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TRANSVERSALITY FOR PIECEWISE LINEAR MANIFOLDS

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WE PROVE three transversality theorems in the piecewise linear category. For the standard definitions and properties of this category see [12]. All maps considered will be piecewise linear, all manifolds compact, and all submanifolds locally flat (which is always the case for codimension $\geqq 3$ by [11]). We say M is a *proper* submanifold of Q if the boundary $\dot{M} \subset \dot{Q}$ and the interior $\mathring{M} \subset \mathring{Q}$.

The main result of this paper (Theorem 1) says that if M, P are proper submanifolds of Q then we can ambient isotop M until it is transversal to P.

Perhaps we should straightway point out some inherent difficulties. We do not assume that P has a normal bundle in Q (or, equivalently, a normal microbundle). As yet the existence of normal bundles in the piecewise linear category is an open question. Haefliger and Wall [5] have proved that normal bundles exist in the stable range, but Hirsch [6] has shown that normal disc bundles do not always exist in the unstable range, and this gives weight to the conjecture that normal bundles also may not always exist.

If P did have a normal bundle in Q, then one could slide M along the fibres until it was transversal. This essentially is the geometrical idea behind Thom's original transversality theorem [8] for smooth maps, and behind Williamson's extension [10] to piecewise linear maps.

However, we are interested in the case where P may not have a normal bundle, and therefore we do not assume anything about normal bundles. Also we are primarily interested in ambient isotoping embeddings to be transversal, rather than homotoping maps, although in Theorem 2 we do deduce a result about maps.

Given $M, P \subset Q$, if we want to isotop M transversal to P, then the following method of attack at once suggests itself. Choose a triangulation K of Q in which M and P appear as subcomplexes. Let K^* denote the dual cell complex of K, and attempt to isotop M into the m-skeleton of K^*, where m is the dimension of M. But this is not always possible, because if it were one could infer that M always had a normal disc bundle in Q contradicting Hirsch's result [6].

Therefore we cannot isotop M into the m-skeleton of K^*. Instead we have to isotop M step by step so as to be transversal to each simplex of K. In other words our proof is by bare hands—the subtlety lying in the interplay between the linear and the piecewise linear. If one uses only the piecewise linear structure, then one runs into a difficulty illustrated by the following example.

The folded disc. Let D be a folded disc crossing an interval I in Euclidean 3-space (E^3) as shown in Fig. 1.

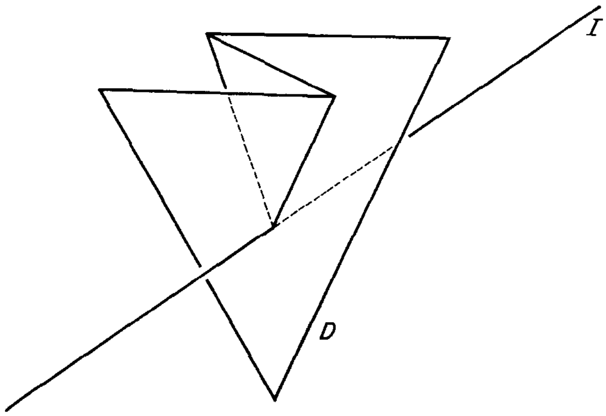


FIG. 1.

This picture is piecewise linearly homeomorphic to a standard linear disc in E^3 together with a perpendicular line through its centre, consequently D and I are transversal in E^3. If we now multiply by an extra dimension, we obtain $D \times I$ crossing $I \times I$ transversally in E^4. However, on tilting $I \times I$ upwards a little keeping $I \times 0$ fixed the transversality is destroyed, since the intersection of $D \times I$ with $I \times I$ becomes three concurrent lines and is no longer a manifold. With this example in mind it is easy to manufacture the following more disheartening situation. Let Δ^q be a q-simplex and S^{m-1}, S^{p-1} spheres crossing transversally in its boundary. Let D^m, D^p be discs formed by joining the spheres to two points in general position in the interior of Δ^q. Then D^m and D^p may cross transversally at all interior points, yet fail to be transversal at their boundaries.

So as not to meet with this kind of difficulty in the inductive step of our proof, we shall introduce the notion of M being *transimplicial* to the triangulation K of the ambient manifold Q. Being transimplicial is roughly the opposite of being a subcomplex. It is not a piecewise linear invariant, but rather is a technical device introduced for the purposes of proof; it uses not only the piecewise linear structure but also the local linear structure of K, and consequently is a stronger property than transversality. With this extra structure we are able to produce (transimplicial) Theorems 4 and 5 that have our main (transversality) result, Theorem 1, as a corollary.

The same techniques are used in Theorem 2 to extend the result from embeddings to maps : any map $f: M \to Q$ is homotopic to a map g transversal to the submanifold P of Q, and the cobordism class of $g^{-1}P$ depends only on the homotopy class of f. It should be noted that in the analogous differential setting [8], the set of all transversal maps is open in the function space, whereas this is not true in piecewise linear theory (we have no derivatives to "control" local movement). This defect accounts for our more directly geometrical approach.

We should point out that although Theorem 5 is a relative transimplicial theorem, we have no corresponding relative transversality theorem. This omission is discussed at the end of the paper.

Our third main result, Theorem 3, can be thought of as an existence theorem for *quotient regular neighbourhoods* (analogous to quotient vector bundles)—the inherent difficulty here being that in a regular neighbourhood there are no convenient fibres to play with. More precisely, given manifolds $M \subset P \subset Q$, we produce a fourth manifold N in Q that cuts P transversally along M.

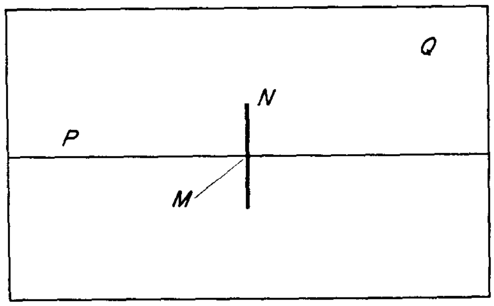


FIG. 2.

At the end of the paper we show how this result can be used to construct induced regular neighbourhoods, and Whitney sums. However, we are unable to prove any uniqueness theorems for these constructions.

We should like to acknowledge an unpublished paper by V. Poenaru and one of us, which contained incomplete proofs of some of the results below.

CONTENTS

§1. THE MAIN THEOREMS

Firstly we give a precise definition of what we mean by transversality. Let M, P be two proper submanifolds of the manifold Q. Denote by E^n n-dimensional Euclidean space and by E^n_+ the closed half space obtained by restricting the first coordinate to be non-negative.

Definition 1. The submanifolds M, P are *transversal at the point* $x \in \mathring{M} \cap \mathring{P}$ (respectively $\dot{M} \cap \dot{P}$) if there is a coordinate neighbourhood $h : E^q \to Q$ $(h : E^q_+ \to Q)$ of x in Q such that $h^{-1}M$, $h^{-1}P$ are two linear subspaces of E^q (E^q_+) in general position.

M and P are *transversal* if they are transversal at all points of $M \cap P$.

It follows immediately that if M, P are transversal in Q, then $M \cap P$ is a proper submanifold of dimension $m + p - q$, which is locally flat in both M and P.

THEOREM 1. *If Q is a manifold with proper submanifolds M and P, then M can be ambient isotoped transversal to P by an arbitrarily small ambient isotopy of Q.*

We want an analogous definition and theorem for maps. For simplicity we confine ourselves to closed manifolds, although there are similar results for bounded manifolds.

Definition 2. (i) Let M, P, Q be closed manifolds, with P a submanifold of Q. Let $f: M \to Q$ be an embedding; we say that the embedding f is *transversal* to P if fM and P are transversal as submanifolds.
(ii) Now suppose $f: M \to Q$ is an arbitrary piecewise linear map. We say that the map f is *graph-transversal* to P if its graph

$$\Gamma f: M \to M \times Q$$

is transversal to $M \times P$ as an embedding. Two properties follow at once.
(A) If $f: M \to Q$ is an embedding that is transversal to P as an embedding, then it is graph-transversal to P as a map. In other words graph-transversality is a generalisation.
(B) If $f: M \to Q$ is a map that is graph-transversal to P then $f^{-1}P$ is a locally-flat submanifold of M of codimension $q - p$. This is because the homeomorphism $\Gamma f: M \to (\Gamma f)M$ maps $f^{-1}P$ onto $(\Gamma f)M \cap (M \times P)$, which is a locally flat submanifold of dimension $m + (m + p) - (m + q)$ by the remark above.

THEOREM 2. *Given closed manifolds M, P, Q with $P \subset Q$, and given a map $f: M \to Q$, then there exists an arbitrarily close homotopic map g that is graph-transversal to P. The inverse image $g^{-1}P$ is a locally flat submanifold of M of codimension $q - p$, and the cobordism class $\{g^{-1}P\}$ depends only on the homotopy class $[f]$.*

Remark. All our results in this paper concern manifolds; a subsequent paper by one of us will deal with polyhedra [2]. In particular a stronger definition of transversality for maps will be given in [2], and a strengthened version of Theorem 2 proved.

THEOREM 3. *Given manifolds $M \subset P \subset Q$, both inclusions being proper, then there exists a fourth manifold N, contained in Q, that intersects P transversally in M.*

Remark. N will not be a proper submanifold of Q, because in general the boundary $\dot{N} \not\subset \dot{Q}$. However it will be proper in the neighbourhood of M, and so the definition of transversality of N and P makes sense.

We proceed now with the business of setting up sufficient machinery to prove Theorems 1, 2 and 3.

§2. (p, q)-DISC FIBERINGS

The ideas introduced in this section will be of fundamental importance throughout the rest of the paper. Let X, Y, Z be polyhedra, and let D^n denote a standard n-dimensional disc with centre 0.

Definition 3. A map $g : Y \to Z$ will be said to be locally a q-disc fibering at $y \in Y$, or more briefly *$F(q)$ at y*, if there exists a neighbourhood N of gy in Z and an embedding $\psi : N \times D^q \to Y$ onto a neighbourhood of y, such that the diagram

$$\begin{array}{ccc} N \times D^q & \xrightarrow{p_1} & N \\ \downarrow \psi & & \downarrow i \\ Y & \xrightarrow[g]{} & Z \end{array}$$

is commutative. Here p_1 denotes projection onto the first factor, and i the inclusion of N in Z.

Definition 4. The pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be locally a (p, q)-disc fibering at $x \in X$, abbreviated to *$F(p, q)$ at x*, if there exists a neighbourhood N of gfx in Z, embeddings $\varphi : N \times D^p \to X$, $\psi : N \times D^q \to Y$ onto neighbourhoods of x, fx respectively, and a map $k : D^p, 0 \to D^q, 0$ such that

$$\begin{array}{ccccc} N \times D^p & \xrightarrow{1 \times k} & N \times D^q & \xrightarrow{p_1} & N \\ \downarrow \varphi & & \downarrow \psi & & \downarrow i \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

commutes.

Note: (i) We can choose φ so that $\varphi(gfx, 0) = x$.

(ii) There is a natural generalisation to sequences of maps of greater length.

(iii) If the *pair* $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at $x \in X$, then the composition $X \xrightarrow{gf} Z$ is $F(p)$ at x.

(iv) The same diagram shows that the pair f, g is also $F(p, q)$ at all points in some neighbourhood of x.

We prove three basic lemmas.

LEMMA 1. (Restriction). *Suppose $X \to Y \to Z$ is $F(p, q)$ at $x \in X$, where $gfx \in Z_0$, a subpolyhedron of Z. Let $Y_0 = g^{-1}Z_0$, $X_0 = f^{-1}Y_0$. Then $X_0 \xrightarrow{f|X_0} Y_0 \xrightarrow{g|Y_0} Z_0$ is also $F(p, q)$ at x.*

Proof. By restriction.

LEMMA 2. (Glueing). *Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, let Z_i $i = 1, \ldots, t$ be subpolyhedra of Z, and suppose $\bigcup_{i=1}^t Z_i$ is a neighbourhood of gfx in Z. Let $Y_i = g^{-1}Z_i$, $g_i = g \mid Y_i$, $X_i = f^{-1}Y_i$ and $f_i = f \mid X_i$. Then $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at x if and only if each $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ is $F(p, q)$ at x.*

Proof. Given that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at x, restriction shows each $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ to be $F(p, q)$ at x.

Conversely, suppose we are given for each i a neighbourhood N_i of gfx in Z_i, embeddings $\varphi_i : N_i \times D^p \to X_i$, $\psi_i : N_i \times D^q \to Y_i$ and a map $k_i : D^p, 0 \to D^q, 0$ such that

$$\begin{array}{ccccc} N_i \times D^p & \xrightarrow{1 \times k_i} & N_i \times D_q & \xrightarrow{p_1} & N_i \\ \downarrow \varphi_i & & \downarrow \psi_i & & \downarrow \subset \\ X_i & \xrightarrow[f_i]{} & Y_i & \xrightarrow[g_i]{} & Z_i \end{array}$$

commutes.

