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On the postulation of s^d fat points in $\mathbb{P}^d$

Laurent Evain

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Abstract

In connection with his counter-example to the fourteenth problem of Hilbert, Nagata formulated a conjecture concerning the postulation of r fat points of the same multiplicity in $\mathbb{P}^2$ and proved it when r is a square. Iarrobino formulated a similar conjecture in $\mathbb{P}^d$. We prove Iarrobino's conjecture when r is a dth power. As a corollary, we obtain new counter-examples modeled on those by Nagata.

1. Introduction

What is the dimension $l(d, \delta, \mu_1, \dots, \mu_r)$ of the sub-vector space of $k[X_0, \dots, X_d]$ containing the homogeneous polynomials of degree δ that vanish at general points $p_1, \dots, p_r \in \mathbb{P}^d$ with order $\mu_1, \dots, \mu_r$? This question remains open as soon as $d \geqslant 2$ and has numerous consequences (see [1,3,6,9,10] for instance).

The question was raised by Nagata in connection with his answer to the fourteenth problem of Hilbert [7]. He gave an example of a linear action on a finite-dimensional vector space such that the algebra of polynomial invariants is not finitely generated. The key point in the proof, which Nagata called the "fundamental lemma," is the equality $l(2, 4m, m_1 = m, \dots, m_{16} = m) = 0$.

E-mail address: laurent.evain@univ-angers.fr.

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When the dimension of the ambiant projective space is $d = 2$ and the number of points is $r \leqslant 9$, the dimension $l(d, \delta, \mu_1, \dots, \mu_r)$ is well known [8]. As for the remaining cases $r > 9$, Nagata formulated the following conjecture:

$$l(2, \delta, \underbrace{\mu, \dots, \mu}_{r \text{ times}}) = l\big(2, \delta, \mu^r\big) = 0 \quad \text{if } \delta \leqslant \sqrt{r}\mu,$$

and proved it when r is a square. This conjecture is of particular interest since it crystallizes the difficulties. Indeed, the expected dimension $l(2, \delta, \mu^r)$ is $\max(0, v(2, \delta, \mu^r))$ where

$$v\big(2, \delta, \mu^r\big) = \frac{(\delta + 2).(\delta + 1)}{2} - r.\frac{\mu.(\mu + 1)}{2}$$

is the so-called virtual dimension. With any known method, the hardest cases are the cases with r fixed, $\mu \gg r$ and the degree δ is such that the virtual dimension is zero. An immediate estimate shows that the critical δ for which the virtual dimension is zero is asymptotically equivalent to $\sqrt{r}\mu$. It follows that the hardest cases correspond to Nagata's conjecture. Nagata proved himself this conjecture when r is a square.

Leaving the two-dimensional case for the general case, there is still a conjecture for the dimension $l(d, \delta, \mu_1, \dots, \mu_r)$, due to Iarrobino [4] (see also [5]). Facing the critical cases too, he derived from his conjecture a generalization of Nagata's conjecture:

Conjecture 1. *Let (r, d) be a couple of integers with*

- $d \geqslant 2$,
- $r \geqslant \max(d + 5, 2^d)$,
- $(r, d) \notin \{(7, 2), (8, 2), (9, 3)\}$.

If $\delta < \sqrt[d]{r}\mu$ then $l(d, \delta, \mu^r) = 0$.

In the 2-dimensional case however, this is not exactly Nagata's conjecture. Indeed, Nagata's conjecture is very slightly stronger, since the condition on δ is $\delta \leqslant \sqrt{r}\mu$, not $\delta < \sqrt{r}\mu$, and this difference turned out to be very important in the applications (in Nagata's counter-example to the fourteenth problem of Hilbert, or in [1] for instance). Replacing carelessly the strict inequality by a large inequality is not possible since the cases $(r, d) = (8, 3)$ and $(r, d) = (9, 2)$ would obviously contradict the statement. Nevertheless, excluding these cases, one can formulate the conjecture as follows:

Conjecture 2. *Let (r, d) be a couple of integers with*

- $d \geqslant 2$,
- $r \geqslant \max(d + 5, 2^d)$,
- $(r, d) \notin \{(7, 2), (8, 2), (9, 2), (8, 3), (9, 3)\}$.

If $\delta \leqslant \sqrt[d]{r}\mu$ then $l(d, \delta, \mu^r) = 0$.

Let us call this conjecture the large critical conjecture in opposition to the conjecture by Iarrobino which we shall call the strict critical conjecture.

The goal of this paper is to prove that the large critical conjecture holds when the number of points is a power with exponent the dimension of the ambiant projective space:

Theorem 3. *Let k be an algebraically closed field of characteristic zero. Let $d \geqslant 2$ be an integer, r be an integer such that $r = s^d$ for some $s \geqslant 2$. Suppose moreover that $(r, d) \notin \{(4, 2), (9, 2), (8, 3)\}$. Then*:

$$l\big(d, \delta, \mu^r\big) = 0 \quad \text{if } \delta \leqslant s\mu.$$

As a corollary, we obtain new counter-examples to the fourteenth problem of Hilbert. Indeed, replacing the fundamental lemma of Nagata with our theorem, one can mimic step by step the construction of Nagata (with a few minor and easy changes) to exhibit a new example. In concrete terms, each couple (s, d) of the theorem gives a new fundamental lemma and a new counter-example. The example associated with the couple (s, d) is an action of the affine group $G_a^{s^d-d-1}$ on a vector space of dimension $2s^d$:

