y$. It is clear that admissibility and boundedness imply that $a(f) = \lim_{x \in S} f(x)$ provided the limit exists (in the sense of net convergence [13]).

Corollary 1. *For a Banach space Y the following are equivalent*:

(a) *Y is complemented in its second dual space.*
(b) *For every commutative semigroup S there is an invariant average for $B(S, Y)$.*
(c) *For every directed set S there is an admissible assignment in $B(S, Y)$.*

Proof. It remains to show that (a) implies (c). Suppose $\pi : Y^{**} \to Y$ is a bounded linear projection and let $(S, \leqslant)$ be a directed set. Take a ultrafilter U refining the Fréchet (= order) filter on S and define $a : B(S, Y) \to Y$ by

$$a(f) = \pi\left({}^{*}\text{weak} - \lim_{U(x)} f(x)\right).$$

The definition makes sense because of the *weak compactness of balls in Y^{**}. It is clear that a is an admissible assignment, with $\|a\| \leqslant \|\pi\|$. □

3. The role of local convexity

In this section we analyze to what extent the results obtained so far depend on the local convexity of the range space. Recall from [7] that a quasi-norm on a (real or complex) vector space X is a non-negative real-valued function on X satisfying:

- $\|x\| = 0$ if and only if $x = 0$;
- $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$;
- $\|x + y\| \leqslant \Delta(\|x\| + \|y\|)$ for some fixed $\Delta \geqslant 1$ and all $x, y \in X$.

A quasi-normed space is a vector space X together with a specified quasi-norm. On such a space one has a (linear) topology defined as the smallest linear topology for which the set $B_X = \{x \in X\colon\ \|x\| \leqslant 1\}$ (the unit ball of X) is a neighborhood of 0. In this way, X becomes a locally bounded space (i.e., it has a bounded neighborhood of 0); and, conversely, every locally bounded topology on a vector space comes from a quasi-norm. A quasi-Banach space is a complete quasi-normed space.

Of course, every Banach space is a quasi-Banach space, but there are important examples of quasi-Banach spaces which are not (isomorphic to) Banach spaces. Let us mention the L_p spaces and the Hardy classes H^p for $0 < p < 1$.

Proposition 1. *Suppose Y is a quasi-Banach space such that $B(\mathbb{Z}, Y)$ admits an invariant average. Then Y is (isomorphic to) a Banach space.*

Proof. Notice that Y is isomorphic to a Banach space if and only if there is a constant M such that

$$\left\| \sum_{i=1}^{n} x_i \right\| \leqslant M \left(\sum_{i=1}^{n} \|x_i\| \right)$$

for all $x_i \in Y$.

Suppose m is a linear average for $B(\mathbb{Z}, Y)$. Take $x_i \in Y$, $1 \leqslant i \leqslant n$, and define $f : \mathbb{Z} \to Y$ by $f(k) = x_i$ if $k \equiv i$ modulo n. From linearity, invariance and consistency of m, it follows that $m(f) = (1/n) \sum_{i=1}^{n} x_i$. Hence,

$$\frac{1}{n} \cdot \left\| \sum_{i=1}^{n} x_i \right\| \leqslant \|m\| \cdot \max_{1 \leqslant i \leqslant n} \|x_i\|$$

holds for each n and all x_i. A straightforward induction shows that, in fact, one has

$$\left\| \sum_{i=1}^{n} x_i \right\| \leqslant \|m\| \cdot \left(\sum_{i=1}^{n} \|x_i\| \right)$$

for all $x_i \in Y$. So Y is a Banach space and the proof is complete. □

Thus, the following example shows that the implication (c) ⇒ (b) of Corollary 1 may fail if one allows quasi-Banach spaces.

Example 1. *A non-locally convex quasi-Banach space Y such that $B(S, Y)$ has admissible assignments for every nested set S.*

Proof. Let us recall that a quasi-Banach space is said to be a pseudo-dual space if there is a linear topology τ weaker than the quasi-norm topology which makes compact the unit (hence every) ball. Suppose Y is a pseudo-dual space. Then, for every nested set $(S, \leqslant)$ there is an admissible assignment for $B(S, Y)$. Indeed, let U be an ultrafilter stronger than the Fréchet filter on S and put

$$a(f) = \tau - \lim_{U(s)} f(s).$$

Clearly, a is an admissible assignment of norm one.

