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Pure Powers in Recurrence Sequences and Some Related Diophantine Equations*

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We prove that there are only finitely many terms of a non-degenerate linear recurrence sequence which are qth powers of an integer subject to certain simple conditions on the roots of the associated characteristic polynomial of the recurrence sequence. Further we show by similar arguments that the Diophantine equation $ax^{2t}+bx^ty+cy^2+dx^t+ey+f=0$ has only finitely many solutions in integers x, y, and t subject to the appropriate restrictions, and we also treat some related simultaneous Diophantine equations. © 1987 Academic Press, Inc.

1. INTRODUCTION

In [15] the authors proved that if a, b, c, and d are integers with b^2-4ac and acd non-zero and if x, y, and t are integers with $|x|$ and t larger than one satisfying

$$ax^{2t}+bx^ty+cy^2=d, \tag{1}$$

then the maximum of $|x|$, $|y|$, and t is less than a number which is effectively computable in terms of a, b, c, and d. Let r_1 and r_2 be integers with $r_1^2+4r_2$ non – zero. Let u_0 and u_1 be integers and put

$$u_n=r_1u_{n-1}+r_2u_{n-2}, \tag{2}$$

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for $n = 2, 3, \ldots$. Then, for $n \geqslant 0$,

$$u_n = a\alpha^n + b\beta^n, \tag{3}$$

where α and β are the two roots of $x^2 - r_1 x - r_2$ and

$$a = \frac{u_0\beta - u_1}{\beta - \alpha}, \qquad b = \frac{u_1 - u_0\alpha}{\beta - \alpha}.$$

The sequence of integers $(u_n)_{n=0}^{\infty}$ is a binary recurrence sequence. It is said to be non-degenerate if $ab\,\alpha\beta \neq 0$ and α/β is not a root of unity. In the course of proving our result concerning Eq. (1) we showed that a non-degenerate binary recurrence sequence contains only finitely many terms u_n, defined as in (3), which are pure powers whenever α and β are units or equivalently whenever $|r_2| = 1$. We also established in [15] the following more general result. Let d be a non-zero integer and let u_n, defined as in (3), be the nth term of a non-degenerate binary recurrence sequence. If

$$dx^q = u_n, \tag{4}$$

for integers x and q larger than one, then the maximum of x, q, and n is less than a number which is effectively computable in terms of a, α, b, β, and d. Independently, Pethö [12] proved that if in (2) we suppose that r_1 and r_2 are coprime and (4) holds for integers x and q larger than one, then the maximum of x, q, and n is less than a number which is effectively computable in term of a, α, b, β, and the greatest prime factor of d. Let c be an integer and let u_n, defined as in (3), be the nth term of a non-degenerate binary recurrence sequence. In [16] Stewart showed that if $|r_2| = 1$ and

$$x^q + c = u_n,$$

for integers n, x, and q with $|x| > 1$, $n \geqslant 0$, and $q \geqslant 3$, then the maximum of n, $|x|$, and q is less than a number which is effectively computable in terms of a, α, b, β, and c. Further if $|r_2| = 1$ and

$$x^2 + c = u_n,$$

for integers n and x with $|x| \geqslant 1$ and $n \geqslant 0$, then the maximum of n and $|x|$ is less than a number which is effectively computable in terms of a, α, b, β, and c provided that $c^2 \neq 4ab$ when $r_2 = -1$ and that $c^2 \neq \pm 4ab$ when $r_2 = 1$; the preceding provisions were overlooked in [16]. Just as the study of Eq. (1) was related to the study of pure powers in binary recurrence sequences there is a generalization of Eq. (1) related to the above result. In particular, we are able to prove the following result.

THEOREM 1. *Let a, b, c, d, e, and f be integers. Put $D=b^2-4ac$ and $\Delta=4acf+bde-ae^2-cd^2-fb^2$ and assume that $D\Delta\neq 0$. If x, y, and t are integers with $|x|>1$ and $t>2$ satisfying*

$$ax^{2t}+bx^ty+cy^2+dx^t+ey+f=0, \tag{5}$$

then the maximum of $|x|$, $|y|$, and t is less than a number which is effectively computable in terms of a, b, c, d, e, and f. Further, if $e^2\neq 4cf$ and x and y are integers satisfying

$$ax^4+bx^2y+cy^2+dx^2+ey+f=0, \tag{6}$$

then the maximum of $|x|$ and $|y|$ is less than a number which is effectively computable in terms of a, b, c, d, e, and f.

The hypothesis $D\Delta\neq 0$ is clearly required in the statement of Theorem 1. To see that the additional hypothesis $e^2\neq 4cf$ is require when $t=2$, observe that if $e^2=4cf$ and $b=0$ then (6) is equivalent to

$$(ax^2+d)x^2=-c(y+e/2c)^2;$$

note that $c\neq 0$ since $D\neq 0$. Thus, if $2c$ divides e, to obtain infinitely many pairs of integers x, y satisfying (6) it suffices to find infinitely many pairs of integers x, t satisfying

$$ax^2+ct^2=-d, \tag{7}$$

and to put $y=xt-e/2c$ for each such pair. Plainly there are infinitely many such choices of a, c, d, and e for which (7) has infinitely many pairs of solutions x and t and for which $D\Delta\neq 0$. In particular, we may take $a=-1$, $d=1$, c a positive integer which is not a square, and $e=2c$.

Let a, b, c, d, a_1, b_1, c_1, and d_1 be integers with $ac\,da_1c_1d_1\neq 0$, $b_1^2\neq 4a_1c_1$, $b^2\neq 4ac$ and such that the roots α_1 and α_2 of $a_1x^2+b_1x+c_1$ are not roots of ax^2+bx+c. In [15] the authors also showed that if x, y, z, and q are integers with q and z larger than one for which

$$a_1x^2+b_1xy+c_1y^2=d_1 \tag{8}$$

and

$$ax^2+bxy+cy^2=dz^q, \tag{9}$$

then the maximum of $|x|$, $|y|$, $|z|$, and q is less than a number which is effectively computable in terms of a, b, c, d, a_1, b_1, c_1, and d_1. This extended earlier work of Mordell [9] who proved, with the above hypotheses and $q=2$, that the simultaneous Eq. (8) and (9) have only finitely many solutions in integers x, y, and z. We are now able to generalize this result considerably.

THEOREM 2. *Let a, b, c, d, e, and f be integers and let $F(t, v)$ be a binary form with integer coefficients and degree at least one. Put $D = b^2 - 4ac$ and $\Delta = 4acf + bde - ae^2 - cd^2 - fb^2$ and assume $D\Delta f \neq 0$. Suppose that $F(t, 1)$ has a simple root α such that $a\alpha^2 + b\alpha + c \neq 0$ and $4f(a\alpha^2 + b\alpha + c) \neq (d\alpha + e)^2$. If x, y, z, s, and q are integers with $s \neq 0$, $q > 1$, and $|z| > 1$, for which*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{10}$$

and

$$F(x, y) = sz^q, \tag{11}$$

then the maximum of $|x|$, $|y|$, $|z|$, $|s|$, and q is less than a number which is effectively computable in terms of a, b, c, d, e, f, the greatest prime factor of s and the binary form F.

For the proof of Theorem 2 we employ Lemma 6 together with a result of Baker on the solutions of the hyperelliptic equation in an algebraic number field. We remark that if we use a result of Brindza [4] in place of the above-mentioned result of Baker, it is possible to show that the condition $4f(a\alpha^2 + b\alpha + c) \neq (d\alpha + e)^2$ may be omitted if q is greater than 2. Lemma 6, which is a slight generalization of Lemma 6 of [15], yields some information on qth powers in general linear recurrence sequences.

