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The (q, h)-Laplace transform on discrete time scales

Mohamad Rafi Segi Rahmat

School of Applied Mathematics, The University of Nottingham Malaysia Campus, Jalan Broga, 43500 Semenyih, Selangor Darul Ehsan, Malaysia

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ABSTRACT

In this paper, we develop a (q, h)-Laplace transform on specific time scales. We show that the transform is reduced for $h = 0$ to the q-Laplace transform, reduce for $q = 1$ to the h-Laplace transform and reduced for $q = h = 1$ to the Z-transform. Finally, we employ the (q, h)-Laplace transform to produce some key results.

1. Introduction and preliminaries

Laplace transform is one of the finest tools available to solve linear differential equations with constant coefficients and certain integral equations, while the Z-transform, which can be considered as a discrete version of the Laplace transform, is well suited for linear recurrence relations and certain summation equations. The theory of Laplace transforms on time scales, which is intended to unify and to generalize the continuous and discrete cases, was initiated by Hilger [10] and then developed by Bohner et al. [4,5,7]. In [8], the authors introduce the concept of h-Laplace and q-Laplace transforms on discrete time scales $h\mathbb{Z} = \mathbb{T}_h$ and $q^{\mathbb{N}_0} = \mathbb{T}_q$, respectively. Many interesting results were obtained including the convolution and inverse Laplace transform. In a very recent paper [9], Ćermák and Nechvátal introduce the discrete time scale $\mathbb{T}_{(q,h)}$ which unifies the time scales $\mathbb{T}_h$ and $\mathbb{T}_q$ in their attempt to introduce (q, h)-fractional calculus.

In this paper, we develop (q, h)-Laplace transform on a special discrete time scale $\mathbb{T}^{t_0}_{(q,h)}$ for $t_0 > 0$, which can be reduced to the h-Laplace transform (the case $q = 1$) and q-Laplace transform (the case $h = 0$) or to the Z-transform (the case $q = h = 1$) (see [11]).

For the convenience of readers, we provide some basic concepts concerning the delta calculus on time scales. For more details, one may refer to [1,4,6]. By time scale $\mathbb{T}$ we understand any nonempty, closed subset of reals with the ordering inherited from reals. Thus the reals $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, the h-numbers $h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with fixed $h > 0$, and the q-numbers $q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with fixed $q > 1$ are examples of time scales.

For any $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

The symbol $f^{\Delta}(t)$ is called the delta derivative (Δ-derivative) of the function $f : \mathbb{T} \to \mathbb{C}$ at $t \in \mathbb{T}^{\kappa}$. Considering discrete time scales (i.e., such that $\mu(t) \neq 0$ for $t \in \mathbb{T}$) $f^{\Delta}(t)$ exist for all $t \in \mathbb{T}$ and they are given by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

E-mail address: Mohd.Rafi@nottingham.edu.my.

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The kth order delta derivatives f^{Δ^k} are defined recursively by

$$f^{\Delta^0} = f \quad \text{and} \quad f^{\Delta^{k+1}} = (f^{\Delta^k})^{\Delta} \quad \text{for} \quad k \in \mathbb{N}_0.$$

A function $F : \mathbb{T} \to \mathbb{C}$ is called a Δ-antiderivative of a function $f : \mathbb{T} \to \mathbb{C}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. Hence, the Δ-integral of f over the time scale interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T}, a \le t \le b\}$, $a, b \in \mathbb{T}$ is defined by

$$\int_a^b f(t)\Delta t := F(b) - F(a).$$

It is known that considering discrete time scales this delta integral exists and can be calculated (provided $a < b$) via the formula

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} f(t)\mu(t). \tag{1}$$

A function $p : \mathbb{T} \to \mathbb{C}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive functions $f : \mathbb{T} \to \mathbb{C}$ will be denoted by $\mathcal{R}$. The set $\mathcal{R}$ forms an Abelian group under the addition $\oplus$ defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \quad \forall t \in \mathbb{T}$$

and the additive inverse of p in this group is denoted by $\ominus p$, and it is given by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)}, \quad \forall\, t \in \mathbb{T}.$$

For later reference we cite the following existence and uniqueness result for initial value problem.

Suppose $p \in \mathcal{R}$. Fix $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{C}$. The unique solution of the initial value problem (IVP)

$$y^{\Delta} = p(t)y, \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t, t_0)y_0,$$

(see [4, Theorem 2.62]). The function $e_p(\cdot, t_0)$ is called the delta exponential function. For details and results concerning the delta exponential function, we refer to the book [4].