Theorem 4. *Let a_{ij} $(i = 0, \dots, d,\ j = 1, \dots, s^d)$ be the coordinates of s^d generic points of $\mathbb{P}^d$. Let V be the vector space of dimension s^d and $V^* \subset V$ be the set of vectors orthogonal to the $d + 1$ vectors $(a_{i1}, \dots, a_{is^d})$. Let G be the set of linear transformations σ of $\operatorname{Spec} k[x_1, \dots, x_{s^d}, t_1, \dots, t_{s^d}]$ such that*

- $\sigma(t_i) = t_i$,
- $\sigma(x_i) = x_i + b_i t_i$

for some $(b_1, \dots, b_{s^d}) \in V^*$. Then the algebra of elements of $k[x_1, \dots, x_{s^d}, t_1, \dots, t_{s^d}]$ invariant under G is not finitely generated.

As mentioned, the proof of Theorem 4 is a straightforward generalization of Nagata's proof [9] and we refer to this paper for it.

Our method to prove Theorem 3 is an induction on the dimension of the ambiant projective space. The formulation of the theorem does not suggest such an induction; however, using the notion of collision of fat points, we transform the statement of the theorem into a combinatorial statement and we perform the induction on the combinatorial statement (see Remark 19).

Remark 5. It seems that Theorem 3 leaves the cases $(r, d) = (4, 2)$, $(r, d) = (9, 2)$ and $(r, d) = (8, 3)$ untreated. However, these cases are completely understood. Indeed, by [8] for $(r, d) = (4, 2)$ and $(9, 2)$, and by Proposition 20 for $(r, d) = (8, 3)$, we have $l(d, \delta, \mu^r) = \max\big(0, \binom{\delta+d}{d} - r.\binom{d+\mu-1}{d}\big)$.

If the characteristic of the base field is arbitrary, we can forget the parts of the proof which use the hypothesis on the characteristic and we still have the strict critical conjecture:

Theorem 6. *Let $d \geqslant 2$ be an integer and let r be a dth-power. If $\delta < \sqrt[d]{r}\mu$ then $l(d,\delta,\mu^r) = 0$.*

2. Stratifications on the Hilbert scheme

In this section, we explain the strategy of the proof: we define locally closed subschemes $C(E_1,\dots,E_i)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^d)$ and we reduce the proof to an incidence between these subschemes.

Monomial subschemes

A staircase E in $\mathbb{N}^d$ is a subset whose complementary $\mathbb{N}^d - E$ verifies

$$\big(\mathbb{N}^d - E\big) + \mathbb{N}^d \subset \mathbb{N}^d - E.$$

A staircase E being fixed, let $I^E \subset k[[x_1,\dots,x_d]]$ (respectively $I^E \subset k[x_1,\dots,x_d]$) be the ideal whose elements are the series (respectively the polynomials)

$$\sum c_{\alpha_1\alpha_2\dots\alpha_d} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d} = \sum c_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$$

verifying $c_{\underline{\alpha}} = 0$ if $\underline{\alpha} \in E$. A zero-dimensional subscheme Z of $\mathbb{P}^d$ supported by a point q is said to be monomial with staircase E if it is defined by the ideal I^E in a suitable formal neighborhood $\mathrm{Spec}\, k[[x_1,\dots,x_d]] \hookrightarrow \mathbb{P}^d$ of q.

A fat point of multiplicity m is by definition a monomial subscheme defined by the regular staircase R_m:

$$R_m := \big\{(\alpha_1,\dots,\alpha_d) \text{ s.t. } \alpha_1 + \dots + \alpha_d < m\big\}.$$

Subschemes of $\mathrm{Hilb}(\mathbb{P}^d)$

If $E_1,\dots,E_i$ are finite staircases in $\mathbb{N}^d$, we denote by $C(E_1,\dots,E_i)$ the reduced subscheme of $\mathrm{Hilb}\,\mathbb{P}^d$ whose points parametrize the subschemes Z of $\mathbb{P}^d$ which are the disjoint union of i distinct monomial subschemes with staircases $E_1,\dots,E_i$. In symbols $Z = \coprod Z_j$, where Z_j is monomial with staircase E_j. It is known by [2] that $C(E_1,\dots,E_i) \subset \mathrm{Hilb}\,\mathbb{P}^d$ is a locally closed irreducible subscheme. In particular it has a generic point G, which parametrizes a subscheme Z_G whose ideal is denoted by I_{Z_G}. We denote by $l(d,\delta,E_1,\dots,E_i) = h^0(I_{Z_G}(\delta))$ the number of independent hypersurfaces of degree δ in $\mathbb{P}^d$ containing Z_G.

Iarrobino's conjecture and incidence between strata

The theorem we want to prove can obviously be reformulated as:

Theorem 7. *Let* $r = s^d$ *and* $\delta \leqslant s\mu$. *Then* $l(d, \delta, \overbrace{R_\mu, \dots, R_\mu}^{r\ times}) = 0$ *if* $(s, d) \notin \{(1, d), (2, d), (3, 2)\}$ *and if the characteristic of the base field is zero.*

The following proposition reduces the proof of the theorem to the computation of the closure of $C(R_\mu, \dots, R_\mu)$.

Proposition 8. *Let* $E_1, \dots, E_i \subset \mathbb{N}^d$ *be staircases. Suppose that there exists a staircase* F *with* $F \supset R_{\delta+1}$ *and* $C(F) \subset \overline{C(E_1, \dots, E_i)}$, *then* $l(d, \delta, E_1, \dots, E_i) = 0$.