Let $r_1, \ldots, r_k$ and $u_0, \ldots, u_{k-1}$ be integers and put

$$u_n = r_1 u_{n-1} + \cdots + r_k u_{n-k}, \tag{12}$$

for $n = k, k+1, \ldots$. The sequence $(u_n)_{n=0}^{\infty}$ is a linear recurrence sequence. We shall assume that $k \geqslant 1$ and that the terms of $(u_n)_{n=0}^{\infty}$ do not satisfy a relation of the form (12) with fewer terms; in particular there does not exist an integer l with $l < k$ and integers $s_1, \ldots, s_l$ such that

$$u_n = s_1 u_{n-1} + \cdots + s_l u_{n-l},$$

for $n = l, l+1, \ldots$. It is well known (see page 62 of [7]) that for $n \geqslant 0$,

$$u_n = f_1(n)\, \alpha_1^n + \cdots + f_t(n)\, \alpha_t^n, \tag{13}$$

where $f_1, \ldots, f_t$ are non-zero polynomials in n with degrees less than $l_1, \ldots, l_t$ respectively and with coefficients from $\mathbb{Q}(\alpha_1, \ldots, \alpha_t)$, where $\alpha_1, \ldots, \alpha_t$ are the distinct roots of the characteristic polynomial of the sequence

$$x^k - r_1 x^{k-1} - \cdots - r_k$$

and $l_1, \ldots, l_t$ are their respective multiplicities. Note that $\alpha_1, \ldots, \alpha_t$ are non-zero since r_k is non-zero by the minimality of (12). We shall say that the sequence $(u_n)_{n=0}^{\infty}$ is non-degenerate if $t>1$ and α_i/α_j is not a root of unity for $1 \leqslant i < j \leqslant t$. Observe that this definition is consistent with our earlier definition of non-degenerate binary recurrence sequences. We shall be interested in linear recurrence sequences $(u_n)_{n=0}^{\infty}$ with u_n defined as in (13) for which $f_1(n)$ is a non-zero constant, λ_1 say. Thus

$$u_n = \lambda_1 \alpha_1^n + f_2(n)\,\alpha_2^n + \cdots + f_t(n)\,\alpha_t^n. \tag{14}$$

Let α be a real algebraic number larger than one from a field K of degree D over the rational numbers. Further let d, a, and b be non-zero numbers from K and let δ be a positive real number. In Lemma 6 of [15] the authors showed that if

$$dx^q = a\alpha^n + b,$$

with $|b| < \alpha^{n(1-\delta)}$ and with x, q, and n integers larger than one, then q is less than a number which is effectively computable in terms of D, d, a, α, and δ only. As a consequence we showed that if d is a non-zero integer, u_n is the nth term of a non-degenerate linear recurrence sequence, as in (14), $|\alpha_1| > |\alpha_j|$ for $j = 2, \ldots, t$, $u_n - \lambda_1 \alpha_1^n$ is non-zero, and

$$dx^q = u_n, \tag{15}$$

for integers x and q larger than one, then q is less than a number which is effectively computable in terms of d and the sequences $(u_n)_{n=0}^{\infty}$. Kiss [5] proved that in fact q is less than a number which is effectively computable in terms of the greatest prime factor of d and the sequence $(u_n)_{n=0}^{\infty}$. Kiss [5] also showed that if we further assume that $|\alpha_2| > |\alpha_j|$ for $j = 3, \ldots, t$ and that $|\alpha_2| > 1$ then, in place of (15), we have

$$|dx^q - u_n| > e^{c_1 n},$$

for integers x and q with x larger than one, provided that n and q are larger than n_1, where c_1 and n_1 are positive numbers which are effectively computable in terms of the greatest prime factor of d and the sequence $(u_n)_{n=0}^{\infty}$. If we make no assumption on the size of $|\alpha_2|$ it is still possible to conclude that the distance between u_n and the nearest qth power, for q sufficiently large, eventually tends to infinity exponentially with n provided that $\lambda_1 \alpha_1^n$ is not the qth power of an integer for n sufficiently large. This follows from our next result, which is a consequence of Lemma 6.

THEOREM 3. *Let δ be a positive real number and let P be a positive* *integer. Let* u_n, *defined as in* (14), *be the* nth *term of a non-degenerate linear recurrence sequence and assume that*

$$|\alpha_1| > |\alpha_j|, \qquad \text{for} \quad j = 2, ..., t.$$

There exists a real number C_0, *which is effectively computable in terms of* δ, P, *and the sequence* $(u_n)_{n=0}^{\infty}$, *such that if* s, x, q, *and* n *are non-zero integers with the greatest prime factor of* s *less than* P, $|x| > 1$, $q > C_0$, $n > 0$, *and* $sx^q \neq \lambda_1 \alpha_1^n$, *then*

$$|sx^q - u_n| > |\alpha_1|^{n(1-\delta)}. \tag{16}$$

While C_0 is effectively computable it is in general rather large as we employ estimates for linear forms in the logarithms of algebraic numbers due to Baker in the proof of Theorem 3. We are able to reduce considerably the size of C_0 by employing an extension of Roth's theorem due to Lang [6].

THEOREM 4. *Let* u_n, *defined as in* (14), *be the* nth *term of a non-degenerate linear recurrence sequence and assume that*

$$|\alpha_1| > |\alpha_2| \geqslant |\alpha_j|, \qquad \textit{for} \quad j = 3, ..., t.$$

Let γ *be a real number with* $\gamma > 1$ *and*

$$|\alpha_1| > \gamma > |\alpha_2|,$$

let d *be the degree of* α_1 *over the rationals, and let* P *be a positive integer. There exists a number* C_1 *such that if* s, x, q, *and* n *are non-zero integers with the greatest prime factor of* s *less than* P, $|x| > 1$, $n > C_1$, $sx^q \neq \lambda_1 \alpha_1^n$, *and*

$$q > (d \log |\alpha_1|)/\log(|\alpha_1|/\gamma), \tag{17}$$

then

$$|sx^q - u_n| > \gamma^n. \tag{18}$$

Taking $u_n = 2^n + 1$ for $n = 0, 1, 2, ...$ we see that the restriction $sx^q \neq \lambda_1 \alpha_1^n$ in Theorem 4 is certainly required. Further, put $u_n = (\sqrt{2} + 1)^n + (\sqrt{2} - 1)^n$ for $n = 0, 1, 2, ...$ and observe that for any positive integer q

$$\begin{aligned} u_n^q - u_{nq} &= q((\sqrt{2}+1)^{q-2})^n \\ &\quad + \binom{q}{2}((\sqrt{2}+1)^{q-4})^n + \cdots + q((\sqrt{2}-1))^{q-2})^n. \end{aligned}$$

Therefore if $q \geqslant 2$ and $\gamma > (\sqrt{2}+1)^{(q-2)/q}$, the inequality

$$|x^q - u_n| < \gamma^n$$

has infinitely many solutions in positive integers n and x; hence, we cannot replace condition (17) by the condition

$$q > (2-\varepsilon)\log|\alpha_1|/\log(|\alpha_1|/\gamma),$$

for any $\varepsilon > 0$.

Because of the ineffective nature of Lang's result we are not able to give an effectively computable number C_1 such that (18) holds for all integers n with $n > C_1$. Theorems 3 and 4 yield information on the equation

$$u_n = dx^q + T(x), \tag{19}$$

where $T(x)$ is a polynomial with integer coefficients having height H and degree r, considered by Nemes and Pethö [10, 11]. Let u_n be defined as in (14) and assume

$$|\alpha_1| > |\alpha_2| > |\alpha_j|, \qquad \text{for} \quad j = 3, ..., t,$$

with $\alpha_2 \neq \pm 1$. Using Lemma 6 of [15], Nemes and Pethö [10] showed that there are positive numbers C_2, C_3, and C_4 which are effectively computable in terms of d, H, and the sequence $(u_n)_{n=0}^{\infty}$ such that if n, x, and q are integers with $n > C_2$, $|x| > 1$, and $q > 1$ for which (19) holds and if $r < C_3 q$ then $q < C_4$. Further, in the special case u_n is the nth term of a non-degenerate binary recurrence sequence and u_n satisfies a relation as in(2) with $|r_2| = 1$. Nemes and Pethö [11] were able to show that if q is a fixed integer larger than one and Eq. (19) has infinitely many solutions in integers n and x, then $T(x)$ can be characterized in terms of the Chebyshev polynomials. By means of Theorem 4 we are able to obtain further information on solutions of (19).

COROLLARY 1. *Let u_n, defined as in* (14), *be the* nth *term of a non-degenerate linear recurrence sequence and assume that*

$$|\alpha_1| > |\alpha_2| > |\alpha_j|,$$

for $j = 3, ..., t$. Let d be the degree of α_1 over $\mathbb{Q}$ and let $T(x)$ be a polynomial with integer coefficients and degree r; we take $r = 0$ if $T(x)$ is the zero polynomial. If α_1 and α_2 are multiplicatively independent and $\alpha_2 \neq \pm 1$ then there are only finitely many integers n, x, and q with $n \geqslant 0$, $|x| > 1$, and

$$q > \max\left(\frac{d\log|\alpha_1|}{\log(|\alpha_1|/\max(1, |\alpha_2|))}, d + r\right)$$

for which

$$u_n = x^q + T(x).$$

As a special case of this result note that if $(u_n)_{n=0}^{\infty}$ is a non-degenerate binary recurrence sequence whose characteristic polynomial has roots which are multiplicatively independent with one root inside the unit circle, then for any integer c the equation

$$u_n = x^q + c$$

has only finitely many solutions in integers n, x, and q with $n \geqslant 0$, $|x| > 1$, and $q > 2$. Thus the distance from, for example,

$$u_n = (2+\sqrt{7})^n + (2-\sqrt{7})^n$$

to the closest pure power larger than 2 tends to infinity with n. We remark that if the coefficients of the characteristic polynomial of a non-degenerate binary recurrence sequence are relatively prime then the roots of the polynomial are multiplicatively independent.