The following concepts are introduced and investigated in [7].

Definition 1.1. Suppose $x : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is a locally Δ-integrable function, i.e., it is Δ-integrable over each compact subinterval of $[t_0, \infty)_{\mathbb{T}}$. Then the delta Laplace transform of x is defined by

$$\mathcal{L}_{\Delta}\{x\}(z) := \int_{t_0}^{\infty} x(t)e_{\ominus z}(\sigma(t), t_0)\Delta t \quad \text{for } z \in \mathcal{D}\{x\}, \tag{2}$$

where $\mathcal{D}\{x\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper Δ-integral exists.

Definition 1.2. For a given function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, its shift (or delay) $\hat{f}(t, s)$ is defined as the solution of the problem

$$\begin{aligned} &\hat{f}^{\Delta_t}(t, \sigma(s)) = -\hat{f}^{\Delta_s}(t, s), \quad t, s \in \mathbb{T}, t \ge s \ge t_0, \\ &\hat{f}(t, t_0) = f(t), \quad t \in \mathbb{T}, t \ge t_0. \end{aligned} \tag{3}$$

The unique solution of (3) is given by

$$\hat{f}(t, s) = \sum_{k=0}^{\infty} h_k(t, s) f^{\Delta^k}(t_0), \tag{4}$$

where the monomials $h_k(t, s)$ are defined in [4].

Definition 1.3. For given functions $f, g : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, their convolution $f * g$ is defined by

$$(f * g)(t) := \int_{t_0}^{t} \hat{f}(t, \sigma(s))g(s)\Delta s, \quad t \in \mathbb{T}, t \ge t_0, \tag{5}$$

where $\hat{f}$ is the shift of f introduced in Definition 1.2.

2. The (q, h)-Laplace transform

The most significant discrete time scales are those originating from arithmetic and geometric sequence of reals, namely

$$\mathbb{T}_h^{t_0} = \{t_0 + kh : k \in \mathbb{Z}\}, h > 0 \quad \text{and} \quad \mathbb{T}_q^{t_0} = \{t_0 q^k : k \in \mathbb{Z}\} \cup \{0\}, \quad q > 1,$$

respectively, where $t_0 \in \mathbb{R}$. These sets form the basis for the study of h-calculus and q-calculus.

In [8], the authors have introduced the concepts of h-Laplace and q-Laplace transforms. In this section, we introduce the concept of (q, h)-Laplace transform which will be the unification of the h-Laplace and q-Laplace transform in the literature.

Consider the two-parameter time scale (see [9]) given by

$$\mathbb{T}_{(q,h)}^{t_0} := \{t_0 q^k + [k]_q h : k \in \mathbb{Z}\} \cup \left\{ \frac{h}{1-q} \right\}, \tag{6}$$

where $t_0 \in \mathbb{R}$, $q \geq 1$, $h \geq 0$, $q + h > 1$. Of course, $\mathbb{T}_{(q,h)}^{t_0} = \mathbb{T}_q^{t_0}$ provided $h = 0$ and $\mathbb{T}_{(q,h)}^{t_0} = \mathbb{T}_h^{t_0}$ provided $q = 1$ (in this case we put $h/(1-q) = -\infty$). It is clear that, for $t \in \mathbb{T}_{(q,h)}^{t_0}$, we have

$$\sigma(t) = qt + h \quad \text{and} \quad \mu(t) = (q-1)t + h.$$

Let $t \in \mathbb{T}_{(q,h)}^{t_0}$ and $f : \mathbb{T}_{(q,h)}^{t_0} \to \mathbb{C}$. Then the delta (q, h)-derivative of f at t is defined by

$$f^{\Delta_{(q,h)}}(t) := \frac{f(qt+h) - f(t)}{(q-1)t + h}. \tag{7}$$

It follows that for any complex number z, the initial value problem

$$y^{\Delta_{(q,h)}}(t) = zy(t), \quad t \in \mathbb{T}_{(q,h)}^{t_0}, \quad y(s) = 1$$

takes the form

$$y(qt+h) = \big(1 + (q't + h)z\big)\, y(t), \quad t \in \mathbb{T}_{(q,h)}^{t_0}, \quad y(s) = 1,$$

where $q' = q - 1$. Hence $e_z(t, s)$ has ($z \neq -1/(q't + h)$) the form

$$e_z(t, s) = \prod_{r \in [s,t)} \big(1 + (q'r + h)z\big) \quad \text{for all } t, s \in \mathbb{T}_{(q,h)}^{t_0}.$$