Proof. By semi-continuity of the cohomology $l(d, \delta, E_1, \dots, E_i) \leqslant l(d, \delta, F)$ and $l(d, \delta, F) \leqslant l(d, \delta, R_{\delta+1})$ since $F \supset R_{\delta+1}$. Since obviously $l(d, \delta, R_{\delta+1}) = 0$, the vanishing of $l(d, \delta, E_1, \dots, E_i)$ follows from the last two inequalities. $\square$

3. Elementary incidences

The previous section explained that the theorems would follow from incidences between the various subschemes $C(E_1, \dots, E_j)$. The goal of this section is to exhibit such incidences.

Let $E \subset \mathbb{N}^d$ be a finite staircase and $i \in \{1, \dots, d\}$ be an integer. There exists a unique "height" function

$$h_{E,i} : \mathbb{N}^{d-1} \to \mathbb{N}$$

such that

$$(a_1, \dots, a_d) \in E \quad \Leftrightarrow \quad a_i < h_{E,i}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d).$$

Conversely, a function h is the height function of some staircase if and only if $h(a+b) \leqslant h(a)$ for any $(a, b) \in \mathbb{N}^{d-1} \times \mathbb{N}^{d-1}$. If $E_1, \dots, E_j$ are staircases, the sum of $E_1, \dots, E_j$ along the ith coordinate is the staircase $S_i(E_1, \dots, E_j)$ characterized by its height function

$$h_{S_i(E_1, \dots, E_j), i} = \sum_{k=1}^{j} h_{E_k, i}.$$

Proposition 9. *Let* $E_1, \dots, E_j$ *be staircases and* $k \in \{1, \dots, j\}$. *Then* $\overline{C(E_1, \dots, E_j)} \supset C(S_i(E_1, \dots, E_k), E_{k+1}, \dots, E_j)$.

Proof. This is a straightforward generalization of [2, Proposition 5.1.2]. $\square$

Let $(a_1, \dots, a_d) \in (\mathbb{N}^*)^d$ and let E be a staircase. We denote by $(a_1, \dots, a_d).E$ the staircase "obtained from E" by the linear map

$$(x_1, \dots, x_d) \mapsto (a_1 x_1, \dots, a_d x_d).$$

Concretely, this is the smallest staircase satisfying the relation:

$$(m_1,\dots,m_d)\in E \quad\Rightarrow\quad \bigl(a_1(m_1+1)-1,\dots,a_d(m_d+1)-1\bigr)\in(a_1,\dots,a_d).E.$$

This is a staircase of cardinal $a_1.a_2\dots a_d.\#E$. Denote by $a.E$ the staircase $(a,a,\dots,a).E$.

Proposition 10. *Let $E, E_1,\dots,E_j$ be staircases. Then*:

$$\overline{C(\underbrace{E,\dots,E}_{\prod a_i \text{ times}},E_1,\dots,E_j)}\supset C\bigl((a_1,\dots,a_d).E,E_1,\dots,E_j\bigr).$$

Proof. By induction on the number of a_i's which are not equal to one. If all the a_i's but one are equal to one, the statement follows from the previous proposition since

$$(1,\dots,1,a_i,1,\dots,1).E=S_i(\underbrace{E,\dots,E}_{a_i \text{ times}}).$$

For the general case, one can suppose by symmetry that $a_1\neq 1$. Applying several times—namely $a_2a_3\cdots a_d$ times—this first step, we get

$$\overline{C(\underbrace{E,\dots,E}_{\prod a_i \text{ times}},E_1,\dots,E_j)}\supset C\bigl(\underbrace{(a_1,1,\dots,1).E,\dots,(a_1,1,\dots,1).E}_{a_2\cdots a_d \text{ times}},E_1,\dots,E_j\bigr)$$

and, by induction,

$$\overline{C(\underbrace{(a_1,1,\dots,1).E,\dots,(a_1,1,\dots,1).E}_{a_2\cdots a_d \text{ times}},E_1,\dots,E_j)}$$

contains

$$C\bigl((1,a_2,\dots,a_d).(a_1,1,\dots,1).E,E_1,\dots,E_j\bigr)=C\bigl((a_1,\dots,a_d).E,E_1,\dots,E_j\bigr).$$

The expected inclusion follows immediately. $\square$

In particular, when $a_1=a_2=\cdots=a_d=s$, we get:

Proposition 11. *Let $E, E_1,\dots,E_j$ be staircases. Then*:

$$\overline{C(\underbrace{E,\dots,E}_{s^d \text{ times}},E_1,\dots,E_j)}\supset C(s.E,E_1,\dots,E_j).$$

Definition 12. Let $\Delta=(\delta_1,\dots,\delta_d)$ be a primitive vector in $\mathbb{Z}^d$ such that there exist i,j satisfying $\delta_i\delta_j<0$. Let $E\subset\mathbb{N}^d$ be a subset. We denote by $\Delta(E)\subset\mathbb{N}^d$ the unique subset verifying the following two conditions:

- for any line L in $\mathbb{R}^d$ with direction Δ, the sets $E \cap L$ and $\Delta(E) \cap L$ are equipotent,
- $\forall i \in \mathbb{N}$, $\forall (n, p) \in (\mathbb{N}^d)^2$, $n \in \Delta(E)$ and $p = n + i\Delta \Rightarrow p \in \Delta(E)$.

