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Coverings and truncations of graded self-injective algebras ☆

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ABSTRACT

Let Λ be a graded self-injective algebra. We describe its smash product $\Lambda \# k\mathbb{Z}^*$ with the group $\mathbb{Z}$, its Beilinson algebra and their relationship. Starting with Λ, we construct algebras with finite global dimension, called τ-slice algebras, we show that their trivial extensions are all isomorphic, and their repetitive algebras are the same as $\Lambda \# k\mathbb{Z}^*$. There exist τ-mutations similar to the BGP reflections for the τ-slice algebras.

1. Introduction

In [3], it is proved that the derived category $D(\operatorname{coh}\mathbb{P}^{n-1})$ of the coherent sheaves of a projective space is equivalent to the stable category $\underline{\mathrm{gr}} \wedge V$ of the graded modules of the exterior algebra (called BGG correspondence). Koszul duality between Artin–Schelter regular Koszul algebra and the self-injective Koszul algebra [21,15] generalizes BGG correspondence to non-commutative setting [16,17, 13,11]. It is also known that the derived category of the coherent sheaves of a projective line is equivalent to the derived category of the Kronecker algebra, which is hereditary and of finite global dimension [14]. Recently, Chen proves in [5] that for a well graded self-injective algebra, the category of its graded modules is equivalent to the category of the graded modules over the trivial extension algebra of its Beilinson algebra, which is of finite global dimension. As a consequence, the derived category of its Beilinson algebra is equivalent to the stable category of its trivial extension [5]. So the case of Kronecker algebra can be generalized, and the BGG correspondence is extended to a derived category of algebra of finite global dimension, and we get equivalences of triangulated categories

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as follows

$$D^b(\operatorname{coh} X) \simeq \underline{\operatorname{gr}}\, \Lambda \simeq D^b(\Lambda').$$

Here the left side is the bounded derived category of the quasi-coherent sheaves of non-commutative projective space, middle is the stable category of a graded self-injective algebra Λ and the right side is the bounded derived category of an algebra Λ' of finite global dimension. According to [5], the right equivalence is also well known [8,9], when we start with Λ'. It follows from [8] that $D^b(\Lambda')$ is also equivalent to $\underline{\operatorname{mod}}\, \widehat{\Lambda}'$, the stable category of finitely generated modules over the repetitive algebra $\widehat{\Lambda}'$ of Λ', as triangulated categories.

This paper mainly studies the algebras appearing on the right side of the equivalences of the triangulated categories above. Starting with a graded self-injective algebra Λ, we are interested in the algebras Λ' of finite global dimension. Our approach is similar to the classical approach initiated in [12], and developed, e.g., in [20,18], using repetitive algebras and coverings in study self-injective algebra. Our aim is to find out how to construct the algebras Λ' of finite global dimension from Λ, and how such algebras are related. Coverings and truncations related to the Nakayama functor play key roles in our approach.

Let Λ be a graded self-injective algebra over an algebraically closed field k, and let $\mathcal{N} = D\operatorname{Hom}_\Lambda(\ ,\Lambda)$ be the Nakayama functor. $\mathcal{N}$ is an auto-equivalence on the category of Λ-modules, and it induces a permutation τ on the vertex set of the Gabriel quiver of Λ, which we call *Nakayama translation*. It also induces Nakayama automorphism on the category of non-isomorphic indecomposable projective Λ-modules. When the group G generated by the Nakayama automorphism acts freely, Λ is a regular covering of its orbit algebra with respect to the group G. This orbit algebra is a weakly symmetric algebra, that is, graded self-injective with trivial Nakayama translation.

To find Λ', we first go to the smash product $\Lambda \# k\mathbb{Z}^*$ of Λ with the group $\mathbb{Z}$, whose bound quiver is described as the separated directed quiver of Λ. The Beilinson algebra of Λ defined in [5] is the first candidate for Λ'. We describe the bound quiver of the Beilinson algebra of Λ as some truncation of the bound quiver of $\Lambda \# k\mathbb{Z}^*$. We also show that the orbit algebra Λ^T of $\Lambda \# k\mathbb{Z}^*$ with respect to the Nakayama functor is a twisted trivial extension of the Beilinson algebra and that $\Lambda \# k\mathbb{Z}^*$ is exactly the repetitive algebra of the Beilinson algebra.

There are more such algebras Λ' for a given graded self-injective algebra Λ, in addition to the Beilinson algebra. These algebras are obtained by truncating the bound quiver of $\Lambda \# k\mathbb{Z}^*$ mimic the complete slice in the tilting theory [10,12]. We call the truncated bound quivers complete τ-slices and the algebras obtained τ-slice algebras. We also introduce the τ-mutations for the τ-slices algebras, mimic the Bernstein–Gel'fand–Poromarev (BGP) reflections. Each connected component of the Beilinson algebra defined in [5] is a τ-slice algebra. We show that all the τ-slice algebras have equivalent derived categories, by showing that they all have the same trivial extensions, and the same repetitive algebra $\Lambda \# k\mathbb{Z}^*$.

The τ-mutation is not a direct generalization of the BGP reflection, we need to go to the Koszul dual for it. We show that if Λ^T is Koszul, then all its τ-slice algebras are also Koszul. If Λ is of Loewy length 3, all the τ-slice algebras are Koszul and the τ-mutation induces the BGP reflection on their Koszul duals.

The paper is organized as follows. In Section 2, we recall basic notions on bound quivers, path algebras etc., to fix terminology. We also give a description on the graded self-injective algebra of Loewy length $l+1$ using bound quiver with Nakayama translation. In Section 3, we study the orbit algebra of a graded self-injective algebra Λ with respect to the Nakayama functor. We show that the orbit algebra with respect to the Nakayama functor is graded self-injective and Λ is a covering of the orbit algebra when the group generated by the Nakayama automorphism acts freely. In Section 4, we introduce separated directed quiver $\overline{Q}$ for the bound quiver Q of a graded self-injective algebra Λ. We show that this quiver is the bound quiver of the smash product $\Lambda \# k\mathbb{Z}^*$ and the group generated by the Nakayama automorphism acts freely on the indecomposable projective $\Lambda \# k\mathbb{Z}^*$-modules. We also introduce specially truncated quiver of the separated directed quiver, and discuss some basic properties of these quivers. In Section 5, we show that the bound quiver of the Beilinson algebra of Λ is the total specially truncated quiver. We also show that the orbit algebra of $\Lambda \# k\mathbb{Z}^*$ with respect to the group generated by the Nakayama automorphism is isomorphic to a twisted trivial extension of the Beilinson algebra, and that $\Lambda \# k\mathbb{Z}^*$ is the repetitive algebra of the Beilinson algebra of Λ. In Section 6, we introduce τ-slices, τ-slice algebras and τ-mutations. The action of the τ-mutations on the τ-slices is transitive. We prove that the trivial extensions of the τ-slice algebras are isomorphic when they are τ-mutations one another. Such isomorphism induces equivalence of the derived categories of the τ-slice algebras, and τ-mutation can be regarded as generalization of tilting process. We also show that all the τ-slice algebras have the same repetitive algebra, $\Lambda \# k\mathbb{Z}^*$. In Section 7, we prove that for a self-injective algebra with vanishing radical cube, the Yoneda algebra of the τ-mutation of a τ-slice algebra is exactly the BGP reflection of its Yoneda algebra. We also discuss the Koszulity of the τ-slice algebras of a graded self-injective algebra.

2. Preliminaries

Throughout this paper, k is an algebraically closed field, all the algebras are basic k algebras and all the modules are usually left modules. By a quiver we usually mean a bound quiver $Q = (Q_0, Q_1, \rho)$, that is, a quiver with the vertex set Q_0, the arrow set Q_1 and the relation set ρ. ρ is a set of linear combinations of paths of length larger or equal to 2. We use the same notation Q for both the quiver and the bound quiver, denote by kQ the path algebra of Q and by $k(Q) = kQ/(\rho)$ the algebra given by the bound quiver Q, that is, the quotient algebra of the path algebra of Q modulo the ideal generated by the relations. A path is called a *bound path* if its image in $k(Q)$ is nonzero.

An algebra Λ is called a graded algebra in this paper if $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \cdots$ as a direct sum of vector spaces, Λ_0 is semi-simple basic algebra, a direct sum of $|Q_0|$-copies of k, and $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for all $i, j \geqslant 0$. By Gabriel's theorem, Λ is given by its bound quiver $Q = (Q_0, Q_1, \rho)$, that is, $\Lambda \simeq kQ/(\rho)$. Write $\mathbf{r} = \Lambda_1 + \Lambda_2 + \cdots$. Let e_i be the idempotent corresponding to the vertex i of Q, then $\{e_i \mid i \in Q_0\}$ is a complete set of orthogonal primitive idempotents of Λ. We have that $1 = \sum_{i \in Q_0} e_i$ when $|Q_0|$ is finite. Let $E(\Lambda) = \mathrm{Ext}_\Lambda(\Lambda_0, \Lambda_0)$ be its Yoneda algebra.

We now characterize the graded self-injective algebras using bound quivers. In this paper, *a graded self-injective algebra* is a locally finite-dimensional graded algebra for which each indecomposable projective module is injective. We say that a bound quiver $Q = (Q_0, Q_1, \rho)$ is *homogeneous* provided that each of the paths appearing in a given linear combination of ρ has the same length. Two relation sets ρ and ρ' of a given quiver Q are said to be equivalent if they generate the same ideal in the path algebra kQ. In this case, we also say that two bound quivers (Q_0, Q_1, ρ) and (Q_0, Q_1, ρ') are equivalent. Clearly, equivalent bound quivers define isomorphic algebras.

Since path algebra is graded, we obviously have the following proposition.

Proposition 2.1. *$k(Q)$ is a graded algebra if and only if Q is equivalent to a homogeneous bound quiver.*

Fix an integer $l \geqslant 1$, a homogeneous bound quiver Q is said to be *stable of Loewy length $l+1$* if there is a permutation τ on the vertex set of the quiver, such that the following conditions are satisfied.

1. The maximal bound paths of Q have the same length l;
2. For each vertex i, there is a maximal bound path from τi to i;
3. There is no bound path of length l from τi to j for any $j \neq i$;
4. Any two maximal bound paths starting at the same vertex are linearly dependent.

The permutation τ is called the *Nakayama translation* of the stable bound quiver Q.

We have the following theorem characterizing the bound quiver of a graded self-injective algebra.

Theorem 2.2. *Let $\Lambda = k(Q)$ be the algebra given by a bound quiver Q, then Λ is a graded self-injective algebra with Loewy length $l+1$ if and only if Q is equivalent to a stable bound quiver of Loewy length $l+1$.*

Proof. Assume that $\Lambda = k(Q)$ is the algebra given by a stable bound quiver Q of Loewy length $l+1$ with Nakayama translation τ. $\mathbf{r}$ is its radical, and we have that $\mathbf{r}^{l+1} = 0$ since maximal bound paths in Q have the same length l. Let e_i be the idempotent corresponding to the vertex i, and let S_i be the simple module corresponding to i, $P(i)$ be its projective cover and $I(i)$ be its injective envelope. For a maximal bound path p, $\mathbf{r}p = 0$, so they span the socle of Λ. Since maximal bound paths starting at i end at $\tau^{-1}i$, and any two of them are linearly dependent, we see that the socle of $P_i \simeq \Lambda e_i$ is isomorphic to $S_{\tau^{-1}i}$. So each indecomposable projective has a simple socle and $\operatorname{soc} P_i \simeq S_{\tau^{-1}i} = P_{\tau^{-1}i}/\mathbf{r}P_{\tau^{-1}i}$. By considering the maximal bound paths ending at i, one gets that indecomposable injective has a simple top and $I_i/\mathbf{r}I_i \simeq S_{\tau i}$. This implies that $P_i \simeq I_{\tau^{-1}i}$ for each $i \in Q_0$, and Λ is a graded self-injective algebra of Loewy length $l+1$.

Let $Q = (Q_0, Q_1, \rho)$ be a bound quiver of a graded self-injective algebra Λ. We may assume that Q is homogeneous. Let τ be the permutation of Q_0 induced by the Nakayama functor of $\operatorname{mod}\Lambda$. Then τ sends each vertex i to the vertex τi corresponding to the top of the injective envelope of the simple S_i. Since projectives have the same Loewy length, say l, by [15], so the maximal bound paths of Q have the same length l. Since the indecomposable projective with top S_i is the indecomposable injective with socle $S_{\tau^{-1}i}$ for each vertex i, there is a maximal bound path from τi to i which is a multiple of each bound path of length l from τi to i. We also see that there is no bound path of length l from τi to j for any $j \neq i$. This shows that Q is a stable bound quiver of Loewy length $l+1$. $\square$

So the Nakayama translation is induced by the Nakayama functor.

A *walk* from a vertex i to a vertex j in a quiver is a sequence of paths $p_1, \dots, p_r$ in Q satisfies the following conditions:

1. r is odd, the length $l(p_t) > 0$ for $t = 2, \dots, r-1$, (here we allow the length of p_1 and p_r to be zero);
2. i is the starting vertex of p_1 and j is the ending vertex of p_r, p_{2t} and p_{2t+1} have the same starting vertex and p_{2t-1} and p_{2t} have the same ending vertex.

When all the paths in a walk are bound paths, we call this walk a *bound walk*.

Now consider a stable bound quiver. By embedding a bound path in maximal ones, one sees that for a bound path from vertex i to vertex j, there is a bound path from j to $\tau^{-1}i$ and a bound path from τj to i. For a walk in a stable bound quiver with two nontrivial bound paths starting from vertices i and i' and ending at vertex j, there is a walk with two nontrivial bound paths from τj to i and i'. Dually, for a walk with two nontrivial bound paths starting at vertex i and ending at vertices j and j', there is a bound walk with two nontrivial bound paths from j and j' to $\tau^{-1}i$. In a finite connected stable bound quiver, τ is periodic, we have the following lemma.