Now, for $t, s \in \mathbb{T}_{(q,h)}^{t_0}$, set $t = q_h^n = t_0 q^n + [n]_q h$ and $s = q_h^m = t_0 q^m + [m]_q h$. Then, it can be easily verified that

$$e_z(q_h^n, q_h^m) = \prod_{k=m}^{n-1} \big(1 + (q' q_h^k + h)z\big).$$

Example 2.1. For the case $\mathbb{T}_{(1,h)}^0 = \mathbb{T}_h = h\mathbb{Z}$, and for all $t, s \in h\mathbb{Z}$, we have

$$\begin{aligned} e_z(t, s) &= \prod_{r \in [s,t)} (1 + hz) \\ &= \prod_{k=s/h}^{t/h} (1 + hz) \\ &= (1 + hz)^{\frac{t-s}{h}} \end{aligned}$$

and in the case $\mathbb{T}_{(q,0)}^1 = \mathbb{T}_q = q^{\mathbb{N}_0}$, by letting $t = q^n$ and $s = q^m$ with $n, m \in \mathbb{N}_0$, we have

$$\begin{aligned} e_z(q^n, q^m) &= \prod_{r \in [q^n, q^m)} \big(1 + q'rz\big) \\ &= \prod_{k=m}^{n-1} (1 + q'q^k z) \quad \text{if} \quad n \geq m \\ &= \prod_{k=n}^{m-1} \frac{1}{(1 + q'q^k z)} \quad \text{if } n \leq m. \end{aligned}$$

Next, for $t \in \mathbb{T}_{(q,h)}^{t_0}$ and $\mu(t) = (q-1)t + h$, we have

$$\ominus z(t) = -\frac{z}{1 + \mu(t)z} = -\frac{z}{1 + (q't + h)z},$$

where $q' = q - 1$ so that the initial value problem

$$y^{\Delta_{(q,h)}} = (\ominus z)(t)y(t), \quad t \in \mathbb{T}^{t_0}_{(q,h)}, \quad y(s) = 1$$

takes the form

$$y(qt+h) = \frac{1}{1+(q't+h)z}y(t), \quad t \in \mathbb{T}^{t_0}_{(q,h)}, \quad y(s) = 1.$$

Hence $e_{\ominus z}(t,s)$ has (again $z \neq -1/(q't+h)$) the form

$$e_{\ominus z}(t,s) = \prod_{r\in[s,t)} \frac{1}{1+(q'r+h)z} \quad \text{for all } t,s \in \mathbb{T}^{t_0}_{(q,h)}.$$

Consequently, for any function $x : [s,\infty)_{\mathbb{T}^{t_0}_{(q,h)}} \to \mathbb{C}$, its Laplace transform has the form

$$\begin{aligned}
\mathcal{L}\{x\}(z) &= \int_s^\infty x(t)e_{\ominus z}(\sigma(t),s)\Delta t \\
&= \sum_{t\in[s,\infty)} \mu(t)x(t)e_{\ominus z}(\sigma(t),s) \\
&= \sum_{t\in[s,\infty)} \frac{(q't+h)x(t)}{\prod\limits_{r\in[s,\sigma(t))}(1+(q'r+h)z)}.
\end{aligned}$$

Thus, we arrived to the following definition.

Definition 2.2. If $x : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{C}$ is a function, then its (q,h)-Laplace transform is defined by

$$X_{(q,h)}(z) = \mathcal{L}_{(q,h)}\{x\}(z) := \sum_{n=0}^{\infty} \frac{(q'q_h^n + h)x(q_h^n)}{\prod\limits_{k=0}^{n}\left(1+(q'q_h^k+h)z\right)} \tag{8}$$

where $q_h^j = t_0q^j + [j]_q h$ for $j \in \mathbb{Z}$, and for those values of $z \neq -\frac{1}{q'q_h^k+h}$ for which this series converges.

For some particular time scales, we have the following examples.

Example 2.3. Clearly, for $s,t \in \mathbb{T}^0_{(1,h)} = \mathbb{T}_h = h\mathbb{Z}$, we have $\sigma(t) = t+h$, $q' = 0$ and $1_h^k = kh$. Hence if $x : \mathbb{T}^0_{(1,h)} \to \mathbb{C}$, then relation (8) becomes

$$\begin{aligned}
X_h(z) = \mathcal{L}_h\{x\}(z) &= \sum_{n=0}^{\infty} \frac{hx(nh)}{\prod\limits_{k=0}^{n}(1+hz)} \\
&= h\sum_{n=0}^{\infty} \frac{x(nh)}{(1+hz)^{n+1}} \\
&= \frac{h}{1+hz}\sum_{n=0}^{\infty} \frac{x(nh)}{(1+hz)^n},
\end{aligned}$$

as defined in [8].