Triangulate Z so that gfx is a vertex and each N_i is a subcomplex. †Let $K = \overline{st}(gfx, Z)$, then each simplex $A \in K$ is contained in some N_i. Consider a conewise expansion

$$gfx = K_0 \nearrow K_1 \nearrow \dots \nearrow K_m = K$$

each K_i being a cone, vertex gfx.

Let $K_{i,\varepsilon}$ denote the cone K_i shrunk by ε, and D^p_ε, D^q_ε the discs D^p, D^q shrunk by ε.

We shall define, inductively on j, a number $\varepsilon_j > 0$, embeddings $\Phi_j : K_{j,\varepsilon_j} \times D^p_{\varepsilon_j} \to X$, $\Psi_j : K_{j,\varepsilon_j} \times D^q_{\varepsilon_j} \to Y$ and a map $k : D^p_{\varepsilon_j}, 0 \to D^q_{\varepsilon_j}, 0$ such that

$$\begin{array}{ccccc} K_{j,\varepsilon_j} \times D^p_{\varepsilon_j} & \xrightarrow{1 \times k} & K_{j,\varepsilon_j} \times D^q_{\varepsilon_j} & \xrightarrow{p_1} & K_{j,\varepsilon_j} \\ \downarrow{\scriptstyle \Phi_j} & & \downarrow{\scriptstyle \Psi_j} & & \downarrow{\scriptstyle \subset} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

commutes.

Begin, for $j = 0$, with $\varepsilon_0 = 1$ and $\Phi_0 = \varphi_i | gfx \times D^p$, $\Psi_0 = \psi_i | gfx \times D^q$, $k = k_i$, for some chosen i.

(Without loss of generality we may assume $k(D^p_\varepsilon) \subset D^q_\varepsilon$ for all ε such that $0 \leqq \varepsilon \leqq 1$, for if not proceed as follows. Choose λ, $0 < \lambda \leqq 1$, such that D^p_λ is contained in the star of the origin in some triangulation of D^p with respect to which k is simplicial. Then $k(D^p_{\lambda\varepsilon}) \subset D^q_\varepsilon$ for all $\varepsilon \in [0, 1]$. Let $\Lambda : D^p \to D^p_\lambda$ be the shrinking map, and replace k, Φ_0 by $k\Lambda$ and $\Phi_0(1 \times \Lambda)$ respectively.)

Inductive step, $j \to j + 1$.
Suppose $K_{j+1} = K_j \cup A$, let $L = K_j \cap A$ and $\rho : A \to L$ be a retraction. Choose r such that $A \subset N_r$. Given $a \in A$, $u \in D^p$, $v \in D^q$, define $\varphi_{r,a} : D^p \to X$ and $\psi_{r,a} : D^q \to Y$ by

$$\varphi_{r,a}(u) = \varphi_r(a, u)$$
$$\psi_{r,a}(v) = \psi_r(a, v).$$

Now $\varphi_r(L \times D^p)$ is a neighbourhood of x in $f^{-1}g^{-1}L$, and moreover Φ_j maps

$$\begin{cases} L_{\varepsilon_j} \times D^p_{\varepsilon_j} \text{ into } f^{-1}g^{-1}L \\ gfx \times 0 \text{ to } x. \end{cases}$$

Also $\psi_r(L \times D^q)$ is a neighbourhood of fx in $g^{-1}L$, and Ψ_j maps

$$\begin{cases} L_{\varepsilon_j} \times D^q_{\varepsilon_j} \text{ into } g^{-1}L \\ gfx \times 0 \text{ to } fx. \end{cases}$$

Therefore there is a positive ε, $\varepsilon \leqq \varepsilon_j$, such that

$$\Phi_j(L_\varepsilon \times D^p_\varepsilon) \subset \varphi_r(L \times D^p)$$
$$\Psi_j(L_\varepsilon \times D^q_\varepsilon) \subset \psi_r(L \times D^q).$$

† Let v be a vertex of a complex K; we denote the open, closed star of v in K by $st(v, K)$ $\overline{st}(v, K)$, respectively.

Choose then $\varepsilon_{j+1} = \varepsilon$ and define

$$\Phi_{j+1}(z, u) = \begin{cases} \Phi_j(z, u) \text{ on } K_{j,\varepsilon} \times D^p_\varepsilon \\ \varphi_{r,z}\varphi^{-1}_{r,\rho z}\Phi_j(\rho z, u) \text{ on } A_\varepsilon \times D^p_\varepsilon. \end{cases}$$

$$\Psi_{j+1}(z, v) = \begin{cases} \Psi_j(z, v) \text{ on } K_{j,\varepsilon} \times D^q_\varepsilon \\ \psi_{r,z}\psi^{-1}_{r,\rho z}\Psi_j(\rho z, v) \text{ on } A_\varepsilon \times D^q_\varepsilon. \end{cases}$$

In both cases we have agreement on the overlap, because here $\rho z = z$. Our map Φ_{j+1} is piecewise linear on $A_\varepsilon \times D^p_\varepsilon$ because it is the composition

$$A_\varepsilon \times D^p_\varepsilon \xrightarrow{r_\rho \times 1} A_\varepsilon \times L_\varepsilon \times D^p_\varepsilon \xrightarrow{1 \times \Phi_j} A_\varepsilon \times X \xrightarrow{1 \times \varphi_r^{-1}} A_\varepsilon \times L \times D^p \xrightarrow{\text{projn}} A_\varepsilon \times D^p \xrightarrow{\varphi_r} X.$$

Similarly for Ψ_{j+1}.

We are left to show the commutativity of

$$\begin{array}{ccccc} A_\varepsilon \times D^p_\varepsilon & \xrightarrow{1 \times k} & A_\varepsilon \times D^q_\varepsilon & \xrightarrow{p_1} & A_\varepsilon \\ \downarrow{\scriptstyle \Phi_{j+1}} & & \downarrow{\scriptstyle \Psi_{j+1}} & & \downarrow{\scriptstyle \subset} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

For the right hand square, if $a \in A_\varepsilon$, $v \in D^p_\varepsilon$, then

$$\begin{aligned} g\Psi_{j+1}(a, v) &= g\psi_{r,a}\psi^{-1}_{r,\rho a}\Psi_j(\rho a, v) \\ &\in g\psi_{r,a}(D^q) \\ &= a \\ &= p_1(a, v). \end{aligned}$$

In the left hand square, for $a \in A_\varepsilon$, $u \in D^p_\varepsilon$, we have

$$\begin{aligned} \Psi_{j+1}(1 \times k)(a, u) &= \Psi_{j+1}(a, ku) = \psi_{r,a}\psi^{-1}_{r,\rho a}\Psi_j(\rho a, ku) \\ &= \psi_{r,a}\psi^{-1}_{r,\rho a}\Psi_j(1 \times k)(\rho a, u) \\ &= \psi_{r,a}\psi^{-1}_{r,\rho a}f\Phi_j(\rho a, u) \text{ by inductive hypothesis} \\ &= \psi_{r,a}\psi^{-1}_{r,\rho a}f\varphi_{r,\rho a}\varphi^{-1}_{r,a}\Phi_{j+1}(a, u) \\ &= \psi_{r,a}k_r\varphi^{-1}_{r,a}\Phi_{j+1}(a, u) \quad \text{since} \quad \begin{array}{ccc} D^p & \xrightarrow{k_r} & D^q \\ \downarrow{\scriptstyle \varphi_{r,\rho a}} & & \downarrow{\scriptstyle \psi_{r,\rho a}} \\ X & \xrightarrow[f]{} & Y \end{array} \quad \text{commutes,} \\ &= f\Phi_{j+1}(a, u) \quad \text{since} \quad \begin{array}{ccc} D^p & \xrightarrow{k_r} & D^q \\ \downarrow{\scriptstyle \varphi_{r,a}} & & \downarrow{\scriptstyle \psi_{r,a}} \\ X & \xrightarrow[f]{} & Y \end{array} \quad \text{commutes} \end{aligned}$$

This completes the inductive step $j \to j + 1$.

Eventually, at the end of the induction, we obtain a commutative diagram

$$\begin{array}{ccccc} K_\varepsilon \times D^p_\varepsilon & \xrightarrow{1\times k} & K_\varepsilon \times D^q_\varepsilon & \xrightarrow{p_1} & K_\varepsilon \\ \downarrow{\scriptstyle \Phi_m} & & \downarrow{\scriptstyle \Psi_m} & & \downarrow{\scriptstyle \subset} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z, \end{array}$$

where $\varepsilon = \varepsilon_m$. Since K_ε is a neighbourhood of gfx in Z, this shows that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at x, and so completes the proof of Lemma 2.

LEMMA 3. (Composition). *Is $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at $x \in X$ and $Z \xrightarrow{h} W$ is $F(n)$ at gfx, then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ is $F(n+p, n+q, n)$ at x.*

Proof. We have a neighbourhood N' of gfx in Z, embeddings φ', ψ' and a map k which give rise to a commutative diagram—

$$\begin{array}{ccccc} N' \times D^p & \xrightarrow{1\times k} & N' \times D^q & \xrightarrow{p_1} & N' \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \subset} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

Choose a neighbourhood N of $hgfx$ in W and an embedding $e : N \times D^n \to Z$ onto a neighbourhood of gfx *in* N' such that

$$\begin{array}{ccc} N \times D^n & \xrightarrow{p_1} & N \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \subset} \\ Z & \xrightarrow[h]{} & W \end{array}$$

commutes.

Define $\psi : N \times D^n \times D^q \to Y$ by

$$\psi(t, u, v) = \psi'(e(t, u), v)$$

and $\varphi : N \times D^n \times D^p \to X$ by

$$\varphi(t, u, v) = \varphi'(e(t, u), v).$$

Then

$$\begin{array}{ccccccc} N \times D^n \times D^p & \xrightarrow{1\times 1\times k} & N \times D^n \times D^q & \xrightarrow{\text{projn}} & N \times D^n & \xrightarrow{p_1} & N \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \subset} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & W \end{array}$$

commutes as required.

COROLLARY. *With the same hypotheses, $X \xrightarrow{f} Y \xrightarrow{hg} W$ is $F(n+p, n+q)$ at x.*

§3. TRANSIMPLICIAL MAPS

Let Q be a manifold, and K a triangulation of Q. If A is an a-dimensional simplex of K, let

$$L^A = lk(A, K)$$

denote the link of A in K. Then† $AL^A = \overline{st}(A, K)$. Let v be a vertex of A, and

$$s^A : AL^A \to vL^A$$

denote the simplicial map defined as the join of $A \to v$ to the identity on L^A.

Let M be another manifold, and $f: M \to Q$ be a map. Given a point x of M, let A be the unique simplex of K such that $fx \in \mathring{A}$.

Definition 5. We say that the map f is *transimplicial to K at x* if the pair

$$f^{-1}AL^A \xrightarrow{f} AL^A \xrightarrow{s^A} vL^A$$

is $F(m + a - q, a)$ at x. If this is the case for all $x \in M$, we say f is *transimplicial to K*.

Note 1. Our definition is independent of the choice of v (by an application of the composition lemma).

Note 2. The restriction and glueing lemmas of the previous section show that equivalent to Definition 5 is : for every principal simplex $AB \in K$, the pair $f^{-1}AB \xrightarrow{f} AB \xrightarrow{s^A} vB$ is $F(m + a - q, a)$ at x.

Note 3. Often it will be convenient to use the idea of a submanifold (i.e. the image of an embedding rather than the embedding itself) being transimplicial to a triangulation. The definition is the obvious one. Given a manifold Q, submanifold M and triangulation K of Q, we say M is transimplicial to K at $x \in M$ if the pair

$$M \cap AL^A \subset AL^A \xrightarrow{s^A} vL^A$$

is $F(m + a - q, a)$ at x, where $x \in \mathring{A}$, $A \in K$, and we use the above notation. Therefore, if (D^a, D^{m+a-q}) denotes an unknotted disc pair, we need a neighbourhood N of v in vL^A and an embedding

$$\varphi : N \times D^a, N \times D^{m+a-q} \to AL^A, M \cap AL^A$$

onto a neighbourhood of x, such that

$$s^A\varphi = \text{projection} : N \times D^a \to N.$$

Figure 4 illustrates the situation.

Note 4. The concept is designed to cut out the folding phenomenon described in our introduction. We illustrate in Fig. 3 a non-transimplicial embedding of a 2-disc in 3-dimensions. The disc lies in the star of a 1-simplex, and has a fold running down to a point in the 1-simplex.

† We denote the join of two complexes K and L by KL.

