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Distance in Digraphs

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Abstract—The (directed) distance $\vec{d}(u,v)$ from a vertex u to a vertex v in a strong digraph D is the length of a shortest u-v path in D. Although this is the standard distance in digraphs, it is not a metric. Two other distances in digraphs are introduced, each of which is a metric. The maximum distance $\mathrm{md}(u,v)$ between two vertices u and v in a strong digraph is defined as $\mathrm{md}(u,v) = \max\{\vec{d}(u,v), \vec{d}(v,u)\}$. The sum distance $\mathrm{sd}(u,v)$ is defined as $\mathrm{sd}(u,v) = \vec{d}(u,v) + \vec{d}(v,u)$. Several results and problems concerning these metrics and such parameters as centers, medians, and peripheries are described.

Keywords—Digraph, Directed distance.

1. DISTANCE IN GRAPHS

The standard *distance* $d(u,v)$ between vertices u and v in a connected graph G is the length of a shortest u-v path in G. This distance is a metric, that is, it satisfies the following three properties:

(1) $d(u,v) \geq 0$ for all vertices u and v, and $d(u,v) = 0$, if and only if $u = v$;
(2) $d(u,v) = d(v,u)$ for all vertices u and v (the *symmetric* property); and
(3) $d(u,w) \leq d(u,v) + d(v,w)$ for all vertices u, v, and w (the *triangle inequality*).

The *eccentricity* $e(v)$ of a vertex v is the distance between v and a vertex farthest from v. The *radius* $\mathrm{rad}\, G$ and *diameter* $\mathrm{diam}\, G$ are defined by

$$\mathrm{rad}\, G = \min_{v \in V(G)} e(v) \quad \text{and} \quad \mathrm{diam}\, G = \max_{v \in V(G)} e(v).$$

The vertices of the graph G of Figure 1 are labeled with their eccentricities. Consequently, $\mathrm{rad}\, G = 2$ and $\mathrm{diam}\, G = 4$ for this graph G.

A simple but useful result concerning radius and diameter is recalled in the next theorem. Since the proof is informative, we include it.

THEOREM 1. *For every connected graph G,* $\mathrm{rad}\, G \leq \mathrm{diam}\, G \leq 2\,\mathrm{rad}\, G$.

PROOF. The first inequality follows from the definitions of radius and diameter. In order to establish the second inequality, let u and w be vertices such that $d(u,w) = \mathrm{diam}\, G$, and let v be a vertex with $e(v) = \mathrm{rad}\, G$. Then

$$\mathrm{diam}\, G = d(u,w) \leq d(u,v) + d(v,w) \leq e(v) + e(v) = 2\ \mathrm{rad}\ G. \qquad \blacksquare$$

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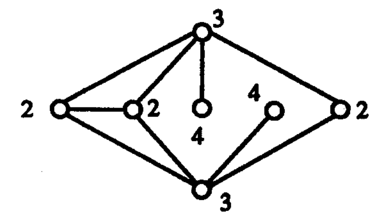

Figure 1.

In fact, if we have any metric on graphs, and define eccentricity, radius, and diameter, as above, in terms of this metric, then the corresponding Theorem 1 follows.

A vertex v is called a *central vertex* if $e(v) = \text{rad } G$, and the *center* $C(G)$ is the subgraph of G induced by its central vertices. The graph of Figure 1 is shown again in Figure 2, along with its center.

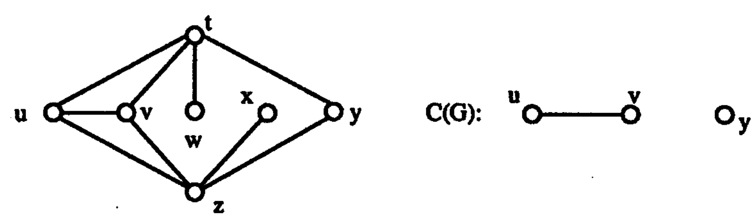

Figure 2.

A basic result on centers was established by Harary and Norman [1]. A *block* of a graph G is either a subgraph induced by a bridge or a maximal 2-connected subgraph of G. Other graph theory terms can be found in [2].

THEOREM 2. *(See [1].) The center of every connected graph G lies in a single block of G.*

The following result is due to Hedetniemi and appears in [3].