Example 2.4. For $s,t \in \mathbb{T}^1_{(q,0)} = \mathbb{T}_q = q^{\mathbb{N}_0}$, we have $\sigma(t) = qt$. Hence if $x : \mathbb{T}^1_{(q,0)} \to \mathbb{C}$, then

$$\begin{aligned}
X_q(z) = \mathcal{L}_q\{x\}(z) &= \sum_{t\in[1,\infty)} \frac{x(t)(q't)}{\prod\limits_{r\in[s,\sigma(t))}(1+q'rz)} \\
&= q'\sum_{t\in[0,\infty)} \frac{q^nx(q^n)}{\prod\limits_{\tau\in[q^{n_0},q^{n+1})}(1+q'sz)} \\
&= q'\sum_{n=n_0}^{\infty} \frac{q^nx(q^n)}{\prod\limits_{k=n_0}^{n}(1+q'q^kz)}
\end{aligned}$$

(see [8]).

Example 2.5. For $s, t \in \mathbb{T}^{t_0}_{(1,1)} = \mathbb{Z}$, we have $\sigma(t) = t + 1$. Hence if $x : \mathbb{T}^{t_0}_{(1,1)} \to \mathbb{C}$,

$$\begin{aligned} X_{(1,1)}(z) &= \sum_{n=0}^{\infty} \frac{x(n)}{\prod_{k=0}^{n}(1+z)} \\ &= \sum_{n=0}^{\infty} \frac{x(n)}{(1+z)^{n+1}} \\ &= \frac{Z\{x\}(z+1)}{1+z} \end{aligned}$$

where $Z\{x\}(z) = \sum_{t=0}^{\infty} \frac{x(t)}{z^t}$ is the classical Z-transform (see e.g., [11]).

Remark 2.6. Let $t = q_h^k = t_0 q^k + [k]_q h$, then (8) becomes

$$\mathcal{L}_{(q,h)}\{x\}(z) := (q' t_0 + h) \sum_{n=0}^{\infty} \frac{q^n x(t_0 q^n + [n]_q h)}{\prod_{k=0}^{n}\left[1 + q^k (q' t_0 + h) z\right]}. \tag{9}$$

Let us set

$$P_n(z) = \prod_{k=0}^{n} \left(1 + (q' q_h^k + h) z\right) \quad n \in \mathbb{N}_0, \tag{10}$$

which is a polynomial in z of degree $n + 1$. It can be easily verified that the relations

$$P_n(z) - P_{n-1}(z) = (q' q_h^n + h) z P_{n-1}(z) \tag{11}$$

and

$$\frac{1}{P_{n-1}(z)} - \frac{1}{P_n(z)} = z \frac{(q' q_h^n + h)}{P_n(z)}, \quad n \in \mathbb{N}_0 \tag{12}$$

hold, where $P_{-1}(z) = 1$.

Let $\alpha_k = -\frac{1}{q' q_h^k + h}$, $k \in \mathbb{N}_0$. For any $\delta > 0$ and $k \in \mathbb{N}_0$, we set

$$D_\delta^k = \{z \in \mathbb{C} : |z - \alpha_k| < \delta\}$$

and

$$\Omega_\delta = \mathbb{C} \setminus \bigcup_{k=0}^{\infty} D_\delta^k = \{z \in \mathbb{C} : |z - \alpha_k| \geq \delta, \forall\, k \in \mathbb{N}_0\}.$$

Lemma 2.7. *For any $z \in D_\delta^k$ and $q_h^k = t_0 q^k + [k]_q h$,*

$$|P_n(z)| \geq [(t_0 q' + h)\delta]^{n+1} q^{\frac{n(n+1)}{2}}, \quad n \in \mathbb{N}_0 \cup \{-1\}.$$