Lemma 2.3. *Let Q be a finite connected stable bound quiver. For each pair of vertices i and i' which are connected by a walk in the stable bound quiver, there is a (unbound) path in Q from i to i'.*

We will need the following results, see Lemma 2.1 of [6].

Lemma 2.4. *Let Λ be a self-injective algebra, let Q be its bound quiver and τ be the Nakayama translation on Q. Then for any $i, j \in Q_0$ if $e_j \Lambda e_i \neq 0$, we have a non-degenerate bilinear form $e_{\tau^{-1}i}\Lambda e_j \otimes e_j \Lambda e_i \to k$ satisfying the multiplicative property, that is $(xy, z) = (x, yz)$.*

If Λ is graded with Loewy length $l+1$, the bilinear form is restricted to a non-degenerate one on $e_{\tau^{-1}i}\Lambda_{l-t} e_j \otimes e_j \Lambda_t e_i \to k$ for $0 \leqslant t \leqslant l$, whenever $e_j \Lambda_t e_i \neq 0$.

So we have that $e_{\tau^{-1}i}\Lambda_{l-1} e_j \simeq D e_j \Lambda_1 e_i$ and $e_j \Lambda_1 e_i \simeq e_{\tau j}\Lambda_1 e_{\tau i}$ as vector spaces. As a corollary of Lemma 2.4, we have the following corollary:

Corollary 2.5. *Let Λ be a graded self-injective algebra with Loewy length $l+1$ and Nakayama translation τ, let Q be its quiver. Then for any i, j in Q_0, we have:*

1. *The number of arrows from* i *to* j *is* $\dim_k e_{\tau^{-1}i}\Lambda_{l-1}e_j$;
2. *The number of arrows from* i *to* j *and the number of arrows from* τi *to* τj *are the same.*

3. Nakayama translation and orbit algebra

Following [2], a k-additive category $\mathcal{C}$ is called *locally bounded*, if it satisfies the following conditions:

1. For each object P in $\mathcal{C}$, $\mathrm{End}(P)$ is local;
2. For each pair P, P' of objects in $\mathcal{C}$, $\dim_k \mathrm{Hom}(P, P')$ is finite;
3. Distinct objects in $\mathcal{C}$ are non-isomorphic;
4. For each object P in $\mathcal{C}$, there are only finite many object P' in $\mathcal{C}$ such that $\mathrm{Hom}(P, P') \neq 0$ or $\mathrm{Hom}(P', P) \neq 0$.

Let $\mathcal{C}, \mathcal{D}$ be locally bounded categories. A k-linear functor $F : \mathcal{C} \to \mathcal{D}$ is called a *covering functor* if,

1. F is surjective on objects.
2. For each object P in $\mathcal{C}$, F; induces isomorphisms

$$\bigoplus_{P' \in F^{-1}(Q)} \mathrm{Hom}\big(P', P\big) \to \mathrm{Hom}\big(Q, F(P)\big)$$

and

$$\bigoplus_{P' \in F^{-1}(Q)} \mathrm{Hom}\big(P, P'\big) \to \mathrm{Hom}\big(F(P), Q\big).$$

If a group G of k-automorphisms on $\mathcal{C}$ acts freely on the objects and $\mathcal{D}$ is equivalent to its orbit category, then we call the covering F regular (or Galois) and call G its group.

Let Λ and Λ' be two k-algebras. If there are locally bounded categories $\mathcal{C}, \mathcal{D}$ such that with the naturally defined multiplications,

$$\Lambda \simeq \bigoplus_{P, P' \in \mathcal{C}} \mathrm{Hom}\big(P, P'\big) \quad \text{and} \quad \Lambda' \simeq \bigoplus_{D, D' \in \mathcal{D}} \mathrm{Hom}\big(D, D'\big)$$

as algebras and there is a covering functor $F : \mathcal{C} \to \mathcal{D}$, then we say that F is *a covering* from Λ to Λ'. If F is regular with group G, we called covering from Λ to Λ' regular with group G.

Let $\mathcal{P}$ be a small k-additive category. Let G be a group of autofunctors of $\mathcal{P}$ and let M be an object in $\mathcal{P}$. Assume that G acts freely on the objects. Define the *orbit algebra* $O(G, M)$ of M with respect to G to be the vector space

$$O(G, M) = \bigoplus_{F \in G} \mathrm{Hom}(FM, M)$$

with the multiplication defined as follows: For $F, F' \in G$, and for any $f \in \mathrm{Hom}(FM, M)$, $g \in \mathrm{Hom}(F'M, M)$,

$$f \cdot g = f \circ Fg.$$

$O(G, M)$ is an associative k algebra. Clearly, $O(G, P) = \mathrm{End}\, P$ when G is trivial.

Let Λ be a self-injective algebra and let $\mathcal{N}$ be the Nakayama functor on $\mathrm{mod}\,\Lambda$. Let $\mathcal{P} = \mathcal{P}(\Lambda)$ be the category of projective Λ-modules, and let $\mathrm{ind}\,\mathcal{P}$ be the category with objects the non-isomorphic indecomposable projective Λ-modules. $\mathcal{N}$ induces an automorphism on the category $\mathrm{ind}\,\mathcal{P}$, denoted it by the same $\mathcal{N}$. $\mathcal{N}$ also induces an automorphism ν on the algebra Λ. Both are called Nakayama automorphism.

Let G be the group generated by $\mathcal{N}$, and assume that G acts freely on the objects in $\mathrm{ind}\,\mathcal{P}$. An object P of $\mathcal{P}$ is called *basic G-orbit generator* if its indecomposable summands are taken from different orbits and we have that $\mathcal{P} = \mathrm{add}\, GP$ for its orbit $GP = \{\mathcal{N}^t P \mid t \in \mathbb{Z}\}$. Let $P = \bigoplus_{i\in I} P_i$ be a basic G-orbit generator, with P_i indecomposable. Denote by $\Lambda^{\mathcal{N}} = O(G, P)$ the orbit algebra of P with respect to G.

The following theorem tells us that $\mathcal{N}$ induces a covering from Λ to the orbit algebra $\Lambda^{\mathcal{N}} = O(G, P)$.

Theorem 3.1. *Assume that G acts freely on the indecomposable objects in $\mathcal{P}$. Then $\mathcal{N}$ induces a regular covering N from Λ to $\Lambda^{\mathcal{N}}$ with the group G, $\Lambda^{\mathcal{N}}$ is a graded self-injective algebra whose Nakayama functor induces trivial Nakayama translation on the bound quiver $Q^{\mathcal{N}}$ of $\Lambda^{\mathcal{N}}$.*

Proof. It follows directly from the definition and [1,4] that $\mathcal{N}$ induces a regular covering N from Λ to $\Lambda^{\mathcal{N}}$ with the group G. Since $P = \sum_{i\in I} P_i$ is a basic G-orbit generator of $\mathcal{P}(\Lambda)$, the object set of $\mathrm{ind}\,\mathcal{P}$ is $\{\mathcal{N}^t P_i \mid i \in I,\ t \in \mathbb{Z}\}$.

$$\mathrm{Hom}_\Lambda(\Lambda, \Lambda) = \bigoplus_{\mathcal{N}^t, \mathcal{N}^{t'} \in G} \bigoplus_{i,j\in I} \mathrm{Hom}_\Lambda\big(\mathcal{N}^t P_i, \mathcal{N}^{t'} P_j\big).$$

Let $\mathcal{D}$ be the category with objects the G-orbits $\{[P_i] \mid i \in I\}$ of indecomposable projective Λ-modules, with hom-sets

$$\begin{aligned}\mathrm{Hom}\big([P_j], [P_i]\big) &= \bigoplus_{P''\in[P_j]} \mathrm{Hom}_\Lambda\big(P'', P_i\big)\\ &= \bigoplus_{\mathcal{N}^t\in G} \mathrm{Hom}_\Lambda\big(\mathcal{N}^t P_j, P_i\big).\end{aligned}$$

The composition of the morphisms is defined as follows

$$f \cdot g = f \circ \mathcal{N}^t g$$

for $f \in \mathrm{Hom}_\Lambda(\mathcal{N}^t P_j, P_i)$, $g \in \mathrm{Hom}_\Lambda(\mathcal{N}^{t'} P_k, P_j)$. Clearly, $\mathrm{Hom}([P_j], [P_i]) \simeq \bigoplus_{\mathcal{N}^t\in G} \mathrm{Hom}_\Lambda(P_j, \mathcal{N}^t P_i)$. Since the $\mathbb{Z}$-grading of Λ is induced by the radical filtration, we have that for each pair P, P' of projectives, $\mathrm{Hom}_\Lambda(P, P') = \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}_\Lambda(P, P')_s$, here $\mathrm{Hom}_\Lambda(P, P')_s$ is the subspace of degree s homomorphism. Since $\mathcal{N}$ preserves the degree of homomorphism and $\Lambda \simeq \mathrm{Hom}_\Lambda(\Lambda, \Lambda)$, we have $\Lambda_s \simeq \mathrm{Hom}_\Lambda(\Lambda, \Lambda)_s$.

$$\begin{aligned}\Lambda^{\mathcal{N}} = \mathcal{O}(G, P) &\simeq \bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{D}}\big([P_j], [P_i]\big)\\ &= \bigoplus_t \bigoplus_{i,j\in I} \mathrm{Hom}_\Lambda\big(\mathcal{N}^t P_i, P_j\big)\\ &= \bigoplus_{s=0}^{l} \bigoplus_t \bigoplus_{i,j\in I} \mathrm{Hom}_\Lambda\big(\mathcal{N}^t P_i, P_j\big)_s.\end{aligned}$$

So the $\mathbb{Z}$-grading of Λ induces a $\mathbb{Z}$-grading on $\Lambda^{\mathcal{N}}$.

Write e_P for the identity of an object P, then $P = \Lambda e_P$ for an indecomposable projective Λ-module P, and $[P] = \Lambda^{\mathcal{N}} e_{[P]}$ for an indecomposable object $[P]$ in $\mathcal{D}$. Let $\operatorname{Hom}_\Lambda(P, P')_s$ denote the subspace of degree s homomorphisms. By Lemma 2.4, $\operatorname{Hom}_\Lambda(P', P)_s = e_{P'}\Lambda_s e_P \simeq De_{\mathcal{N}P}\Lambda_{l-s}e_{P'} = D\operatorname{Hom}_\Lambda(\mathcal{N}P, P')_{l-s}$ for any P, P'. For an indecomposable projective $\Lambda^{\mathcal{N}}$-module $\Lambda^{\mathcal{N}} e_{[P]}$,

$$\begin{aligned}
\Lambda^{\mathcal{N}} e_{[P]} &= \operatorname{Hom}_{\Lambda^{\mathcal{N}}}\bigl(\Lambda^{\mathcal{N}}, \Lambda^{\mathcal{N}} e_{[P]}\bigr) \\
&= \bigoplus_s \bigoplus_{[P']} \operatorname{Hom}_{\Lambda^{\mathcal{N}}}\bigl(\Lambda^{\mathcal{N}} e_{[P']}, \Lambda^{\mathcal{N}} e_P\bigr)_s \\
&= \bigoplus_s \bigoplus_v \bigoplus_{P_j} \operatorname{Hom}_{\mathcal{C}}(\Lambda e_{\mathcal{N}^v P'}, \Lambda e_P)_s \\
&= \bigoplus_s \bigoplus_v \bigoplus_{P_j} D\operatorname{Hom}_{\mathcal{C}}(e_{\mathcal{N}P}\Lambda, e_{\mathcal{N}^v P'}\Lambda)_{l-s} \\
&\simeq D\bigoplus_s \bigoplus_v \bigoplus_{P_j} \operatorname{Hom}_{\mathcal{C}}(e_{\mathcal{N}P}\Lambda, e_{\mathcal{N}^v P'}\Lambda)_{l-s} \\
&\simeq D\bigoplus_s \bigoplus_v \bigoplus_{P_j} \operatorname{Hom}_{\mathcal{C}}(e_{\mathcal{N}^{v+1} P}\Lambda, e_{P'}\Lambda)_{l-s} \\
&= D\bigoplus_s \bigoplus_v \bigoplus_{P_j} \operatorname{Hom}_{\mathcal{D}}\bigl(e_{[\mathcal{N}P]}\Lambda^{\mathcal{N}}, e_{[P']}\Lambda^{\mathcal{N}}\bigr)_{l-s} \\
&= De_{[\mathcal{N}P]}\Lambda^{\mathcal{N}}.
\end{aligned}$$

Thus $\Lambda^{\mathcal{N}} e_{[P]}$ is injective, so $\Lambda^{\mathcal{N}}$ is self-injective of Loewy length $l+1$. It also follows from this that the Nakayama translation on the quiver $Q^{\mathcal{N}}$ of $\Lambda^{\mathcal{N}}$ is identity. $\square$

We see that $\Lambda^{\mathcal{N}}$ is a weakly symmetric algebra, and call it *the weakly symmetric algebra* of Λ. Now we assume that the bound quiver of Λ is $Q = (Q_0, Q_1, \rho)$. Assume that G acts freely on Λ, that is, G acts freely on $\operatorname{ind}\mathcal{P}(\Lambda)$. Then Λ is a regular covering of $\Lambda^{\mathcal{N}}$ with group G. The vertex set of the quiver $Q^{\mathcal{N}}$ of $\Lambda^{\mathcal{N}}$ is the set of the orbits of vertices of Q_0 under the Nakayama translation τ. G can be regarded as the group generated by τ. It follows from Lemma 2.4 that $\dim_k \Lambda_1 e_i = \dim_k \Lambda_1 e_{\tau^r i}$, and $\dim_k e_j\Lambda_1 = \dim_k e_{\tau^r j}\Lambda_1$. So the number of arrows starting or ending at each vertex of Q and its images in $Q^{\mathcal{N}}$ are the same. We have the following description of the quivers.