THEOREM 3. *(See [3].) For every graph G, there is a connected graph H with $C(H) \cong G$.*

The *appendage number* $A(G)$ of a graph G is the minimum number of vertices that must be added to G to produce a graph H whose center is isomorphic to G. The proof of Theorem 3 shows that $A(G) \leq 4$ for every graph G, and this bound is sharp.

A vertex v of G is a *peripheral vertex* if $e(v) = \text{diam } G$, and the *periphery* $P(G)$ of G is the subgraph induced by the peripheral vertices. For the graph G of Figure 2, $P(G)$ consists of two isolated vertices, namely, w and x. Bielak and Syslo [4] characterized those graphs that are peripheries of a connected graph.

THEOREM 4. *(See [4].) Let G be a graph. There exists a connected graph H such that $P(H) \cong G$, if and only if*

(i) *every vertex of G has eccentricity 1, or*
(ii) *no vertex of G has eccentricity 1.*

The center is only one concept that was introduced to describe the "middle" of a graph. We mention a second of these. The *distance* (or *status*) $d(v)$ of a vertex v in a connected graph G is defined as

$$d(v) = \sum_{u \in V(G)} d(v, u).$$

A vertex x is called a *median vertex* if $d(x) = \min_{v \in V(G)} d(v)$. The *median* $M(G)$ of G is the subgraph induced by the median vertices.

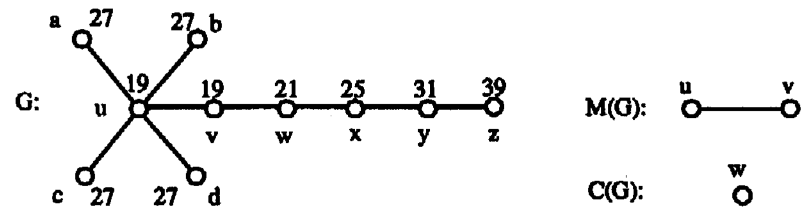

Figure 3.

The distances of the vertices of the graph G of Figure 3 are indicated in that figure, as well as the median and center of the graph.

Slater [5] established the analogue to Theorem 3 for medians.

THEOREM 5. *(See [5].) For every graph G, there exists a connected graph H such that $M(H) \cong G$.*

Hendry [6] showed that the results of Theorems 3 and 5 could be accomplished simultaneously.

THEOREM 6. *(See [6].) For every two graphs G_1 and G_2, there exists a connected graph H such that $C(H) \cong G_1$ and $M(H) \cong G_2$.*

In the proof of Theorem 6, Hendry constructed a graph H where $C(H)$ and $M(H)$ are disjoint, in fact, $d(C(H), M(H)) = 1$, where the distance between two subgraphs F_1 and F_2 of a connected graph G is defined as

$$d(F_1, F_2) = \min\{d(v_1, v_2) \mid v_1 \in V(F_1), v_2 \in V(F_2)\}.$$

Hendry's result may have come as a bit of a surprise, not only because he showed that any two graphs could be the center and median of the same graph, but that the center and the median need not even overlap. Holbert [7] then extended Hendry's result further.

THEOREM 7. *(See [7].) For every two graphs G_1 and G_2 and positive integer k, there exists a connected graph H with $C(H) \cong G_1$, $M(H) \cong G_2$, and $d(C(H), M(H)) = k$.*

At the other extreme, Novotny and Tian [8] showed that the center and median of a graph may overlap in any possible way.

THEOREM 8. *(See [8].) For two graphs G_1 and G_2 and every graph K that is isomorphic to an induced subgraph of G_1 and G_2, there exists a connected graph H with $C(H) \cong G_1$, $M(H) \cong G_2$, and $C(H) \cap M(H) \cong K$.*

2. DISTANCE IN DIGRAPHS

We now turn to the main topic of this paper: distance in digraphs. The standard distance in digraphs is directed distance. Let D be a strong digraph. The (*directed*) *distance* $\vec{d}(u, v)$ from a vertex u to a vertex v in D is the length of a shortest u-v (directed) path. This distance is *not* a metric, however. Although it satisfies the first property and the third property (the triangle inequality), it does not satisfy the symmetric property, in general. Indeed, $\vec{d}(u, v) = \vec{d}(v, u)$ for all pairs u, v of vertices of D if and only if D is a symmetric digraph, that is, if and only if D is a graph! Nevertheless, we can define some of the familiar concepts which emanate from distance in graphs.