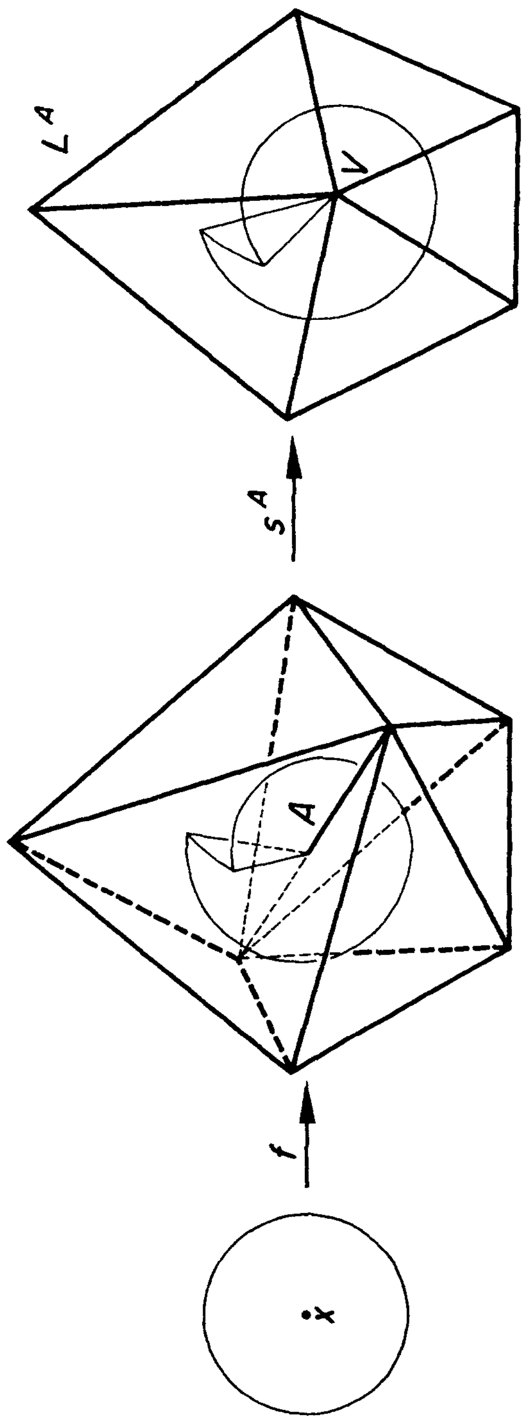


FIG. 3.

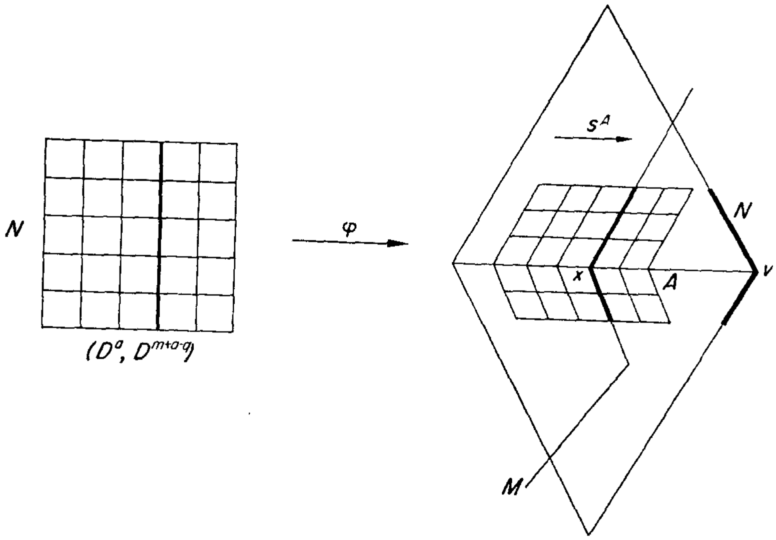


FIG. 4.

The embedding f fails to be transimplicial at x, because if it were, then the composition $s^A f$ would be $F(0)$, i.e. would be an embedding; but it is not an embedding because it is three-to-one where the fold gets flattened down.

Notice that if we move the fold point into the interior of a 3-simplex, then the embedding does become transimplicial. In fact this is the geometric idea behind our main proof. Given an embedding $M \to Q$ and a triangulation K of Q, we cannot isotop M into the m-skeleton of K^* (by Hirsch's result [6]), but nevertheless we shall show that we can push the worst fold and kink points into top dimensional simplexes, and so make M transimplicial to K.

Note 5. To prove the theorems in this paper we need only consider transimplicial embeddings rather than transimplicial maps. However, maps are just as easy to handle as embeddings at this stage, and several of the more general results that we prove for maps will be useful in [2].

LEMMA 4. (Openness). *If f is transimplicial to K at $x \in M$, then f is transimplicial to K at each point in some neighbourhood of x.*

Proof. Using the previous notation, the pair

$$f^{-1}AL^A \xrightarrow{f} AL^A \xrightarrow{s^A} vL^A$$

is $F(m + a - q, a)$ at x. By the openness of disc fiberings, there is a neighbourhood U of x in M such that this pair is $F(m + a - q, a)$ at all points of U. Let $y \in U$ and suppose $fy \in \mathring{B}$, $B \in K$; then A is a face of B and consequently $BL^B \subset AL^A$; let $B = AC$. By restriction the pair $f^{-1}BL^B \xrightarrow{f} BL^B \xrightarrow{s^A} vCL^B$ is $F(m + a - q, a)$ at y. But $s^{vC} : vCL^B \to vL^B$ is $F(b - a)$ at $s^A fy$, and $s^{vC}s^A = s^B : BL^B \to vL^B$. Therefore by the corollary to Lemma 3

$$f^{-1}BL^B \xrightarrow{f} BL^B \xrightarrow{s^B} vL^B$$

is $F(m + b - q, b)$ at y, completing the proof.

LEMMA 5. *For any subdivision K' of K, f transimplicial to K' imples f transimplicial to K.*

Proof. Given $x \in M$, suppose $fx \in \mathring{A}'$, where $A' \in K'$ and $\mathring{A}' \subset \mathring{A}$, $A \in K$. Let v' be a vertex of A', v a vertex of A, $L' = lk(A', K')$ and $L = lk(A, K)$. Then $s^A : AL \to vL$ induces a linear (i.e. each simplex is mapped linearly) map $\lambda : v'L' \to vL$ which makes the following diagram commute

$$\begin{array}{ccccc} {}^{-1}A'L' & \xrightarrow{f} & A'L' & \xrightarrow{s^{A'}} & v'L' \\ \cap & & \cap & & \downarrow \lambda \\ f^{-1}AL & \xrightarrow{f} & AL & \xrightarrow{s^A} & vL. \end{array}$$

Since f is transimplicial to K' the pair $f^{-1}A'L' \to A'L' \to v'L'$ is $F(m + a' - q, a')$ at x.

If we show that λ is $F(a - a')$ at v', then $f^{-1}AL \to AL \to vL$ is $F(m + a - q, a)$ by composition, and so the lemma follows. Therefore it remains to show that λ is $F(a - a')$ at v'.

K is contained in some Euclidean space E. Let F be the decomposition space of E consisting of all a-planes parallel to A, and let $g : E \to F$ be the natural map. Then g embeds vL in F because A is joinable to L. Similarly g' embeds $v'L'$ in F', where $g' : E \to F'$ is the natural map onto the decomposition space of all a'-planes parallel to A'. We have a commutative diagram

$$\begin{array}{ccc} v'L' & \xrightarrow{\lambda} & vL \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ F' & \xrightarrow{\mu} & F \end{array}$$

where μ is the natural map. Since μ is linear it is $F(a - a')$ everywhere.

Let $N = g(vL)$, $N' = g'(v'L')$. Then N' is a neighbourhood of $g'v'$ in $\mu^{-1}N$ because $A'L'$ is a neighbourhood of x in AL. Therefore $\mu : N' \to N$ is $F(a - a')$ at $g'v'$ by restriction. Therefore $\lambda : v'L' \to vL$ is $F(a - a')$ at v', and the proof of Lemma 5 is complete.

Let P be a proper submanifold of Q, and let K be a triangulation of the pair Q, P; in other words K is a triangulation of Q in which P appears as a subcomplex K_1.

LEMMA 6. (Consistency). *If M is a proper submanifold of Q that is transimplicial to K, then M is transversal to P.*

Proof. Given $x \in M \cap P$, suppose $x \in \mathring{A}$, $A \in K_1$. Let $L = lk(A, K)$, $L_1 = lk(A, K_1)$ and v be a vertex of A. Since M is transimplicial to K we have, with the usual notation, a commutative diagram:

$$\begin{array}{ccccc} N \times D & \xrightarrow{1 \times k} & N \times D_* & \xrightarrow{\text{projection}} & N \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \subset} \\ M \cap AL & \xrightarrow[\subset]{} & AL & \xrightarrow[s^A]{} & vL, \end{array}$$

where $D = D^{m+a-q}$ and $D_* = D^a$. Let $N_1 = N \cap vL_1$. Since Q, P is a locally flat manifold pair, we can choose N such that N, N_1 is an unknotted ball pair. The above left hand square can be rewritten:

$$\begin{array}{ccc} N \times D & \xrightarrow{1 \times k} & N \times D_* \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ M & \xrightarrow[\subset]{} & Q. \end{array}$$

Since M is locally flat in Q, we know that $N \times kD$ is locally flat at $(v, 0)$ in $N \times D_*$, and therefore that kD is locally flat at 0 in D_*. Meanwhile N_1 is locally flat at v in N. Therefore $N \times kD$ and $N_1 \times D_*$ are transversal at $(v, 0)$ in $N \times D_*$. Taking the image under ψ we deduce that M and P are transversal at x in Q. This is true for all $x \in M \cap P$, and so M, P are transversal.

We shall require triangulations of our manifolds that possess a certain local linearity property.

Definition 6. A combinatorial manifold K, of dimension q, is called *Brouwer* if:

(i) For each $A \in \mathring{K}$ there is a linear embedding $\overline{st}(A, K) \to E^q$.

(ii) For each $A \in \dot{K}$ there is a linear embedding $\overline{st}(A, K), \overline{st}(A, \dot{K}) \to E^q_+, E^{q-1}$.

Notes: 1. If only (ii) holds we say K is *Brouwer at the boundary.*

2. Not every combinatorial manifold is Brouwer, see Cairns [4].

3. Any subdivision of a Brouwer manifold is Brouwer.

The following lemma is due, in a sharpened form, to Whitehead [9].

LEMMA 7. (a) *Any combinatorial manifold K has a Brouwer subdivision K'.*

(b) *If K is already Brouwer at the boundary, we can choose K' such that $\dot{K}' = \dot{K}$.*

Proof. (a) Choose an atlas of q-simplexes $f_i : \Delta \to K$, $1 \leqq i \leqq r$, that cover K in the sense that every point has some $f_i \Delta$ as a closed neighbourhood. Now produce K' by subdividing so that all the f_i are simultaneously simplicial (using [12], Theorem 1].

(b) If K is already Brouwer at the boundary, we can confine our attention to a subatlas not meeting $\dot{K}$ that covers every simplex not meeting $\dot{K}$. In order to make the subatlas simplicial, it is not necessary to subdivide any simplex on the boundary.

The main burden of this paper will be to prove the following two theorems.

THEOREM 4. *If $f: M \to Q$ is an embedding between closed manifolds, and K any triangulation of Q, then f can be ambient isotoped, by an arbitrarily small ambient isotopy, to an embedding g that is transimplicial to K.* This theorem is in fact true for maps (see [2]). We now give a relative version.

THEOREM 5. *Let P be a proper submanifold of Q, and J a Brouwer triangulation of the boundary $\dot{Q}$, $\dot{P}$. Let $f: M \to Q$ be a proper embedding such that $f \mid \dot{M}$ is transimplicial to J. Then there exists an extension of J to a Brouwer triangulation K of Q, P, and an arbitrarily small ambient isotopy keeping $\dot{Q}$ fixed carrying f into an embedding g that is transimplicial to K.*

Remark. Let K be an arbitrary extension of J to a Brouwer triangulation of Q, P. Then although $f \mid \dot{M}$ is transimplicial to J, it may well happen that f is not transimplicial to K at points of $\dot{M}$. For example, let D be a disc properly embedded in a tetrahedron T as shown in Fig. 5. Then $\dot{D}$ is transimplicial to $\dot{T}$, but the fold ensures that D is not transimplicial to T at the boundary point x.