Corollary 3.2. *Assume that G acts freely on Λ. As quivers, Q is a regular covering of $Q^{\mathcal{N}}$ with the group G generated by the Nakayama translation of Q.*

4. Smash product $\Lambda \# k\mathbb{Z}^*$, separated directed quiver and special truncated quiver

Covering theory is very important in representation theory, especially in the study of self-injective algebra [12,20,18]. In this section, we study a universal covering for a graded self-injective algebra Λ, its smash product with the infinite cyclic group $\mathbb{Z}$. We will describe the bound quiver of the smash product and study its properties.

Starting with a stable bound quiver $Q = (Q_0, Q_1, \rho)$ of Loewy length $l+1$, say, of Λ, we construct a directed quiver $(\overline{Q}_0, \overline{Q}_1)$ as follows.

Vertex set:

$$\overline{Q}_0 = \bigl\{(i, n) \bigm| i \in Q_0,\ n \in \mathbb{Z}\bigr\};$$

Arrow set:

$$\overline{Q}_1 = \{(\alpha, n) : (i, n) \to (j, n+1) \mid \alpha : i \to j \in Q_1,\ n \in \mathbb{Z}\}.$$

If $p = \alpha_s \cdots \alpha_1$ is a path in Q, define $p[n] = (\alpha_s, n+s-1) \cdots (\alpha_1, n)$ for each $n \in \mathbb{Z}$. Define relations

$$\bar{\rho} = \{\zeta[n] \mid \zeta \in \rho,\ n \in \mathbb{Z}\}$$

here $\zeta[n] = \sum_t a_t p_t[n]$ for each $\zeta = \sum_t a_t p_t \in \rho$. $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \bar{\rho})$ is a locally finite bound quiver if Q is so. We call $\overline{Q}$ the *separated directed quiver* of the stable bound quiver Q.

By the definition, one sees easily that the following holds.

Proposition 4.1. *The quiver $\overline{Q}$ contains no oriented cycle.*

We show that this bound quiver gives exactly the smash product of Λ with the infinite cyclic group $\mathbb{Z}$.

Let $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_l$ be a graded self-injective algebra of Loewy length $l+1$. Let Q be its bound quiver. Recall that the smash product $\Lambda \# k\mathbb{Z}^*$ of Λ with $\mathbb{Z}$ is the free Λ module with basis $\mathbb{Z}^* = \{\delta_n \mid n \in \mathbb{Z}\}$, and the multiplication is defined by

$$x\delta_n y\delta_m = xy_{n-m}\delta_m$$

for $x, y \in \Lambda$, $y = \sum_{t=0}^{l} y_t$ with $y_t \in \Lambda_t$. This is an infinite-dimensional algebra without the unit.

Since δ_n centralize Λ_0, if $\{e_i \mid i \in Q_0\}$ is a complete set of orthogonal primitive idempotents of Λ, $\{e_i\delta_n \mid i \in Q_0,\ n \in \mathbb{Z}\}$ is a complete set of orthogonal primitive idempotents of $\Lambda \# k\mathbb{Z}^*$.

Assume that $0 \neq x \in e_j \Lambda_t e_i$, $0 \neq y \in e_{j'} \Lambda_{t'} e_{i'}$ are homogeneous elements of degree t and t', respectively, then $y\delta_m x\delta_n = yx\delta_n \neq 0$ if and only if $j = i'$, $0 \neq yx$ and $m = n+t$. Especially, if $0 \neq x \in e_j \Lambda_1 e_i$ is homogeneous element of degree 1 in Λ, then $e_{j'}\delta_m x \delta_{m'} e_{i'} \delta_n = e_{j'} x e_{i'} \delta_n \neq 0$ if and only if $i = i'$, $j = j'$, $n = m'$ and $m = n+1$. We see that $\Lambda \# k\mathbb{Z}^*$ is a locally finite-dimensional algebra whenever Λ is so. $(\Lambda_1 + \cdots + \Lambda_l)k\mathbb{Z}^*$ is a nil ideal of $\Lambda \# k\mathbb{Z}^*$ and $\Lambda \# k\mathbb{Z}^*/(\Lambda_1 + \cdots + \Lambda_l)k\mathbb{Z}^*$ is semi-simple. Write $e_i\delta_n$ as $e_{(i,n)}$, and for a homogeneous element $x \in \Lambda_t$, write $x\delta_n$ as $x[n]$. We see that for $\alpha : i \to j \in Q_1$, $\alpha\delta_n = \alpha[n] : (i, n) \to (j, n+1)$ is an arrow in the quiver of $\Lambda \# k\mathbb{Z}^*$. This shows that the Gabriel quiver of $\Lambda \# k\mathbb{Z}^*$ is exactly $(\overline{Q}_0, \overline{Q}_1)$.

Clearly, if $\alpha_l \cdots \alpha_1$ is a path in quiver Q, then $\alpha_l[n+l-1] \cdots \alpha_2[n+1]\alpha_1[n] = \alpha_l \cdots \alpha_2\alpha_1[n]$ is a path in quiver $\overline{Q}$. Also, we have that $\sum_s a_s p_s = 0$ for paths p_s of Q and $a_s \in k$ if and only if for all n, we have $\sum_s a_s p_s[n] = 0$ in $\Lambda \# k\mathbb{Z}^*$. Thus, a path $\alpha_l \cdots \alpha_1$ is a maximal bound path of the bound quiver Q if and only if for all n, $\alpha_l[n+l-1] \cdots \alpha_2[n+1]\alpha_1[n] = \alpha_l \cdots \alpha_2\alpha_1[n]$ is a maximal bound path of $\overline{Q}$. So we see that a maximal bound path in $\overline{Q}$ starting at $(\tau i, n-l)$ ends at (i, n). Especially, maximal bound paths starting at the same vertex end at the same vertex, and any two of them are linearly dependent. Define $\bar{\tau}(i, n) = (\tau i, n-l)$, this is a permutation on the vertex set of the quiver $\overline{Q}$. This shows that $\Lambda \# k\mathbb{Z}^*$ is given by the relations $\bar{\rho} = \{\zeta[n] \mid \zeta \in \rho,\ n \in \mathbb{Z}\}$. And $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \bar{\rho})$ is a stable bound quiver with the Nakayama translation $\bar{\tau}$. So we have the following theorem.

Theorem 4.2. *If Λ is a graded self-injective algebra of Loewy length $l+1$ with bound quiver Q. Then:*

1. *$\Lambda \# k\mathbb{Z}^*$ is a self-injective algebra of Loewy length $l+1$ with the bound quiver $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \bar{\rho})$.*
2. *The Nakayama translation $\bar{\tau}$ of $\overline{Q}$ is defined by $\bar{\tau}(i, n) = (\tau i, n-l)$.*
3. *$\overline{Q}$ is a locally finite stable bound quiver of Loewy length $l+1$ if Q is so.*

We will write τ for $\bar{\tau}$ when no confusion appears.

The quiver of $\overline{Q}$ is different from the usual quiver $\mathbb{Z}Q$ used in representation theory of algebras, and it is usually not connected. Assume that the lengths of minimal oriented cycles in Q are $l_1, \dots, l_r$, and let $d = \gcd(l_1, \dots, l_r)$ be their greatest common divisor. The number of connected components of $\overline{Q}$ is given below.

Proposition 4.3. *Let Q be a finite connected stable bound quiver. Then $\overline{Q}$ has d connected components.*

If Q contains a loop, then $\overline{Q}$ is connected.

The proposition follows easily from the following lemma.

Lemma 4.4. *For any vertex j of Q, (j, m') and (j, m'') are in the same connected component of $\overline{Q}$ if and only if $m' - m'' \equiv 0 \bmod d$.*

Proof. If both (j, m') and (j, m'') are in the same connected component, then by Lemma 2.3, there is some vertex (i, n) such that there is a path from (i, n) to (j, m') and a path from (i, n) to (j, m''), and there is a path from (j, m) to (i, n) for some integer n, m. Clearly, $m < n < \min\{m', m''\}$. We see that the paths from (j, m) to (j, m') and (j, m'') are got from oriented cycles of Q, hence $d | m' - m$ and $d | m'' - m$, so $d | m' - m''$ and $m' - m'' \equiv 0 \bmod d$.

Now assume that $m' - m'' \equiv 0 \bmod d$, we may assume that $m' - m'' = \sum_{t=1}^{r} s_t l_t$ for integers $s_t \in \mathbb{Z}$. We can choose a vertex i_t from each minimal oriented cycle q_t and a path p_t from j to i_t in Q. The walk

$$p_r^{-1} q_r^{s_r} p_r \cdots p_2^{-1} q_2^{s_2} p_2 p_1^{-1} q_1^{s_1} p_1$$

in Q gives rise to a walk in $\overline{Q}$ from (j, m'') to (j, m'), so (j, m') and (j, m'') are in the same connected component of $\overline{Q}$. $\square$

Taking a vertex $i \in Q_0$, denote by $(\overline{Q}, i)$ the connected component of $\overline{Q}$ containing the vertex $(i, 0)$. If there exist paths p, q of the same length from i and i', respectively, to the same ending vertex j, then $(\overline{Q}, i) = (\overline{Q}, i')$. If there is an arrow $\alpha : i_0 \to j_0$ in Q, then we have an isomorphism from $(\overline{Q}, i_0)$ to $(\overline{Q}, j_0)$ sending (i'', n) to $(i'', n - 1)$. Since Q is connected, we have the following proposition.

Proposition 4.5. *All the connected components of $\overline{Q}$ are isomorphic.*

Fix a connected component $(\overline{Q}, i_0)$, we now study its truncations with respect to the Nakayama translation.

The bound quiver obtained by taking the vertex set

$$\left(Q^N, i_0\right)_0 = \left\{(j, n) \in (\overline{Q}, i_0) \,\middle|\, 0 \leqslant n \leqslant l - 1\right\}$$

together with all the arrows in $(\overline{Q}, i_0)$ among these vertices and the induced relations is called a *specially truncated quiver* of $\overline{Q}$ with i_0 as a source and is denoted by (Q^N, i_0).

Note that for all t, we have that $(Q^N, i_0) \simeq (Q^N, \tau^t i_0)$. Let d be the greatest common divisor of the lengths of the oriented cycles of Q. Take a path $i_1 \to i_2 \to \cdots \to i_d$ of length $d - 1$ in Q, then for any vertex j of Q, we have that $(Q^N, j) \simeq (Q^N, i_t)$ for some $1 \leqslant t \leqslant d$. So the union $\bigcup_{1 \leqslant t \leqslant d} (Q^N, i_t)$ is independent of the choice of the path. The full subquiver Q^N of $\overline{Q}$ with the vertex set

$$Q_0^N = \left\{(i, n) \,\middle|\, i \in Q_0,\ 0 \leqslant n \leqslant l - 1\right\}$$

is called *the total specially truncated quiver* of Q. We use the same notations (Q^N, i_0) and Q^N for the bound quivers with induced relations. Clearly the following proposition holds.

Proposition 4.6. *Q^N is isomorphic to a disjoint union of specially truncated quivers*

$$Q^N \simeq \bigcup_{1 \leqslant t \leqslant d} \big(Q^N, i_t\big),$$

for any path $i_1 \to i_2 \to \cdots \to i_d$ of length $d-1$ in Q.

The following is obvious.

Proposition 4.7. *For any vertex $(j, 0)$ of $(\overline{Q}, i_0)$, we have $(\overline{Q}, i_0) = (\overline{Q}, j)$ and $(Q^N, i_0) = (Q^N, j)$.*

Write $\mathcal{P} = \mathcal{P}(\Lambda \# k\mathbb{Z}^*)$ for the category of finitely generated indecomposable projective $\Lambda \# k\mathbb{Z}^*$-modules. Let $P_{i,n} = \Lambda \# k\mathbb{Z}^* e_{(i,n)}$ be the indecomposable projective $\Lambda \# k\mathbb{Z}^*$-modules corresponding to the vertex (i, n). $k(Q^N, i_0) = \operatorname{End}_{\mathcal{P}} \bigoplus_{(i,n)\in(Q^N, i_0)_0} P_{i,n}$ is given by the bound quiver (Q^N, i_0). Clearly, for all t, $k(Q^N, i_0) \simeq k(Q^N, \tau^t i_0)$. The algebra $k(Q^N, i_0)$ is called a *specially truncated algebra*. If Q is the bound quiver of a graded self-injective algebra Λ, we also call this algebra a *specially truncated algebra of Λ*.

Fix a path $i_1 \to i_2 \to \cdots \to i_d$ of length $d-1$ in Q, write

$$\Lambda^N = \bigoplus_{1 \leqslant t \leqslant d} k\big(Q^N, i_t\big),$$

then up to isomorphism, Λ^N is independent of the choice of the path, and it is given by the bound quiver Q^N. We call this algebra *the total specially truncated algebra*.