Therefore, for an arbitrary number $R > 0$, there exists a positive number $n_0 = n_0(R, \delta, q, h)$ such that

$$|P_n(z)| \geq R^{n+1}$$

for all $n \geq n_0$ and $z \in \Omega_\delta$. In particular,

$$\lim_{n\to\infty} P_n(z) = \infty \quad \forall\, z \in \Omega_\delta.$$

Proof. Let $q_h^k = t_0 q^k + [k]_q h$. For any $z \in \Omega_\delta$, we have

$$\begin{aligned} |P_n(z)| &= \left| \prod_{k=0}^{n} \left(1 + (q' q_h^k + h) z\right) \right| = \left| \prod_{k=0}^{n} (q' q_h^k + h)(z - \alpha_k) \right| \\ &\geq \prod_{k=0}^{n} (q' q_h^k + h)\delta = \prod_{k=0}^{n} q^k (t_0 q' + h)\delta \end{aligned}$$

$$= [(t_0 q' + h)\delta]^{n+1} \prod_{k=0}^{n} q^k$$
$$= [(t_0 q' + h)\delta]^{n+1} q^{\frac{n(n+1)}{2}}.$$

Note that

$$|P_n(z)| \ge [(t_0 q' + h)\delta q^{\frac{n}{2}}]^{n+1}.$$

Since, $q > 1$, we can choose for any given number $R > 0$ a positive integer $n_0 = n_0(R, \delta, q, h)$ such that

$$(t_0 q' + h)\delta q^{\frac{n}{2}} \ge R \quad \text{for all } n \ge n_0.$$

Hence, we have

$$|P_n(z)| \ge R^{n+1}. \quad \square$$

Example 2.8. Let $x(t) = 1$ for $t \in \mathbb{T}^{t_0}_{(q,h)}$. Then, using (12), we have

$$\begin{aligned}
\mathcal{L}_{(q,h)}\{1\}(z) &= \sum_{n=0}^{\infty} \frac{(q' q_h^n + h)}{\prod\limits_{k=n_0}^{n} (1 + q' q^k z)} = \sum_{n=0}^{\infty} \frac{(q' q_h^n + h)}{P_n(z)} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} \left\{ \frac{1}{P_{n-1}(z)} - \frac{1}{P_n(z)} \right\} \qquad \text{(A telescoping series)} \\
&= \frac{1}{z} \lim_{n\to\infty} \left[1 - \frac{1}{P_m(z)} \right] = \frac{1}{z}.
\end{aligned}$$

Let $\mathcal{F}_\delta$ be the class of functions $x : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{C}$ for which the (q, h)-Laplace transform exist and satisfying the condition

$$\sum_{n=0}^{\infty} [(t_0 q' + h)\delta]^{-n} q^{\frac{-n(n-1)}{2}} |x(q_h^n)| < \infty.$$

Theorem 2.9. *For any $x \in \mathcal{F}_\delta$, the series in (8) converges uniformly with respect to z in the region Ω_δ, and therefore its sum $X_{(q,h)}(z)$ is an analytic function in Ω_δ.*

Proof. The proof follows from Lemma 2.7. $\square$

The following theorems are evident.

Theorem 2.10. *Suppose that $x \in \mathcal{F}_\delta$. Then*

$$\mathcal{L}_{(q,h)}\{x^{\Delta_{(q,h)}}\}(z) = z X_{(q,h)}(z) - x(t_0), \tag{13}$$
$$\mathcal{L}_{(q,h)}\{x^{\Delta^2_{(q,h)}}\}(z) = z^2 X_{(q,h)}(z) - z x(t_0) - x^{\Delta}(t_0). \tag{14}$$

Proof. Using the Definition 2.2, we have

$$\begin{aligned}
\mathcal{L}_{(q,h)}\{x^{\Delta_{(q,h)}}\}(z) &= \sum_{n=0}^{\infty} \frac{(q' t + h) x^{\Delta_{(q,h)}}(q_h^n)}{P_n(z)} = \sum_{n=0}^{\infty} \frac{x(q_h^{n+1}) - x(q_h^n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(q_h^{n+1})}{P_n(z)} - \sum_{n=0}^{\infty} \frac{x(q_h^n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(q_h^{n+1})}{P_{n+1}(z)} (1 + (q' q_h^{n+1} + h) z) - \sum_{n=0}^{\infty} \frac{x(q_h^n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(q_h^{n+1})}{P_{n+1}(z)} + z \sum_{n=0}^{\infty} \frac{(q' q_h^{n+1} + h) x(q_h^{n+1})}{P_{n+1}(z)} - \sum_{n=0}^{\infty} \frac{x(q_h^n)}{P_n(z)} \\
&= -\frac{x(q_h^0)}{P_0(z)} + z \sum_{n=1}^{\infty} \frac{(q' q_h^n + h) x(q_h^n)}{P_n(z)}
\end{aligned}$$

$$= -\frac{x(q_h^0)}{P_0(z)} - (q'q_h^0 + h)z\frac{x(q_h^0)}{P_0(z)} + z\sum_{n=0}^{\infty}\frac{(q'q_h^n + h)x(q_h^n)}{P_n(z)}$$

$$= -(1 + (q'q_h^0 + h)z)\frac{x(q_h^0)}{P_0(z)} + zX_{(q,h)}(z) = -x(t_0) + zX_{(q,h)}(z).$$