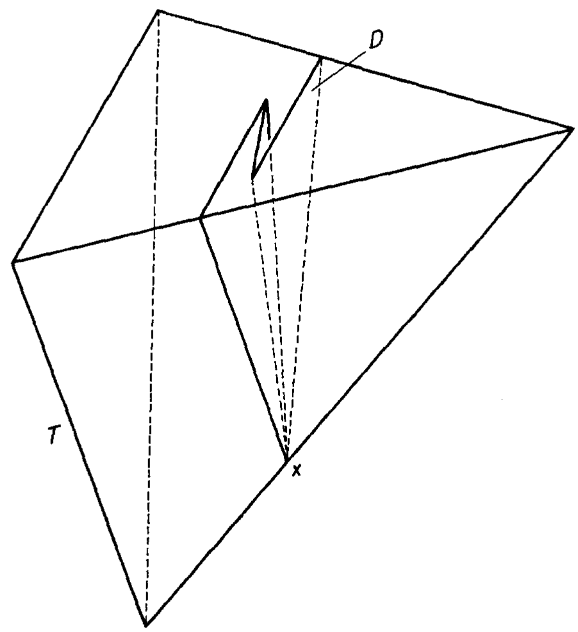


Fig. 5.

In our proof of Theorem 5, we get round this difficulty by using a collaring technique to construct a particular extension K relative to which such folds are straightened out.

Before proving these transimplicial results, we give applications in the form of proofs of our transversality theorems.

§4. PROOF OF THEOREM 1

We are given proper submanifolds M, P of Q, and have to ambient isotop M transversal to P.

By Lemma 7, it is possible to choose a Brouwer triangulation of the pair $\dot{Q}, \dot{P}$. Apply Theorem 4 to ambient isotop $\dot{M}$ transimplicial to J, and extend this ambient isotopy from $\dot{Q}$ to the whole of Q by [7] Addendum (2.2). Suppose the effect of this isotopy has been to move M to $M_1 \subset Q$, then $\dot{M}_1$ is transimplicial to J. We are now in a position to apply Theorem 5. This provides:

(a) an extension of J to a Brouwer triangulation K of the pair Q, P.

(b) an arbitrarily small ambient isotopy which moves M_1 transimplicial to K whilst keeping $\dot{Q}$ fixed.

Reference to Lemma 6 shows that the composition of our two isotopies produces the required result.

§5. PROOF OF THEOREM 2

We are given closed manifolds M, and $P \subset Q$, together with a map $f: M \to Q$ which we want to homotop graph-transversal to P. The graph $\Gamma f: M \to M \times Q$ is an embedding.

Choose Brouwer triangulations K_1 of M and K_2 of Q, P, and let K_3 be a subdivision of the cell complex $K_1 \times K_2$ triangulating $M \times Q$, $M \times P$. Using Theorem 4, ambient isotop Γf into an embedding F that is transimplicial to K_3.

LEMMA 8. *We can choose F so that the composition*

$$M \xrightarrow{F} M \times Q \xrightarrow{p_1} M$$

is a homeomorphism, where p_1 is the projection.

The proof of this lemma is postponed, it can be found directly following the proof of Theorem 4.

Meanwhile, let $e = (p_1 F)^{-1}$, the inverse homeomorphism. Define $G = (e \times 1)F : M \to M \times Q$, and let g denote the composition

$$M \xrightarrow{G} M \times Q \xrightarrow{p_2} Q.$$

Then g is homotopic to f and $G = \Gamma g$, the graph of g.

The triangulation K_3 of $M \times Q$ is really a homeomorphism $t : K_3 \to M \times Q$. Let K denote the triangulation

$$(e \times 1)t : K_3 \to M \times Q.$$

Then since $e \times 1$ maps $M \times P$ to itself, K is also a triangulation of $M \times Q$, $M \times P$. Now F is transimplicial to the triangulation K_3, and since we have applied the homeomorphism $e \times 1$ to both embedding and triangulation, we deduce that G is transimplicial to K. Therefore by Lemma 6 we know G is transversal to $M \times P$. Hence g is graph-transversal to P, because $\Gamma g = G$, and consequently $g^{-1}P$ is a locally flat submanifold of M of codimension $q - p$.

It remains to show the invariance of the cobordism class $\{g^{-1}P\}$. There were two choices involved in the above construction, namely those of triangulation and isotopy. Let K_*, g_* arise from different choices. Then g, g_* are connected by a homotopy $h : M \times I \to Q$ say.

The graph

$$\Gamma h : M \times I \to M \times I \times Q$$

is a proper embedding, whose restriction to the boundary

$$\Gamma g \cup \Gamma g_* : \partial(M \times I) \to \partial(M \times I \times Q)$$

is transimplicial to the Brouwer triangulation $K \cup K_*$ of $\partial(M \times I \times Q)$. By Theorem 5 extend $K \cup K_*$ to a triangulation of $M \times I \times Q$, $M \times I \times P$ and ambient isotop Γh, keeping the boundary fixed, to a transimplicial embedding H, say.

By Lemma 6 H is transversal to $M \times I \times P$, and so $H^{-1}(M \times I \times P)$ is an $(m + 1 + p - q)$-dimensional submanifold of $M \times I$ with boundary $g^{-1}P \cup (-g_*^{-1}P)$, the minus sign referring to orientation. In other words $g^{-1}P$ and $g_*^{-1}P$ are cobordant. If f_* is homotopic to f then the same g will do for both, and so the cobordism class $\{g^{-1}P\}$ depends only upon the homotopy class $[f]$.

Remark. There is a small but subtle point here. If f happened to be already graph-transversal to P we could not infer that $f^{-1}P \in \{g^{-1}P\}$, because f might not be transimplicial to any triangulation, and so we could not use the relative transimplicial Theorem 5, as in the proof above. Nor do we have a relative transversal theorem to use instead (see the end of the paper).

§6. PROOF OF THEOREM 3

We are given manifolds $M \subset P \subset Q$, with both inclusions proper, and need to construct a "perpendicular" manifold N. Begin as for Theorem 1, combining the results of Theorems 4 and 5 to obtain a triangulation J of P and an ambient isotopy of P moving M to M_1, where M_1 is transimplicial to J. By [7] Corollary (2.3) extend the ambient isotopy of P to give an ambient isotopy of the whole of Q. Extend J to a triangulation K of Q, this is possible since P is proper and locally flat in Q (see [3]). Let K' denote a first derived of K mod J.

For each simplex $A \in J$, let

$$L_A = \{\text{simplexes } B \in K' : AB \in K' \text{ and } B \cap J = \varnothing\}.$$

Define

$$X = \bigcup_{A \in J} (A \cap M_1) L_A,$$

the joins being made linearly inside the simplexes of K. Note firstly that the dimension of X is $m + q - p$. X need not be a manifold, however we shall show that it is a manifold "near" M_1.

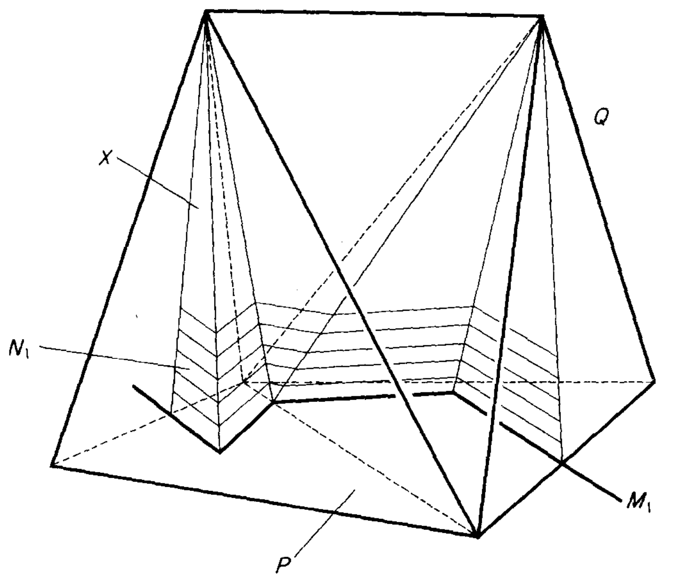


FIG. 6.

For $x \in M_1$, suppose $x \in \mathring{A}$, $A \in J$, and write $L^P = lk(A, J)$, $L^Q = lk(A, K')$. Let v be a vertex of A. Since M_1 is transimplicial to J, the pair

$$M_1 \cap AL^P \subset AL^P \xrightarrow{s^A} vL^P$$

is $F(m + a - p, a)$ at x. This implies that

$$X \cap AL^Q \subset AL^Q \xrightarrow{sA} vL^Q$$

is also $F(m + a - p, a)$ at x. So as not to interrupt the main line of argument, we ask the reader to temporarily accept this implication; a proof will be given following Lemma 12. We have therefore a neighbourhood D^{q-a} of v in vL^Q and an embedding of

$$D^{q-a} \times D^{m+a-p} = D^{m+q-p}$$

onto a neighbourhood of x in X. Consequently there is a neighbourhood N_1 of M_1 in X (for example take a second derived neighbourhood) which is an $(m + q - p)$-manifold and transimplicial to K'. By Lemma 6 N_1 is transversal to P in Q. By construction $N_1 \cap P = M$ Now reverse the original ambient isotopy of Q to obtain the required manifold N.

§7. THE t-SHIFT OF AN EMBEDDING

For the proof of Theorem 4 we shall use a sequence of special local moves (first introduced in [12] Chapter 6) called t-shifts. The parameter t concerns dimension, and the construction involves choice of local coordinate systems (i.e. replacing the piecewise linear structure by local linear structures) and choices of points in general position.

Suppose $f: M \to Q$ is a proper embedding between manifolds. By Lemma 7, we can find triangulations K_1, K_2 of M, Q such that $f: K_1 \to K_2$ is simplicial and K_2 Brouwer. If $K_1^{(2)}$, $K_2^{(2)}$ denote the barycentric second derived complexes of K_1, K_2, then $f: K_1^{(2)} \to K_2^{(2)}$ remains simplicial.

Let T_1 be a t-simplex of K_1 such that $\mathring{T}_1 \subset \dot{M}$, and let $T_2 = fT_1$. Take a simplicial neighbourhood of T_2 modulo its boundary in $K_2^{(2)}$ (i.e. this consists of all closed simplexes of $K_2^{(2)}$ which meet the interior of T_2) and call the resulting q-ball B_2. Let $B_1 = f^{-1}B_2$, this is an m-ball (it is in fact the corresponding simplicial neighbourhood of T_1 mod $\dot{T}_1$ in $K_1^{(2)}$). For $i = 1$, 2 let $\hat{T}_i$ denote the barycentre of T_i, and let $S_i = \dot{B}_i$. Then the polyhedron $|B_i| = |\hat{T}_i S_i|$, although of course as a complex B_i is a subdivision of $\hat{T}_i S_i$.

Denote by $f_T : B_1 \to B_2$ the restriction of f. Thus f_T is the join of the two maps $\hat{T}_1 \to \hat{T}_2$, $S_1 \to S_2$. The idea is to construct another embedding $g_T : B_1 \to B_2$ that agrees with f_T on the boundary $\dot{B}_1$, and is ambient isotopic to f_T keeping the boundary $\dot{B}_2$ fixed. We shall give the explicit construction below; it will be apparent that g_T can be chosen arbitrarily close to f_T, and the ambient isotopy made arbitrarily small.

Define a new embedding of M in Q by

$$g = \begin{cases} f \text{ on } M - B_1 \\ g_T \text{ on } B_1. \end{cases}$$

Then g is ambient isotopic to f. We call the move $f \to g$ a *local t-shift* with respect to the triangulation K_2.

Construction of the local shift. Choose a linear embedding h of $\overline{st}(T_2, K_2)$ in E^q (this is possible since K_2 is Brouwer), then h embeds B_2 linearly in E^q.

Let X denote the combinatorial q-ball hB_2, $Y = \dot{X}$, and $v = h\hat{T}_2$. Choose a point $w \in E^q$ near v which satisfies:

(i) $w \in st(v, X)$

(ii) w and Y are joinable

(iii) w is in general position with respect to the vertices of X.

Define a homeomorphism $j: X \to X$ as the join of the identity on Y to the map $v \to w$. Thus $h^{-1}jh$ is a homeomorphism of the ball B_2 which keeps its boundary fixed. Define $g_T = h^{-1}jhf_T$. Then g_T is ambient isotopic to f_T keeping $\dot{B}$ fixed in view of:

ALEXANDER'S LEMMA. *Any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.*

Suppose we now let T_1 run over a sequence of "interior" t-simplexes of K_1, then the corresponding balls $\{B_1\}$ overlap only in their boundaries, on which the $\{g_T\}$ agree with f, and therefore with each other. Consequently the resulting embeddings, and ambient isotopies, may be combined to give an embedding g ambient isotopic to f. We call $f \to g$ a *global t-shift* or, more briefly, a t-shift.