Example 4.8. The following stable bound quiver (Q, ρ) defines a graded self-injective algebra Λ of Loewy length 4. The quiver Q:

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha_1"] \arrow[r, bend left, "\beta_1"] & 2 \arrow[loop above, "\alpha_2"] \arrow[l, bend left, "\gamma_1"] \arrow[r, bend left, "\beta_2"] & 3 \arrow[loop above, "\alpha_3"] \arrow[l, bend left, "\gamma_2"] \end{tikzcd},$$

and relation set

$$\rho = \big\{\alpha_i^2 \bigm| i = 1, 2, 3\big\} \cup \{\beta_2\beta_1, \gamma_1\gamma_2\}$$
$$\cup \{\beta_i\alpha_i - \alpha_{i+1}\beta_i,\ \gamma_i\alpha_{i+1} - \alpha_i\gamma_i\ \beta_1\gamma_1 - \gamma_2\beta_2 \mid i = 1, 2\},$$

with trivial Nakayama translation, $\tau i = i$ for $i = 1, 2, 3$. Its separated directed quiver $\overline{Q}$ has only one connected component:

$$
\begin{tikzcd}[column sep=large]
{} \arrow[d, dash, dashed] & {} \arrow[d, dash, dashed] & {} \arrow[d, dash, dashed] \\
(1,-1) \arrow[d, "{(\alpha_1,-1)}"'] \arrow[dr, "{(\beta_1,-1)}" near start] & (2,-1) \arrow[d, "{(\alpha_2,-1)}"'] \arrow[dl, "{(\gamma_1,-1)}"' near start] \arrow[dr, "{(\beta_2,-1)}" near start] & (3,-1) \arrow[d, "{(\alpha_3,-1)}"'] \arrow[dl, "{(\gamma_2,-1)}"' near start] \\
(1,0) \arrow[d, "{(\alpha_1,0)}"'] \arrow[dr, "{(\beta_1,0)}" near start] & (2,0) \arrow[d, "{(\alpha_2,0)}"'] \arrow[dl, "{(\gamma_1,0)}"' near start] \arrow[dr, "{(\beta_2,0)}" near start] & (3,0) \arrow[d, "{(\alpha_3,0)}"'] \arrow[dl, "{(\gamma_2,0)}"' near start] \\
(1,1) \arrow[d, "{(\alpha_1,1)}"'] \arrow[dr, "{(\beta_1,1)}" near start] & (2,1) \arrow[d, "{(\alpha_2,1)}"'] \arrow[dl, "{(\gamma_1,1)}"' near start] \arrow[dr, "{(\beta_2,1)}" near start] & (3,1) \arrow[d, "{(\alpha_3,1)}"'] \arrow[dl, "{(\gamma_2,1)}"' near start] \\
(1,2) \arrow[d, "{(\alpha_1,2)}"'] \arrow[dr, "{(\beta_1,2)}" near start] & (2,2) \arrow[d, "{(\alpha_2,2)}"'] \arrow[dl, "{(\gamma_1,2)}"' near start] \arrow[dr, "{(\beta_2,2)}" near start] & (3,2) \arrow[d, "{(\alpha_3,2)}"'] \arrow[dl, "{(\gamma_2,2)}"' near start] \\
(1,3) \arrow[d, dash, dashed] & (2,3) \arrow[d, dash, dashed] & (3,3) \arrow[d, dash, dashed] \\
{} & {} & {}
\end{tikzcd}
$$

with relation set:

$$
\begin{aligned}
\overline{\rho} = {} & \bigl\{(\alpha_i, n+1)(\alpha_i, n) \bigm| i = 1, 2, 3;\ n \in \mathbb{Z}\bigr\} \cup \bigl\{(\beta_2, n+1)(\beta_1, n),\ (\gamma_1, n+1)(\gamma_2, n) \bigm| n \in \mathbb{Z}\bigr\} \\
& \cup \bigl\{(\beta_i, n+1)(\alpha_i, n) - (\alpha_{i+1}, n+1)(\beta_i, n), \\
& \quad (\gamma_i, n+1)(\alpha_{i+1}, n) - (\alpha_i, n+1)(\gamma_i, n) \bigm| i = 1, 2;\ n \in \mathbb{Z}\bigr\}.
\end{aligned}
$$

The Nakayama translation is defined by $\overline{\tau}(i, n) = (i, n-3)$ for $i = 1, 2, 3$ and $n \in \mathbb{Z}$. Its specially truncated quiver $Q^N = (Q^N, 1)$ is

$$
\begin{tikzcd}[column sep=large]
(1,0) \arrow[d, "{(\alpha_1,0)}"'] \arrow[dr, "{(\beta_1,0)}" near start] & (2,0) \arrow[d, "{(\alpha_2,0)}"'] \arrow[dl, "{(\gamma_1,0)}"' near start] \arrow[dr, "{(\beta_2,0)}" near start] & (3,0) \arrow[d, "{(\alpha_3,0)}"'] \arrow[dl, "{(\gamma_2,0)}"' near start] \\
(1,1) \arrow[d, "{(\alpha_1,1)}"'] \arrow[dr, "{(\beta_1,1)}" near start] & (2,1) \arrow[d, "{(\alpha_2,1)}"'] \arrow[dl, "{(\gamma_1,1)}"' near start] \arrow[dr, "{(\beta_2,1)}" near start] & (3,1) \arrow[d, "{(\alpha_3,1)}"'] \arrow[dl, "{(\gamma_2,1)}"' near start] \\
(1,2) & (2,2) & (3,2)
\end{tikzcd}
$$

with relation set

$$
\begin{aligned}
\rho^N = {} & \bigl\{(\alpha_i, n+1)(\alpha_i, n) \bigm| i = 1, 2, 3;\ n = 0, 1\bigr\} \cup \bigl\{(\beta_2, n+1)(\beta_1, n),\ (\gamma_1, n+1)(\gamma_2, n) \bigm| n = 0, 1\bigr\} \\
& \cup \bigl\{(\beta_i, n+1)(\alpha_i, n) - (\alpha_{i+1}, n+1)(\beta_i, n), \\
& \quad (\gamma_i, n+1)(\alpha_{i+1}, n) - (\alpha_i, n+1)(\gamma_i, n) \bigm| i = 1, 2;\ n = 0, 1\bigr\}.
\end{aligned}
$$

5. The Beilinson algebra, its trivial extension and repetitive algebra

In [5], Chen introduces the Beilinson algebra and shows that the category of graded modules of a well graded self-injective algebra is equivalent to the category of the graded modules of the trivial extension of its Beilinson algebra. We now describe the Beilinson algebra of a graded self-injective algebra, its trivial extension and its repetitive algebra, using algebras and the bound quivers introduced in the last section.

Let $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_l$ be a basic graded self-injective algebra. The Beilinson algebra of Λ, defined in [5], is the algebra the form

$$b(\Lambda) = \begin{pmatrix} \Lambda_0 & \Lambda_1 & \cdots & \Lambda_{l-2} & \Lambda_{l-1} \\ 0 & \Lambda_0 & \cdots & \Lambda_{l-3} & \Lambda_{l-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Lambda_0 & \Lambda_1 \\ 0 & 0 & \cdots & 0 & \Lambda_0 \end{pmatrix}.$$

We have the following theorem.

Theorem 5.1. *Let Q be the bound quiver of a graded self-injective algebra Λ. Then Q^N is the bound quiver of its Beilinson algebra.*

Proof. Since Λ is naturally graded by the lengths of paths, Λ_0 is a vector spaces with the primitive idempotents (trivial paths) as its basis, and Λ_t has a basis consisting of paths of length t.

So we have that

$$b(\Lambda) = b(\Lambda)_0 + b(\Lambda)_1 + \cdots + b(\Lambda)_{l-1},$$

with

$$b(\Lambda)_t = \begin{pmatrix} 0 & \cdots & 0 & \Lambda_t & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & \Lambda_t \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

for $t = 0, 1, \ldots, l-1$. The Jacobson radical

$$\mathbf{r}b(\Lambda) = b(\Lambda)_1 + \cdots + b(\Lambda)_{l-1},$$

$$b(\Lambda)/\mathbf{r}b(\Lambda) \simeq \begin{pmatrix} \Lambda_0 & 0 & \cdots & 0 \\ 0 & \Lambda_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_0 \end{pmatrix},$$

$$\mathbf{r}b(\Lambda)/\mathbf{r}^2 b(\Lambda) \simeq \begin{pmatrix} 0 & \Lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let ψ be the map from the path algebra kQ^N to $b(\Lambda)$ sending the path $p[n]$ of length t of Q^N to the matrix of a single path p of Λ at the $(l+1-n+t, l+1-n)$ position. Then $\psi(e_{(i,n)})$ is the matrix with a single idempotent e_i of Λ at the $(l+1-n, l+1-n)$ position, for $n = 1, \dots, l$, and $\psi(\alpha, n) = \alpha[n]$ is the matrix with a single arrow α of Λ at the $(l+1-(n+1), l+1-n)$ position, for $n = 1, \dots, l-1$. We see easily that ψ defines an epimorphism from kQ^N to $b(\Lambda)$. Clearly $\sum_t a_t \psi(p_t[n]) = 0$ in $b(\Lambda)$ if and only if $\sum_t a_t p_t = 0$ in Λ. So the relations of $b(\Lambda)$ are identified with those of the bound quiver Q^N. This shows that a bound quiver of $b(\Lambda)$ is exactly the same as the total specially truncated quiver Q^N. $\square$

This also shows that $b(\Lambda) \simeq \Lambda^N$.

Consider the category $\operatorname{ind}\mathcal{P} = \operatorname{ind}\mathcal{P}(\Lambda \# k\mathbb{Z}^*)$. Let G be the group generated by the Nakayama automorphism $\mathcal{N}$, G acts freely on the objects of $\operatorname{ind}\mathcal{P}$. For any positive integer r, let $G(r) = (\mathcal{N}^r)$ be the subgroup of G generated by $\mathcal{N}^r$, G_r acts freely on the objects of $\operatorname{ind}\mathcal{P}$, too. Let $\mathcal{P}_r$ be a finitely generated basic G_r-orbit generator. Let $\Lambda^{T,r} = O(G_r, P_r)$ be the orbit algebra. Similar to Theorem 3.1, we see that $\Lambda^{T,r}$ is a graded self-injective algebra of Loewy length $l+1$ and $\Lambda \# k\mathbb{Z}^*$ is a Galois covering of $\Lambda^{T,r}$ with the group G_r. The Nakayama automorphism of $\operatorname{ind}\mathcal{P}(\Lambda^{T,r})$ acts freely and has order r. Let $Q^{T,r}$ be the bound quiver of $\Lambda^{T,r}$. Let $\Lambda^T = \Lambda^{T,1}$ and $Q^T = Q^{T,1}$, then $\Lambda^T = k(Q^T)$. By Theorem 3.1, we have

Proposition 5.2. *$\Lambda^{T,r}$ is a regular covering of Λ^T with the group $\mathbb{Z}/r\mathbb{Z}$.*

Clearly $\Lambda^{T,r}$ is an intermediate covering of Λ^T for $r > 1$. The vertex set of $Q^{T,r}$ is the set of the orbits of the vertices of $\overline{Q}$ with respect to the group generated by the rth power of Nakayama translation. It follows from Corollary 2.5 that the number of arrows from i to j is the same as the number of the arrows from τi to τj, so we may extend τ to a bijective map on the arrows, and τ is extended to a quiver automorphism of $\overline{Q}$. The arrows of $Q^{T,r}$ are regarded as the orbits of the arrows under τ^r. So the quiver of $Q^{T,r}$ is the orbit quiver of $\overline{Q}$, obtained from the separated directed quiver $\overline{Q}$ by identifying the vertices and arrows in a τ^r-orbit, respectively. So the vertices (j, m) and $(\tau^{-rt} j, m + rtl)$ of $\overline{Q}$ are identified in $Q^{T,r}$ for all integers t.

We now turn to the $r = 1$ case. The following proposition follows easily from Corollary 2.5.

Proposition 5.3. *Each connected component of the quiver Q^T is obtained from a connected component of a specially truncated quiver by adding an arrow from (j, l) to $(i, 0)$ for each member in a maximal set of linearly independent bound paths from $(i, 0)$ to (j, l).*

Example 5.4. For the stable bound quiver Q in Example 4.8, the quiver Q^T and relations are as follows

$$\begin{tikzcd}
(1,0) \arrow[r, dash, dashed] \arrow[d, "(\alpha_1{,}0)"'] \arrow[dr, "(\gamma_1{,}0)"] & (2,0) \arrow[r, dash, dashed] \arrow[dl, "(\beta_1{,}0)"'] \arrow[d, "(\alpha_2{,}0)"'] \arrow[dr, "(\gamma_2{,}0)"] & (3,0) \arrow[dl, "(\beta_2{,}0)"'] \arrow[d, "(\alpha_3{,}0)"'] \\
(1,1) \arrow[d, "(\alpha_1{,}1)"'] \arrow[dr, "(\gamma_1{,}1)"] & (2,1) \arrow[dl, "(\beta_1{,}1)"'] \arrow[d, "(\alpha_2{,}1)"'] \arrow[dr, "(\gamma_2{,}1)"] & (3,1) \arrow[dl, "(\beta_2{,}1)"'] \arrow[d, "(\alpha_3{,}1)"'] \\
(1,2) \arrow[d, "(\alpha_1{,}2)"'] \arrow[dr, "(\gamma_1{,}2)"] & (2,2) \arrow[dl, "(\beta_1{,}2)"'] \arrow[d, "(\alpha_2{,}2)"'] \arrow[dr, "(\gamma_2{,}2)"] & (3,2) \arrow[dl, "(\beta_2{,}2)"'] \arrow[d, "(\alpha_3{,}2)"'] \\
(1,0) \arrow[r, dash, dashed] & (2,0) \arrow[r, dash, dashed] & (3,0)
\end{tikzcd}$$

and

$$\begin{aligned}\rho^T = &\{\alpha_i, n+1)(\alpha_i, n) \mid i = 1,2,3;\ n \in \mathbb{Z}/3\mathbb{Z}\} \\ &\cup \{(\beta_2, n+1)(\beta_1, n),\ (\gamma_1, n+1)(\gamma_2, n) \mid n \in \mathbb{Z}/3\mathbb{Z}\} \\ &\cup \{(\beta_i, n+1)(\alpha_i, n) - (\alpha_{i+1}, n+1)(\beta_i, n), \\ &\quad (\gamma_i, n+1)(\alpha_{i+1}, n) - (\alpha_i, n+1)(\gamma_i, n) \mid i = 1,2;\ n \in \mathbb{Z}/3\mathbb{Z}\}.\end{aligned}$$

Let Λ be an algebra and M be a Λ-bimodule. Recall the trivial extension $\Lambda \ltimes M$ of Λ by M is the algebra defined on the vector spaces $\Lambda \oplus M$ with the multiplication defined by

$$(a, x)(b, y) = (ab, ay + xb),$$

for $a, b \in \Lambda$ and $x, y \in M$. A trivial extension $\Lambda \ltimes M$ is always graded by taking $(\Lambda \ltimes M)_0 = \Lambda$ and $(\Lambda \ltimes M)_1 = M$ (see [5,8]). So Λ is a subalgebra of $\Lambda \ltimes M$. Assume that $\Lambda = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_l$ is a graded algebra, and $M = M_0 + \cdots + M_l$ is graded and generated at degree 0. By taking $(\Lambda \ltimes M)_t = \Lambda_t + M_{t-1}$ for $t = 0, \ldots, l+1$, $\Lambda \ltimes M$ becomes a graded algebra such that the degrees of the homogeneous elements of Λ are preserved.