Our next result may be viewed as a p-adic analogue of Theorem 3. Let K be a field of finite degree over $\mathbb{Q}$ and let $\mathfrak{p}$ be a prime ideal of the ring of algebraic integers of K. For any element α in K we denote by $\operatorname{ord}_{\mathfrak{p}} \alpha$ the order to which $\mathfrak{p}$ divides the principal ideal generated by α.

THEOREM 5. *Let u_n, defined as in* (14), *be the* nth *term of a non-degenerate linear recurrence sequence and put $K = \mathbb{Q}(\alpha_1, ..., \alpha_t)$. Let $\mathfrak{p}$ be a prime ideal in the ring of algebraic integers of K lying above the prime p and assume that*

$$\operatorname{ord}_{\mathfrak{p}} \alpha_1 < \operatorname{ord}_{\mathfrak{p}} \alpha_j,$$

for $j = 2, ..., t$. If s, x, q, and n are integers with $s \neq 0$, $|x| > 1$, $(p, q) = 1$, $n \geqslant 0$, $sx^q \neq \lambda_1 \alpha_1^n$, and

$$sx^q = u_n, \tag{20}$$

then q is less than a number which is effectively computable in terms of the greatest prime factor of s, p, and the sequence $(u_n)_{n=0}^{\infty}$.

We remark that if $sx^q = \lambda_1 \alpha_1^n$ and (20) holds then

$$f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n = 0. \tag{21}$$

Clearly if $|\alpha_2| > |\alpha_j|$ for $j = 3, ..., t$ all solutions of (21) are less than C_5, a number which is effectively computable in terms of $\alpha_2, ..., \alpha_t$ and $f_2, ..., f_t$. In this case the conditions $n \geqslant 0$ and $sx^q \neq \lambda_1 \alpha_1^n$ in the statement of

Theorem 5 can be replaced by the condition $n > C_5$, or alternatively, since $|x| > 1$, the condition $sx^q \neq \lambda_1 \alpha_1^n$ may be dropped. In general, Eq. (21) has only finitely many solutions by the Skolem–Mahler theorem [8] and so the conditions $n \geqslant 0$ and $sx^q \neq \lambda_1 \alpha_1^n$ may be replaced by the condition that n be sufficiently large.

Let us recall some facts about valuations. Let $\{p_1, p_2, ...\}$ be the set of prime numbers. Let $|\ |_{p_0}$ denote the ordinary absolute value on $\mathbb{Q}$ and let $|\ |_{p_i}$ denote the p_i-adic value on $\mathbb{Q}$ normalized so that $|p_i|_{p_i} = p_i^{-1}$ for $i = 1, 2, ...$ Let K be a field of finite degree over $\mathbb{Q}$ and let v be a non-trivial valuation on K. Then v restricted to $\mathbb{Q}$ is equivalent to $|\ |_{p_i}$ for some $i \geqslant 0$. We shall suppose that v is normalized so that

$$|a|_v = |a|_{p_i}, \tag{22}$$

for all a in $\mathbb{Q}$. Let K_v be the completion of K at v, let $\mathbb{Q}_{p_i}$ be the completion of $\mathbb{Q}$ at p_i, and put

$$\|\gamma\|_v = |\gamma|_v^{N_v}, \tag{23}$$

for all γ in K, where N_v is the degree of K_v over $\mathbb{Q}_{p_i}$. Let V be the set of non-trivial valuations v, normalized as in (22), on K. Then, for all non-zero elements γ in K, we have

$$\prod_{v \in V} \|\gamma\|_v = 1. \tag{24}$$

Combining Theorems 3 and 5 with Lang's generalization of Roth's theorem we are able to prove the following result.

THEOREM 6. *Let $(u_n)_{n=0}^{\infty}$ be a non-degenerate linear recurrence sequence with u_n defined as in (14) and put $K = \mathbb{Q}(\alpha_1, ..., \alpha_t)$. Let $v_1, ..., v_r$ be inequivalent valuations on K normalized as in (22) and suppose*

$$|\alpha_1|_{v_i} > |\alpha_j|_{v_i},$$

for $j = 2, ..., t$ and $i = 1, ..., r$. Put $\theta_i = \max\{\|\alpha_2\|_{v_i}, ..., \|\alpha_t\|_{v_i}\}$ for $i = 1, ..., r$ with $\|\ \|_{v_i}$ defined as in (23) and let b be an integer with $1 \leqslant b \leqslant t$ for which

$$|\alpha_b| \geqslant |\alpha_j|,$$

for $j = 1, ..., t$. Let D be the degree of K over $\mathbb{Q}$ and let P be a positive integer. The equation

$$sx^q = u_n$$

has only finitely many solutions in integers s, x, q, and n, with the greatest prime factor of s less than P, $|x|>1$, $n \geqslant 0$, and

$$q > \frac{D \log |\alpha_b|}{\log(\prod_{i=1}^{r} (\|\alpha_1\|_{v_i}/\theta_i))}.$$

Theorem 6 has the following consequence.

COROLLARY 2. *Let $(u_n)_{n=0}^{\infty}$ be a non-degenerate linear recurrence sequence with u_n defined as in (14) and let P be a positive integer. Assume that $f_2(n)$ is a non-zero constant, $t=3$, and that $|\alpha_1|=|\alpha_2|=|\alpha_3|$. Then the equation*

$$sx^q = u_n$$

has only finitely many solutions in integers s, x, q, and n with the greatest prime factor of s at most P, $|x|>1$, $q>2$, and $n \geqslant 0$.

We remark that with the above hypotheses $|u_n| \to \infty$ as $n \to \infty$ by Lemma 5. Thus a non-degenerate ternary recurrence sequence, the roots of whose characteristic polynomial have the same absolute value, contains only finitely many qth powers of integers for $q>2$. In particular let d be a positive square free integer and let a and b be non-zero integers with $a \neq \pm b$ if $d=1$ and $a \neq \pm b$ and $a \neq \pm 3b$ if $d=3$. Then there are only finitely many integers n such that

$$((a+b\sqrt{-d})^2)^n + ((a-b\sqrt{-d})^2)^n + (a^2+db^2)^n$$

is the qth power of an integer with $q>2$. The hypothesis $q>2$ in Corollary 2 cannot be replaced by $q>1$ since, for example, for all $n \geqslant 0$,

$$((2+i)^n + (2-i)^n)^2 = (3+4i)^n + (3-4i)^n + 2 \cdot 5^n.$$

2. PRELIMINARY LEMMAS

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be non-zero algebraic numbers. Let $K = \mathbb{Q}(\alpha_1, ..., \alpha_n)$ and denote the degree of K over $\mathbb{Q}$ by D. Let $A_1, ..., A_n$ be upper bounds for the heights of $\alpha_1, ..., \alpha_n$, respectively; the height of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in its minimal polynomial. We assume that A_n is at least 4. Further let $b_1, ..., b_{n-1}$ be rational integers with absolute values at most B, and let b_n be a non-zero rational integer with absolute value at most B'. We assume that B' is at least 3. Put

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

where the logarithms are assumed to have their principal values. In 1973 Baker proved the following result; take $\delta = 1/B'$ in Theorem 1 of [2].

LEMMA 1. *If $\Lambda \neq 0$ then $|\Lambda| > \exp(-C(\log B' \log A_n + B/B'))$, where C is a positive number which is effectively computable in terms of n, D, and $A_1, ..., A_{n-1}$ only.*

In 1976 van der Poorten established the following p-adic analogue of Baker's theorem; take $\delta = 1$ in Theorem 3 of [13].

LEMMA 2. *Let $\wp$ be a prime ideal of K lying above the rational prime p and assume that b_n is not divisible by p. If $\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$ is non-zero, then*

$$\operatorname{ord}_{\wp}(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < C(\log B' \log A_n + B/B'),$$

where C is a positive number which is effectively computable in terms of n, D, $A_1, ..., A_{n-1}$, and p only.

We shall also require the following result, due to Baker, which gives bounds for the solutions of the hyperelliptic equation. Let θ be an algebraic number. We denote by $\|\theta\|$ the maximum of the absolute value of the conjugates of θ over $\mathbb{Q}$.

LEMMA 3. *Let K be an algebraic number field of degree d over $\mathbb{Q}$. Let $a_n, a_{n-1}, ..., a_0$, and b be algebraic numbers from K with $a_n b \neq 0$, and let m and n be positive integers with $m \geq 2$. Further let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with at least 3 simple roots. All solutions in algebraic integers x, y, from K of*

$$by^m = f(x)$$

satisfy $\max\{\|x\|, \|y\|\} < C$, where C is a number which is effectively computable in terms of b, $a_0, a_1, ..., a_n$, and K.

Proof. When K is the field of rational numbers the result follows from Theorems 1 and 2 of [1]. The generalization to an algebraic number field K follows directly as is indicated by Theorems 4.1 and 4.2 of [3].