To be more explicit, the set $L \cap \mathbb{N}^d$ is finite by hypothesis on Δ. If $m_1 < m_2 < \cdots < m_j$ are its elements, ordered by the relation

$$m_{i_1} < m_{i_2} \quad \Leftrightarrow \quad \exists i \in \mathbb{N},\ m_{i_1} = m_{i_2} + i\Delta, \tag{$<$}$$

then $\Delta(E) \cap L = \{m_1, \dots, m_k\}$, where $k = \#(E \cap L)$.

Proposition 13. *Let $\Delta = (\delta_1, \dots, \delta_d) \in \mathbb{Z}^d$ be a vector such that*

- $\exists i$, $\delta_i = 1$,
- $\forall k$, $k \neq i \Rightarrow \delta_k \leqslant 0$,
- $\exists j \neq i$, $\delta_j \neq 0$.

Then for every staircase E, $\Delta(E)$ is a staircase. Moreover, we have in characteristic zero the incidence:

$$\overline{C(E)} \supset C\bigl(\Delta(E)\bigr).$$

Proof. Suppose by symmetry that $\delta_1 = 1$. Let

$$\begin{aligned} \Phi : k[x_1, \dots, x_d] &\to k[x_1, \dots, x_d]\left[t, \frac{1}{t}\right] \\ x_1 &\mapsto t x_1 + x_2^{-\delta_2} x_3^{-\delta_3} \dots x_d^{-\delta_d} \\ x_i &\mapsto x_i \quad \text{if } i \neq 1. \end{aligned}$$

The ideal

$$I(t) = k[x_1, \dots, x_d]\left[t, \frac{1}{t}\right]\Phi\bigl(I^E\bigr)$$

defines a subscheme

$$F \subset \bigl(\mathbb{A}^1 - \{0\}\bigr) \times \mathbb{A}^d$$

whose fiber over each $t \in \mathbb{A}^1 - \{0\}$ is a monomial subscheme with staircase E. In particular, F is flat over $\mathbb{A}^1 - \{0\}$. The closure $\overline{F} \subset \mathbb{A}^1 \times \mathbb{A}^d$ is defined by the ideal $J(t) = I(t) \cap k[x_1, \dots, x_d, t]$ and it is flat over $\mathbb{A}^1$.

We want to prove the equality $J(0) = I^{\Delta(E)}$, using a natural graduation.

Let $\varphi_1, \dots, \varphi_{d-1} : \mathbb{Z}^d \to \mathbb{Z}$ be independent linear forms which vanish on Δ. Consider the multi-graduation D defined by:

$$D : \text{Monomials of } k[x_1, \dots, x_d] \to \mathbb{Z}^{d-1}$$
$$\underline{x}^{\underline{\alpha}} \mapsto \big(\varphi_1(\underline{\alpha}), \dots, \varphi_{d-1}(\underline{\alpha})\big).$$

The conditions on Δ imply that, for all $\underline{z} = (z_1, \dots, z_{d-1}) \in \mathbb{Z}^{d-1}$, the sub-vector space $k[x_1, \dots, x_d]_{\underline{z}} \subset k[x_1, \dots, x_d]$ containing the elements of degree $\underline{z}$ has finite dimension. Note that $J(t)$ is a graded ideal, i.e.,

$$J(t) = \bigoplus_{\underline{z} \in \mathbb{Z}^{d-1}} J_{\underline{z}}(t)$$

where

$$J_{\underline{z}}(t) = J(t) \cap k[x_1, \dots, x_d]_{\underline{z}}[t].$$

In particular, to compute $J(0) = \lim_{t \to 0} J(t)$, it suffices to compute the limit of its graded parts in the Grassmannians $G(l, k[x_1, \dots, x_d]_{\underline{z}})$, where $l = \dim J_{\underline{z}}(t)$, $t \neq 0$. Let $m_1 < \dots < m_k$ be the monomials of $k[x_1, \dots, x_d]_{\underline{z}}$, where the order is given by the relation $(<)$ above. Let us admit temporarily the inclusion

$$m_{k-l+1}, m_{k-l+2}, \dots, m_k \in J_{\underline{z}}(0). \tag{$*$}$$

Then $J_{\underline{z}}(0)$ is the vector space generated by $m_{k-l+1}, m_{k-l+2}, \dots, m_k$ for dimensional reasons and $J(0) = I^{\Delta(E)}$ since these two graded ideals have the same graded parts. In particular $J(0)$ is an ideal generated by monomials and the set $\Delta(E)$ of monomials which are not in $J(0)$ is a staircase. Moreover, replacing the coordinates $x_1, \dots, x_d$ of $\mathbb{A}^d$ by any local system of coordinates, one shows by the same computation that any closed point of $C(\Delta(E), E_1, \dots, E_j)$ is a limit of points which are in $\overline{C(E, E_1, \dots, E_j)}$. This gives the incidence between the strata.