Formula (14) is obtained by applying (13) to the second (q, h)-derivative. □

3. The convolution theorem

For the case $\mathbb{T} = \mathbb{T}^{t_0}_{(q,h)}$, the shifting problem (3) takes the form

$$(q's + h)[\hat{f}(qt + h, qs + h) - \hat{f}(t, qs + h)] + (q't + h)[\hat{f}(t, qs + h) - \hat{f}(t, s)] = 0, \tag{15}$$

where $q' = q - 1$ and $t, s \in \mathbb{T}^{t_0}_{(q,h)}$, $t \geq s \geq t_0$,

$$\hat{f}(t, t_0) = f(t), \quad t \in \mathbb{T}^{t_0}_{(q,h)}, t \geq t_0,$$

where $f : [t_0, \infty)_{\mathbb{T}^{t_0}_{(q,h)}} \to \mathbb{C}$ is a given function.

To determine the unique solution of (15), we need the following notations (see, [2,3,9,12,13]). For $q > 1$ and $h > 0$,

$$[k]_q = \frac{q^k - 1}{q - 1}, \qquad [k]_1 = k \quad k \in \mathbb{N}_0,$$

$$[0]_q! = 1, \quad [n]_q! = \prod_{k=1}^{n}[k]_q,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!} = \prod_{i=1}^{n}\frac{q^{m-i+1} - 1}{q^i - 1}, \quad m, n \in \mathbb{N}_0,$$

$$(t - s)_q^{(n)} = t^n\prod_{k=0}^{n-1}(1 - q^k s/t),$$

$$t_h^{(n)} = \prod_{k=0}^{n-1}(t + kh).$$

The (q, h)-analogue of the power function [9] is defined as

$$(t - s)^{(n)}_{(q,h)} = \begin{cases} (\tilde{t} - \tilde{s})_q^{(n)}, & \text{for } q > 1, h \geq 0; \\ (t - s)_h^{(n)}, & \text{for } q = 1, h > 0, \end{cases}$$

where $\tilde{t} := t + \frac{h}{q-1}$ and $\tilde{s} := s + \frac{h}{q-1}$.

The (q, h)-calculus monomials are given by

$$h_n(t, s) = \frac{(t - s)^{(n)}_{(q,h)}}{[n]_q!}, \quad t, s \in \mathbb{T}^{t_0}_{(q,h)}. \tag{16}$$

Indeed, for $h = 0$ and $q > 1$, we have

$$h_n(t, s) = \frac{(t - s)_q^{(n)}}{[n]_q!} = \prod_{k=0}^{n-1}\frac{t - q^k s}{\sum\limits_{i=0}^{k} q^i}$$

as shown in [4].

For $q = 1$ and $h > 0$, we have

$$h_n(t, s) = \frac{(t - s)^{(n)}}{n!} = \prod_{k=0}^{n-1}\left(\frac{t - s + kh}{k + 1}\right).$$

In the case of $q = h = 1$, we obtain

$$h_n(t, s) = \frac{(t - s)^{(n)}}{n!} = \prod_{k=0}^{n-1}\left(\frac{t - s + k}{k + 1}\right) = \binom{t - s}{k}.$$

For $t_1, t_2 \in \mathbb{T}^{t_0}_{(q,h)}$ with $t_1 \geq t_2$, we define the positive integer $k = k(t_1, t_2)$ by the relation $t_1 = \sigma^k(t_2)$ (in this case the symbol σ^k means the kth iteration of the σ).

Corollary 3.1. *Let $t, s \in \mathbb{T}^{t_0}_{(q,h)}$ and $t > s$ be such that $t = q^k s + [k]_q h$ for some $k \in \mathbb{Z}$. Then, the following formula holds:*

$$h_n(t,s) = \begin{bmatrix} k(t,s) \\ n \end{bmatrix}_q (\mu(s))^n q^{\binom{n}{2}}, \tag{17}$$

where

$$k(t,s) = \begin{cases} \log_q \dfrac{\mu(t)}{\mu(s)}, & \text{for } q > 1, h \geq 0; \\ \dfrac{t-s}{h}, & \text{for } q = 1, h > 0. \end{cases}$$