The *eccentricity* $e(v)$ of a vertex v in a digraph D is the distance from v to a vertex farthest from v. The *radius* $\operatorname{rad} D$ of D is the minimum eccentricity among the vertices of D, while the *diameter* $\operatorname{diam} D$ is the maximum eccentricity. In the strong digraph D of Figure 4, the eccentricity of each vertex is indicated.

We note that $\operatorname{rad} D = 2$ and $\operatorname{diam} D = 5$ for the digraph D of Figure 4. Consequently, the familiar inequality $\operatorname{diam} D \leq 2 \operatorname{rad} D$ does not hold for directed distance in strong digraphs. We see where the proof of Theorem 1 fails in this instance. This can be done with the aid of the

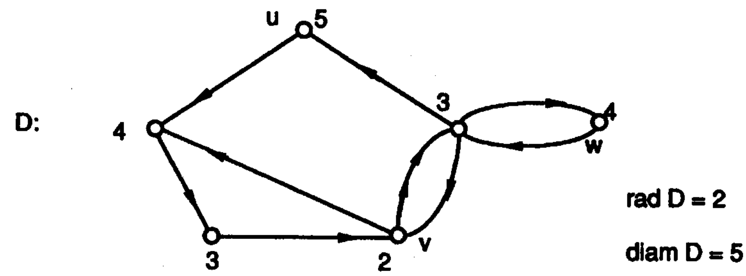

Figure 4.

digraph D of Figure 4. For this digraph, $\vec{d}(u,w) = \operatorname{diam} D$ and $e(v) = 2$. Proceeding as in the proof of Theorem 1, we have

$$\operatorname{diam} D = \vec{d}(u,w) \leq \vec{d}(u,v) + \vec{d}(v,w) \leq e(u) + e(v) = \operatorname{diam} D + \operatorname{rad} D.$$

Because the distance in graphs is symmetric, $d(u,v) = d(v,u)$ and we can write $d(u,v) \leq e(v) = \operatorname{rad} G$, but such is not the case for directed distance in digraphs. Indeed, all we can conclude there is the less-than-exciting inequalities $\operatorname{rad} D \leq \operatorname{diam} D \leq \operatorname{diam} D + \operatorname{rad} D$. Hence, a useful property of radius and diameter that holds for graphs is lost when the distance under study is not a metric. With this thought in mind, we turn to the problem of defining distance in other ways so that a metric is produced.

In a connected graph G, the familiar distance satisfies each of the following because it is symmetric:

(1) $d(u,v) = \min\{d(u,v), d(v,u)\}$,
(2) $d(u,v) = \max\{d(u,v), d(v,u)\}$,
(3) $d(u,v) = a \cdot d(u,v) + b \cdot d(v,u)$, $a, b \geq 0$, $a + b = 1$.

These rather simple observations suggest the possibility of defining new distances in strong digraphs in terms of directed distance, namely, to define

(1) $d(u,v) = \min\{\vec{d}(u,v), \vec{d}(v,u)\}$,
(2) $d(u,v) = \max\{\vec{d}(u,v), \vec{d}(v,u)\}$, or
(3) $d(u,v) = a \cdot \vec{d}(u,v) + b \cdot \vec{d}(v,u)$, $a, b \geq 0$, $a + b = 1$.

We first consider the possibility of defining $d(u,v) = \min\{\vec{d}(u,v), \vec{d}(v,u)\}$. This distance, though symmetric, does not satisfy the triangle inequality. For the vertices u, v, and w in the strong digraph D of Figure 5,

$$3 = d(u,w) > d(u,v) + d(v,w) = 2.$$

Consequently, definition (1) does not offer us the desired properties.

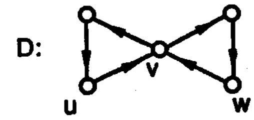

Figure 5.

We now turn to definition (2), namely, $d(u,v) = \max\{\vec{d}(u,v), \vec{d}(v,u)\}$. Certainly, this distance satisfies the first two properties required of a metric. It also satisfies the triangle inequality. To see this, let u, v, and w be any three vertices of a strong digraph D, and suppose

$\max\{\vec{d}(u,w), \vec{d}(w,u)\} = \vec{d}(u,w)$. Then

$$\begin{aligned} d(u,w) &= \max\left\{\vec{d}(u,w), \vec{d}(w,u)\right\} = \vec{d}(u,w) \\ &\leq \vec{d}(u,v) + \vec{d}(v,w) \\ &\leq \max\left\{\vec{d}(u,v), \vec{d}(v,u)\right\} + \max\left\{\vec{d}(v,w), \vec{d}(w,v)\right\} \\ &= d(u,v) + d(v,w). \end{aligned}$$

Thus, distance as defined in (2) is a metric. For future reference, we denote this distance by md (maximum distance), that is, $\mathrm{md}(u,v) = \max\{\vec{d}(u,v), \vec{d}(v,u)\}$.