We shall want to perform a succession of t-shifts, one for each value of t, $\dim K_1 \geqq t \geqq 0$. But after the first shift the resulting embedding will no longer be simplicial with respect to K_1, K_2. However, in the construction of a shift, our initial assumption that f be simplicial was a luxury rather than a necessity, and the construction can be adapted as follows. Suppose $r > t$, $e: K_1 \to K_2$ simplicial, and that we perform an r-shift $e \to f$. Then given a t-simplex $T_1 \in K_1$:

(a) f maps T_1 linearly onto a t-simplex $T_2 \in K_2$.

(b) If B_2 is as above, and if $B_1 = f^{-1}B_2$, then B_1 is an m-ball and $f^{-1}\dot{B}_2 = \dot{B}_1$.

(c) $f_T: B_1 \to B_2$ is the join of $\dot{B}_1 \to \dot{B}_2$ to $\hat{T}_1 \to \hat{T}_2$.

Property (a) is satisfied because the r-shift does not move the $(r-1)$-skeleton, and properties (b) and (c) follow from property (i) of w in each local r-shift.

With the amount of structure contained in (a), (b) and (c) we can construct a local t-shift $f \to g$ exactly as before. Only one minor modification is needed, and that is in property (iii) for the point w: for this choose subdivisions such that $B_1' \to B_2'$ is simplicial, let X' be the corresponding subdivision of X, and choose w in general position with respect to the vertices of X'. The remainder of the construction is unaltered.

In this way we can construct t-shifts for all t, $m = \dim K_1 \geqq t \geqq 0$, in descending order, because for each t-simplex, the preceding higher dimensional shifts preserve the structure (a), (b) and (c).

§8. PROOF OF THEOREM 4

Let X be a *combinatorial* q-ball, with boundary Y, linearly embedded in E^q, and S^{m-1} an $(m-1)$-sphere in Y. Suppose that Y is joinable to the interior point w of X; in other words X and wY have the same underlying polyhedron. We have the following two lemmas.

LEMMA 9. *If S^{m-1} is transimplicial to Y at y, then wS^{m-1} is transimplicial to X at y.*

LEMMA 10. *If S^{m-1} is a subcomplex of Y, and if w is in general position with respect to the vertices of X, then wS^{m-1} is transimplicial to X at all interior points of X.*

Proof of 9. Suppose $y \in \mathring{A}$, $A \in Y$. Let v be a vertex of A, $L = lk(A, X)$, $L_1 = lk(A, Y)$ and s the simplicial map $AL \to vL$. We know that $S^{m-1} \cap AL_1 \subset AL_1 \xrightarrow{s} vL_1$ is $F(m + a - q, a)$ at y: i.e. there is a neighbourhood N_1 of v in vL_1 and a commutative diagram

$$\begin{array}{ccccc} N_1 \times D_1^{m+a-q} & \xrightarrow{1 \times \subset} & N_1 \times D_1^a & \xrightarrow{\text{projn}} & N_1 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \subset} \\ S^{m-1} \cap AL_1 & \xrightarrow{\subset} & AL_1 & \xrightarrow{s} & vL_1 \end{array}$$

where φ_1 embeds $N_1 \times D_1^a$, $N_1 \times D_1^{m+a-q}$ as neighbourhoods of y in AL_1, $S^{m-1} \cap AL_1$ respectively. Since w is joinable to Y, every ray radiating from w meets Y in a unique point. The same is true for points near w. Thus any ray near wy and parallel to wy *also* meets Y in a unique point. Therefore given a neighbourhood V of y in Y, there exists a neighbourhood U of y in X such that projection parallel to wy gives a map $r : U \to V$. Now choose V, U sufficiently small so that $V \subset \varphi_1(N_1 \times D_1^a)$ and $U \subset AL$. Define $\theta : U \to D_1^a$ as the composition

$$U \xrightarrow{r} \varphi_1(N_1 \times D_1^a) \xleftarrow{\varphi_1} N_1 \times D_1^a \xrightarrow{\text{projn}} D_1^a.$$

Then $s \times \theta : U \to vL \times D_1^a$ is piecewise linear and onto a neighbourhood of $v \times 0$ in $vL \times D_1^a$. Moreover, $s \times \theta$ is an embedding, for suppose u_1, u_2 have the same image under $s \times \theta$. Since $su_1 = su_2$ the interval $u_1 u_2$ is parallel to A. Therefore the interval $(ru_1)(ru_2)$ is also parallel to A and of the same length, consequently the points $\varphi_1^{-1}ru_1$, $\varphi_1^{-1}ru_2$ have the same first coordinate in $N_1 \times D_1^a$. Since $\theta u_1 = \theta u_2$, they also have the same last coordinate. Therefore they are equal, giving $ru_1 = ru_2$, and so $u_1 = u_2$. Thus $s \times \theta$ is an embedding as required.

Choose neighbourhoods N of v in vL, D^a of 0 in D_1^a, D^{m+a-q} of 0 in D_1^{m+a-q} such that

$$N \times D^a \subset (s \times \theta)U, \text{ and}$$

$$D^{m+a-q} \subset D^a.$$

Define $\varphi : N \times D^a \to AL$ by $\varphi = (s \times \theta)^{-1} | N \times D^a$. By construction

$$\begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{1 \times \subset} & N \times D^a & \xrightarrow{\text{projn}} & N \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \subset} \\ wS^{m-1} \cap AL & \xrightarrow{\subset} & AL & \xrightarrow{s} & vL \end{array}$$

commutes, showing wS^{m-1} transimplicial to X at y.

Proof of 10. (See Fig. 7). Since w is in general position it must lie in the interior of a principal simplex of X, hence trivially wS^{m-1} is transimplicial to X at w. Given an interior point x of wS^{m-1}, $x \neq w$, suppose that $x \in \mathring{A}$ where A is a simplex of X (we may assume $\dim A < q$, otherwise the lemma is again trivial). Let $L = lk(A, X)$. We need to show that

$$wS^{m-1} \cap AL \subset AL \xrightarrow{s} vL$$

is $F(m + a - q, a)$ at x. Denote by $[A]$ the linear subspace of E^q spanned by A. Then $w \notin [A]$, by the general position of w. Let $[wx]$ meet Y in y, where $y \in \mathring{C}$, $C \in Y$. Again using the general position of w, we infer that $[A]$ and $[C]$ together span E^q. Therefore $[wA] \cap C$ is a convex linear cell, containing y in its interior, of dimension $(a + 1 + c - q)$. Call this cell E.

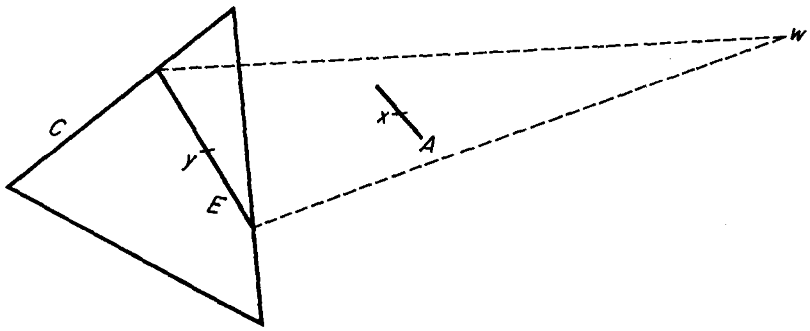


FIG. 7.

Let $L_1 = lk(C, S^{m-1})$, $L_2 = lk(C, Y)$. Then EL_1, EL_2 are respectively $m + a - q$, a — balls.

Let $\rho : C \to [E]$ denote orthogonal projection, and V be the neighbourhood $(\rho^{-1}E)L_2$ of y in Y. Let $\bar{\rho} : V \to EL_2$ be the join of ρ to the identity on L_2. As in the proof of the previous lemma any ray parallel and sufficiently close to wx meets Y in a unique point, and therefore there exists a neighbourhood U of x in X such that projection parallel to wx gives a map $r : U \to V$. We can choose U sufficiently small so that $U \subset AL$. Let θ be the composition

$$U \xrightarrow{r} V \xrightarrow{\bar{\rho}} EL_2.$$

Then θ is a projection in a direction complementary to the projection

$$U \xrightarrow{\subset} AL \xrightarrow{s} vL.$$

Therefore the product

$$s \times \theta : U \to vL \times EL_2$$

is a piecewise linear embedding onto a neighbourhood of $v \times y$ in $vL \times EL_2$. Choose neighbourhoods N of v in vL, D^{m+a-q} of y in EL_1, D^a of y in EL_2, such that $D^{m+a-q} \subset D^a$ and $N \times D^a \subset (s \times \theta)U$. Define $\psi : N \times D^a \to AL$ by $\psi = (s \times \theta)^{-1} | N \times D^a$. By construction we have a commutative diagram

$$\begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{\subset} & N \times D^a & \xrightarrow{\text{projn}} & N \\ \downarrow \psi & & \downarrow \psi & & \downarrow \subset \\ wS^{m-1} \cap AL & \xrightarrow{\subset} & AL & \xrightarrow{s} & vL \end{array}$$

and therefore the proof of Lemma 10 is complete.

We shall also need:

LEMMA 11. *Let M, Q be closed manifolds, and $f : M \to Q$ an embedding. Suppose B_2 is a q-ball contained in Q such that $(B_2, B_2 \cap fM)$ is a (q, m)-ball pair. Let $B_1 = f^{-1}(B_2 \cap fM)$, and let K be a triangulation of Q, B_2. Then if x is a point of $\dot{B}_1$ such that both*

$$|B_1 : B_1 \to B_2, \text{ and}$$
$$f | M - \mathring{B}_1 : M - \mathring{B}_1 \to Q - \mathring{B}_2$$

are transimplicial to K at x, then f is transimplicial to K at x.

Proof. A straightforward application of the glueing lemma. (Of course in saying $f | B_1 : B_1 \to B_2$ is transimplicial to K, we mean that it is transimplicial to the subcomplex of K triangulating B_2; similarly for the statement about $f | M - \mathring{B}_1$. Where no confusion can arise this abbreviation will be constantly used.)

Inductive proof of Theorem 4. Recall the statement of Theorem 4. We are given an embedding $f: M \to Q$ between closed manifolds, together with a triangulation K of Q, and we have to ambient isotop f to g such that g is transimplicial to K.

Choose a triangulation K_1 of M and a subdivision K_2 of K so that $f: K_1 \to K_2$ is simplicial and K_2 is Brouwer. Let K_1^t denote the t-skeleton of K_1, and $K_2^{(2)}$ the barycentric second derived of K_2. We shall produce inductively a sequence of embeddings of M in Q

$$f = g_{m+1}, g_m, \dots, g_0 = g$$

such that
(i) g_t is transimplicial to $K_2^{(2)}$ at points of $K_1 - K_1^{t-1}$, and
(ii) g_t is ambient isotopic to g_{t+1} by an arbitrarily small ambient isotopy.
Application of Lemma 5 shows that the final embedding g is transimplicial to K.

Beginning of induction. Apply a local m-shift to f, with respect to K_2, for each m-simplex of K_1. Define g_m to be the embedding which results from the global m-shift. Then (ii) is satisfied. Let A_1 be an m-simplex of K_1 and $A_2 = fA_1$. It is sufficient to show that g_m is transimplicial to $K_2^{(2)}$ at points of $\mathring{A}_1$. Recall the local m-shift process. Using the notation of the previous section, we have

$$g_m = h^{-1}jhf : \hat{A}_1\mathring{A}_1 \to \hat{A}_2 S_2.$$

By Lemma 10, $jhfA_1$ is transimplicial to X at all interior points. Therefore, since the property of being transimplicial is preserved under an isomorphism, $g_m A_1$ is transimplicial to $K_2^{(2)}$ at points of $g_m \mathring{A}_1$ as required.

Inductive step. Assume that, as a result of r-shifts for $m \geqq r > t$, we have

$$g_m, \dots, g_{t+1}$$

satisfying (i) and (ii).

Apply a local t-shift to g_{t+1}, with respect to K_2, for each t-simplex of K_1, and define g_t as the embedding resulting from the global t-shift. Again (ii) is immediately satisfied, and in proving (i) it is sufficient to examine the effect of a local shift, say that associated with $T_1 \in K_1$. We again use the notation of the previous section. Then:

$$g_t = g_{t+1} \text{ on } M - \mathring{B}_1, \text{ and}$$
$$g_t = h^{-1}jhg_{t+1} : B_1 \to B_2.$$

We claim that g_t is transimplicial to $K_2^{(2)}$ at points of
(a) $K_1 - K_1^t$, and
(b) $\mathring{T}_1$.

By the inductive hypothesis and restriction,

$$g_t : M - \mathring{B}_1 \to Q - \mathring{B}_2$$

is transimplicial to $K_2^{(2)}$ at points of $K_1 - K_1^t$. It remains to show

$$g_t : B_1 \to B_2$$

transimplicial to $K_2^{(2)}$ at all points except those of $\dot{T}_1$.