It remains to show $(*)$. Let $n_1 = x^{\underline{\alpha}(1)}, \dots, n_l = x^{\underline{\alpha}(l)}$ be the monomials of $I^E \cap k[x_1, \dots, x_d]_{\underline{z}}$, where $\underline{\alpha}(i) = (\alpha_1(i), \dots, \alpha_d(i))$. The ideal $I(t)$ contains the monomials

$$\Phi(n_i) = \big(tx_1 + x_2^{-\delta_2} x_3^{-\delta_3} \dots x_d^{-\delta_d}\big)^{\alpha_1(i)} x_2^{\alpha_2(i)} \dots x_d^{\alpha_d(i)}.$$

Since the degree of m_i in x_1 is $k - i$, this equality can be rewritten as:

$$\Phi(n_i) = \sum_{j=0}^{\alpha_1(i)} \binom{\alpha_1(i)}{j} t^j m_{k-j} = \sum_{j=0}^{k-1} \binom{\alpha_1(i)}{j} t^j m_{k-j}$$

with the usual convention $\binom{\alpha_1(i)}{j} = 0$ if $j > \alpha_1(i)$. If N and M are the column matrices whose entries are respectively $\Phi(n_i)$, $i \in \{1, \dots, l\}$, and $t^j m_{k-j}$, $j \in \{0, \dots, k-1\}$, if P is the matrix whose coefficient P_{ij} is $\binom{\alpha_1(i)}{j}$, the above equality writes down $N = PM$. Take the first l columns of P to get a square matrix

$$Q = \begin{pmatrix} 1 & \alpha_1(1) & \binom{\alpha_1(1)}{2} & \dots & \binom{\alpha_1(1)}{l-1} \\ \dots & \dots & & & \\ 1 & \alpha_1(l) & \binom{\alpha_1(l)}{2} & \dots & \binom{\alpha_1(l)}{l-1} \end{pmatrix}.$$

Since the coefficients in the third column are polynomials of degree 2 in α_1, one can replace the third column by a linear combination of the first three columns so that the ith element in the third column becomes $\alpha_1(i)^2$. Similarly, after suitable operations on the columns, the ith element in the fourth, fifth column... becomes $\alpha_1(i)^3, \alpha_1(i)^4, \dots$. The resulting matrix is a Van Der Monde matrix in the $\alpha_1(i)$'s. In characteristic zero, its determinant is not zero since the $\alpha_1(i)$'s are distinct. In particular Q is invertible.

The ideal $I(t)$ contains the elements which are the coefficients of the matrix $Q^{-1}N = Q^{-1}PM$. Using that the identity is a submatrix of $Q^{-1}P$ by construction, the ith element in this column matrix is $c_i(t) = t^{i-1}m_{k-i+1} + R$ where R is a polynomial dividable by t^i. Thus, $\frac{c_i(t)}{t^{i-1}} \in J(t)$ and, as expected, $J(0)$ contains $\frac{c_i(t)}{t^{i-1}}(0) = m_{k-i+1}$ for $i \in \{1, \dots, l\}$. □

If we have a finite set of monomial subschemes, we can specialize the first one and leave the remaining subschemes unchanged. Thus, we get as a corollary of the previous proposition:

Proposition 14. *Let $\Delta = (\delta_1, \dots, \delta_d) \in \mathbb{Z}^d$ be a vector such that*

- $\exists i, \delta_i = 1$,
- $\forall k, k \neq i \Rightarrow \delta_k \leqslant 0$,
- $\exists j \neq i, \delta_j \neq 0$.

Then for every set of staircases $E, E_1, \dots, E_j$, we have in characteristic zero the incidence:

$$\overline{C(E, E_1, \dots, E_j)} \supset C\big(\Delta(E), E_1, \dots, E_j\big).$$

3.1. Combinatorial properties of Δ

We give here some combinatorial properties of the map $E \mapsto \Delta(E)$ that we will use later on.

Lemma 15. *Let E and F be two subsets of $\mathbb{N}^d$ and $\Delta = (\delta_1, \dots, \delta_d) \in \mathbb{Z}^d$ be a direction satisfying the properties of the preceding proposition. Suppose that for every line L with direction Δ, we have the inequality on cardinals*:

$$\#\{E \cap L\} \geqslant \#\{F \cap L\}$$

then $\Delta(E) \supset \Delta(F)$.

Proof. We must show for every line L the inclusion $\Delta(E) \cap L \supset \Delta(F) \cap L$. This is obvious since, using the m_i's introduced after Definition 12, $\Delta(E) \cap L = \{m_1, \ldots, m_{\#\{E\cap L\}}\}$ and $\Delta(F) \cap L = \{m_1, \ldots, m_{\#\{F\cap L\}}\}$. $\square$

Applying this lemma to the following E and to $F = R_\mu$, noticing that $\Delta(R_\mu) = R_\mu$, we get:

Lemma 16. *Let R_μ be a regular staircase, $m \in R_\mu$, $P \subset \mathbb{N}^d$ a subset such that $P \cap R_\mu = \emptyset$ and $E = R_\mu \cup P - \{m\}$. If there exists $i \in \mathbb{Z}$ such that $m + i\Delta \in P$, then $\Delta(E) \supset R_\mu$.*

Lemma 17. *Let (s, d) be a couple of integers with $d \geqslant 2$, $s \geqslant 2$, and $(s, d) \notin \{(2, 2), (2, 3), (3, 2)\}$. Then there exists $(\Delta_d, \ldots, \Delta_1) \in (\mathbb{Z}^d)^d$ such that*

- $\forall i$, *Δ_i verifies the conditions of Proposition* 13,
- $\forall \mu > 0$, $\Delta_d(\Delta_{d-1}(\ldots(\Delta_1(s.R_\mu)))) \supset R_{s\mu+1}$.

Remark 18. More precisely, it will follow from the proof that the choice of the Δ_i depend on s in the following way.