The *converse* D' of a digraph D is that digraph produced by reversing the direction of every arc of D. The maximum distance between two vertices u and v in a strong digraph D may then also be defined as $\mathrm{md}(u,v) = \max\{\vec{d}_D(u,v), \vec{d}_{D'}(u,v)\}$.

We now consider definition (3) of distance, that is, define $d(u,v) = a \cdot \vec{d}(u,v) + b \cdot \vec{d}(v,u)$, where $a, b \geq 0$ and $a + b = 1$. First, we observe that

$$\begin{aligned} d(u,w) &= a \cdot \vec{d}(u,w) + b \cdot \vec{d}(w,u) \\ &\leq a\left(\vec{d}(u,v) + \vec{d}(v,w)\right) + b\left(\vec{d}(w,v) + \vec{d}(v,u)\right) \\ &= \left[a \cdot \vec{d}(u,v) + b \cdot \vec{d}(v,u)\right] + \left[a \cdot \vec{d}(v,w) + b \cdot \vec{d}(w,v)\right] \\ &= d(u,v) + d(v,w), \end{aligned}$$

so that the triangle inequality holds.

In order for the symmetric property to hold, we must have

$$a \cdot \vec{d}(u,v) + b \cdot \vec{d}(v,u) = a \cdot \vec{d}(v,u) + b \cdot \vec{d}(u,v),$$

for each pair u, v of vertices of a strong digraph D. Consequently,

$$(a-b) \cdot \left(\vec{d}(u,v) - \vec{d}(v,u)\right) = 0,$$

for each pair u, v of vertices of D. Therefore, if D is not a symmetric digraph (that is, a graph), then $a = b = 1/2$. Thus, if we define

$$d(u,v) = \frac{1}{2}\vec{d}(u,v) + \frac{1}{2}\vec{d}(v,u) = \frac{1}{2}\vec{d}_D(u,v) + \frac{1}{2}\vec{d}_{D'}(u,v),$$

a metric results. Since multiplying $d(u,v)$ by a positive constant still produces a metric, we define the *sum distance* $\mathrm{sd}(u,v) = \vec{d}(u,v) + \vec{d}(v,u)$.

We now consider these two metrics on strong digraphs in more detail.

3. MAXIMUM DISTANCE IN DIGRAPHS

For vertices u and v in a strong digraph D, the maximum distance between u and v was defined by $\mathrm{md}(u,v) = \max\{\vec{d}(u,v), \vec{d}(v,u)\}$. As we have seen, this is a metric. The *m-eccentricity* $\mathrm{me}(v)$ of a vertex v of D is defined as $\mathrm{me}(v) = \max_{u \in V(D)}\{\mathrm{md}(v,u)\}$; while the *m-radius* of D is $m\,\mathrm{rad}\,D = \min_{v \in V(D)}\{\mathrm{me}(v)\}$ and the *m-diameter* is $m\,\mathrm{diam}\,D = \max_{v \in V(D)}\{\mathrm{me}(v)\}$. Since the m-distance is a metric, we have the following result.

THEOREM 9. *For every strong digraph D,*

$$m\,\mathrm{rad}\,D \leq m\,\mathrm{diam}\,D \leq 2m\,\mathrm{rad}\,D.$$

In a natural way, we can define the center of a strong digraph D (with respect to this metric). The *m-center* of D is the subdigraph induced by the vertices of D having minimum m-eccentricity, that is, by those vertices v with $\mathrm{me}(v) = m\,\mathrm{rad}\,D$. The following result was established in [9].

THEOREM 10. *(See [9].) The m-center of every nontrivial strong digraph D lies in a single block of the underlying graph of D.*

An analogue to Theorem 3 was found in [9] for asymmetric digraphs.