Trivial extension of Λ by its dual $D\Lambda$ is called the trivial extension of Λ and we denote it by $\iota(\Lambda) = \Lambda \ltimes D\Lambda$.

Let σ be an automorphism of Λ. Let M be a Λ-bimodule. Define the twist M^σ of M as the bimodule with M as the vector space. The left multiplication is the same as M, and the right multiplication is twisted by σ, that is, defined by $xb = x\sigma(b)$ for all $x \in M^\sigma$ and $b \in \Lambda$. Define the twisted trivial extension $\iota_\sigma(\Lambda) = \Lambda \ltimes D\Lambda^\sigma$ to be the trivial extension of Λ by the twisted Λ-bimodule $D\Lambda^\sigma$.

Let Q be a stable bound quiver of Loewy length $l+1$ and let $\overline{Q}$ be its separated directed quiver. Let $Q' = (Q^N, i_0)$ be a specially truncated quiver of $\overline{Q}$, and let $Q^T(i_0)$ be a connect component of Q^T containing $(i_0, 0)$. Let $\Lambda' = k(Q')$ and $\Lambda^{T'} = k(Q^T(i_0))$.

Lemma 5.5. *Λ' is a subalgebra of $\Lambda^{T'}$.*

Proof. We observe that $Q^T(i_0)$ is obtained from Q' by adding certain arrows from vertices $(i, l-1)$ to $(j, 0)$ and relations with paths containing the new arrows. Since the relations added does not concern any path in Q', $\Lambda' = k(Q')$ is a subalgebra of $k(Q^T(i_0))$. $\square$

Regard Q' as a bound subquiver of $Q^T(i_0)$ with the same vertex set, index the vertices as (j, m) with $0 \leqslant m \leqslant l-1$. $\Lambda^{T'}$ is a graded self-injective algebra with Loewy length $l+1$. Write $\Lambda^{T'} = \Lambda_0^{T'} + \Lambda_1^{T'} + \cdots + \Lambda_{l-1}^{T'} + \Lambda_l^{T'}$, where $\Lambda_t^{T'}$ is the homogeneous component of degree t.

Let $\Lambda' = \Lambda'_0 + \Lambda'_1 + \cdots + \Lambda'_{l-1}$, where Λ'_t is the homogeneous component of degree t. Let $\iota(\Lambda')$ be the trivial extension of Λ'. By Lemma 5.5, Λ' is subalgebra both of $\iota(\Lambda')$ and of $\Lambda^{T'}$. Clearly, $\Lambda'_0 \simeq \iota(\Lambda')_0 \simeq \Lambda_0^{T'}$ is semi-simple algebra, and we identify them with Λ'_0. Write $e_{(i,t)}$ for the primitive idempotent corresponding to the vertex (i, t).

For any vertices $(i, t_1), (j, t_2) \in Q^T(i_0)_0$ with $0 \leqslant t_1 < t_2 \leqslant l-1$, we have

$$e_{(j,t_2)}\iota(\Lambda')_{t_2-t_1} e_{(i,t_1)} = e_{(j,t_2)}\Lambda'_{t_2-t_1} e_{(i,t_1)} = e_{(j,t_2)}\Lambda^{T'}_{t_2-t_1} e_{(i,t_1)}.$$

The vertex set of $Q^T(i_0)$ is the same as that of Q', decompose $\Lambda^{T'}$ accordingly,

$$\begin{aligned}\Lambda^{T'} &= \bigoplus_{(i,t_1),(j,t_2) \in Q^T(i_0)_0,\, t_1 \leqslant t_2} e_{(j,t_2)}\Lambda^{T'}_{t_2-t_1} e_{(i,t_1)} \oplus \bigoplus_{(i,t_1),(j,t_2) \in Q^T(i_0)_0,\, t_1 \leqslant t_2} e_{(i,t_1)}\Lambda^{T'}_{l+t_1-t_2} e_{(j,t_2)} \\ &= \Lambda' \oplus \bigoplus_{(i,t_1),(j,t_2) \in (Q^N, i_0)_0,\, t_1 \leqslant t_2} e_{(i,t_1)}\Lambda^{T'}_{l+t_1-t_2} e_{(j,t_2)}.\end{aligned}$$

Let

$$M = \bigoplus_{(i,t_1),(j,t_2)\in(Q^N,i_0)_0,\, t_1\leqslant t_2} e_{(i,t_1)} \Lambda^{T'}_{l+t_1-t_2} e_{(j,t_2)}.$$

Lemma 5.6. *M is a Λ'-bimodule and $\Lambda^{T'}$ is a trivial extension of Λ' by M.*

Proof. Clearly, M is a Λ'-bimodule.

Since for any i, j, each bound path in $Q^T(i_0)$ passes through the arrows from $(i, l-1)$ to $(j, 0)$ at most once and each path in M passes through some arrow from $(i, l-1)$ to $(j, 0)$ at least once. So for any elements $x, y \in M$, we have $xy = 0$ in $\Lambda^{T'}$. Hence $\Lambda^{T'}$ is a trivial extension of Λ' by M. □

Lemma 5.7. *M is isomorphic to $D\Lambda'^{\sigma}$ for some automorphism σ of Λ'.*

Proof. By Theorem 3.1, $\Lambda^{T'}$ is self-injective. From its proof we see that all the projectives of $\Lambda^{T'}$ have the same Loewy length as those of $\Lambda \# k\mathbb{Z}^*$, which is the same as the Loewy length of the projectives of Λ. So $\Lambda^{T'}$ is well graded in the sense of [5], and the lemma follows from Lemma 2.5 of [5]. □

So we get the following theorem immediately.

Theorem 5.8. *Let Λ' be the algebra given by the bound quiver $Q' = (Q^N, i_0)$ and let $\iota(\Lambda')$ be the trivial extension of Λ'. Let $\Lambda^{T'}$ be the orbit algebra of a connected component of $\Lambda \# k\mathbb{Z}^*$ containing $e_{(i_0,0)}$ with respect to the Nakayama functor. Then there is an automorphism σ of Λ' such that*

$$\Lambda^{T'} \simeq \iota_\sigma\big(\Lambda'\big).$$

As a corollary, we have that $\Lambda^T \simeq \iota_\sigma(\Lambda^N)$ for some automorphism σ of Λ^N.

Note that $I = \{(i, n) \mid i \in Q_0,\ n \in \mathbb{Z}\}$ is the index set of a complete set of orthogonal primitive idempotents in $\Lambda \# k\mathbb{Z}^*$. Set $I[s] = \{(i, n) \mid i \in Q_0,\ sl \leqslant n < (s+1)l\}$ for $s \in \mathbb{Z}$. Let $e_{I[s]} = \sum_{(i,n)\in I[s]} e_{(i,n)}$, and let

$$M[s,t] = e_{I[s]}\Lambda \# k\mathbb{Z}^* e_{I[t]} = \sum_{(i,n)\in I[s]} \sum_{(j,m)\in I[t]} e_{(i,n)}\Lambda \# k\mathbb{Z}^* e_{(j,m)}$$

for all $s, t \in \mathbb{Z}$. By Theorem 4.2, maximal bound path in $\Lambda \# k\mathbb{Z}^*$ has length l, and each path of length larger than l is zero. Since $e_{(j,n)}\Lambda \# k\mathbb{Z}^* e_{(i,m)}$ is spanned by paths of length $n - m$, thus $e_{(j,n)}\Lambda \# k\mathbb{Z}^* e_{(i,m)} = 0$ if $n - m < 0$ or $n - m > l$. This leads to the following lemma.

Lemma 5.9. *$\Lambda \# k\mathbb{Z}^* = \bigoplus_{s,t\in\mathbb{Z}} M[s,t]$ as vector spaces.*

$M[s,t] = 0$ if $s \neq t, t+1$.

$M[s,t]M[s,t] = 0$ for all $s \neq t$.

Let $\Gamma[s] = M[s,s] = e_{I[s]}\Lambda \# k\mathbb{Z}^* e_{I[s]}$, it is an algebra and it has the unit $e_{I[s]}$ if Λ is finite-dimensional. It follows easily that

Lemma 5.10.

$$\Lambda \# k\mathbb{Z}^* \simeq \begin{pmatrix} \ddots & & & & \mathbf{0} & \\ & \Gamma[s+1] & M[s+1,s] & & & \\ & & \Gamma[s] & M[s,s-1] & & \\ & & & \Gamma[s-1] & & \\ \mathbf{0} & & & & & \ddots \end{pmatrix}.$$

Let $Q[s]$ be the bound quiver with vertex set $I[s]$ and induced relations. Then $Q[s]$ is a shift of $Q[0] = Q^N$ and $\Gamma[s] = k(Q[s])$. The shifts induce isomorphisms

$$\psi_{s,t} : \Gamma[s] \to \Gamma[t]$$

between these algebras, and $M[s,t]$ is a $\Gamma[s]$–$\Gamma[t]$-bimodule. Using Lemma 2.4, similar to the argument in the proof of Theorem 3.1, we have

Lemma 5.11. *$M[s+1,s] \simeq D\Gamma[s]$ as right $\Gamma[s]$-module and $M[s+1,s] \simeq D\Gamma[s+1]$ as left $\Gamma[s+1]$-module.*

Now identify $\Gamma[s]$ with $\Gamma[0] = \Lambda^N$ for all s, using the above isomorphisms, then $M[s+1,s]$ are identified with $D\Lambda^N$, for all s. The following theorem follows from Lemma 5.10.

Theorem 5.12. *$\Lambda \# k\mathbb{Z}^* \simeq \widehat{\Lambda^N}$ is the repetitive algebra of Λ^N.*

Since $\Lambda^N = \bigoplus_i (\Lambda^N, i)$, where i runs over the vertices of a path of length $d-1$. So we have $\Lambda \# k\mathbb{Z}^* = (\bigoplus_i (\Lambda^N, i))\widehat{\ } = \bigoplus_i \widehat{(\Lambda^N, i)}$. Obviously, $\widehat{(\Lambda^N, i)}$ is an algebra with bound quiver $(\overline{Q}, i)$ for each i, so these direct summands of $\Lambda \# k\mathbb{Z}^*$ are isomorphic. Since $\overline{Q}$ has d connected components, so we have that.

Proposition 5.13. *$\Lambda \# k\mathbb{Z}^* = \bigoplus_i \widehat{(\Lambda^N, i)}$ is a direct sum of d isomorphic algebras.*

It follows from Lemma 2.1 of [5] that Λ, Λ^T and $\iota(\Lambda')$ have equivalent categories of graded modules. We have the following theorem from Lemma 5.10.

Theorem 5.14. *Let Λ be a finite-dimensional graded self-injective algebra over k. Then we have equivalences among the following triangulated categories*:

1. $\mathcal{D}^b(\Lambda^N)$, *the bounded derived category of the category* $\operatorname{mod} \Lambda^N$ *of finitely generated Λ^N-modules.*
2. $\underline{\operatorname{gr}}\, \iota(\Lambda^N)$, *the stable category of finitely generated graded $\iota(\Lambda^N)$-modules.*
3. $\underline{\operatorname{gr}}\, \Lambda$, *the stable category of finitely generated graded Λ-modules.*
4. $\underline{\operatorname{mod}}\, \Lambda \# k\mathbb{Z}^*$, *the stable category of finitely generated $\Lambda \# k\mathbb{Z}^*$-modules.*

Proof. Since in our case, the degree 0 part of the algebra Λ, Λ_0 is semi-simple and hence of finite global dimension, the equivalence of the first three categories follows from Lemma 2.1 and Theorem 1.1 of [5]. The equivalence of the first and the last categories follows from Theorem 5.12 and Theorem II 4.9 of [8]. □

6. τ-Slices, τ-slice algebras and τ-mutations

In the last section, we discuss the Beilinson algebra defined in [5] using bound quiver and Nakayama translation. Beilinson algebra is a solution to the problem of finding algebras of finite global dimension whose derived categories are equivalent to the stable category of a given graded self-injective algebra. Now we give a systematical way of finding algebras with this property, and investigate the interrelation among these algebras.

Let Q be a stable bound quiver with only finite many τ-orbits, and let $(\overline{Q}, i_0)$ be a separated directed quiver of Q. Let Q' be a full bound subquiver of $(\overline{Q}, i_0)$. Q' is called a *τ-slice* of Q if it has the following property:

(a) For each vertex v of $(\overline{Q}, i_0)$, the intersection of the τ-orbit of v and the vertex set of Q' is a single-point set.

A τ-slice Q' is called a *path complete τ-slice*, if it also satisfies the following property:

(b) Q' is path complete in the sense that for each path $p: v_0 \to v_1 \to \cdots \to v_t$ of $(\overline{Q}, i_0)$ with v_0 and v_t in Q', the whole path p lies in Q'.

Path complete τ-slice is a generalization of complete slice introduced in [10,12]. By Theorem 6.6, it will also be called *complete τ-slice* (see definition below).