- $s > 3$: $\Delta_1 = (0, \ldots, 0, 1, -s + 1)$, $\Delta_2 = (0, \ldots, 0, -s + 2, 1)$, $\Delta_i = (0, \ldots, 0, 1, -1, -1, 0, \ldots, 0)$ for $i \geqslant 3$, where the 1 is on the position of index $1 + d - i$.
- $s = 3$: $\Delta_1 = (0, \ldots, 0, 1, -2, 0)$, $\Delta_2 = (0, \ldots, 0, -3, 0, 1)$, $\Delta_3 = (0, \ldots, 0, 0, 1, -2)$, $\Delta_i = (0, \ldots, 0, 1, -1, -1, 0, \ldots, 0)$ for $i \geqslant 4$.
- $s = 2$: $\Delta_1 = (0, \ldots, 0, 1, -1, -1, -1)$, $\Delta_2 = (0, \ldots, 0, -1, 1, -1, 0)$, $\Delta_3 = (0, \ldots, 0, -1, 0, 1, -1)$, $\Delta_4 = (0, \ldots, 0, -1, -1, 0, 1)$, $\Delta_i = (0, \ldots, 0, 1, -1, -1, 0, \ldots, 0)$ for $i \geqslant 5$.

Proof. We proceed by induction on d. Considering the couples (s, d) involved in the proposition, we have to initialize the induction with the cases $(s > 3, d = 2)$, $(s = 3, d = 3)$ and $(s = 2, d = 4)$.

Initial cases. If $d = 2$, $s > 3$, then one can take $\Delta_1 = (1, -s + 1)$ and $\Delta_2 = (-s + 2, 1)$. When $s = 3$, $d = 3$, we must find Δ_1, Δ_2, Δ_3 such that

$$\Delta_3\big(\Delta_2\big(\Delta_1(3.R_\mu)\big)\big) \supset R_{3\mu+1}.$$

The $\frac{(\mu+1)(\mu+2)}{2}$ elements of the difference

$$R_{3\mu+1} - 3.R_\mu = \big\{(3x, 3y, 3z),\ x + y + z = \mu\big\}$$

are shown in Fig. 1 with $\mu = 2$. Taking $\Delta_1 = (1, -2, 0)$, we have:

$$R_{3\mu+1} - \Delta_1(3.R_\mu) = \big\{(0, 3y, 3z),\ y + z = \mu\big\}.$$

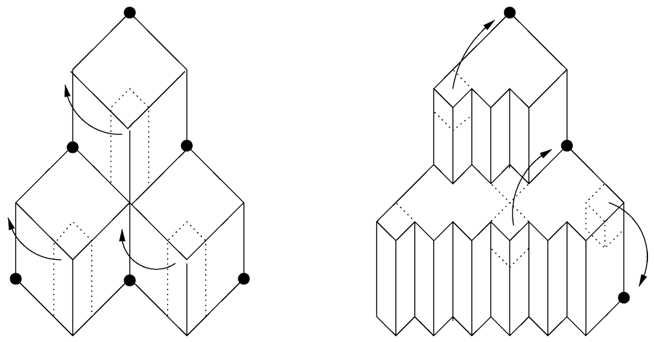

Fig. 1.

Finally, taking $\Delta_2 = (-3, 0, 1)$ and $\Delta_3 = (0, 1, -2)$,

$$\Delta_3\big(\Delta_2\big(\Delta_1(3.R_\mu)\big)\big) \supset R_{3\mu+1},$$

as expected.

Consider now the last initial case ($s = 2, d = 4$). By definition,

$$2R_\mu = \big\{(x, y, z, t) \text{ s.t. } [x/2] + [y/2] + [z/2] + [t/2] < \mu\big\},$$

where [ ] stands for the integral part. If $P \subset \mathbb{N}^4$ is a subset, we denote by P^i the subset of P containing the elements (x, y, z, t) such that i elements among (x, y, z, t) are odd and we put $S_m = R_{m+1} - R_m$. With these notations, easy considerations on the parities of (x, y, z, t) give the equality:

$$2R_\mu = R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}\big) \amalg S^3_{2\mu+1} \amalg S^4_{2\mu+2}.$$

To compute $\Delta(2R_\mu)$, we note that we can define Δ on subsets in such a way that if $E = \coprod E_i$ is a disjoint union, then $\Delta(E) = \coprod \Delta(E_i)$. Indeed, by construction of the map Δ, if L is a line in $\mathbb{R}^d$ with direction Δ, then $E \cap L$ and $\Delta(E) \cap L$ are two totally ordered sets of the same finite cardinality, hence there is a unique increasing one-to-one correspondence between $E \cap L$ and $\Delta(E) \cap L$. If $e \in E$ and L is the line with direction Δ passing through e, $\Delta(e)$ is the image of e through this correspondence. We let $\Delta(E_i) = \bigcup_{e_i \in E_i} \Delta(e_i)$.