THEOREM 11. *(See [9].) For every asymmetric digraph D, there exists a strong asymmetric digraph H such that* $\mathrm{mC}(H) \cong D$. *Furthermore, there exists such a digraph H whose order exceeds that of D by at most 4.*

For an asymmetric digraph D, we define the *maximum appendage number* $\mathrm{mA}(D)$ of D as the minimum number of vertices that must be added to D to produce a strong asymmetric digraph H whose m-center is isomorphic to D. By Theorem 11, $\mathrm{mA}(D) \leq 4$ for every asymmetric digraph D.

There are many asymmetric digraphs D for which $\mathrm{mA}(D) = 0$; for example, every vertex-transitive asymmetric digraph (such as a directed cycle) has this property. If $\mathrm{mA}(D) \neq 0$ and H is a strong asymmetric superdigraph of minimum order containing D as its m-center, then $V(H)-V(D)$ contains all vertices with maximum m-eccentricity. Since there are at least two vertices having maximum m-eccentricity, it follows that $\mathrm{mA}(D) \neq 1$ for all asymmetric digraphs.

The following result is not difficult to prove.

THEOREM 12. *Let D be an asymmetric digraph with* $2 \leq \mathrm{mA}(D) \leq 3$. *If H is an asymmetric digraph of minimum order with* $\mathrm{mC}(H) \cong D$, *then* $m \operatorname{diam} H = m \operatorname{rad} H + 1$.

The next result will allow us to show that the upper bound $\mathrm{mA}(D) \leq 4$ cannot be improved, in general. For a vertex v in a digraph D, the *out-neighborhood* $N^+(v)$ is the set of vertices of D adjacent from v, while the *in-neighborhood* $N^-(v)$ is the set of vertices adjacent to v.

THEOREM 13. *Let D be an asymmetric digraph with* $\mathrm{mA}(D) = 2$ *and let H be an asymmetric digraph of minimum order with* $\mathrm{mC}(H) \cong D$. *If* $\vec{d}_H(v,w) = m \operatorname{diam} H$, *then for all* $x \in N^+(v)$ *and* $y \in N^-(w)$, *any shortest x-y path lies entirely in D.*

PROOF. Observe that $m \operatorname{diam} H = \max\{\mathrm{me}(x) \mid x \in V(H)\} = \mathrm{md}(y,z)$, where $y,z \in V(H)-V(D)$. Since $\mathrm{mA}(D) = 2$ and $\mathrm{md}(v,w) = m \operatorname{diam} H$, it follows that $V(H) = V(D) \cup \{v,w\}$ and $\mathrm{me}_H(u) = m \operatorname{rad} H < m \operatorname{diam} H$ for all $u \in V(D)$. Suppose, to the contrary, that there exist vertices $x \in N^+(v)$ and $y \in N^-(w)$ such that a shortest x-y path P contains v or w. If v lies on P, then $\vec{d}(x,y) = \vec{d}(x,v) + \vec{d}(v,y)$. Since $(y,w) \in E(D)$ and $(v,x) \in E(D)$, it follows that $\vec{d}(y,w) = 1$ and $\vec{d}(x,v) > 1$. Therefore,

$$\begin{aligned} m \operatorname{diam} H = \vec{d}(v,w) &\leq \vec{d}(v,y) + \vec{d}(y,w) \\ &< \vec{d}(x,v) + \vec{d}(v,y) = \vec{d}_H(x,y) \\ &\leq \mathrm{md}(x,y) \leq \mathrm{me}_H(x), \end{aligned}$$

which contradicts the fact that $\mathrm{me}(x) < m \operatorname{diam} H$. The proof is similar if w lies on a shortest x-y path in H. ∎

The next result shows that the upper bound $\mathrm{mA}(D) \leq 4$ is sharp.

THEOREM 14. *Let D be the (empty) digraph consisting of* $n \geq 2$ *isolated vertices. Then* $\mathrm{mA}(D) = 4$.

PROOF. It is clear that $\mathrm{mA}(D) \neq 0$. Since D is acyclic, it follows by Theorem 13, that $\mathrm{mA}(D) \geq 3$. Since $\mathrm{mA}(D) \leq 4$, it suffices to prove that $\mathrm{mA}(D) \neq 3$. Suppose, to the contrary, that $\mathrm{mA}(D) = 3$. Let H be an asymmetric digraph with $V(H) = V(D) \cup \{u,v,w\}$ and $\mathrm{mC}(H) \cong D$. We assume, without loss of generality, that $\vec{d}_H(u,v) = m \operatorname{diam} H$ and P is a shortest u-v path in H. Since $E(D) = \emptyset$ and $|V(H) - V(D)| = 3$, P contains at most two vertices of D. On the other hand, since $m \operatorname{diam} H \geq 3$, P contains at least one vertex of D. Therefore, $\vec{d}_H(u,v) = 3$ or 4.