Let K be a field of finite degree over $\mathbb{Q}$ and let V be the set of non-trivial valuations v, normalized as in (22), on K. For any β in K we define $H_K(\beta)$ by

$$H_K(\beta) = \prod_{v \in V} \max(1, \|\beta\|_v).$$

The following generalization of Roth's theorem is due to Lang.

LEMMA 4. *Let K be a field of finite degree over $\mathbb{Q}$. Let V be the set of non-trivial valuations v, normalized as in* (22), *on K and let S be a finite subset of V. For each v in S let α_v be non-zero and algebraic over K and assume that v is extended to the algebraic closure of K in some way. Let ε be a positive real number. There is a positive real number C which depends upon ε and α_v for $v \in S$ such that*

$$\prod_{v \in S} \min(1, \|\alpha_v - \beta\|_v) \prod_{v \in S} \min(1, \|\beta\|_v)$$

$$\times \prod_{v \in S} \min(1, \|\beta^{-1}\|_v) > \frac{C}{(H_K(\beta))^{2+\varepsilon}},$$

for all elements β in K which are non-zero and different from α_v for v in S.

Proof. This follows from Theorem 1.1, page 160, together with remarks (iv) and (v), page 161 of [6].

LEMMA 5. *Let $\alpha_1, ..., \alpha_t$ be non-zero algebraic numbers and let $f_1(n), ..., f_t(n)$ be polynomials which are not identically zero with coefficients which are algebraic numbers. Put $v_n = f_1(n)\alpha_1^n + \cdots + f_t(n)\alpha_t^n$, for $n = 0, 1, 2, ...$ If α_i/α_j is not a root of unity for $1 \leqslant i < j \leqslant t$ then $v_n = 0$ for only finitely many integers n.*

Proof. See [8 or 7], page 59.

LEMMA 6. *Let α be a real algebraic number larger than one from a field K of degree D over Q. Let s be a non-zero integer, let a and b be non-zero numbers from K, and let δ be a positive real number. If*

$$sx^q = a\alpha^n + b, \tag{25}$$

with $|b| < \alpha^{n(1-\delta)}$ and with x, q, and n integers larger than one, then q is less than C, a number which is effectively computable in terms of the greatest prime factor of s, D, a, α, and δ only.

Proof. Let $c_1, c_2, ...$ be positive numbers which are effectively computable in terms of the greatest prime factor of s, D, a, α, and δ only. We shall assume that n is larger than c_1, where c_1 is chosen sufficiently large to ensure the validity of the subsequent arguments. Note that if $n < c_1$ and (25) holds then $q < c_2$ as required since x is an integer larger than one.

From (25) we have

$$\frac{sx^q}{a\alpha^n} = 1 + \frac{b}{a\alpha^n}, \tag{26}$$

so

$$1 - (|a|\, \alpha^{\delta n})^{-1} \leqslant |s|\, |a|^{-1}\, \alpha^{-n} x^q \leqslant 1 + (|a|\, \alpha^{\delta n})^{-1}.$$

For n sufficiently large $(|a|\,\alpha^{\delta n})^{-1} < 1/2$. On taking logarithms and recalling that $|\log(1+y)| \leqslant y$ and $|\log(1-y)| \leqslant 2y$ for $0 \leqslant y \leqslant 1/2$, we find that

$$|\log|s| - \log|a| - n\log\alpha + q\log x| < c_3\alpha^{-\delta n}. \tag{27}$$

We have $|s| = p_1^{r_1} \cdots p_k^{r_k}$ with $k \geqslant 0$ and $p_1, ..., p_k$ prime numbers. Note that the maximum of $r_1, ..., r_k$ is at most $c_4 n$. Put

$$\begin{aligned}\Lambda = r_1 \log p_1 + \cdots + r_k \log p_k - \log|a|\\ - n\log\alpha + q\log x.\end{aligned}$$

By (26) and the fact that $b \neq 0$ we see that $\Lambda \neq 0$. We may now apply Lemma 1 with $B' = q$ and B the maximum of $r_1, ..., r_k$ and n to obtain

$$|\Lambda| > \exp(-c_5(\log q \log x + (n/q))). \tag{28}$$

Comparing (27) and (28) we find that

$$c_6 n < \log q \log x + n/q.$$

Certainly we may assume that $q > 2c_6$ and therefore

$$c_7 n < \log q \log x.$$

On the other hand, by (25),

$$\log|s| + q\log x < c_8 n$$

hence

$$q \log x < c_9 \log q \log x.$$

Since x is at least 2 we conclude that $q < c_{10}$ as required.

3. Proof of Theorem 1

Let $c_1, c_2, ...$ denote positive numbers which are effectively computable in terms of a, b, c, d, e, and f. Let us first assume that (5) holds for integers x, y, and t with $|x| > 1$ and $t > 1$.

If a and c are both zero then, since D is non-zero, b is non-zero. Thus, from (5),

$$(by + d)x^t = -ey - f. \tag{29}$$

If $by + d = 0$ then, from (29), $ey + f = 0$ and so $de - fb = 0$, contradicting the assumption $\Delta \neq 0$. Thus $by + d \neq 0$ and so by (29), $|x^t| < c_1$ hence the maximum of $|x|$ and t is at most c_1. Further $(bx'+e)y=-f-dx'$ and, since $\Delta \neq 0$, $bx'+e \neq 0$ hence $|y|<c_2$. Therefore the theorem holds if a and c are both zero. We shall assume henceforth that at least one of a and c is non-zero.

If a is non-zero put

$$X=2ax'+by+d \text{ and } Y=y+(bd-2ae)/D. \tag{30}$$

Then (5) is equivalent to

$$X^2-DY^2=M, \tag{31}$$

where $M=4a\,\Delta/D$. Further, if D is less than zero or if D is the square of a non-zero integer then, by (31), $|X|$ and $|Y|$ are at most c_3 hence $|x|$, $|y|$, and t are at most c_4. Thus if a is non-zero we may assume that D is positive and not the square of an integer, hence that c is non-zero. On the other hand, if c is non-zero then arguing as above we may deduce that a is non-zero. Therefore we may assume that both a and c are non-zero and that D is positive and not the square of an integer.

Since $a\,\Delta$ is non-zero, M is non-zero and thus, by (31), $X-\sqrt{D}\,Y$ is non-zero. Let ε denote the fundamental unit in $\mathbb{Q}(\sqrt{D})$. Define n to be that integer for which $1 \leqslant |(X-\sqrt{D}\,Y)\varepsilon^{-n}|<\varepsilon$ and put $\pi_1=(X-\sqrt{D}Y)\varepsilon^{-n}$. Then

$$X-\sqrt{D}Y=\pi_1\varepsilon^n. \tag{32}$$

Let σ denote the non-trivial element of the Galois group of $\mathbb{Q}(\sqrt{D})$ over $\mathbb{Q}$ and apply it to both sides of (32) to obtain

$$X+\sqrt{D}\,Y=\sigma(\pi_1)\,\sigma(\varepsilon)^n.$$

Put $\pi_2=\sigma(\pi_1)$ and observe that the heights of π_1 and π_2 are at most c_5 since $1 \leqslant |\pi_1|<\varepsilon$ and $\pi_1\pi_2=(\varepsilon\sigma(\varepsilon))^{-n}M$. Further

$$2X=\pi_1\varepsilon^n+\pi_2\sigma(\varepsilon)^n, \tag{33}$$

and

$$-2\sqrt{D}Y=\pi_1\varepsilon^n-\pi_2\sigma(\varepsilon)^n. \tag{34}$$

By (30),

$$2DX-2b\,DY=4a\,Dx'+R,$$

where $R=4a(be-2cd)$. Thus, from (33) and (34),

$$4a\,Dx'=(D+b\sqrt{D})\,\pi_1\varepsilon^n+(D-b\sqrt{D})\,\pi_2\sigma(\varepsilon)^n-R. \tag{35}$$

Notice that $\varepsilon > 1$ and $0 < |\sigma(\varepsilon)| < 1$. Further $(D + b\sqrt{D})\pi_1$ and $(D - b\sqrt{D})\pi_2$ are non-zero, since $ac\,D \neq 0$, and have heights at most c_6. If $n > c_7$ then $|(D - b\sqrt{D})\,\pi_2\sigma(\varepsilon)^n - R| < \varepsilon^{n/2}$ and we may apply Lemma 6 with $\delta = 1/2$ to deduce that $t < c_8$ while if $n < -c_9$ then $|(D + b\sqrt{D})\,\pi_1\varepsilon^n - R| < |\sigma(\varepsilon)|^{n/2}$ and, again by Lemma 6, $t < c_{10}$. Finally, if $|n| < c_{11}$ then, since $|x| > 1$, we conclude from (35) that $t < c_{12}$.