For then (b) is automatically taken care of, and (a) follows at once by application of Lemma 11. Our aim is accomplished using Lemmas 9 and 10. By Lemma 10, $jhg_{t+1} B_1$ is transimplicial to X', and therefore to X, at all interior points. Consequently

$$h^{-1}jhg_{t+1} B_1 = g_t B_1$$

is transimplicial to $K_2^{(2)}$ at all points in its interior. Before the move we see by restriction that $hg_{t+1}\dot{B}_1$ is transimplicial to Y except at points of $hg_{t+1}\dot{T}_1$. Therefore, since j keeps Y fixed, Lemma 9 shows $jhg_{t+1} B_1$ transimplicial to X at all points of $jhg_{t+1}(\dot{B}_1 - \dot{T}_1)$. Consequently $g_t B_1 \subset B_2$ is transimplicial to $K_2^{(2)}$ at points of $g_t(\dot{B}_1 - \dot{T}_1)$, and the induction is complete.

Proof of Lemma 8. Let us recall and simplify the statement of Lemma 8. We are given two closed manifolds M, Q. Let $\mathscr{E}$ denote the set of embeddings $e : M \to M \times Q$ with the property that the composition

$$M \xrightarrow{e} M \times Q \xrightarrow{\text{Projn}} M$$

is a homeomorphism. In particular if $f : M \to Q$ is an arbitrary map, then its graph $\Gamma f \in \mathscr{E}$. Let K_1, K_2 be Brouwer triangulations of M, Q and let K_3 be a simplicial subdivision of the convex linear cell complex $K_1 \times K_2$. Then Lemma 8 follows from:

LEMMA 8*. *Given $e \in \mathscr{E}$, there exists $e' \in \mathscr{E}$ transimplicial to K_3 and ambient isotopic to e.*

Proof. By Theorem 4 we can ambient isotop e to e' transimplicial to K_3. The only thing left is to make sure $e' \in \mathscr{E}$, and this is achieved by taking care over the t-shifts. The ambient isotopy e to e' consists of a finite sequence of local shifts.

$$e \to e_1 \to e_2 \to \dots \to e_r = e'.$$

We proceed by induction on the number of local shifts. This begins trivially since $e \in \mathscr{E}$. Suppose we have managed to ensure $e_i \in \mathscr{E}$, and consider the local shift $e_i \to e_{i+1}$. It takes place inside a ball AL, where $A \in K_3'$, K_3' some subdivision of K_3, and $L = lk(A, K_3')$. Since K_3' is a subdivision of $K_1 \times K_2$, there exist simplexes $A_1 \in K_1$, $A_2 \in K_2$ such that

$$AL \subset st(A_1, K_1) \times st(A_2, K_2).$$

Also, since K_1, K_2 are both Brouwer, we can choose linear embeddings $h_1 : st(A_1, K_1) \to E^m$, $h_2 : st(A_2, K_2) \to E^q$. We shall use the linear embedding

$$h = h_1 \times h_2 : AL \to E^m \times E^q$$

in order to construct the shift.

In detail, if $X = h(AL)$ and $v = h\hat{A}$, then $X = v\dot{X}$. Choose w in general position in $\mathring{X}$ sufficiently near v such that $X = w\dot{X}$. Define $j : X \to X$ by mapping $v \to w$, keeping $\dot{X}$ fixed, and joining linearly. Use $h^{-1}jh : AL \to AL$ to define the shift $e_i \to e_{i+1}$.

Now let $M_0 = e_i^{-1}(AL) \subset M$, and let $Z = he_i M_0$. Then Z is an m-cell, and $Z \subset X$, $\dot{Z} \subset \dot{X}$, $Z = v\dot{Z}$. Let $\pi : E^m \times E^q \to E^m$ denote the projection. Then since $e_i \in \mathscr{E}$, π embeds Z as an m-cell in E^m, and

$$\pi Z = (\pi v)(\pi \dot{Z}).$$

We now choose w sufficiently close to v such that

$$\pi Z = (\pi w)(\pi \dot{Z}).$$

As a consequence, although $e_i M_0 \neq e_{i+1} M_0$, nevertheless the projection $M \times Q \to M$ will map both $e_i M_0$ and $e_{i+1} M_0$ homeomorphically onto the same m-cell in M. Then $e_{i+1} \in \mathscr{E}$, and the inductive step is complete.

We end this section by filling the gap left in the proof of Theorem 3. For this we need:

LEMMA 12. *Let E be a simplex, F a principal face of E, v the vertex opposite F, and W a submanifold of F. If W is transimplicial to F at a point x, then vW is transimplicial to E at x.*

Proof. By exactly the same technique as was used for Lemma 9.

COROLLARY. *Let F, C be simplexes, and W a submanifold of F. If W is transimplicial to F, then CW is transimplicial to CF at points of W.*

Proof. Join successively to the vertices of C, applying the lemma at each step.

Recall the proof of Theorem 3. With the previous notation, we needed to show that for any point $x \in M_1$,

$$X \cap AL^Q \subset AL^Q \xrightarrow{s} vL^Q$$

is $F(m + a - p, a)$ at x.

Given $B \in L^Q$, write $AB = CF$ where $F = AB \cap J$ and C is the face of AB opposite F. Since M_1 is transimplicial to J, we have by restriction $M_1 \cap F$ transimplicial to F. But $X \cap AB = C(M_1 \cap F)$ and so by the Corollary above $X \cap AB$ is transimplicial to AB at x. In other words

$$X \cap AB \subset AB \xrightarrow{s} vB$$

is $F(m + a - p, a)$ at x. Therefore by glueing (Lemma 2) over all $B \in L^Q$, we have the desired result. This completes the proof of Theorem 3.

§9. PROOF OF THEOREM 5

It is necessary to do a considerable amount of preparatory work.

Collars. Let Q be a manifold with boundary. A *collar* c_Q of Q is an embedding

$$c_Q : \dot{Q} \times I \to Q$$

such that $c(x, 0) = x$ for all $x \in \dot{Q}$. Any compact manifold has a collar, and any two collars are ambient isotopic keeping the boundary fixed ([12], Theorem 13).

Given a proper embedding $f: M \to Q$ then by [12], Lemma 24 we can choose collars c_M, c_Q of M, Q that are *compatible* with f, that is to say the following diagram commutes

$$\begin{array}{ccc} \dot{M} \times I & \xrightarrow{c_M} & M \\ \Big\downarrow{\scriptstyle (f|\dot{M})\times 1} & & \Big\downarrow{\scriptstyle f} \\ \dot{Q} \times I & \xrightarrow{c_Q} & Q. \end{array}$$

In particular if P is a proper submanifold of Q, then we can choose compatible collars, that is to say $c_P = c_Q | \dot{P} \times I$.

Suppose we are now given a collar c_Q of Q and a triangulation J of the boundary $\dot{Q}$. If Q_1 denotes the image of c_Q, then we can extend J to a triangulation of the collar Q_1, in a canonical way, as follows. $J \times I$ is a convex linear cell complex, which has a canonical simplicial subdivision, $(J \times I)'$ say, obtained by starring in order of decreasing dimension all simplexes $A \times 1$, $A \in J$. The resulting triangulation

$$(J \times I)' \to \dot{Q} \times I \xrightarrow{c_Q} Q_1$$

is called the *canonical extension* of J to the collar. The canonical extension is functorial in the following sense. Let P be a proper submanifold of Q, and suppose we are given compatible collars c_Q, c_P and a triangulation J of $\dot{Q}$, $\dot{P}$. If Q_1, P_1 denote the images of c_Q, c_P, then the canonical extension of J to Q_1 is a triangulation of the pair Q_1, P_1 and the restriction to P_1 is the canonical extension of the restriction of J to $\dot{P}$.

LEMMA 13. *Let P be a proper submanifold of Q. Given a triangulation J of $\dot{Q}$, $\dot{P}$ then there exists an extension of J to a triangulation K of Q, P. Further, if J is Brouwer then K can be chosen to be Brouwer.*

Proof. Choose compatible collars c_Q, c_P, let Q_1, P_1 denote their images, and let $Q_2 = \overline{Q - Q_1}$, $P_2 = \overline{P - P_1}$. Let $(J \times I)'$ be the canonical extension of J to Q_1 and let J' denote the subcomplex triangulating the inside of the collar, $\dot{Q}_2$.

Choose any triangulation L of Q_2, P_2. Then both J' and $\dot{L}$ triangulate $\dot{Q}_2$, and so they have a common subdivision, say $J'' = \dot{L}'$ (see [12] Lemma 4). These subdivisions extend uniquely to subdivisions $(J \times I)''$, L' of $(J \times I)'$, L without introducing any more vertices. Identifying $J'' = \dot{L}'$, the union $K = (J \times I)'' \cup L'$ gives a triangulation of Q, P and provides the required extension of J.

Finally, if J is Brouwer then so is the canonical extension to the collar. Therefore K is Brouwer at the boundary, and so by Lemma 7(b) we can choose a Brouwer subdivision K' that also extends J.

Relative t-shifts. In proving Theorem 5 we shall need to be more precise in our t-shift process; recall the considerable choice available for the position of the point w. The necessary accuracy is expressed in the following lemma.

Let M, Q be manifolds and K a triangulation of Q. Given a map $f: M \to Q$ let

$$T_K^f = \{x \in M : f \text{ is transimplicial to } K \text{ at } x\}.$$

LEMMA 14. *Suppose $f: M \to Q$ is a proper embedding, K a Brouwer triangulation of Q, and $K^{(2)}$ a second derived of K. Let K_1 be a triangulation of M, and K_2 a subdivision of $K^{(2)}$*

such that $f: K_1 \to K_2$ *is simplicial. Let T be a t-simplex of* K_1 *such that* $T \subset \dot{M}$, *and* $f \to g$ *the associated local t-shift made in the local linear structure of K. If the shift is sufficiently small then* $T_K^f \subset T_K^g$.

Remark. The proof of Lemma 14 is long, and more complicated than our corresponding work in the proof of Theorem 4. The difficulty is that we are in a situation where the glueing lemma is no longer applicable.

Proof of Lemma 14. Since f is a proper embedding we know $fT \subset \mathring{Q}$. Define, as before, B_2 to be a simplicial neighbourhood of fT modulo its boundary in $K_2^{(2)}$, and $B_1 = f^{-1}B_2$. Now

$$\begin{aligned} fB_1 \subset B_2 &\subset \overline{st}(fT, K_2) \\ &\subset \overline{st}(u'', K^{(2)}) \text{ for some vertex } u'' \in \mathring{K}^{(2)} \\ &\subset st(u, K) \quad \text{ for some vertex } u \in \mathring{K}. \end{aligned}$$

Therefore the problem is localised both with respect to K and K_2. Using the Brouwer property of K choose a linear embedding

$$h : \overline{st}(u, K) \to E^q.$$

Then h automatically embeds B_2 linearly in E^q. The local shift may now be defined as before; in particular we write

$$f_0 = hf : f^{-1}\overline{st}(u, K) \to E^q, \text{ and}$$
$$g_0 = jhf : f^{-1}\overline{st}(u, K) \to E^q.$$

Remark. The above construction explains our reason for calling this section "relative t-shifts". We are t-shifting f with respect to the triangulation K_2, but with the reservation that we do so relative to the local linear structure of K.

Suppose f is transimplicial to K at $x \in M$, we want to ensure that g is also. If $x \notin B_1$, the result is trivial because some neighbourhood of x is not moved by the shift. Also if $x \in \mathring{B}_1$, application of Lemma 10 as in the proof of Theorem 4 shows g transimplicial to $K_2^{(2)}$, and therefore to K, at x.

Therefore there remains the case $x \in \dot{B}_1$; *here* $fx = gx$. Let A be the simplex of K such that $fx \in \mathring{A}$, and let $L^A = lk(A, K)$. Then $AL^A \subset \overline{st}(u, K)$. Define $E^a = [hA]$, the linear subspace of E^q spanned by hA, and $E^{q-a} = E^q//E^a$, the decomposition space whose points are a-dimensional linear subspaces of E^q parallel to E^a. Let $\pi : E^q \to E^{q-a}$ be the natural projection and $\pi^* : E^q \to E^a$ the orthogonal projection (see Fig. 8).