Let $\overline{Q}$ be a directed stable bound quiver of Loewy length $l+1$ with Nakayama translation τ, let $v \in Q_0$. We define the *τ-hammock H^v starting at v* as the full subquiver with the vertex set

$$H_0^v = \{u \in Q_0 \mid \text{there is a bound path from } v \text{ to } u\}.$$

Dually we define the *τ-hammock H_v ending at v*. H^v is the support of the projective cover of the simple corresponding to the vertex v and H_v is the support of the injective envelope of the simple corresponding to the vertex v of the algebra $k(\overline{Q})$. The τ-hammock starting at τv coincides with the τ-hammock ending at v, that is

$$H^{\tau v} = H_v.$$

A vertex v of a τ-slice Q' is called *τ-initial* provided that $Q_0' \cap H_0^v = H_0^v \setminus \{\tau^{-1} v\}$ for the vertices. A vertex v is called *τ-terminal* provided that $Q_0' \cap H_{v,0} = H_{v,0} \setminus \{\tau v\}$ for the vertices.

A τ-slice Q' in $(\overline{Q}, i_0)$ is called a *complete τ-slice* of Q if it has the following properties:

1. Each source of Q^S is τ-initial;
2. Each sink of Q^S is τ-terminal;
3. Assume that $v \to u$ is an arrow in $(\overline{Q}, i_0)$. If v is a vertex of Q^S, then either u or τu is a vertex of Q^S; and if u is a vertex of Q^S, then either v or $\tau^{-1} v$ is a vertex of Q^S.

We prove that the path complete τ-slice is the same as the complete τ-slice, when there is only finite many τ-orbits in Q. We first prove that a path complete τ-slice is a complete τ-slice.

Proposition 6.1. *Let Q be a stable bound quiver. If a full subquiver Q^S of $(\overline{Q}, i_0)$ is a path complete τ-slice, then it is a complete τ-slice.*

Proof. Assume that Q^S is a path complete τ-slice. Let v be a source of Q^S. For any vertex u, if there is an arrow $u \to v$ in $(\overline{Q}, i_0)$, then u is not a vertex of Q^S. By (a), there is some t such that $\tau^t u$ is a vertex of Q^S. If $t \geqslant 0$, there is a path $\tau^t u \to \cdots \to u \to v$ in $(\overline{Q}, i_0)$, so u is a vertex of Q^S, a contradiction. So $t < 0$, and we have $\tau^{-1} u$ is a vertex of Q^S by (b). All the vertices in H^v except for $\tau^{-1} v$ lie in the paths from v to $\tau^{-1} u$ for some u with an arrow $u \to v$ in $(\overline{Q}, i_0)$. It follows from

(b) again, that we have that all the vertices in H^v except for $\tau^{-1}v$ are in Q^S. So (1) holds for a path complete τ-slice.

Similar argument shows that (2) also holds for a path complete τ-slice.

Now assume that $v \to u$ is an arrow in $(\overline{Q}, i_0)$. Assume that v is a vertex of Q^S. There is an integer t such that $\tau^t u$ is a vertex of Q^S, by (a). Since there is always a path from $\tau^t u$ to $\tau^{t'} u$ whenever $t > t'$, so $t = 0$ or 1. This proves the first assertion of (3), the second assertion is proved similarly. $\square$

Assume that Q^S is a complete τ-slice in $(\overline{Q}, i_0)$. Let r be an integer. The full bound quiver $Q^S(r)$ with the vertex set

$$Q^S(r)_0 = \left\{(j, m-r) \,\middle|\, (j, m) \in Q^S_0\right\}$$

is a complete τ-slice in $(\overline{Q}, i'_0)$ for some vertex i'_0 and it is isomorphic to Q^S as quivers. We call $Q^S(r)$ *a shift of* Q^S. So we see that up to shift, a complete τ-slice is independent of the choice of the component of $\overline{Q}$. The number $d(Q^S) = \max\{n \mid (i,n) \in Q^S\} - \min\{n \mid (i,n) \in Q^S\}$ is called *the depth* of Q^S. If Q is finite, then Q^S is finite and contains no oriented cycle, we may shift it in $\overline{Q}$ such that $\min\{n \mid (i,n) \in Q^S\} = 0$.

We have that $d(Q^S) \geqslant l-1$, and when $d(Q^S) = l-1$, we call Q^S an *initial τ-slice*. Clearly, we have the following proposition by shifting.

Proposition 6.2. *Q^S is an initial complete τ-slice if and only if $Q^S \simeq (Q^N, i_0)$ for some vertex i_0.*

Let Q^S be a complete τ-slices in $(\overline{Q}, i_0)$ and let (i, m) be a sink of Q^S. Define the *τ-mutation* $s_i^-(Q^S)$ of Q^S at i as the full bound subquiver in $(\overline{Q}, i_0)$ obtained by replacing the vertex (i, m) by its Nakayama translation $(\tau i, m-l)$. Dually, for a source (j, m) of S, we define the *τ-mutation* $s_j^+(Q^S)$ of Q^S at j as the full bound subquiver of $(\overline{Q}, i_0)$ obtained by replacing the vertex (j, m) by its inverse Nakayama translation $(\tau^{-1} j, m+l)$.

Example 6.3. The following are some τ-hammocks of the directed stable bound quiver $\overline{Q}$ in Example 4.8.

$H^{(1,0)} = H_{(1,3)}$:

$$\begin{tikzcd}
(1,0) \arrow[d, "{(\alpha_1,0)}"'] \arrow[dr, "{(\beta_1,0)}"] & \\
(1,1) \arrow[d, "{(\alpha_1,1)}"'] \arrow[dr, "{(\beta_1,1)}"] & (2,1) \arrow[dl, "{(\gamma_1,1)}"'] \arrow[d, "{(\alpha_2,1)}"] \\
(1,2) \arrow[d, "{(\alpha_1,2)}"'] & (2,2) \arrow[dl, "{(\gamma_1,2)}"'] \\
(1,3) &
\end{tikzcd}$$

$H^{(2,0)} = H_{(2,3)}$:

$$\begin{tikzcd}
 & (2,0) \arrow[dl, "{(\gamma_1,0)}"'] \arrow[d, "{(\alpha_2,0)}"] \arrow[dr, "{(\beta_2,0)}"] & \\
(1,1) \arrow[d, "{(\alpha_1,1)}"'] \arrow[dr, "{(\beta_1,1)}"] & (2,1) \arrow[dl, "{(\gamma_1,1)}"'] \arrow[d, "{(\alpha_2,1)}"] \arrow[dr, "{(\beta_2,1)}"] & (3,1) \arrow[dl, "{(\gamma_2,1)}"'] \arrow[d, "{(\alpha_3,1)}"] \\
(1,2) \arrow[dr, "{(\beta_1,2)}"] & (2,2) \arrow[d, "{(\alpha_2,2)}"] & (3,2) \arrow[dl, "{(\gamma_2,2)}"'] \\
 & (2,3) &
\end{tikzcd}$$

$H^{(3,0)} = H_{(3,3)}$:

$$\begin{tikzcd}
 & (3,0) \arrow[dl, "{(\gamma_2,0)}"'] \arrow[d, "{(\alpha_3,0)}"] \\
(2,1) \arrow[d, "{(\alpha_2,1)}"'] \arrow[dr, "{(\beta_2,1)}"] & (3,1) \arrow[dl, "{(\gamma_2,1)}"'] \arrow[d, "{(\alpha_3,1)}"] \\
(2,2) \arrow[dr, "{(\beta_2,2)}"] & (3,2) \arrow[d, "{(\alpha_3,2)}"] \\
 & (3,3)
\end{tikzcd}$$

We remark that the arrow $(\alpha_t, 1)$ annihilates the elements in the indecomposable projective module which the τ-hammock $H^{t,0}$ supports. When such arrows are remove, the second hammock becomes a cube, and the first and the third became halves of cubes, with the Nakayama translation sends the vertices $(1,3)$, $(2,3)$ and $(3,3)$ to the other vertices of the diagonals of the cubes from them. Some τ-mutations of Q^N are as follows

$s_1^- Q^N$:

$$\begin{tikzcd}
(1,-1) \arrow[d, "{(\alpha_1,-1)}"'] \arrow[dr, "{(\beta_1,-1)}"] & & \\
(1,0) \arrow[d, "{(\alpha_1,0)}"'] \arrow[dr, "{(\beta_1,0)}"] & (2,0) \arrow[dl, "{(\gamma_1,0)}"'] \arrow[d, "{(\alpha_2,0)}"'] \arrow[dr, "{(\beta_2,0)}"] & (3,0) \arrow[dl, "{(\gamma_2,0)}"'] \arrow[d, "{(\alpha_3,0)}"] \\
(1,1) \arrow[dr, "{(\beta_1,1)}"] & (2,1) \arrow[d, "{(\alpha_2,1)}"'] \arrow[dr, "{(\beta_2,1)}"] & (3,1) \arrow[dl, "{(\gamma_2,1)}"'] \arrow[d, "{(\alpha_3,1)}"] \\
& (2,2) & (3,2)
\end{tikzcd}$$

$s_3^+ s_1^- Q^N$:

$$\begin{tikzcd}
(1,-1) \arrow[d, "{(\alpha_1,-1)}"'] \arrow[dr, "{(\beta_1,-1)}"] & & \\
(1,0) \arrow[d, "{(\alpha_1,0)}"'] \arrow[dr, "{(\beta_1,0)}"] & (2,0) \arrow[dl, "{(\gamma_1,0)}"'] \arrow[d, "{(\alpha_2,0)}"'] \arrow[dr, "{(\beta_2,0)}"] & \\
(1,1) \arrow[dr, "{(\beta_1,1)}"] & (2,1) \arrow[d, "{(\alpha_2,1)}"'] \arrow[dr, "{(\beta_2,1)}"] & (3,1) \arrow[dl, "{(\gamma_2,1)}"'] \arrow[d, "{(\alpha_3,1)}"] \\
& (2,2) \arrow[dr, "{(\beta_2,2)}"] & (3,2) \arrow[d, "{(\alpha_3,2)}"] \\
& & (3,3)
\end{tikzcd}$$

Clearly, we have the following lemma.

Lemma 6.4. *A τ-mutation of a complete τ-slice in $(\overline{Q}, i_0)$ is again a complete τ-slice in $(\overline{Q}, i_0)$.*
If (i,m) is a sink of S, then $s_i^+ s_i^- Q^S = Q^S$, and if (i,m) is a source of Q^S, then $s_i^- s_i^+ Q^S = Q^S$.

The following lemma follows easily from induction on the depth of the complete τ-slices and on the number of pairs of vertices which reach the maximal depth.

Lemma 6.5. *Let Q be a stable bound quiver with finitely many τ-orbits, and let Q^S be a complete τ-slice of Q. Then there is a sequence $\sigma_1, \dots, \sigma_r$ of τ-mutations such that $\sigma_r \cdots \sigma_1 Q^S$ is an initial complete τ-slice (Q^N, i_0).*

Now we prove that complete τ-slice coincides with path complete τ-slice.

Theorem 6.6. *Let Q be a stable bound quiver with only finite many τ-orbits. Then τ-slice is complete τ-slice if and only if it is path complete τ-slice.*

Proof. By Proposition 6.1, a path complete τ-slice is a complete τ-slice. We need only to prove that a complete τ-slice is a path complete one.

It is obvious that a complete τ-slice is path complete if and only if its τ-mutation is so.

Let Q^S be a complete τ-slice. Assume that $p: v_1 \to \cdots \to v_r$ is a path in $(\overline{Q}, i_0)$ with v_1, v_r in Q^S. If not all the vertices of p are in Q^S, we may assume that there are t_1, t_2 with $1 < t_1 \leqslant t_2 < r$, such that v_{t_1-1} and v_{t_2+1} are in Q^S, but v_t is not in Q^S for $t_1 \leqslant t \leqslant t_2$.

We prove by induction on $t_2 - t_1$ that this will lead to a contradiction. If $t_1 = t_2 = t$, then by the property (3) of complete τ-slice, both τv_t and $\tau^{-1} v_t$ are in Q^S, this leads to a contradiction to property (a) of a τ-slice. Assume that $t_2 - t_1 > 0$. Since $(\overline{Q}, i_0)$ is directed and it has only finitely many τ orbits, we may assume that v_1 is a source of Q^S, by taking τ-mutations if necessary. By property (3) of a complete τ-slice, τv_{t_1} is a vertex of Q^S, and by taking τ-mutations if necessary, we may assume that τv_{t_1} is initial, and take τv_{t_1} as v_1. Let $v_1 = (i,m)$, the τ-mutation $\sigma_i^+ Q^S$ of Q^S is also a complete τ-slice. $p': v_2 \to \cdots \to v_r$ is a path in $(\overline{Q}, i_0)$ with $v_2, \dots, v_{t_1}, v_{t_2+1}$ in $\sigma_i^+ Q^S$, but v_t are not vertices of $\sigma_i^+ Q^S$ for $t_1 + 1 \leqslant t \leqslant t_2$. Since $t_2 - (t_1 + 1) < t_2 - t_1$, it follows from induction that this leads to a contradiction.

This proves that for any path $p: v_1 \to \cdots \to v_r$ in $(\overline{Q}, i_0)$ with v_1, v_r in Q^S, all the vertices of p are in Q^S. Thus Q^S is a path complete τ-slice. $\square$

A complete τ-slice in $(\overline{Q}, i'_0)$ is a bound quiver with the relations induced from the relations of $(\overline{Q}, i'_0)$. The algebra defined by a complete τ-slice is called a *τ-slice algebra* of Q. If Q is the bound quiver of a finite-dimensional graded self-injective algebra Λ. A τ-slice algebra of a complete τ-slice in $\overline{Q}$ is also called a *τ-slice algebra of Λ*.

Since a separated directed quiver contains no oriented cycle, so does its subquiver. The following proposition follows from the path completeness of a complete τ-slice.

Proposition 6.7. *A τ-slice algebra of Λ is subalgebra of $\Lambda \# k\mathbb{Z}^*$ of finite global dimension.*

Let Q^S be a complete τ-slice and let σ be a τ-mutation defined on Q^S. If Λ' and Λ'' are τ-slice algebras defined by Q^S and σQ^S, respectively, Λ'' is called a *τ-mutation of Λ'*, and write it as

$$\Lambda'' = \sigma \Lambda'.$$

The following theorem asserts that the trivial extensions of τ-slice algebras are invariant under the τ-mutations.