Denote $\sqrt{D}((D + b\sqrt{D})\,\pi_1\varepsilon^n - (D - b\sqrt{D})\,\pi_2\sigma(\varepsilon)^n)$ by z and observe that z is an algebraic integer in $\mathbb{Q}(\sqrt{D})$ which is invariant under σ. Thus z is a rational integer. Further,

$$\begin{aligned} z^2 = D((D + b\sqrt{D})\,\pi_1\varepsilon^n + (D - b\sqrt{D})\,\pi_2\sigma(\varepsilon)^n)^2 \\ -4\,D(D^2 - b^2\,D)\,\pi_1\pi_2(\varepsilon\sigma(\varepsilon))^n. \end{aligned} \tag{36}$$

Recall that $\pi_1\pi_2(\varepsilon\sigma(\varepsilon))^n = M$. Therefore, by (35) and (36), $z^2 = f(x)$, where

$$f(x) = 16a^2\,D^3x^{2t} + 8a\,D^2Rx^t + D(R^2 - 4D(D - b^2)M).$$

Put $g(u) = 16a^2\,D^3u^2 + 8a\,D^2Ru + D(R^2 - 4D(D - b^2)M)$. Since $ac\,D\,\Delta$ is non-zero the two roots of g are distinct. Since one of the roots of g is non-zero f has at least t simple zeros and so, for $t > 2$, we may apply Lemma 3 to conclude that $|x| < c_{13}$ and hence, by (35), that $|n| < c_{14}$. Further, by (30) and (34), $|y| < c_7$. Similarly if $t = 2$ and both roots of g are non-zero f has four simple zeros and we may apply Lemma 3 as above. The additional hypothesis $e^2 \neq 4cf$ ensures that $D(R^2 - 4D(D - b^2)M)$ is non-zero and hence that both roots of g are non-zero, and this completes the proof.

4. Proof of Theorem 2

Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of a, b, c, d, e, f, the greatest prime factor of s, and the binary form F. As in the proof of Theorem 1 we may assume that a and c are non-zero and that D is positive and not the square of an integer. Further, put

$$X = 2ax + by + d \qquad \text{and} \qquad Y = y + (bd - 2ae)/D,$$

so that (10) is equivalent to

$$X^2 - DY^2 = M,$$

where $M = 4a\,\Delta/D$. Finally, define ε, σ, π_1, and π_2 as in the proof of Theorem 1.

From (35),

$$x = \left(1 + \frac{b}{\sqrt{D}}\right)\frac{\pi_1}{4a}\varepsilon^n + \left(1 - \frac{b}{\sqrt{D}}\right)\frac{\pi_2}{4a}\sigma(\varepsilon)^n + \frac{2cd - be}{D}, \tag{37}$$

while, from (30) and (34),

$$y = -\frac{\pi_1}{2\sqrt{D}}\varepsilon^n + \frac{\pi_2}{2\sqrt{D}}\sigma(\varepsilon)^n + \frac{2ae-bd}{D}. \tag{38}$$

Let h be the degree of α over the rationals and let $f(t, v)$ be the binary form of degree h for which $f(t, 1)$ is the minimal polynomial of α over the rationals. Since α is a root of $F(t, 1)$ we have

$$F(t, v) = f(t, v)\, f_1(t, v),$$

where $f_1(t, v)$ is a binary form with integer coefficients. Since α is a simple root of $F(t, 1)$ we see that the binary forms $f(t, v)$ and $f_1(t, v)$ have no common linear factor in their factorizations over the complex numbers. Plainly the greatest common divisor of x and y divides f, and f is non-zero. Therefore the greatest common divisor of $f(x, y)$ and $f_1(x, y)$ is at most c_1. Thus there are non-zero integers m, s_1, and z_1 with $|m|$ and the greatest prime factor of s_1 at most c_2 such that

$$mf(x, y) = s_1 z_1^q. \tag{39}$$

Put

$$A_1 = mf\left(\left(1 + \frac{b}{\sqrt{D}}\right)\frac{1}{4a}, -\frac{1}{2\sqrt{D}}\right)\pi_1^h$$

and

$$A_2 = mf\left(\left(1 - \frac{b}{\sqrt{D}}\right)\frac{1}{4a}, \frac{1}{2\sqrt{D}}\right)\pi_2^h.$$

It follows from (37) and (38) that

$$mf(x, y) = A_1\varepsilon^{hn} + B + A_2\sigma(\varepsilon)^{hn},$$

where

$$\max\{|B|, |\sigma(B)|\} \leqslant c_3\varepsilon^{(h-1)|n|}. \tag{40}$$

Note that A_1 and A_2 are non-zero since $f(t, 1)$ is the minimal polynomial of α and, by assumption, $a\alpha^2 + b\alpha + c \neq 0$. Further if $|n| > c_4$ then, by (40), $A_1\varepsilon^{hn} + B \neq 0$ and $B + A_2\sigma(\varepsilon)^{hn} \neq 0$. Thus if $n > c_5$,

$$mf(x, y) = A_1\varepsilon^{hn} + B_1,$$

with $A_1 B_1 \neq 0$ and $|B_1| < (\varepsilon^h)^{(1-1/2h)n}$. Therefore, if $n > c_5$ and $|z_1| > 1$, then on applying Lemma 6 with $\alpha = \varepsilon^h$ and $\delta = 1/2h$ we conclude that $q < c_6$. On the other hand, if $-n > c_7$ then

$$mf(x, y) = A_2(\sigma(\varepsilon)^{-h})^{|n|} + B_2,$$

with $A_2 B_2 \neq 0$ and $|B_2| < |\sigma(\varepsilon)^{-h}|^{(1-1/2h)|n|}$, and on applying Lemma 6 we conclude that $q < c_8$. Note, by (37) and (38) that if $-c_7 \leqslant n \leqslant c_5$ then $\max\{|x|, |y|\} < c_9$ hence, from (11), $\max\{|z|, |s|, q\} < c_{10}$ and the theorem holds. Therefore we may assume that $|z_1| = 1$ or that $|z_1| > 1$ and $q < c_{10}$. If $|z_1| = 1$ put $q_1 = 2$ and otherwise put $q_1 = q$. Further, put $s_1 = s_2 s_3^{q_1}$, where s_2 and s_3 are integers, s_2 is not divisible by the q_1th power of a prime, and s_1 and s_2 have the same sign. Then $|s_2| < c_{11}$ and, by (39),

$$mf(x, y) = s_2 z_2^{q_1}, \tag{41}$$

where $z_2 = s_3 z_1$. Let $\alpha = \alpha_1, \alpha_2, ..., \alpha_h$ be the conjugates of α over $\mathbb{Q}$ and let v be the coefficient of t^h in $f(t, 1)$. Let $r \equiv hn \pmod{q_1}$ with $0 \leqslant r < q_1$. Multiply both sides of (41) by ε^{hn} to obtain

$$mv((\varepsilon^n(x - \alpha_1 y)) \cdots (\varepsilon^n(x - \alpha_h y))) = \varepsilon^r s_2 z_3^{q_1},$$

where $z_3 = \varepsilon^{[hn/q_1]} z_2$. If $n \equiv 0 \pmod 2$ put $k = 0$, while if $n \equiv 1 \pmod 2$ put $k = 1$. By (37) and (38), for $i = 1, ..., h$,

$$\varepsilon^n(x - \alpha_i y) = \gamma_{1,i}(\varepsilon^{[n/2]})^4 + \gamma_{2,i}(\varepsilon^{[n/2]})^2 + \gamma_{3,i},$$

where

$$\gamma_{1,i} = \left(\left(1 + \frac{b}{\sqrt{D}}\right)\frac{1}{4a} + \frac{\alpha_i}{2\sqrt{D}}\right)\pi_1 \varepsilon^{2k},$$

$$\gamma_{2,i} = \left(\frac{2cd - be}{D} - \alpha_i \frac{(2ae - bd)}{D}\right)\varepsilon^k,$$

and

$$\gamma_{3,i} = \left(\left(1 - \frac{b}{\sqrt{D}}\right)\frac{1}{4a} - \frac{\alpha_i}{2\sqrt{D}}\right)\pi_2(\varepsilon\sigma(\varepsilon))^n.$$

Thus the hyperelliptic equation

$$mv \prod_{i=1}^{h} (\gamma_{1,i} T^4 + \gamma_{2,i} T^2 + \gamma_{3,i}) = \varepsilon^r s_2 z^{q_1}$$

has a solution $T = \varepsilon^{[n/2]}$ and $Z = z_3$. Since $q_1 \geqslant 2$, if the polynomial $f(T) = mv \prod_{i=1}^{h} (\gamma_{1,i} T^4 + \gamma_{2,i} T^2 + \gamma_{3,i})$ has at least three simple zeros then by

Lemma 3 $\max\{\|\varepsilon^{[n/2]}\|, \|z_3\|\} < c_{12}$. But then $|n| < c_{13}$ hence, by (37) and (38), $\max(|x|, |y|) < c_{14}$ and so by (11), $\max(|s|, |z|, q) < c_{15}$ as required. Therefore, to complete our proof it suffices to show that $f(T)$ has at least three simple zeros. Put $g_i(U) = \gamma_{1,i}U^2 + \gamma_{2,i}U + \gamma_{3,i}$, for $i = 1, ..., h$, and observe that $f(T)$ has $4h$, hence at least three, simple zeros provided that $g_i(U)$ has two distinct non-zero roots for $i = 1, ..., h$ and that $g_i(U)$ and $g_j(U)$ have no common root for $i \leqslant i < j \leqslant h$.