Since f is transimplicial to K at x, the pair

$$f^{-1}AL^A \xrightarrow{f_0} E^q \xrightarrow{\pi} E^{q-a}$$

is $F(m + a - q, a)$ at x. Therefore if $y = f_0 x$, $z = \pi y$, there is a neighbourhood N of z in E^{q-a} (which we may take to be a simplex), and embeddings φ, ψ onto neighbourhoods of x, y in $f^{-1}AL^A$, E^q respectively, such that the following diagram commutes

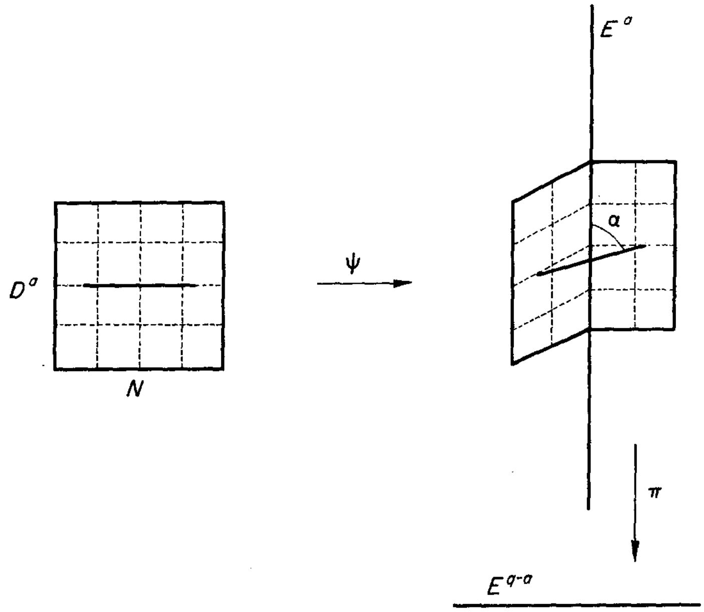


FIG. 8.

$$\begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{1 \times k} & N \times D^a & \xrightarrow{\text{projn}} & N \\ \downarrow \varphi & & \downarrow \psi & & \downarrow \subset \\ f^{-1}AL^A & \xrightarrow[f_0]{} & E^q & \xrightarrow[\pi]{} & E^{q-a}. \end{array}$$

Call E^a "the *vertical*". Given two points $y_1, y_2 \in E^q$, let $\alpha(y_1, y_2)$ denote the angle that the vector $y_1 y_2$ makes with the vertical. More precisely

$$\alpha(y_1, y_2) = \tan^{-1}\left(\frac{d(\pi y_1, \pi y_2)}{d(\pi^* y_1, \pi^* y_2)}\right)$$

$$0 \leqq \alpha \leqq \pi/2$$

where d denotes Euclidean distance.

SUBLEMMA 1. *There exists $\alpha_0 > 0$ such that given any two distinct points $x_1, x_2 \in N$ and any $y \in D^a$, then*

$$\alpha(\psi(x_1, y), \psi(x_2, y)) \geqq \alpha_0.$$

Proof. We chose N to be a simplex and we can regard D^a as a simplex, therefore $N \times D^a$ is a convex linear cell. Let J be a simplicial subdivision of $N \times D^a$ such that $\psi : J \to E^q$ is linear.

Case (i). Suppose $(x_1, y), (x_2, y)$ both lie in a simplex $S \in J$. Then their images $\psi(x_1, y)$, $\psi(x_2, y)$ lie in $\psi(S \cap (N \times y))$, which is a convex linear cell in E^q. This cell is embedded in E^{q-a} by π (because $\pi\psi : N \times D^a \to N$ is the projection), and therefore it makes an angle $\alpha_S > 0$ (independent of y since $\psi \mid S$ is linear) with the vertical. Let $\alpha_0 = \min(\alpha_S : S \in J)$. Then $\alpha(\psi(x_1, y), \psi(x_2, y)) \geqq \alpha_S \geqq \alpha_0$.

Case (ii). (x_1, y) and (x_2, y) do not lie in the same simplex of J. Since $\psi(N \times y) \xrightarrow{\pi} N$ is a homeomorphism, the vector $x_1 x_2 \subset N$ lifts under π^{-1} to an arc, I say, in $\psi(N \times y)$ which joins $\psi(x_1, y)$ to $\psi(x_2, y)$. Then I consists of a finite number of linear segments, all lying in the $(a+1)$-dimensional linear subspace of E^q above $x_1 x_2$, each one of which makes an angle greater than or equal to α_0 with the vertical. Therefore the vector joining the ends of I also makes an angle $\geqq \alpha_0$ with the vertical. This completes Sublemma 1, and we now continue with the proof of Lemma 14.

As before we denote the combinatorial ball hB_2 by X, and its boundary by Y. Recall the homeomorphism $j : X \to X$, defined by moving $f_0\hat{T} = v$ to a suitable point $w = g_0\hat{T}$ in general position with respect to the vertices of X, and joining linearly to Y. Extend j by the identity to the whole of E^q.

SUBLEMMA 2. *Given* $\alpha_0 > 0$, *there exists* $\varepsilon > 0$ *such that if* $\mathrm{d}(f_0\hat{T}, g_0\hat{T}) < \varepsilon$ *then for all* $y_1, y_2 \in E^q$

$$\alpha(y_1, y_2) \geqq \alpha_0 \Rightarrow \alpha(jy_1, jy_2) > 0.$$

Proof. Let S be a simplex of X. Since $j \mid S$ is linear, there exists $\varepsilon_S > 0$ such that if j moves $f_0\hat{T}$ less than ε_S, then any vector in S changes direction by less than α_0. Let $\varepsilon = \min(\varepsilon_S : S \in X)$. Suppose now that j moves $f_0\hat{T}$ by less than ε. Given y_1, y_2 in E^q, the vector $y_1 y_2$ consists of a finite number of segments, each one lying either in some simplex of X or in $E^q - X$. Therefore $j(y_1 y_2)$ is an arc, consisting of a finite number of linear segments each making an angle less than α_0 with $y_1 y_2$. Therefore the vector $(jy_1)(jy_2)$ joining the ends of this arc also makes an angle less than α_0 with $y_1 y_2$. But $y_1 y_2$ makes an angle $\geqq \alpha_0$ with the vertical, and therefore $(jy_1)(jy_2)$ makes an angle >0 with the vertical. This completes Sublemma 2.

We now make our local shift within the ε given by Sublemma 2; it remains to show this ensures g transimplicial to K at x. To do this it is sufficient to construct a commutative diagram

$$\begin{array}{ccccc}
N_* \times D_*^{m+a-q} & \xrightarrow{1 \times k_*} & N_* \times D_*^a & \xrightarrow{\text{projn}} & N_* \\
\downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \subset} \\
g^{-1}AL^A & \xrightarrow[g_0]{} & E^q & \xrightarrow[\pi]{} & E^{q-a}
\end{array}$$

which we now proceed to do. Let $U = j\psi(N \times D^a)$; since $jy = y$, U is a neighbourhood of y in E^q. Define $\theta : U \to D^q$ as the composition

$$U \xleftarrow{j} \psi(N \times D^a) \xleftarrow{\psi} N \times D^a \xleftarrow{\text{projn}} D^a.$$

Then the product $\pi \times \theta : U \to E^{q-a} \times D^a$ is piecewise linear and onto a neighbourhood of $(z, 0)$. We claim that it is an embedding; for given $y_1 \neq y_2 \in U$ with $\theta y_1 = \theta y_2$, then $\alpha(y_1, y_2) > 0$ by Sublemmas 1 and 2, thus $\pi y_1 \neq \pi y_2$. Choose a neighbourhood N_* of z in E^{q-a}, and a disc neighbourhood D_*^a of 0 in D^a such that $N_* \times D_*^a \subset (\pi \times \theta)U$. Define $\psi_* = (\pi \times \theta)^{-1} : N_* \times D_*^a \to E^q$. We have therefore produced the right hand half of our

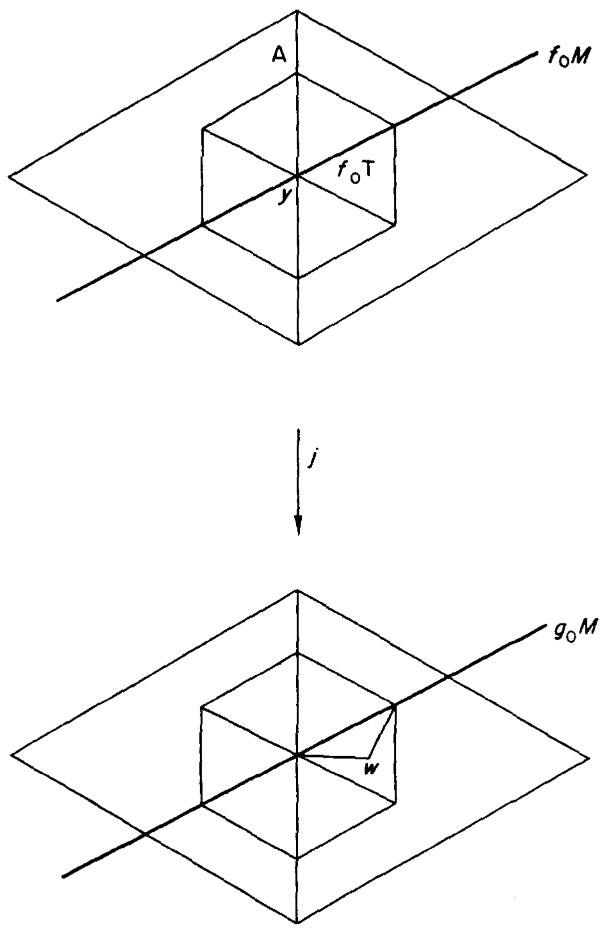


FIG. 9.

diagram. Since $k : D^{m+a-q}, 0 \to D^a, 0$ is an embedding, choose D_*^{m+a-q} as a disc neighbourhood of 0 in $k^{-1}D_*^a$ and define $k_* = k \mid D_*^{m+a-q} : D_*^{m+a-q} \to D_*^a$. Finally we need to define φ_*. It is elementary to check that

$$\psi_*(1 \times k_*)(N_* \times D_*^{m+a-q}) \subset g_0 AL^A,$$

therefore since g_0 is an embedding we can define

$$\varphi_* = g_0^{-1}\psi_*(1 \times k_*) : N_* \times D_*^{m+a-q} \to g_0^{-1}AL^A.$$

We have not finished the proof of Lemma 14 yet: so far we have shown that, given $x \in \dot{B}_1 \cap T_K^f$, then there exists $\varepsilon > 0$, such that if $d(f_0\hat{T}, g_0\hat{T}) < \varepsilon$ then $x \in \dot{B}_1 \cap T_K^g$. Notice that ε depends upon x. Suppose that $x' \in \dot{B}_1 \cap T_K^f$ and x, x' lie in the interior of the same simplex $S \in K_1$.

SUBLEMMA 3. *The same* ε *will do for* x'.

Proof. Choose neighbourhoods V, V' of x, x' in $st(S, K_1)$, such that linear translation by the vector xx' maps V into V'. Let $\lambda : V, x \to V', x'$ denote this linear translation. Since f_0

maps $st(S, K_1)$ linearly into E^q, there are linear translations λ', λ'' of E^q, E^{q-a} respectively such that the diagram

$$\begin{array}{ccccc} V, x & \xrightarrow{f_0} & E^q & \xrightarrow{\pi} & E^{q-a} \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \lambda''} \\ V', x' & \xrightarrow{f_0} & E^q & \xrightarrow{\pi} & E^{q-a} \end{array}$$

is commutative. Recall the commutative diagram

$$\begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{1 \times k} & N \times D^a & \xrightarrow{\text{projn}} & N \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \subset} \\ f^{-1}(AL^A) & \xrightarrow{f_0} & E^q & \xrightarrow{\pi} & E^{q-a} \end{array}$$

expressing the fact that f is transimplicial to K at x. We can choose N, D^{m+a-q} such that im $\varphi \subset V$ (replacing them by subballs if necessary); note that this replacement does not alter the angle α_0 of Sublemma 1. Now replace the three vertical arrows by $\lambda\varphi$, $\lambda'\psi$, λ'' respectively, and we have an expression of the transimpliciality of f to K at x'. Again α_0 is unaltered. Therefore the ε of Sublemma 2 is unaltered. This completes the proof of Sublemma 3, and we now conclude the lemma.

$\dot{B}_1$ is covered by the interiors of a finite number of simplexes of K_1; for each of these choose an ε by Sublemmas 2 and 3, and select the minimum such ε. Therefore if $d(f_0\hat{T}, g_0\hat{T}) < \varepsilon$ then $\dot{B}_1 \cap T_K^f \subset \dot{B}_1 \cap T_K^g$. In other words if the shift is sufficiently small $T_K^f \subset T_K^g$. This completes the proof of Lemma 14.

Proof of Theorem 5. Recall the statement of Theorem 5. We are given a manifold-pair Q, P, a Brouwer triangulation J of the boundary $\dot{Q}$, $\dot{P}$ and a proper embedding $f: M \to Q$ such that $f \mid \dot{M}$ is transimplicial to J. We have to extend J to a Brouwer triangulation K of Q, and ambient isotop f to g keeping $\dot{Q}$ fixed, so that g is transimplicial to K.