Proof. Let $q > 1, h \geq 0$. If $t = q^k s + [k]_q h$, then $\mu(t) = q^k \mu(s)$.
Now,

$$\begin{aligned} (t-s)^{(n)}_{(q,h)} &= (\tilde{t})^n \prod_{j=0}^{n-1} (1 - q^j \tilde{s}/\tilde{t}) \\ &= (q-1)^{-n} (\mu(t))^n \prod_{j=0}^{n-1} (1 - q^{j-k}) \\ &= q^{nk} (\mu(s))^n \prod_{j=0}^{n-1} q^{j-k} \frac{q^{k-j} - 1}{q-1} \\ &= q^{\binom{n}{2}} (\mu(s))^n [k]_q \cdot [k-1]_q \cdots [k-n+1]_q \\ &= q^{\binom{n}{2}} (\mu(s))^n [n]_q! \begin{bmatrix} k \\ n \end{bmatrix}_q. \end{aligned}$$

Hence, for $q > 1, h \geq 0$, we have

$$h_n(t,s) = \begin{bmatrix} \log_q \dfrac{\mu(t)}{\mu(s)} \\ n \end{bmatrix}_q (\mu(s))^n q^{\binom{n}{2}}.$$

The case $q = 1, h > 0$ is an easy exercise. □

The following theorem is due to Čermák and Nechvátal [9], see also [2,3].

Theorem 3.2. *If $k \in \mathbb{Z}^+, f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{C}$ and $t \in \mathbb{T}^{t_0}_{(q,h)}$, $t > h/(1-q)$, then*

$$f^{\Delta^k_{(q,h)}}(t) = q^{-\binom{k}{2}} (\mu(t))^{-k} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} f(\sigma^{k-j}(t)).$$

Finally, we are in the position to present the unique solution of the problem (15).

Theorem 3.3. *Let $t, s \in \mathbb{T}^{t_0}_{(q,h)}$ be such that $t = \sigma^k(s)$ for a suitable $k = k(t,s) \in \mathbb{Z}$. Then, the shift of $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{C}$ is given by*

$$\hat{f}(t,s) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \left(\frac{\mu(s)}{\mu(t_0)} \right)^j \left(1 - \frac{\mu(s)}{\mu(t_0)} \right)^{(k-j)}_{(q,h)} f(\sigma^j(t_0)). \tag{18}$$

Proof. Let $q > 1$, $h \geq 0$ and $t, s \in \mathbb{T}^{t_0}_{(q,h)}$ where $t > s > t_0$ be such that $t = \sigma^a(s)$ and $s = \sigma^b(t_0)$ for some $a, b \in \mathbb{Z}$. Then $\mu(t) = q^a \mu(s)$ and $\mu(s) = q^b \mu(t_0)$. Let $k = a + b$. For brevity we set $s^k_h = q^k s + [k]_q h$. Then, according to (4) and Theorem 3.2, we have

$$\begin{aligned} \hat{f}(t,s) &= \hat{f}(s^k_h, s) = \sum_{m=0}^{k} h_m(s^k_h, s) f^{\Delta^m_{(q,h)}}(t_0) \\ &= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (\mu(s))^m q^{\binom{m}{2}} f^{\Delta^m_{(q,h)}}(t_0) \end{aligned}$$

$$= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q q^{(k-a)m} (\mu(t_0))^m q^{\binom{m}{2}} f^{\Delta^m_{(q,h)}}(t_0)$$

$$= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q q^{(k-a)m} \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\binom{j}{2}} f(\sigma^{m-j}(t_0))$$

$$= \sum_{m=0}^{k} \sum_{j=0}^{m} \begin{bmatrix} k \\ m \end{bmatrix}_q q^{(k-a)m} (-1)^{m-j} \begin{bmatrix} m \\ m-j \end{bmatrix}_q q^{\binom{m-j}{2}} f(\sigma^j(t_0))$$

$$= \sum_{j=0}^{k} \sum_{m=j}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} m \\ m-j \end{bmatrix}_q q^{(k-a)m} (-1)^{m-j} q^{\binom{m-j}{2}} f(\sigma^j(t_0))$$

$$= \sum_{j=0}^{k} \sum_{m=0}^{k-j} \begin{bmatrix} k \\ m+j \end{bmatrix}_q \begin{bmatrix} m+j \\ m \end{bmatrix}_q (-1)^m q^{(k-a)(m+j)+\binom{m}{2}} f(\sigma^j(t_0))$$