We shall first show that $g_i(U)$ has two distinct non-zero roots for $i = 1, ..., h$. To this end it suffices to show that $\gamma_{1,i}\gamma_{3,i} \neq 0$ and $\gamma_{2,i}^2 - 4\gamma_{1,i}\gamma_{3,i} \neq 0$, for $i = 1, ..., h$. Recall that $\pi_1\pi_2(\varepsilon\sigma(\varepsilon))^n = M = (4a\,\Delta)/D$ hence

$$\gamma_{1,i}\gamma_{3,i} = -(a\alpha_i^2 + b\alpha_i + c)\varepsilon^{2k}\,\Delta/D^2,$$

for $i = 1, ..., h$. Since Δ is non-zero and since α_i is a conjugate of α and $a\alpha^2 + b\alpha + c \neq 0$ we have $\gamma_{1,i}\gamma_{3,i} \neq 0$ for $i = 1, ..., h$. Next observe that

$$\gamma_{2,i}^2 - 4\gamma_{1,i}\gamma_{3,i} = \frac{\varepsilon^{2k}}{D}\left((d^2 - 4af)\alpha_i^2 + (2\,de - 4bf)\alpha_i + (e^2 - 4cf)\right),$$

for $i = 1, ..., h$. Since α_i is a conjugate of α and $4f(a\alpha^2 + b\alpha + c) \neq (d\alpha + e)^2$, $\gamma_{2,i}^2 - 4\gamma_{1,i}\gamma_{3,i} \neq 0$, for $i = 1, ..., h$.

For i and j with $1 \leqslant i < j \leqslant h$ put

$$\begin{aligned} G_{i,j} = {} & (\gamma_{1,i}\gamma_{3,j} - \gamma_{1,j}\gamma_{3,i})^2 + (\gamma_{1,i}\gamma_{2,j} - \gamma_{1,j}\gamma_{2,i}) \\ & \times (\gamma_{3,i}\gamma_{2,j} - \gamma_{3,j}\gamma_{2,i}), \end{aligned}$$

and observe that if $g_i(U)$ and $g_j(U)$ have a common root then $G_{i,j} = 0$. However, some calculation reveals that $G_{i,j} = -(\alpha_i - \alpha_j)^2\,\Delta f\varepsilon^{4k}/D^2$. Since $f(t, 1)$ is the minimal polynomial of α, $f(t, 1)$ has no repeated roots and thus $\alpha_i \neq \alpha_j$, hence $G_{i,j}$ is non-zero as required.

5. Proof of Theorem 3

Let $c_1, c_2, ...$ be positive numbers which are effectively computable in terms of δ, P, and the sequence $(u_n)_{n=0}^{\infty}$. We have

$$u_n = \lambda_1\alpha_1^n + f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n.$$

We may assume that α_1 is positive by, if necessary, changing the sign of λ_1. Further since α_1 is an algebraic integer with absolute value strictly larger than all its conjugates, α_1 is real and either $\alpha_1 > 1$ or α_1 is 1. But if $\alpha_1 = 1$ then $t = 1$, contradicting our assumption that the sequence $(u_n)_{n=0}^{\infty}$ is non-degenerate, and so we may assume $\alpha_1 > 1$. Further we may assume, without loss of generality, that $|\alpha_2| \geqslant |\alpha_j|$ for $j = 3, ..., t$. Put

$$d_1 = \max\{\text{degree}(f_j) \mid j = 2, ..., t\}.$$

Then

$$|f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n| \leqslant c_1 n^{d_1} |\alpha_2|^n. \tag{42}$$

We shall now assume that for some non-zero integers s, x, q, and n with the greatest prime factor of s at most P, $|x| > 1$, $n > 0$, and $sz^q \neq \lambda_1 \alpha_1^n$, that (16) does not hold and we shall show that $q < c_2$ as required. Therefore

$$|sx^q - u_n| \leqslant \alpha_1^{n(1-\delta)}, \tag{43}$$

and since

$$|sx^q - u_n| \geqslant |sx^q - \lambda_1 \alpha_1^n| - |f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n|,$$

by (42) and (43),

$$|sx^q - \lambda_1 \alpha_1^n| \leqslant \alpha_1^{n(1-\delta)} + c_1 n^{d_1} |\alpha_2|^n. \tag{44}$$

Put $\theta = 0$ if $|\alpha_2| \leqslant 1$ and $\theta = (\log |\alpha_2|)/\log \alpha_1$ otherwise and put $\delta_1 = \min\{\delta/2, (1-\theta)/2\}$. Then, by (44),

$$|sx^q - \lambda_1 \alpha_1^n| < \alpha_1^{n(1-\delta_1)}, \tag{45}$$

for $n > c_3$. Notice that if $n \leqslant c_3$ then, since $|x|$ is at least 2, $q < c_4$. On the other hand, if $n > c_3$ then (45) holds and since $sx^q \neq \lambda_1 \alpha_1^n$, we may apply Lemma 6 to conclude that $q < c_5$. Our result now follows.

6. Proof of Theorem 4

Let ε be a positive real number and let γ be a real number with $|\alpha_1| > \gamma > |\alpha_2|$ and $\gamma > 1$. Let $c_1, c_2, ...$ be real numbers which depend only on P, $(u_n)_{n=0}^{\infty}$, γ, and ε. We shall assume that s, x, q, and n are non-zero integers with the greatest prime factor of s at most P, $|x| > 1$, $sx^q \neq \lambda_1 \alpha_1^n$, and

$$|sx^q - u_n| \leqslant \gamma^n, \tag{46}$$

and we shall show that if n is greater than c_1 then

$$q \leqslant ((1+2\varepsilon)\, d \log |\alpha_1|)/\log(|\alpha_1|/\gamma).$$

Define d_1 as in (42). Then

$$\begin{aligned}|sx^q - u_n| &\geqslant |sx^q - \lambda_1 \alpha_1^n| - |u_n - \lambda_1 \alpha_1^n| \\ &\geqslant |sx^q - \lambda_1 \alpha_1^n| - c_1 n^{d_1} |\alpha_2|^n,\end{aligned}$$

hence, by (46),

$$0 < |sx^q - \lambda_1 \alpha_1^n| \leqslant 2\gamma^n, \tag{47}$$

for $n > c_2$. Thus, by Theorem 3, $q < c_3$ for $n > c_2$. Therefore we may write $sx^q = s_1 x_1^q$, where s_1 and x_1 are integers with $|s_1| < c_4$ and $x_1 \geqslant 1$.

Consequently

$$1 \leqslant x_1 \leqslant c_5 |\alpha_1|^{n/q}. \tag{48}$$

Put $n_1 = [n/q]$ and $A_1 = \lambda_1 \alpha_1^{n - n_1 q} s_1^{-1}$. Then

$$|sx^q - \lambda_1 \alpha_1^n| = |s_1 \alpha_1^{n_1 q}| \, |(x_1/\alpha_1^{n_1})^q - A_1|. \tag{49}$$

Further

$$|(x_1/\alpha_1^{n_1})^q - A_1| \geqslant c_6 |(x_1/\alpha_1^{n_1}) - A_1^{1/q}|, \tag{50}$$

where $A_1^{1/q}$ is the qth root of A_1 closest to $x_1/\alpha_1^{n_1}$. Applying Lemma 4 with $K = \mathbb{Q}(\alpha_1^{n_1})$ and S the set of Archimedean valuations on K normalized as in (22), together with those normalized non-Archimedean valuations v for which $|\alpha_1^{n_1}|_v < 1$, we obtain

$$\begin{aligned}|(x_1/\alpha_1^{n_1}) - A_1^{1/q}| \geqslant c_7 &\left(\prod_{v \in S} \min(1, \|\alpha_1^{n_1}/x_1\|_v)^{-1} \right) \\ &\times H_K(x_1/\alpha_1^{n_1})^{-(2+\varepsilon)}.\end{aligned}$$

But x_1 is an integer and therefore

$$\begin{aligned}H_K(x_1/\alpha_1^{n_1}) &= \prod_{v \in V} \max(1, \|x_1/\alpha_1^{n_1}\|_v) = \prod_{v \in S} \max(1, \|x_1/\alpha_1^{n_1}\|_v) \\ &= \prod_{v \in S} \min(1, \|\alpha_1^{n_1}/x_1\|_v)^{-1}.\end{aligned}$$

Thus,

$$|(x_1/\alpha_1^{n_1}) - A_1^{1/q}| \geqslant c_7/(H_K(x_1/\alpha_1^{n_1}))^{-(1+\varepsilon)}. \tag{51}$$