We consider the following cases.

CASE 1. Assume that $\vec{d}_H(u,v) = 3$ and $P: u, x, w, v$ for some $x \in V(D)$. Let $y \in V(D) - \{x\}$. Since $\text{me}(z) = 2$ for all $z \in V(D)$, it follows that $\vec{d}(x,y) = 2$. Noting that $\vec{d}(u,v) = 3$, we conclude that $(w,y) \in E(H)$. However, this implies that $\vec{d}(y,w) \geq 3$, a contradiction.

CASE 2. Assume that $\vec{d}_H(u,v) = 3$ and $P: u, w, x, v$ for some $x \in V(D)$. Let $y \in V(D) - \{x\}$. Since H contains a w-y path of length at most 2 and (w,u), $(w,v) \notin E(H)$, it follows that $(w,y) \in E(H)$. Similarly, since H contains a y-v path of length at most 2, it follows that $(y,v) \in E(H)$. Furthermore, since $\vec{d}_H(u,v) = 3$, it follows that $(u,y) \notin E(D)$. Therefore, $\vec{d}_H(u,v) \geq 3$, a contradiction.

CASE 3. Assume that $\vec{d}_H(u,v) = 4$ and $P: u, x, w, y, v$ for some vertices $x, y \in V(D)$. It follows that $(v,u) \notin E(H)$; for otherwise, $m\,\text{diam}\,H = \text{me}_H(w) = \max\{\text{md}(w,u), \text{md}(w,v)\} = 3$, a contradiction. We claim that $\vec{d}_H(v,u) = 2$. Suppose, to the contrary, that $\vec{d}_H(u,v) \geq 3$. Then w lies on every shortest v-u path. Furthermore, since $\vec{d}_H(u,v) \leq 4$, it follows that $\vec{d}_H(w,u) \leq 3$ and $\vec{d}_H(v,w) \leq 3$. By noting that $\vec{d}_H(u,w) = 2$ and $\vec{d}_H(w,v) = 2$, we conclude that $m\,\text{diam}\,H = \text{me}(w) = \max\{\text{md}(w,u), \text{md}(w,v)\} \leq 3$, a contradiction. Therefore, there exists a vertex $z \in V(D) - \{x,y\}$ such that (v,z), $(z,u) \in E(H)$. Since $(x,v) \notin E(H)$ and $\vec{d}_H(x,z) \leq 3$, we have $(w,z) \in E(H)$. Therefore, $\vec{d}_H(z,y) = 4$, a contradiction. ∎

Buckley, Miller and Slater [3] characterized those trees with appendage number 2 and proved that no tree has appendage number 3. We define an asymmetric digraph to be a *(directed) tree* if its underlying graph is a tree. In [10] it was shown that if D is an acyclic asymmetric digraph with $\text{mA}(D) = 2$, then D is a tree. Unlike the situation for graphs, however, a (directed) tree may have appendage number 3. In Figure 6, $\text{mA}(T_1) = 2$ and $\text{mA}(T_2) = 3$.

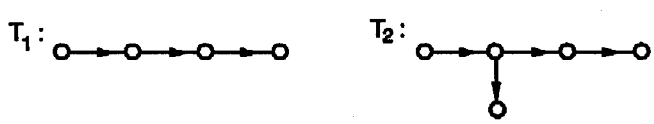

Figure 6.

A vertex v in a strong digraph D is called an *m-peripheral vertex* if $\text{me}(v) = m\,\text{diam}\,D$. The *m-periphery* $\text{mP}(D)$ of D is the subdigraph of D induced by its m-peripheral vertices. Asymmetric digraphs that are m-peripheries of strong asymmetric digraphs were characterized in [9]. This result closely parallels Theorem 4.