Let $\Delta_1 = (1, -1, -1, -1)$. To compute the image $\Delta_1(e)$ of an element e, we make the following observation. If $E \subset \mathbb{N}^4$ can be written as a disjoint union

$$E = R_{j+1} \amalg E_{j+1} \amalg E_{j+2} \amalg \cdots E_{j+k}, \quad \text{with } E_l \subset S_l$$

and if $\Delta_1 = (\Delta_x, \Delta_y, \Delta_z, \Delta_t)$ satisfies $-2(\Delta_x + \Delta_y + \Delta_z + \Delta_t) \geqslant k$, then

$$\Delta_1(e) = \begin{cases} e + \Delta_1 & \text{if } e + \Delta_1 \in \mathbb{N}^4 \setminus E, \\ e & \text{otherwise.} \end{cases}$$

This observation leads to the equality

$$\Delta_1(2R_\mu) = R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}\big) \amalg S^3_{2\mu+1} \amalg \Delta_1\big(S^4_{2\mu+2}\big).$$

If $P \subset \mathbb{N}^4$, we define $P(1, *, \neq 0, e) \subset P$ to be the subset containing the elements (x, y, z, t) with $x = 1$, y any number, $z \neq 0$ and t even. There are obvious generalizations of this notation. With this notation, we have:

$$\begin{aligned}
\Delta_1(2R_\mu) &= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}\big) \amalg S^3_{2\mu+1} \amalg S^0_{2\mu}(\neq 0, *, *, *) \\
&= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, *, *, *)\big) \amalg S^3_{2\mu+1} \\
&= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, *, *, *)\big) \amalg S^3_{2\mu+1}(1, *, *, e) \\
&\quad \amalg S^3_{2\mu+1}(1, *, e, *) \amalg S^3_{2\mu+1}(1, e, *, *) \amalg S^3_{2\mu+1}(\neq 1, *, *, *).
\end{aligned}$$

Let $\Delta_2 = (-1, 1, -1, 0)$, $\Delta_3 = (-1, 0, 1, -1)$, $\Delta_4 = (-1, -1, 0, 1)$. Then,

$$\begin{aligned}
\Delta_2 \circ \Delta_1(2R_\mu) &= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, *, *, *)\big) \amalg \Delta_2\big(S^3_{2\mu+1}(1, *, *, e)\big) \\
&\quad \amalg S^3_{2\mu+1}(1, *, e, *) \amalg S^3_{2\mu+1}(1, e, *, *) \amalg S^3_{2\mu+1}(\neq 1, *, *, *) \\
&= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, *, *, *)\big) \amalg S^0_{2\mu}(0, \neq 0, *, *) \\
&\quad \amalg S^3_{2\mu+1}(1, *, e, *) \amalg S^3_{2\mu+1}(1, e, *, *) \amalg S^3_{2\mu+1}(\neq 1, *, *, *) \\
&= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, 0, *, *)\big) \\
&\quad \amalg S^3_{2\mu+1}(1, *, e, *) \amalg S^3_{2\mu+1}(1, e, *, *) \amalg S^3_{2\mu+1}(\neq 1, *, *, *),
\end{aligned}$$

$$\begin{aligned}
\Delta_3 \circ \Delta_2 \circ \Delta_1(2R_\mu) &= R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, 0, *, *)\big) \\
&\quad \amalg S^3_{2\mu+1}(1, *, e, *) \amalg \Delta_3\big(S^3_{2\mu+1}(1, e, *, *)\big) \\
&\quad \amalg S^3_{2\mu+1}(\neq 1, *, *, *) \\
&\supset R_{2\mu} \amalg \big(S_{2\mu} \setminus S^0_{2\mu}(0, 0, 0, *)\big) \\
&\quad \amalg S^3_{2\mu+1}(1, *, e, *) \amalg S^3_{2\mu+1}(\neq 1, *, *, *),
\end{aligned}$$

$$\begin{aligned}
\Delta_4 \circ \Delta_3 \circ \Delta_2 \circ \Delta_1(2R_\mu) &\supset R_{2\mu} \amalg \big(S_{2\mu} \setminus (0, 0, 0, 2\mu)\big) \\
&\quad \amalg \Delta_4\big(S^3_{2\mu+1}(1, *, e, *)\big) \amalg S^3_{2\mu+1}(\neq 1, *, *, *) \\
&\supset R_{2\mu} \amalg S_{2\mu} \amalg S^3_{2\mu+1}(\neq 1, *, *, *).
\end{aligned}$$

We have obtained the required inclusion $\Delta_4 \circ \Delta_3 \circ \Delta_2 \circ \Delta_1(2R_\mu) \supset R_{2\mu+1}$.

Step from $d-1$ to d. As we will proceed by induction on the dimension d, we precise our notations and we denote by $R_i(d)$ the regular staircase R_i in $\mathbb{N}^d$. Let T_i be the "ith slice" of $s.R_\mu(d)$, i.e.,

$$T_i := \{m \in \mathbb{N}^{d-1} \text{ s.t. } (i,m) \in s.R_\mu(d)\}.$$

Then $T_i = s.R_{\nu(i)}(d-1)$ with $\nu(i) = \max(0, \mu - [\frac{i}{s}])$. When $i < s\mu$, $\nu(i) > 0$ and we can apply the induction to T_i. Using moreover that $s\nu(i) \geqslant s\mu - i$, we get elements $\gamma_1, \dots, \gamma_{d-1} \in \mathbb{N}^{d-1}$ such that,

$$T_i' = \gamma_{d-1}\big(\dots\big(\gamma_1(T_i)\big)\big) \supset R_{s\nu(i)+1}(d-1) \supset R_{s\mu+1-i}(d-1).$$

Let $\Delta_i = (0, \gamma_i) \in \mathbb{N}^d$. The ith slice of the staircase

$$F = \Delta_{d-1}\big(\dots\big(\Delta_1\big(s.R_\mu(d)\big)\big)\big)$$

is T_i'. Summing up, for $i < s\mu$, the ith slice of F strictly contains the ith slice $R_{s\mu+1-i}(d-1)$ of $R_{s\mu+1}(d)$. In particular, F contains all the d-tuples whose sum is $s\mu$ except $(s\mu, 0, 0, \dots, 0)$.