Theorem 6.8. *The trivial extensions of all the τ-slice algebras of a finite stable bound quiver are isomorphic.*

Proof. Let Q^S be a complete τ-slice of a stable bound quiver Q. Then by Lemma 6.5, there is a sequence $\sigma_1, \ldots, \sigma_r$ of τ-mutations such that $Q^S = \sigma_r \sigma_{r-1} \cdots \sigma_1(Q^N, v)$ for an initial complete τ-slice $Q' = (Q^N, v)$.

Let $Q^t = \sigma_t \cdots \sigma_1(Q^N, v)$ and let $\Lambda^t = k(Q^t)$ be the algebra given by the bound quiver Q^t.

Let $\Lambda' = k(Q')$, and let $\Lambda^E = \iota(\Lambda')$ be its trivial extension. Let Q^E be the bound quiver of Λ^E. We use the same notation for the vertex of Q' and Q^E. Take the second indices for vertices of Q^t from $\mathbb{Z}$, and second indices for the vertices of Q^E from $\mathbb{Z}/l\mathbb{Z}$.

Our theorem follows immediately from the following lemma.

Lemma 6.9. *For $t = 0, 1, \ldots, r$, Q^t is a full bound subquiver of Q^E, Q^E is obtained from Q^t by adding an arrow (i, n) to $(j, n+1-l)$ for each member in a maximal set of linearly independent bound paths of length $l-1$ from $(j, n+1-l)$ to (i, n), and*

$$\iota(\Lambda^t) = \iota\big(k\big(Q^t\big)\big) \simeq k\big(Q^E\big) = \Lambda^E.$$

Proof. We prove this lemma by induction on t. Clearly, Q^E is obtained from $Q^0 = Q'$ by adding an arrow from vertices $(i, l-1)$ to $(j, 0)$ for each member in a maximal set of linearly independent paths of maximal length from $(j, 0)$ to $(i, l-1)$. So the lemma holds when $t = 0$.

Assume that $0 < t \leqslant r$ and the lemma holds for $t-1$.

We may assume that $\sigma_t = s^-_{i_0}$, the other case is proved similarly. There is a vertex (i_0, m) such that Q^t is obtained from Q^{t-1} by removing the vertex (i_0, m) with all the arrows ending at (i_0, m) and adding the vertex $(\tau i_0, m-l)$ together with all the arrows starting from $(\tau i_0, m-l)$ in $\overline{Q}$.

Both Q^{t-1} and Q^t have the same set of vertices as Q^E (after identifying i with τi and taking the image of their second indices in $\mathbb{Z}/l\mathbb{Z}$). So we may identify the degree zero part

$$\Lambda^t_0 = \Lambda^{t-1}_0 = \Lambda^E_0.$$

The arrows of the two subquivers are the same except those concerning the vertices $(\tau i_0, m-l)$ and (i_0, m). $(\tau i_0, m-l)$ is a source in Q^t and (i_0, m) is a sink in Q^{t-1}. Let $Q^{t-1,'}$ and $Q^{t,'}$ be the full bound quiver obtained from Q^{t-1}, and respectively Q^t, by removing the vertex (i_0, m), and respectively $(\tau i_0, m-l)$. Then $Q^{t-1,'} = Q^{t,'}$, denote it by Q^*.

By induction, $\Lambda^E = k(Q^E) \simeq \iota(\Lambda^{t-1})$, and Q^E is obtained from Q^{t-1} by adding an arrow from (i,n) to $(j,n-l+1)$ for each member in a maximal set of linearly independent bound paths of length $l-1$ from $(j,n-l+1)$ to (n,i).

Let $\Lambda^* = k(Q^*)$ be the algebra defined by the bound quiver Q^*, it is a subalgebra of both Λ^{t-1} and Λ^t. Denote by $\widetilde{\Lambda} = k(\overline{Q}, \nu)$ the algebra of the bound quiver $(\overline{Q}, \nu)$. It is a direct summand of $\Lambda \# k\mathbb{Z}^*$, so it is locally finite-dimensional self-injective algebra of Loewy length $l+1$. Λ^*, Λ^{t-1} and Λ^t are embedded in $\widetilde{\Lambda}$ as subalgebras.

For the arrows ending at (i_0,m) in $\iota(\Lambda^t)$, by Lemma 2.4, we have that for any $(j,m-1)$

$$\begin{aligned} e_{(\tau i_0,m-l)}\iota\big(\Lambda^t\big)_1 e_{(j,m-1)} &\simeq De_{(j,m-1)}\Lambda^t_{l-1}e_{(\tau i_0,m-l)} \simeq e_{(i_0,m)}\widetilde{\Lambda}_1 e_{(j,m-1)} = e_{(i_0,m)}\Lambda^{t-1}_1 e_{(j,m-1)} \\ &= e_{(i_0,m)}\iota\big(\Lambda^{t-1}\big)_1 e_{(j,m-1)} \simeq e_{(i_0,m)}\Lambda^E_1 e_{(j,m-1)}, \end{aligned}$$

as Λ^t_0-bimodules. Note that $\Lambda^t_0 = \Lambda^E_0$. By identifying the vertex (i_0,m) of Q^{t-1} with the vertex $(\tau i_0, m-l)$ of Q^t, this implies that the number of the arrows from $(j,m-1)$ to (i_0,m) in the quiver of $\iota(\Lambda^t)$ is the same as that of Q^E, and it equals the number of linearly independent bound paths of length $l-1$ of Q^t from $(\tau i_0, m-l)$ to $(j,m-1)$. Since $e_{(i,n)}\Lambda^t_{l-1}e_{(j,n+1-l)} = e_{(i,n)}\Lambda^{t-1}_{l-1}e_{(j,n+1-l)}$ for other pair $(j,n+1-l),(i,n)$ of vertices. It follows from induction that Q^E is obtained from Q^t by adding an arrow (i,n) to $(j,n+1-l)$ for each member in a maximal set of linearly independent bound paths of length $l-1$ from $(j,n+1-l)$ to (i,n).

Denote by M the sub-Λ^*-bimodule of $\iota(\Lambda^t)$ generated by $De_{(j,m-1)}\Lambda^t_{l-1}e_{(\tau i_0,m-l)}$, it is isomorphic to $\bigoplus_{(j,n)\in Q^*_0} e_{(i_0,m)}\widetilde{\Lambda}e_{(j,n)} \simeq e_{(i_0,m)}\Lambda^{t-1}$. Thus Λ^{t-1} is isomorphic to a trivial extension

$$\Lambda^{t-1} \simeq \big(\Lambda^* + e_{(i_0,m)}\iota\big(\Lambda^t\big)_0 e_{(i_0,m)}\big) \ltimes M,$$

and the right-hand side is a subalgebra of $\iota(\Lambda^t)$.

Since Λ^{t-1} and Λ^t are graded algebras whose maximal bound paths having the same length $l-1$. $\Lambda^t_{l-1} = \Lambda^*_{l-1} + e_{(i_0,l)}\Lambda^{t-1}_{l-1}$ and $\Lambda^{t-1}_{l-1} = \Lambda^*_{l-1} + \Lambda^{t-1}_{l-1}e_{(\tau i_0,0)}$, $D\Lambda^{t-1}$ is generated by $D\Lambda^{t-1}_{l-1}$ as Λ^{t-1}-bimodule, and $D\Lambda^t$ is generated by $D\Lambda^t_{l-1}$ as Λ^t-bimodule.

$$\Lambda^E \simeq \iota\big(\Lambda^{t-1}\big) = \Lambda^{t-1} + D\Lambda^{t-1}$$

as Λ^{t-1}-bimodule with multiplication defined naturally (as the trivial extension). So one gets

$$\iota\big(\Lambda^t\big) = \Lambda^* + \Lambda^t e_{(\tau i_0,m-l+1)} + D\Lambda^* + D\Lambda^t e_{(\tau i_0,m-l+1)}$$

as Λ^*-bimodule. Note that $\Lambda^* + D\Lambda^t e_{(\tau i_0,m-l+1)}$ is a subalgebra and

$$\begin{aligned} \Lambda^* + D\Lambda^t e_{(\tau i_0,m-l+1)} &\simeq \Lambda^* + e_{(\tau i_0,m-l+1)} D\Lambda^t_l e_{(\tau i_0,m-l+1)} + M \\ &\simeq \big(\Lambda^* + e_{(i_0,m)}\iota\big(\Lambda^t\big)_0 e_{(i_0,m)}\big) \ltimes M \\ &\simeq \Lambda^{t-1}, \end{aligned}$$

and $\Lambda^t e_{(\tau i_0,m-l+1)} + D\Lambda^*$ is a $(\Lambda^* + e_{(i_0,m)}\iota(\Lambda^t)_0 e_{(i_0,m)}) \ltimes M$-bimodule. Identify the above isomorphic algebras, we get a Λ^{t-1}-bimodule isomorphism

$$\Lambda^t e_{(\tau i_0,m-l+1)} + D\Lambda^* \simeq D\Lambda^{t-1}.$$

The multiplication of $\iota(\Lambda^t)$ defines a trivial extension of $(\Lambda^* + e_{(i_0,m)}\iota(\Lambda^t)_l e_{(i_0,m)}) \ltimes M$ by its bimodule $\Lambda^t e_{(\tau i_0,m-l+1)} + D\Lambda^*$. Thus

$$\begin{aligned}\iota(\Lambda^t) &= \big(\big(\Lambda^* + e_{(i_0,m)}\iota(\Lambda^t)_0 e_{(i_0,m)}\big) \ltimes M\big) \ltimes \big(\Lambda^t e_{(\tau i_0, m-l+1)} + D\Lambda^*\big) \\ &\simeq \Lambda^{t-1} \ltimes D\Lambda^{t-1} = \iota(\Lambda^{t-1}) = \Lambda^E.\end{aligned}$$

This shows that our lemma holds by induction. □

Denote by $\Lambda(i)$ the specially truncated algebra defined by the bound quiver (Q^N, i). As a corollary, we have the following theorem.

Theorem 6.10. *$\iota(\Lambda(i))$ are isomorphic for all $i \in Q_0$.*

According to [8,5], we have an equivalence between the bounded derived categories $\mathcal{D}^b(\Lambda')$ of the τ-slice algebra Λ' and the stable category $\underline{\mathrm{gr}}\,\iota(\Lambda')$ of the finite-dimensional graded modules over its trivial extension.

Let Λ be a graded self-injective algebra. As a corollary of Theorem 6.8 and Corollary 1.2 of [5], we get:

Corollary 6.11. *Let Λ be a finite-dimensional graded self-injective algebra. Then all the τ-slice algebras of Λ have equivalent bounded derived categories, and they are all equivalent to the stable category of graded Λ-modules.*

Let Q^S be a complete τ-slice in $(\overline{Q}, i)$ and let $I^S[r] = \{(\tau^r i, n - lr) \mid (i,n) \in Q_0^S\}$ for $r \in \mathbb{Z}$. Let $e_{I^S[s]} = \sum_{(i,n)\in I[s]} e_{(i,n)}$, and let

$$M^S[s,t] = e_{I^S[s]}\Lambda \# k\mathbb{Z}^* e_{I^S[t]} = \sum_{(i,n)\in I^S[s]} \sum_{(j,m)\in I^S[t]} e_{(i,n)}\Lambda \# k\mathbb{Z}^* e_{(j,m)}$$

for all $s, t \in \mathbb{Z}$. Let $\Gamma^S[s] = M^S[s,s]$. Then $\Gamma^S[s]$ are isomorphic to the τ-slice algebra $\Lambda^S = k(Q^S)$. $M^S[s+1,s]$ are $\Gamma^S[s+1]$–$\Gamma^S[s]$-bimodules and $M^S[t,s] = 0$ for $t \neq s, s+1$, and $M[s,t]M[s,t] = 0$ when $s \neq t$.

Similar to the proofs of Theorem 5.12 and Proposition 5.13, we see that

$$\begin{pmatrix} \ddots & & & & \mathbf{0} \\ & \Gamma^S[s+1] & M^S[s+1,s] & & \\ & & \Gamma^S[s] & M^S[s,s-1] & \\ & & & \Gamma^S[s-1] & \\ \mathbf{0} & & & & \ddots \end{pmatrix}$$

is the direct summand of $\Lambda \# k\mathbb{Z}^*$ corresponding to the component $(\overline{Q}, i)$. Clearly it is isomorphic to the repetitive algebra $\widehat{\Lambda^S}$. So we have the following theorem.

Theorem 6.12. *The repetitive algebra of a τ-slice algebra is a direct summand of $\Lambda \# k\mathbb{Z}^*$.*
All the τ-slice algebras have isomorphic repetitive algebras.