THEOREM 15. *(See [9].) An asymmetric digraph D is isomorphic to the m-periphery of some strong asymmetric digraph H if and only if*

(i) *every vertex of D has m-eccentricity 2, or*
(ii) *no vertex of D has m-eccentricity 2.*

The *m-distance* $\text{md}(v)$ of a vertex v in a strong digraph D is defined as $\text{md}(v) = \sum_{u \in V(D)} \text{md}(v,u)$. The *m-median* $\text{mM}(D)$ of D is the subdigraph induced by those vertices having minimum m-distance. The *m-distance* $\text{md}(H_1, H_2)$ *between* two *subdigraphs* H_1 and H_2 in a digraph H is defined by

$$\text{md}(H_1, H_2) = \min\{\text{md}(u,v) \mid u \in V(H_1),\ v \in V(H_2)\}.$$

The following two results were obtained by Chartrand and Tian [11].

THEOREM 16. *(See [11].) Let D_1 and D_2 be asymmetric digraphs, and let k be a positive integer. Then there is a strong asymmetric digraph H with $\text{mC}(H) \cong D_1$, $\text{mM}(H) \cong D_2$, and $\text{md}(\text{mC}(H), \text{mM}(H)) = k$.*

THEOREM 17. *(See [11].) For every two asymmetric digraphs D_1 and D_2 and every digraph K isomorphic to an induced subdigraph of both D_1 and D_2, there exists a strong asymmetric digraph H such that $mC(H) \cong D_1$, $mM(H) \cong D_2$, and $mC(H) \cap mM(H) \cong K$.*

4. SUM DISTANCE IN DIGRAPHS

The *sum distance* $\mathrm{sd}(u,v)$ between two vertices u and v in a strong digraph D was defined as $\mathrm{sd}(u,v) = \vec{d}(u,v) + \vec{d}(v,u)$. A number of natural concepts can now be defined.

The *s-eccentricity* $\mathrm{se}(v)$ of v in D is $\mathrm{se}(v) = \max_{u \in V(D)} \mathrm{sd}(v,u)$. The *s-radius* of D is s-rad $D = \min_{v \in V(D)} \mathrm{se}(v)$, while the *s-diameter* is s-diam $D = \max_{v \in V(D)} \mathrm{se}(v)$. The *s-center* $\mathrm{sC}(D)$ of D is the subdigraph induced by those vertices with minimum s-eccentricity. Since the sum distance is a metric, we have our usual immediate result.

THEOREM 18. *For every strong digraph D,*

$$s\text{-rad } D \leq s\text{-diam } D \leq 2s\text{-rad } D.$$

The following two (familiar sounding) results were obtained by Tian [12].

THEOREM 19. *(See [12].) The s-center of every nontrivial strong asymmetric digraph D lies in a single block of the underlying graph of D.*

THEOREM 20. *(See [12].) For every asymmetric digraph D, there exists a strong asymmetric digraph H such that* $\mathrm{sC}(H) \cong D$. *Furthermore, there exists such a digraph H whose order exceeds that of D by at most 6.*

For an asymmetric digraph D, the *sum appendage number* $\mathrm{sA}(D)$ is the minimum number of vertices that must be added to D to produce a strong asymmetric digraph H whose s-center is isomorphic to D. By Theorem 20, $\mathrm{sA}(D) \leq 6$ for every asymmetric digraph D. However, it is not known whether 6 is the best upper bound for $\mathrm{sA}(D)$. Indeed, if D is a tournament, then $\mathrm{sA}(D) \leq 4$ (see [12]).

The *s-periphery* $\mathrm{sP}(D)$ of a strong asymmetric digraph D is the subdigraph induced by those vertices of maximum s-eccentricity. The following theorem [12] parallels Theorems 4 and 15.

THEOREM 21. *(See [12].) An asymmetric digraph D is isomorphic to the s-periphery of some strong asymmetric digraph H if and only if*

(i) *every vertex of D has s-eccentricity 3, or*
(ii) *no vertex of D has s-eccentricity 3.*

It should be noted that there exists an asymmetric digraph that is isomorphic to the s-periphery of some strong asymmetric digraph but not to the m-periphery of a strong asymmetric digraph. For example, let D be the digraph of Figure 7. Then s-rad D = s-diam $D = 4$, so $\mathrm{sP}(D) = D$. However, $\mathrm{me}(u) = 2$ and $\mathrm{me}(v) = 3$, so by Theorem 15, D is not the m-periphery of any strong asymmetric digraph.

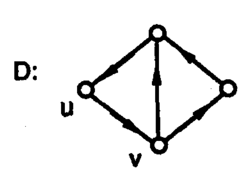

Figure 7.

The s-median of a strong digraph can also be defined in an obvious manner, but to this point, the anticipated results have remained elusive.

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