By the product formula (24), $H_K(\theta) = H_K(\theta^{-1})$ for every non-zero element θ in K. For any algebraic number β we shall denote the height of β

by $H(\beta)$; recall that the height of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in its minimal polynomial. We have (see Schmidt [14, pp. 255–257]), for any non-zero algebraic number β,

$$H_{\mathbb{Q}(\beta)}(\beta) \leqslant CH(\beta),$$

where C is a positive number which is effectively computable in terms of the degree of β only. Thus

$$H_K(x_1/\alpha_1^{n_1}) = H_K(\alpha_1^{n_1}/x) \leqslant c_8 H(\alpha_1^{n_1}/x).$$

Let d_0 denote the degree of $\alpha_1^{n_1}$ over $\mathbb{Q}$. Since $|\alpha_1| \geqslant |\alpha_j|$ for $j = 2, ..., t$ we have, by (48),

$$H(\alpha_1^{n_1}/x) \leqslant c_9 |\alpha_1|^{d_0 n_1},$$

and since $d_0 \leqslant d$,

$$H(\alpha_1^{n_1}/x) \leqslant c_9 |\alpha_1|^{dn/q}. \tag{52}$$

Thus, from (49)–(52),

$$|sx^q - \lambda_1 \alpha_1^n| \geqslant c_{10} |\alpha_1|^{n \quad ((1+\varepsilon) dn/q)},$$

and so, by (47),

$$n \log \gamma \geqslant -c_{11} + n(1 - (1+\varepsilon)\, d/q) \log |\alpha_1|.$$

Since q is at most c_3 and $|\alpha_1|$ is greater than one,

$$n \log \gamma \geqslant n(1 - (1+2\varepsilon)\, d/q) \log |\alpha_1|,$$

hence

$$\frac{(1+2\varepsilon)\, d \log |\alpha_1|}{\log(\alpha_1|/\gamma)} \geqslant q,$$

for $n > c_{12}$. Our result now follows since q is an integer and ε can be arbitrarily small.

7. Proof of Corollary 1

We shall suppose that there are infinitely many integer triples (n, x, q) with $n \geqslant 0$, $|x| > 1$, and

$$q > \max\left(\frac{d \log |\alpha_1|}{\log(|\alpha_1|/\max(1, |\alpha_2|))}, d + r\right) \tag{53}$$

such that

$$u_n = x^q + T(x), \tag{54}$$

and we shall show that this leads to a contradiction. The condition,

$$q > (d \log |\alpha_1|)/\log(|\alpha_1|/\max(1, |\alpha_2|, |\alpha_1|^{r/(d+r)}))$$

is equivalent to (53). Further, since q is an integer, there exists a real number γ with

$$\gamma > \max(1, |\alpha_2|, |\alpha_1|^{r/(d+r)}), \tag{55}$$

such that (53) is equivalent to

$$q > (d \log |\alpha_1|)/\log(|\alpha_1|/\gamma). \tag{56}$$

It follows from Theorem 4, (55), and (56) that either there are infinitely many triples (n, x, q) as above with

$$|x^q - u_n| > \gamma^n, \tag{57}$$

or there are infinitely many such triples (n, x, q) with $x^q = \lambda_1 \alpha_1^n$.

Let $c_1, c_2, \ldots$ denote positive numbers which depend only on $(u_n)_{n=0}^{\infty}$ and T. For n sufficiently large

$$|u_n| = |x^q + T(x)| \geqslant |x|^q - c_1 |x|^r \geqslant \tfrac{1}{2} |x|^q, \tag{58}$$

and, since $|\alpha_1| > |\alpha_2|$,

$$|u_n| \leqslant |\lambda_1| \, |\alpha_1|^n + |f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n| < c_2 |\alpha_1|^n. \tag{59}$$

Thus, from (58) and (59), $|x| \leqslant c_3 |\alpha_1|^{n/q}$, hence

$$|T(x)| \leqslant c_4 |\alpha_1|^{rn/q} \leqslant c_4 |\alpha_1|^{rn/(d+r)}, \tag{60}$$

for n sufficiently large. It follows from (54), (55), and (60) that (57) holds for only finitely many integers n, hence for only finitely many triples (n, x, q) with $n \geqslant 0$, $|x| > 1$, and q satisfying (53). Therefore we have $x^q = \lambda_1 \alpha_1^n$ and so by (54),

$$T(x) = f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n, \tag{61}$$

for infinitely many such triples (n, x, q). Notice that α_2 is a real number since $|\alpha_1| > |\alpha_2| > |\alpha_j|$ for $j = 3, \ldots, t$, and since the conjugates of α_2 over the rationals are in $\{\alpha_1, \ldots, \alpha_t\}$. Further, since $|\alpha_2| > |\alpha_j|$ for $j = 3, \ldots, t$, $f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n$ is non-zero for n sufficiently large. Thus if $|\alpha_2| < 1$ then

$$1 > |f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n| > 0,$$

for n sufficiently large. However, T has integer coefficients and so either $T(x)=0$ or $|T(x)| \geqslant 1$. Therefore, by (61), $|\alpha_2| \geqslant 1$. Furthermore $|\alpha_2| > 1$ since α_2 is real and, by hypothesis, $\alpha_2 \neq \pm 1$.

Let d_2 denote the degree of f_2. For n sufficiently large

$$c_4 n^{d_2} |\alpha_2|^n < |f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n| < c_5 n^{d_2} |\alpha_2|^n., \tag{62}$$

and, by (61),

$$c_6 |x|^r < |T(x)| < c_7 |x|^r. \tag{63}$$

It follows from (62) and (63) that if $x^q = \lambda_1 \alpha_1^n$ then $|x|^r = |\lambda_1|^{r/q} |\alpha_1|^{rn/q}$ hence

$$c_8 n^{d_2} |\alpha_2|^n < |\alpha_1|^{rn/q} < c_9 n^{d_2} |\alpha_2|^n. \tag{64}$$

Let ε be a positive real number. Since there are infinitely many triples (n, x, q) as above with $x^q = \lambda_1 \alpha_1^n$ there exists such a triple (n_0, x_0, q_0) with n_0 sufficiently large that, by (64),

$$(1-\varepsilon)\frac{\log |\alpha_2|}{\log |\alpha_1|} < \frac{r}{q_0} < (1+\varepsilon)\frac{\log |\alpha_2|}{\log |\alpha_1|}.$$

Since ε is arbitrary and r is fixed there exists a positive integer q_1 with

$$r \log |\alpha_1| = q_1 \log |\alpha_2|.$$

Thus $|\alpha_1|$ and $|\alpha_2|$ are multiplicatively dependent and, since α_1 and α_2 are real, α_1 and α_2 are multiplicatively dependent. This contradicts our hypothesis and so establishes the result.

8. Proof of Theorem 5

Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of the greatest prime factor of s, the prime p, and the sequence $(u_n)_{n=0}^{\infty}$. By (14) and (20),

$$sx^q = \lambda_1 \alpha_1^n + f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n.$$

Thus, since $\operatorname{ord}_{\mathfrak{p}}(\alpha_j/\alpha_1) \geqslant 1$ for $j = 2, \ldots, t$,

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}}(sx^q\lambda_1^{-1}\alpha_1^{-n} - 1) &= \operatorname{ord}_{\mathfrak{p}}(\lambda_1^{-1} f_2(n)(\alpha_2/\alpha_1)^n \\ &\qquad + \cdots + \lambda_1^{-1} f_t(n)(\alpha_t/\alpha_1)^n) \\ &\geqslant n - c_1 \log n. \end{aligned} \tag{65}$$

Certainly

$$|u_n| < e^{c_2 n}, \tag{66}$$

for all positive integers n, and thus on writing $s = (-1)^{r_0} p_1^{r_1} \cdots p_k^{r_k}$ with $k \geqslant 0$ and $p_1, ..., p_k$ distinct prime numbers we see that the maximum of r_0, $r_1, ..., r_k$ is at most $c_3 n$. Thus, since $sx^q \neq \lambda_1 \alpha_1^n$ and p does not divide q we may apply Lemma 2 with $B' = q$ to obtain

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}}(sx^q \lambda_1^{-1} \alpha_1^{-n} - 1) &= \operatorname{ord}_{\mathfrak{p}}((-1)^{r_0} p_1^{r_1} \cdots p_k^{r_k} \lambda_1^{-1} \alpha_1^{-n} x^q - 1) \\ &< c_4(\log q \log x + n/q). \end{aligned} \tag{67}$$

Comparing (65) and (67) we find that

$$n - c_1 \log n - c_4 n/q < c_4 \log q \log x. \tag{68}$$

Notice that we may assume $q > c_4/3$ since otherwise our result holds. Similarly we may assume that $c_1 \log n < n/3$ since otherwise $n < c_5$ whence from (20) and (66) $q < c_6$ as required. Thus from (68),

$$n/3 < c_4 \log q \log x. \tag{69}$$

But since $sx^q = u_n$ we have, from (66) and (69),

$$q \log x < c_7 \log q \log x,$$

and thus $q < c_8$ as required.