Recall that a subcategory $\mathcal{T}$ of a triangulated category $\mathcal{C}$ is called a *tilting subcategory* if it generates $\mathcal{C}$ and we have $\mathrm{Hom}(\mathcal{T}, \mathcal{T}[i]) = 0$ for all $i \neq 0$. According to [19], two algebras Λ' and Λ'' of finite global dimension are derived equivalent if and only if there is a tilting subcategory $\mathcal{T} = \mathrm{add}\, T$ of $\mathcal{D}\Lambda'$ for some objects T and $\Lambda'' = \mathrm{End}\, T$. Now let $\sigma_1, \dots, \sigma_r$ be a sequence of τ-mutations. If Λ' is a τ-slice algebra, then $\sigma_r \cdots \sigma_1 \Lambda'$ is also a τ-slice algebra and we have that $\iota(\Lambda') \simeq \iota(\sigma_r \cdots \sigma_1 \Lambda')$. Thus $\mathcal{D}^b(\Lambda')$ and $\mathcal{D}^b(\sigma_r \cdots \sigma_1 \Lambda')$ are equivalent as triangulated category. Hence by [19] there is a tilting object T in $\underline{\mathrm{gr}}\,\iota(\Lambda')$ such that $\sigma_r \cdots \sigma_1 \Lambda' \simeq \mathrm{End}\, T$. So we get:

Corollary 6.13. *Let Λ' and Λ'' be τ-slice algebras of a graded self-injective algebra Λ, then there is a tilting object T in $\underline{\mathrm{gr}}\,\Lambda$, such that $\Lambda' \simeq \operatorname{End} T$.*

7. Koszul duality and τ-mutations

It is natural to investigate the relationship between our τ-mutation and the BGP reflection. They are not related directly, as is shown in the following example. Consider the case $l = 2$. Let Λ be a self-injective algebra with vanishing radical cube whose quiver Q is the double quiver of A_5. It follows from [7] that this quiver is a stable bound quiver of Loewy length 3 with trivial Nakayama translation. Its separated directed quiver $\overline{Q}$ is two copies of $\mathbb{Z}A_5$, with the total special truncation $Q^N = (Q^N, 1) \cup (Q^N, 2)$, where

$$\begin{aligned} \big(Q^N, 1\big) &: (1,0) \to (2,1) \leftarrow (3,0) \to (4,1) \leftarrow (5,0), \\ \big(Q^N, 2\big) &: (1,1) \leftarrow (2,0) \to (3,1) \leftarrow (4,0) \to (5,1). \end{aligned}$$

Let Λ^S be the τ-slice algebra of $Q^S = (Q^N, 1)$. Then its τ-mutation $s^-_{(2,1)}\Lambda^S$ is given by the bound quiver

$$s^-_{(2,1)} Q^S : (1,0) \leftarrow (2,-1) \to (3,0) \to (4,1) \leftarrow (5,0)$$

with a single relation $(2,-1) \to (3,0) \to (4,1)$. If we do BGP reflection for the quiver $Q^S = (Q^N, 1)$ at vertex $(2,1)$, one gets the same quiver without relation.

Note that the above algebras Λ^S and $s^-_{(2,1)}\Lambda^S$ are both Koszul, and we can look at their Koszul duals, which are just the path algebras of Q^S and $s^-_{(2,1)}Q^S$, respectively. So we see that our τ-mutation coincides with the BGP reflection when applied to the Koszul duals of the τ-slice algebras. This is true in general.

Assume that Λ is a self-injective algebra with vanishing radical cube whose bound quiver is Q. In this case $l = 2$. Let Q^S be a complete τ-slice, and let Λ^S be the τ-slice algebra with bound quiver Q^S. Λ^S is an algebra with vanishing radical square, so are its τ-mutations. Q^S is a directed quiver, so its orientation is admissible. Since $\Lambda \# k\mathbb{Z}^*$ is a self-injective algebra with vanishing radical cube, its bound quiver $\overline{Q}$ is a stable translation quiver [7] with $\bar{\tau}$ as the translation. Now let (i, m) be a sink in Q^S, then $\bar{\tau}(i, m) = (\tau i, m - 2)$ is not a vertex of Q^S. We have that $H_{(i,m)}$ forms a mesh in $\overline{Q}$,

$$\begin{tikzcd} & (j_1, m-1) \arrow[dr, "\beta_1"] & \\ (\tau i, m-2) \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_r"'] & \vdots & (i, m) \\ & (j_r, m-1) \arrow[ur, "\beta_r"'] & \end{tikzcd}$$

with $\beta_1, \dots, \beta_r$ arrows in Q^S and $\alpha_1, \dots, \alpha_r$ not in Q^S. The τ-mutation $s^-_{(i,m)}Q^S$ is obtained from Q^S by replacing (i, m) with $(\tau i, m - 2)$ and each β_t with α_t. The BGP reflection acts on a quiver Q at its sink by just reverse all arrows to this vertex. When we identify the vertex (i, m) with $(\tau i, m - 2)$, this is exactly the BGP reflection of quiver Q^S at the sink (i, m). Same argument also works when (i, m) is a source.

Since algebra with radical squared zero is Koszul whose Yoneda algebra is hereditary when its quiver does not contain oriented cycle. Thus their Yoneda algebras $E(\Lambda^S)$ and $E(s^-_{(i,m)}\Lambda^S)$ are hereditary algebras, with the quiver Q^S and $s^-_{(i,m)}(Q^S)$, respectively, without relation.

Use the same notations $s^-_{(i,m)}$, and respectively $s^+_{(i',m')}$ for the BGP reflections on a path algebra at a sink (i,m), and respectively at a source (i',m') of the quiver. We gave the following observation.

Proposition 7.1. *Assume that Λ is a self-injective algebra with vanishing radical cube and Λ^S be a τ-slice algebra of Λ. Then the Yoneda algebra of a τ-mutation of Λ^S at a vertex (i,m) is the BGP reflection of the Yoneda algebra of Λ^S at the same vertex. That is*

$$E\big(s^{\pm}(i,m)\Lambda^S\big) = s^{\pm}(i,m)E\big(\Lambda^S\big).$$

We remark that in the case of Dynkin quivers, the graded self-injective algebras we starting with are not Koszul, but all the τ-slice algebras are Koszul since they have vanishing radical square.

Let Q be the bound quiver of a graded self-injective algebra Λ and $\overline{Q}$ be its separated directed quiver. Now consider the orbit algebra $\Lambda^{T,h} = O((\mathcal{N}^h), P)$ where h is a positive integer, $(\mathcal{N}^h)$ is the group generated by $\mathcal{N}^h$, and P is a basic $(\mathcal{N}^h)$-orbit generator of $\mathcal{P}(\Lambda \# k\mathbb{Z}^*)$. It follows from Proposition 5.2 that $\Lambda^{T,h}$ is a finite regular covering of Λ^T. It is proved in [22] that a finite regular covering of a Koszul self-injective algebra is also Koszul. So we have the following proposition.

Proposition 7.2. *If Λ^T is a Koszul algebra, so is $\Lambda^{T,h}$.*

Assume that Λ^T is Koszul, we are going to prove that all the τ-slice algebras are also Koszul. We need some preparation.

Let $h > 0$ be an integer. A subset U of the vertex set Q_0 of a stable bound quiver Q is called *h-convex* provided that for any $(i,m),(j,n) \in U$, if there is a bound path of length $\leqslant h$ from (i,m) to (j,n) in Q such that all its vertices are in U, then for any bound path of length $\leqslant h$ from (i,m) to (j,n), its vertices are all in U. A full subquiver Q' of Q is called *h-convex* if its vertex set is h-convex in Q. Clearly if Q is h-convex, then it is h'-convex for any $h' \leqslant h$.

Lemma 7.3. *Let Q' be a bound subquiver of a bound quiver Q, and suppose that the length of each path appearing in the relation of Q is less or equal to h. If Q' is h-convex, then $k(Q')$ is a subalgebra of $k(Q)$.*

Proof. Clearly, we have an embedding of path algebras $i : kQ' \to kQ$. Since Q' is h-convex, and the length of each path appearing in the relation of Q is less or equal to h, a relation of Q is either a relation of Q' or its terms contain no paths in Q'. Thus i induces an embedding from $k(Q')$ into $k(Q)$, and our assertion holds. $\square$

The following proposition follows from the path completeness of a complete τ-slice.

Proposition 7.4. *A complete τ-slice is h-convex in $\overline{Q}$ for any h.*

Theorem 7.5. *If Λ is a finite-dimensional self-injective algebra such that Λ^T is Koszul. Then each of it's τ-slice algebra is a Koszul algebra with finite global dimension.*

Proof. Let Λ^S be a τ-slice algebra with the bound quiver Q^S. We may assume that $\min\{n \mid (i,n) \in Q^S_0\} = 0$, by shifting suitably. Assume that the depth of Q^S is $d = \max\{n \mid (i,n) \in Q^S_0\}$. Take a sufficient large positive integer $r > 1$ with $3\max\{d,l\} + 1 < rl$. Embed Q^S into $Q^{T,r}$ as a full bound subquiver.

By Proposition 7.4, Q^S is h'-convex in $\overline{Q}$ for any h', so it is h'-convex in $Q^{T,r}$ for $h' \leqslant (r-1)\max\{d,l\} - 1$. By Proposition 7.2, $\Lambda^{T,r}$ is Koszul, so relations of $Q^{T,r}$ is quadratic. Let $e^S =$

$\sum_{(i,n)\in Q_0^S} e_{(i,n)}$, then by Lemma 7.3, $\Lambda^S = e^S\Lambda^{T,r}e^S \simeq \Lambda^{T,r}/\Lambda^{T,r}(1-e^S)\Lambda^{T,r}$ is both a subalgebra and a quotient algebra of $\Lambda^{T,r}$.

Let M be a finitely generated Λ^S-module which is Koszul as $\Lambda^{T,r}$-module, and let

$$\cdots \xrightarrow{f_{t+1}} P^{(t)} \xrightarrow{f_t} \cdots \xrightarrow{f_1} P^{(0)} \xrightarrow{f_0} M \to 0$$

be a minimal projective resolution of M as a $\Lambda^{T,r}$-module. Then $P^{(t)}$ is finitely generated in degree t. Apply the functor $\mathrm{Hom}_{\Lambda^{T,r}}(\Lambda^{T,r}e^S, \ )$, then we get an exact sequence

$$\begin{aligned}\cdots &\to \mathrm{Hom}_{\Lambda^{T,r}}\big(\Lambda^{T,r}e^S, P^{(t_0)}\big) \to \mathrm{Hom}_{\Lambda^{T,r}}\big(\Lambda^{T,r}e^S, P^{(t_0-1)}\big) \to \cdots\\ &\to \mathrm{Hom}_{\Lambda^{T,r}}\big(\Lambda^{T,r}e^S, P^{(0)}\big) \to M \to 0.\end{aligned}$$

Denote by P^m the direct sum of m-copies of P. Assume that

$$P^{(t)} = \bigoplus_{(i,n)\in Q_0^{T,r}} \big(\Lambda^{T,r}e_{(i,n)}\big)^{m_t(i,n)},$$

for $0 \leqslant m_t(i,n) \in \mathbb{Z}$. We have that $m_0(i,n) \neq 0$ implies (i,n) is in Q^S since M is a Λ^S module, hence $0 \leqslant n \leqslant d$. $m_t(i,n) \neq 0$ implies that $t \leqslant n \leqslant d+t$ since M is Koszul as a $\Lambda^{T,r}$-module, and there is a path of length t from (i,n) to some (i',n') with $m_0(i',n') \neq 0$. Take $t_0 = d+1$, then $m_{t_0}(i,n) = 0$ for $0 \leqslant n \leqslant d$ and $2d+1 < n < rl$. So $\mathrm{Hom}_{\Lambda^{T,r}}(\Lambda^{T,r}e^S, P^{(t_0)}) = 0$. Note that $\mathrm{Hom}_{\Lambda^{T,r}}(\Lambda^{T,r}e_{(i,n)}, \Lambda^{T,r}e_{(i',n')}) \neq 0$ implies that there is a path of length $\leqslant l$ from (i,n) to (i',n') in $Q^{T,r}$. If $\mathrm{Hom}_{\Lambda^{T,r}}(\Lambda^{T,r}e^S, \Lambda^{T,r}e_{(i,n)}) \neq 0$ and $m_t(i,n) \neq 0$ for $0 \leqslant t \leqslant t_0$, then $t < d < (r-2)l$. The $l+d$-convexity of Q^S implies that (i,n) is in Q^S. So for $0 \leqslant t \leqslant t_0$, $\mathrm{Hom}_{\Lambda^{T,r}}(\Lambda^{T,r}e^S, (\Lambda^{T,r}e_{(i,n)})^{m_t(i,n)}) = 0$ for $(i,n) \notin Q_0^S$, and we see that

$$\begin{aligned}\mathrm{Hom}_{\Lambda^{T,r}}\big(\Lambda^{T,r}e^S, P^{(t)}\big) &= \mathrm{Hom}_{\Lambda^{T,r}}\Bigg(\Lambda^{T,r}e^S, \bigoplus_{(i,n)\in Q_0^{T,r}} \big(\Lambda^{T,r}e_{(i,n)}\big)^{m_t(i,n)}\Bigg)\\ &= \bigoplus_{(i,n)\in Q_0^{T,r}} \mathrm{Hom}_{\Lambda^{T,r}}\big(\Lambda^{T,r}e^S, \big(\Lambda^{T,r}e_{(i,n)}\big)^{m_t(i,n)}\big)\\ &= \bigoplus_{(i,n)\in Q_0^{S}} \mathrm{Hom}_{\Lambda^{T,r}}\big(\Lambda^{T,r}e^S, \big(\Lambda^{T,r}e_{(i,n)}\big)^{m_t(i,n)}\big)\\ &\simeq \bigoplus_{(i,n)\in Q_0^{S}} \big(e^S\Lambda^{T,r}e_{(i,n)}\big)^{m_t(i,n)}.\end{aligned}$$

Thus we have an exact sequence:

$$\begin{aligned}0 &\to \bigoplus_{(i,n)\in Q_0^{S}} \big(e^S\Lambda^{T,r}e_{(i,n)}\big)^{m_{t_0-1}(i,n-1)} \to \cdots\\ &\to \bigoplus_{(i,n)\in Q_0^{S}} \big(e^S\Lambda^{T,r}e_{(i,n)}\big)^{m_0(i,n)} \to M \to 0.\end{aligned}$$

This is a projective resolution of M as a Λ^S-module, since $e^S\Lambda^{T,r}e^S \simeq \Lambda^S$. The projective module $\bigoplus_{(i,n)\in Q_0^S}(e^S\Lambda^{T,r}e_{(i,n)})^{m_t(i,n)}$ is generated at degree t. This shows that M is Koszul as a Λ^S-module. Especially Λ^S is a Koszul algebra when Λ^T is so. $\square$

We also have the following corollary.

Corollary 7.6. *Let Λ be a graded self-injective algebra Λ of Loewy length $l+1$ with Λ^T Koszul. Then specially truncated algebra of Λ are all Koszul algebras of global dimension $l-1$.*

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