First choose compatible collars c_Q, c_P of Q, P. Then choose collars c_M, c_Q^* of M, Q compatible with $f: M \to Q$. By [12] Theorem 13 ambient isotop c_Q^* to c_Q keeping $\dot{Q}$ fixed, and suppose that this ambient isotopy carries f to g. The result is that c_M, c_Q are now compatible with g.

Intuitively what we have done so far is unfold M near the boundary, and get rid of the sort of kinks that are illustrated in the diagram of the Remark after Theorem 5. More precisely, we shall describe this unfolding in transimplicial terms, as follows.

Extend the triangulation J to the collars by the canonical extension, which is Brouwer, and then extend further over the rest of the manifolds by Lemma 13 to give a Brouwer triangulation K of Q, P. We claim that g is transimplicial to K at points of $\dot{M}$ (notice that before the unfolding we only knew that $f \mid \dot{M}$ was transimplicial to J at points of $\dot{M}$). To prove this claim we use the compatibility of the collars c_M, c_Q with g, because it then suffices to show that

$$(g \mid \dot{M}) \times 1 : \dot{M} \times I \to \dot{Q} \times I$$

is transimplicial at points of $\dot{M} \times 0$ to the canonical triangulation $(J \times I)'$ of $\dot{Q} \times I$. Now we can use the fact that $g | \dot{M} = f | \dot{M}$, which is transimplicial to J. Given $x \in \dot{M} = \dot{M} \times 0$, suppose $fx \in \mathring{A}$, $A \in J = J \times 0$. Let v be a vertex of A, $L = lk(A, K)$, $L_1 = lk(A, J)$. By the transimpliciality of $f | \dot{M}$ we have a commutative diagram

$$\begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{1 \times k} & N \times D^a & \xrightarrow{\text{projn}} & N \\ \downarrow \varphi & & \downarrow \psi & & \downarrow \subset \\ f^{-1}AL_1 & \xrightarrow{f} & AL_1 & \xrightarrow{s^A} & vL_1 \end{array}$$

Let $U = [\psi(N \times D^a) \times I] \cap AL$, and let $r : \dot{Q} \times I \to \dot{Q}$ be the projection. Define $\theta : U \to D^a$ as the composition

$$U \xrightarrow{r} \psi(N \times D^a) \xleftarrow{\psi} N \times D^a \xrightarrow{\text{projn}} D^a.$$

Then $s^A \times \theta : U \to vL \times D^a$ is a piecewise linear map onto a neighbourhood of $(v, 0)$. Moreover it is an embedding because given $u_1 \neq u_2$ such that $s^A u_1 = s^A u_2$, then $u_1 u_2$ is parallel to A, and so is $(vu_1)(vu_2)$, implying that $\theta u_1 \neq \theta u_2$. Therefore, choosing discs $N_* \subset vL$, $D^a_* \subset D^a$ such that $N_* \times D^a_* \subset (s^A \times \theta)U$, we can define

$$\psi_* = (s^A \times \theta)^{-1} : N_* \times D^a_* \to AL.$$

The required diagram for the transimpliciality of $(f | \dot{M}) \times 1$ at x can now be built up in the usual fashion. Therefore g is transimplicial to K at points of $\dot{M}$.

There remains to isotop g transimplicial on the interior (keeping $\dot{Q}$ fixed) as follows. By Lemma 4 g is transimplicial to K at all points in some open neighbourhood U of $\dot{M}$. Let $K^{(2)}$ be the second barycentric derived of K. Choose a triangulation K_1 of M and a subdivision K_2 of $K^{(2)}$ such that

(a) $g : K_1 \to K_2$ is simplicial, and
(b) if V is the closed simplicial neighbourhood of $\dot{K}_1$ in K_1, then $V \subset U$.

Now perform the t-shifts of Lemma 14 in order of decreasing dimension for all simplexes $T \in K_1$ such that $\mathring{T} \subset M - V$. Then, as in the proof of Theorem 4, we see that g becomes transimplicial to K_2, and therefore to K, at all points of $M - V$. By Lemma 14 g remains transimplicial to K at points of V.

The proof of Theorem 5 is complete.

Remark. The significance of Lemma 14 in the above proof should now be apparent. At the last stage we had an embedding g transimplicial to K at points of $\dot{M}$. If we had just haphazardly made interior shifts of g with respect to some subdivision of K, then we may well have introduced new folds at boundary points, and so lost the transimplicial property there.

§10. RELATIVE TRANSVERSALITY?

We were able to prove relative transimpliciality (in Theorem 5) but not relative transversality. We tried the procedure

transversal on the boundary	$\Rightarrow$	transimplicial on the boundary	$\Rightarrow$	isotop transimplicial on the interior	$\Rightarrow$	transversal on the interior,

and although the second two steps are given by Theorem 5 and Lemma 6, we failed to achieve the first step. Essentially it is a passage from local to global, because transversality is local but transimpliciality is global, in the sense that an atlas is local while a triangulation is global. It is true that given manifolds $M \subset Q$, it is possible to triangulate Q so that M is transimplicial as follows: triangulate Q anyhow, ambient isotop M transimplicial, and then apply the inverse isotopy to move both M and the triangulation back. But the question is whether it is possible to have another manifold as a subcomplex at the same time.

CONJECTURE 1. *Given two transversal submanifolds of Q, then it is possible to triangulate Q so that one is a subcomplex and the other transimplicial.*
Conjecture 1 would supply the missing step to prove:

CONJECTURE 2. (Relative Transversality). *If M, P are proper submanifolds of Q such that $\dot{M}$, $\dot{P}$ are transversal in $\dot{Q}$, then M can be ambient isotoped transversal to P keeping $\dot{Q}$ fixed.*
A special case of Conjecture 2, which in fact turns out to be equivalent to Conjecture 2 is:

CONJECTURE 3. *Transversal spheres S^{m-1}, $S^{p-1} \subset S^{q-1}$ can be spanned by transversal discs D^m, $D^p \subset D^q$.*
Joining linearly to interior points is no good, because if we join them to the same point the discs fail to be transversal at that point, and if we join them to separate points, they fail to be transversal at the boundary (by the folded disc phenomenon). Conjecture 2 would imply:

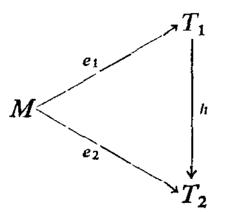
CONJECTURE 4. *If M, Q are closed and $f, g : M \to Q$ are homotopic maps transversal to P, then $f^{-1}P$, $g^{-1}P$ are cobordant.*
Summarising:

Conjecture 1 $\Rightarrow$ Conjecture 2 $\Leftrightarrow$ Conjecture 3 $\Rightarrow$ Conjecture 4.

§11. TUBES

Definition. We use the word *tube* as an abbreviation for the term "abstract regular neighbourhood", which is rather a mouthful. Let M^m be closed. Define a *t-tube on M* to be a manifold T^{m+t} together with a proper (locally flat) embedding $e : M \to T$ such that $T \searrow eM$. In other words T is a regular neighbourhood of a homeomorphic copy of M. We call t the dimension of the tube.

Two tubes are *homeomorphic* if there exists a homeomorphism h making a commutative diagram



Let $\mathscr{T}^t(M)$ denote the set of homeomorphy classes of t-tubes on M, and let $\mathscr{T}(M) = \sum_0^\infty \mathscr{T}^t(M)$.

Remarks. 1. Tubes are the natural analogue in piecewise linear theory of vector bundles in differential theory. The existence and uniqueness of regular neighbourhoods show that any proper embedding $M \subset Q$ determines a unique element of $\mathscr{T}^{q-m}(M)$, which we call the *normal tube.*

2. The important thing about tubes is that, like tubes in ordinary life, they are not fibered. In fact Hirsch's example is a 3-tube on S^4 that cannot be fibered. In some sense the lack of fibering is more "geometrical" because the tube is more homogeneous.

3. In the stable range, $t \geqq m + 2$ Haefliger and Wall [5] have shown that any tube can be fibered with t-discs, and so $\mathscr{T}^t(M)$ coincides with $K_{top}^t(M)$ of piecewise linear microbundle theory.

4. The collapse $T \searrow eM$ determines a homotopy equivalence $\pi : T \to M$ such that $\pi e = 1$. However π is not natural, not unique, and not in general a fibering. The non-naturality of π reveals itself, when it turns out to be no good for defining induced tubes.

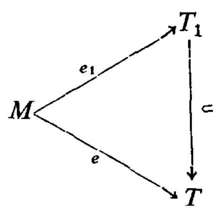
5. There is a trivial tube $0 \in \mathscr{T}^t(M)$ containing $M \times D^t$, and a suspension

$$\mathscr{T}^t(M) \to \mathscr{T}^{t+1}(M)$$

given by product with I, which stabilises in the stable range. To examine the structure of $\mathscr{T}(M)$ more thoroughly we define below subtubes, quotient tubes, induced tubes and Whitney sums.

6. The concept of tube generalises to polyhedra other than manifolds, to give a theory totally different from vector bundle theory, even in the stable range.

Subtubes. Call $e_1 : M \to T_1$ a subtube of $e : M \to T$ if T_1 is a proper (locally flat) submanifold of T such that $T \searrow T_1$ and the diagram



is commutative. Call two subtubes $T_1, T_2 \subset T$ *transversal* if T_1, T_2 intersect transversally in eM. Notice that in this case $t = t_1 + t_2$. We call the class of T_2 the *quotient tube* T/T_1.

COROLLARY TO THEOREM 3. *Quotient tubes exist.*

Question. Are they nnique?

We can question not only whether two such T_2's are unique up to homeomorphism, but whether they are unique up to ambient isotopy, keeping T_1 fixed.

Proof of Corollary. Given $eM \subset T_1 \subset T$, Theorem 3 furnishes a manifold P intersecting T_1 transversally in eM. So far P is not proper. Triangulate everything and let N be a second derived neighbourhood of T_1 in T. Then N is a tube, and T_1 a subtube because $N \searrow T_1$. Also $N \cap P$ is a subtube because $N \searrow (N \cap P) \cup T_1 \searrow N \cap P$, and $N \cap P$ cuts T_1 transversally.

By uniqueness of regular neighbourhoods, there is a homeomorphism $N \to T$ keeping T_1 fixed, and throwing $N \cap P$ onto T_2, say. We have shown T_2 exists.

Quotient normal tubes. Suppose we are given proper embeddings $M^m \subset P^p \subset Q^q$, where M is closed. Define the quotient normal tube on M to be the quotient tube T_Q/T_P where T_P, T_Q are regular neighbourhoods of M in P, Q such that T_P is a subtube of T_Q. Notice that $\dim(T_Q/T_P) = q - p$.

Induced tubes. Given a map $f: M_1 \to M_2$ between closed manifolds and a tube $e_2 : M_2 \to T_2$ on the target, define the *induced tube* on M_1 to be the quotient normal tube of

$$M_1 \xrightarrow{\Gamma f} M_1 \times M_2 \xrightarrow{1 \times e_2} M_1 \times T_2.$$

Notice that the induced tube has the same dimension as the given tube. By the above, induced tubes exist, but we do not know if they are unique.

Remark. Normally induced objects are defined categorically. For example if $\Pi : V_2 \to M_2$ is a vector bundle then the induced vector bundle is the pull-back of

$$\begin{array}{ccc} & & V_2 \\ & & \downarrow{\scriptstyle \pi} \\ M_1 & \xrightarrow{f} & M_2. \end{array}$$

However in the case of tubes π is non-natural, and consequently the pull-back is not in general a manifold. What is natural is the embedding $e_1 : M_1 \to T_1$ of a tube on the source of f, but the push-out of

$$\begin{array}{ccc} T_1 & & \\ \uparrow{\scriptstyle e_1} & & \\ M_1 & \longrightarrow & M_2 \end{array}$$

is again not in general a manifold. Therefore neither pull-backs nor push-outs give induced tubes, and we have to work for them.

Whitney sums. Given tubes $e_1 : M \to T_1$ and $e_2 : M \to T_2$ on the same manifold M, define the Whitney sum $T = T_1 \oplus T_2$ to be the quotient normal tube of

$$M \xrightarrow{\text{diagonal}} M \times M \xrightarrow{e_1 \times e_2} T_1 \times T_2.$$

Notice that $t = t_1 + t_2$, and so the Whitney sum gives a product $\mathscr{T}^{t_1} \times \mathscr{T}^{t_2} \to \mathscr{T}^{t_1+t_2}$. Again we have existence, but uniqueness is unsolved.

Questions. (i) Can T_1, T_2 be embedded transversally in $T_1 \oplus T_2$?
(ii) Is the Whitney sum homeomorphic to the tube induced from $e_1 : M \to T_1$ by $\pi_2 : T_2 \to M$, and vice versa?

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