So, the proof will be complete if we exhibit a non-locally convex pseudo-dual space. Classical examples are the Hardy spaces H^p for $0 < p < 1$ (according to Montel's theorem about normal families of holomorphic functions, the topology of compact convergence makes compact the unit ball of H^p). A simpler example is provided by the sequence space l_p for $0 < p < 1$. Needless to say, $l_p = l_p(\mathbb{N})$ consists of all sequences $f : \mathbb{N} \to \mathbb{K}$ for which the quasi-norm

$$\|f\|_p = \left(\sum_{k=1}^{\infty} \left|f(k)\right|^p\right)^{1/p}$$

is finite. Fix $0 < p < 1$ and consider l_p as a subset of l_1 via the formal identity $l_p \to l_1$. Clearly, $\|f\|_p \leqslant \|f\|_1$ for all $f \in l_p$. Let τ be the restriction to l_p of the *weak topology of l_1 viewed as the space of linear functionals on c_0 (= the space of null sequences with the sup norm). Since c_0 is separable, the *weak topology of l_1 is metrizable on bounded sets, and so is τ. We show that B_p, the unit ball of l_p, is (sequentially) τ-compact. Let $(f_n)_{n=1}^{\infty}$ be a sequence in B_p. Since B_1 is a *weakly compact set and contains B_p we may assume and do that $(f_n)_{n=1}^{\infty}$ converges *weakly to some $f \in B_1$. It remains to see that f belongs to B_p. But *weak convergent sequences in l_1 are pointwise convergent, hence one has $|f_n(k)|^p \to |f(k)|^p$ as $n \to \infty$ for all k. Thus Fatou's lemma yields

$$\|f\|_p^p = \sum_{k=1}^{\infty} \lim_{n\to\infty} \left|f_n(k)\right|^p \leqslant \liminf_{n\to\infty} \left(\sum_{k=1}^{\infty} \left|f_n(k)\right|^p\right) = \liminf_{n\to\infty} \|f_n\|_p^p \leqslant 1,$$

and so f lies is the unit ball of l_p. Since l_p is not locally convex for $0 < p < 1$, the proof is complete. □

4. Admissible assignments and quasi-Banach ultrasummands

In this section, we prove that a quasi-Banach space Y is an ultrasummand if and only if $B(S, Y)$ has admissible assignments for every directed set S.

Let X be a quasi-Banach space, S a (not necessarily directed) set and U an ultrafilter on S. The ultrapower of X with respect to U is the quasi-Banach space obtained taking the quotient of $B(S, X)$ by the subspace

$$N_U = \left\{ f \in B(S, X)\colon\ \lim_{U(s)} \left\|f(s)\right\|_X = 0 \right\}$$

and will be denoted by X_U. The quasi-norm of X_U enjoys the following nice property:

$$\left\|[f]\right\|_{X_U} = \lim_{U(s)} \left\|f(s)\right\|_X,$$

where $[f]$ denotes the class of $f \in B(S, X)$ in X_U. Observe that X_U contains a natural copy of the space X that consists of all (classes of) constant maps $S \to X$.

A quasi-Banach space X is said to be an ultrasummand provided it is complemented in each ultrapower X_U. For Banach spaces this turns out to be equivalent to being complemented in the bidual (a consequence of Corollary 1 and the following result).

Theorem 3. *A quasi-Banach space Y is an ultrasummand if and only if, for every directed set S, the space $B(S,Y)$ has admissible assignments.*

Proof. *Sufficiency.* Let S be a directed set, U be an ultrafilter on S and let Y_U be the corresponding ultrapower. If $P: Y_U \to Y$ is a bounded linear projection and $\pi: Y_U \to Y$ is the natural quotient map, then $P \circ \pi$ is an admissible assignment for $B(S,Y)$.

Necessity. Suppose $Y_U = B(S,Y)/N_U$ is an ultrapower. Consider the set $\Lambda = \{(s,A)\colon\ A \in U,\ s \in A\}$ directed by $(s,A) \leqslant (t,B) \Leftrightarrow B \subset A$. Now, observe that every bounded $f: S \to Y$ extends to a bounded net $\tilde{f}: \Lambda \to Y$ by $\tilde{f}(s,A) = f(s)$. Moreover, $\tilde{f}$ is constant if and only if f is. On the other hand,

$$\lim_{(s,A)\in\Lambda} \tilde{f}(s,A) = \lim_{U(s)} f(s),$$

for all $f \in B(S,Y)$.

Let, finally, $a: B(\Lambda,Y) \to Y$ be an admissible assignment. Then, the map $\tilde{a}: B(S,Y) \to Y$ given by $\tilde{a}(f) = a(\tilde{f})$ is a bounded projection onto Y. Since $\tilde{a}$ obviously vanishes on N_U, $\tilde{a}$ factors throughout Y_U thus given a bounded projection from Y_U onto Y. This completes the proof. □

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