$$= \sum_{j=0}^{k} \sum_{m=0}^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} k-j \\ m \end{bmatrix}_q (-1)^m q^{(k-a)(m+j)+\binom{m}{2}} f(\sigma^j(t_0))$$

$$= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-a)j} \left[\sum_{m=0}^{k-j} \begin{bmatrix} k-j \\ m \end{bmatrix}_q (-q^{(k-a)})^m q^{\binom{m}{2}} \right] f(\sigma^j(t_0))$$

$$= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-a)j} (1 - q^{k-a})_q^{(k-j)} f(\sigma^j(t_0))$$

$$= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \left(\frac{\mu(s)}{\mu(t_0)} \right)^j \left(1 - \frac{\mu(s)}{\mu(t_0)} \right)_q^{(k-j)} f(\sigma^j(t_0)).$$

In the case $q = 1$, $h > 0$ we have $k = (t - t_0)/h$ and the property $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ implies the required formula quite analogously. □

Example 3.4. For $q > 1$, $h = 0$ and $t_0 = 1$, we have $\mu(s) = (q - 1)s$, $\mu(1) = q - 1$ and $\sigma^j(1) = q^j$. Then

$$\hat{f}(s_0^k, s) = \hat{f}(q^k s, s) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \left(\frac{\sigma(s)}{\sigma(1)} \right)^j \left(1 - \frac{\sigma(s)}{\sigma(1)} \right)_q^{(k-j)} f(\sigma^j(t_0))$$
$$= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q s^j (1 - s)_q^{(k-j)} f(q^j).$$

Hence, formula (18) is reduces to the quantum calculus case as shown in [7].

We observe that for $t, s \in \mathbb{T}^{t_0}_{(q,h)}$, if $t = q_h^n$ and $s = q_h^m$ for $m, n \in \mathbb{N}_0$ with $n \geq m$, then we have $\mu(q_h^n) = q^n \mu(t_0)$ and $\mu(q_h^m) = q^m \mu(t_0)$. Thus,

$$\hat{f}(q_h^n, q_h^m) = \sum_{j=0}^{n-m} \begin{bmatrix} n-m \\ j \end{bmatrix}_q q^{mj} (1 - q^m)_{(q,h)}^{(n-m-j)} f(\sigma^j(t_0)). \tag{19}$$

Definition 3.5. For a given functions $f, g : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{C}$, their convolution $f * q$ is defined by

$$(f * g)(q_h^n) = \sum_{k=0}^{n-1} \mu(q_h^k) \hat{f}(q_h^n, q_h^{k+1}) g(q_h^k)$$
$$= \sum_{k=0}^{n-1} (q' t_h^k + h) \left[\sum_{j=0}^{n-k-1} \begin{bmatrix} n-k-1 \\ j \end{bmatrix}_q q^{(k+1)j} (1 - q^{k+1})_{(q,h)}^{(n-k-j-1)} f^{\sigma^j}(t_0) \right] g(q_h^k)$$

where $q' = q - 1$ and $q_h^n = t_0 q^n + [n]_q h$, $n \in \mathbb{Z}$.

Theorem 3.6 (*Convolution Theorem*). *By assuming that $\mathcal{L}_{(q,h)}\{f\}(z)$, $\mathcal{L}_{(q,h)}\{g\}(z)$ and $\mathcal{L}_{(q,h)}\{f*g\}(z)$ exist for given $z \in \mathbb{C}$, we have*

$$\mathcal{L}_{(q,h)}\{f*g\}(z) = \mathcal{L}_{(q,h)}\{f\}(z)\mathcal{L}_{(q,h)}\{g\}(z). \tag{20}$$

Proof. By setting $e_{n,m}(z) = e_z(q_h^n, q_h^m)$ and $\hat{f}_{n,m} = \hat{f}(q_h^n, q_h^m)$, the proof follows analogously as that of Theorem 5.8 in [8]. □

Finally, we state the inverse Laplace transform. The proof follows analogously to Theorem 5.11 of [8].

Theorem 3.7. *Let $x \in \mathcal{F}_\delta$ and $X_{(q,h)}(z)$ be its (q,h)-Laplace transform defined by (8). Then*

$$x(q_h^n) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma X_{(q,h)}(z) \prod_{k=0}^{n-1} (1 + q' q_h^k z) \Delta z \quad \text{for } n \in \mathbb{N}_0,$$

where Γ is any positively oriented closed curve in the region Ω_δ that encloses all the points $\alpha_k = -1/(q' q_h^k + h)$ for $k \in \mathbb{N}_0$.

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