It remains to find Δ_d such that $\Delta_d(F) \supset R_{s\mu+1}(d)$ by an application of Lemma 16.

Note that

$$T_{s\mu-1} = s.R_1(d-1) \supset K = R_2(d-1) \cup (1,1,0,\dots,0).$$

It follows that

$$T'_{s\mu-1} \supset \gamma_{d-1}\big(\dots\big(\gamma_1(K)\big)\big) = K$$

and that the element $z = (s\mu-1, 1, 1, 0, \dots, 0)$ is in F. Let $\Delta_d = (1, -1, -1, 0, \dots, 0)$. Applying Lemma 16 with $m = (s\mu, 0, 0, \dots, 0)$, $E = F$, $P = F - R_{s\mu+1}(d)$, $\Delta = \Delta_d$, $s\mu + 1$ instead of μ, $m + i\Delta = z$, we get the expected inclusion $\Delta_d(F) \supset R_{s\mu+1}(d)$. □

4. Conclusion of the proofs

4.1. Proof of Theorems 3 and 6

Let us denote the stratum $C(E_1, \dots, E_1, \dots, E_r, \dots, E_r)$ by $C(E_1^{n_1}, \dots, E_r^{n_r})$ where n_i is the number of copies of E_i. According to Proposition 8, to conclude the proof of Theorem 3 (respectively of Theorem 6), we must show that, for $s \geqslant 2$, $d \geqslant 2$ and $(s,d) \notin \{(2,2), (2,3), (3,2)\}$ (respectively for $s \geqslant 1$, $d \geqslant 2$) $\overline{C(R_\mu^{s^d})} \supset C(E)$ for some staircase E containing $R_{s\mu+1}$ (respectively containing $R_{s\mu}$). By Proposition 11,

$$\overline{C(R_\mu^{s^d})} \supset C(s.R_\mu).$$

Since $s.R_\mu \supset R_{s\mu}$, this concludes the proof of Theorem 6. As for Theorem 3, taking for E the staircase $\Delta_d(\Delta_{d-1}(\dots\Delta_1(s.R_\mu)))$ constructed in Lemma 17, we have

$$\overline{C(s.R_\mu)} \supset C(E)$$

by Proposition 13. The required inclusion $\overline{C(R_\mu^{s^d})} \supset C(E)$ follows immediately from the last two displayed inclusions.

Remark 19. The above proof relies heavily on Lemma 17, which is the key point. This key lemma is proved by an induction on the dimension.

4.2. The case $r = 8$, $d = 3$

The goal of this section is to compute the postulation of 8 fat points of multiplicity μ in $\mathbb{P}^3$, stated in Remark 5:

Proposition 20. *Let $r = 8, d = 3$ and $v(d,\delta,\mu^r) = \binom{\delta+d}{d} - r\binom{d+\mu-1}{d}$. Then*:

$$l(d,\delta,\mu^r) = \max\bigl(0, v(d,\delta,\mu^r)\bigr).$$

Proof. If Z_G is the generic union of 8 fat points of multiplicity μ, the vector space $H^0(I_{Z_G}(\delta))$ being the kernel of the restriction morphism:

$$H^0\bigl(\mathcal{O}_{\mathbb{P}^d}(\delta)\bigr) \to H^0\bigl(\mathcal{O}_{Z_G}(\delta)\bigr),$$

its dimension $l(d,\delta,\mu^r)$ is at least

$$v\bigl(d,\delta,\mu^r\bigr) = h^0\bigl(\mathcal{O}_{\mathbb{P}^d}(\delta)\bigr) - h^0\bigl(\mathcal{O}_{Z_G}(\delta)\bigr).$$

Let $\Delta = (1,-1,-1)$ and $E = \Delta(2R_\mu)$.

To prove the reverse inequality $l(d,\delta,\mu^r) \leqslant \max(0, v(d,\delta,\mu^r))$, since $\overline{C(R_\mu^{2^d})} \supset C(E)$ by Proposition 11 and Proposition 13, it suffices by semi-continuity to exhibit a subscheme Z in $C(E)$ such that

$$h^0\bigl(I_Z(\delta)\bigr) = \max\bigl(0, v\bigl(d,\delta,\mu^r\bigr)\bigr)$$

for all δ. Let $\mathbb{A}^d = \operatorname{Spec} k[x_1,\dots,x_d] \subset \mathbb{P}^d$ be an affine space and Z be the subscheme of $\mathbb{A}^d$ whose ideal is I^E. By dehomogenization, the vector space $H^0(\mathcal{O}_{\mathbb{P}^d}(\delta))$ is in bijection with the subspace $S_\delta \subset k[x_1,\dots,x_d]$ containing the polynomials of degree at most δ, and $H^0(I_Z(\delta))$ corresponds to $I^E \cap S_\delta$. Now, $\dim I^E \cap S_\delta$ is the number of monomials in $R_{\delta+1}$ which are not in E. Since

$$R_{2\mu} \subset E \subset R_{2\mu+1},$$

this number is 0 if $\delta \leqslant 2\mu - 1$ and $h^0(\mathcal{O}_{\mathbb{P}^d}(\delta)) - \#E$ if $\delta \geqslant 2\mu$. $\square$

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