9. Proof of Theorem 6

Let ε be a positive real number an let $c_1, c_2, ...$ denote positive numbers which depend only on ε, $v_1, ..., v_r$, P, and $(u_n)_{n=0}^{\infty}$. Let s, x, q, and n be integers with the greatest prime factor of s less than P, $|x| > 1$, $n \geqslant 0$, and $q \geqslant 1$ for which $sx^q = u_n$.

If v_1 is an Archimedean valuation then by Theorem 3 we may suppose that $q < c_1$ or that $sx^q = \lambda_1 \alpha_1^n$. If v_1 is non-Archimedean then by Theorem 5 we may suppose that $q < c_2$ or that $sx^q = \lambda_1 \alpha_1^n$; the condition $(p, q) = 1$ in the statement of Theorem 5 does not pose a problem since if p^k divides q then we may replace, if necessary, x by x^{p^k} and q by q/p^k. If $sx^q = \lambda_1 \alpha_1^n$ then

$$f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n = 0,$$

and by Lemma 5 this happens for only finitely many integers n since $(u_n)_{n=0}^{\infty}$ is a non-degenerate recurrence sequence. Similarly, by Lemma 5,

$u_n = 0$ for only finitely many integers n. Let us therefore assume that n is sufficiently large that $sx^q \neq \lambda_1 \alpha_1^n$ and $u_n \neq 0$. Then $q < c_3$ and so we may write $sx^q = s_1 x_1^q$ where s_1 and x_1 are integers with $|s_1| < c_4$ and $x_1 > 1$. Therefore

$$s_1 x_1^q - \lambda_1 \alpha_1^n = f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n.$$

Further, since $|u_n| < n^{c_5} |\alpha_b|^n$ for $n > 1$,

$$1 \leqslant x_1 \leqslant n^{c_5} |\alpha_b|^{n/q}. \tag{70}$$

Put $n_1 = [n/q]$ and $A_0 = \lambda_1 \alpha_1^{n - n_1 q} s_1^{-1}$. Then, for $i = 1, ..., r$,

$$\|s_1 x_1^q - \lambda_1 \alpha_1^n\|_{v_i} = \|s_1 \alpha_1^{n_1 q}\|_{v_i} \|(x_1/\alpha_1^{n_1})^q - A_0\|_{v_i},$$

hence

$$\begin{aligned} \|(x_1/\alpha_1^{n_1})^q - A_0\|_{v_i} &= \|s_1 \alpha_1^{n_1 q}\|_{v_i}^{-1} \|f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n\|_{v_i} \\ &\leqslant c_6 \|\alpha_1\|_{v_i}^{-n} \max_{j = 2,\dots,t} \|f_j(n)\alpha_j^n\|_{v_i} \\ &\leqslant c_6 n^{c_7} (\theta_i / \|\alpha_i / \|\alpha_1\|_{v_i})^n. \end{aligned} \tag{71}$$

Also

$$\|(x_1/\alpha_1^{n_1})^q - A_0\|_{v_i} \geqslant c_8 \|(x_1/\alpha_1^{n_1}) - A_i\|_{v_i}, \tag{72}$$

where A_i is the qth root of A_0 for which $\|(x_1/\alpha_1^{n_1}) - A_i\|_{v_i}$ is minimal.

Let S be the set of all normalized Archimedean valuations on K, the valuations $v_1, ..., v_r$, and all normalized non-Archimedean valuations v such that $\|\alpha_1\|_v < 1$. Put $A_{v_i} = A_i$ for $i = 1, ..., r$ and for $v \in S$ with v different from $v_1, ..., v_r$ put $A_v = 1$ unless $x_1 = \alpha_1^{n_1}$, in which case put $A_v = 2$. Then from (71) and (72),

$$\prod_{v \in S} \min(1, \|(x_1/\alpha_1^{n_1}) - A_v\|_v) \leqslant c_9 n^{rc_7} \left(\prod_{i=1}^{r} (\theta_i / \|\alpha_1\|_{v_i}) \right)^n. \tag{73}$$

Notice that A_0 can assume at most c_{10} possible values since $q < c_3$ and $|s_1| < c_4$ and thus there are at most c_{11} different possible values for A_v with v in S. Further A_v is non-zero and algebraic over K for v in S. Furthermore $x_1/\alpha_1^{n_1}$ is non-zero since u_n is non-zero and $x_1/\alpha_1^{n_1}$ is different from A_v for v in S since $sx^q \neq \lambda_1 \alpha_1^n$. Therefore we may apply Lemma 4 to conclude that

$$\begin{aligned} &\prod_{v \in S} \min(1, \|(x_1/\alpha_1^{n_1}) - A_v\|_v) \\ &\qquad \geqslant c_{12} \left(\prod_{v \in S} \min(1, \|\alpha_1^{n_1}/x_1\|_v)^{-1} \right) (H_K(x_1/\alpha_1^{n_1}))^{-2-\varepsilon}. \end{aligned}$$

As in the proof of Theorem 4 we find that

$$H_K(x_1/\alpha_1^{n_1}) = \prod_{v \in S} \min(1, \|\alpha_1^{n_1}/x_1\|_v)^{-1}.$$

Thus

$$\prod_{v \in S} \min(1, \|(x_1/\alpha_1^{n_1}) - A_v\|_v) \geqslant c_{12}(H_K(x_1/\alpha_1^{n_1}))^{-1-\varepsilon}. \tag{74}$$

Put $K_0 = \mathbb{Q}(\alpha_1^{n_1})$, $D = [K : \mathbb{Q}]$, and $d = [K_0 : \mathbb{Q}]$. We have

$$H_K(x_1/\alpha_1^{n_1}) = H_K(\alpha_1^{n_1}/x_1) = (H_{K_0}(\alpha_1^{n_1}/x))^{D/d}. \tag{75}$$

Again as in the proof of Theorem 4,

$$H_{K_0}(\alpha_1^{n_1}/x_1) \leqslant c_{13} H(\alpha_1^{n_1}/x_1), \tag{76}$$

where $H(\alpha_1^{n_1}/x_1)$ denote the height of $\alpha_1^{n_1}/x_1$. Thus, by (70),

$$H(\alpha_1^{n_1}/x_1) \leqslant c_{14} n^{c_5 d} |\alpha_b|^{nd/q}.$$

Therefore, by (75) and (76),

$$H_K(x_1/\alpha_1^{n_1}) \leqslant c_{15} n^{c_5 D} |\alpha_b|^{nD/q}. \tag{77}$$

We find, from (73), (74), and (77), that

$$\left(\prod_{i=1}^{r} \left(\frac{\|\alpha_1\|_{v_i}}{\theta_i}\right)\right)^n \leqslant c_{16} n^{c_{17}} |\alpha_b|^{(1+\varepsilon)nD/q}.$$

Since ε is arbitrary and q is an integer we have

$$q \leqslant \frac{(D \log |\alpha_b|)}{\log\left(\prod_{i=1}^{r} \left(\frac{\|\alpha_1\|_{v_i}}{\theta_i}\right)\right)},$$

for n sufficiently large. Our result now follows.

10. Proof of Corollary 2

Let $g(x)$ be a polynomial with integer coefficients and let the roots of $g(x)$ be α_1, α_2, and α_3 with multiplicities 1, 1, and m, respectively. Assume that $|\alpha_1| = |\alpha_2| = |\alpha_3|$ and that α_i/α_j is not a root of unity for $1 \leqslant i < j \leqslant 3$. Then exactly one of α_1, α_2, and α_3 is a real number and the other two numbers are complex conjugates, hence of the same multiplicity. Therefore it is no loss of generality to assume that α_1 and α_2 are complex conjugates and that α_3 is a real number. Since $|\alpha_1| = |\alpha_3|$ there is a real number θ such that $\alpha_1 = e^{i\theta}\alpha_3$ and since $\alpha_2 = \bar{\alpha}_1$, $\alpha_2 = e^{-i\theta}\alpha_3$ hence $\alpha_1\alpha_2\alpha_3 = \alpha_3^3$. Since $\alpha_1\alpha_2\alpha_3$ is an integer, α_3^3 is an integer and since α_1/α_3 and α_2/α_3 are not roots of unity, α_3 itself is an integer. In summary, α_3 is an integer and α_1 and α_2 are complex conjugate algebraic integers of degree 2 with $\alpha_1\alpha_2 = \alpha_3^2$. Thus $(x-\alpha_1)(x-\alpha_2) = x^2 + bx + c^2$ with b and c integers. Put $k = (b, c)$. Then α_1/k and α_2/k are algebraic integers since they are the roots of $x^2 + (b/k)x + (c/k)^2$. For any θ in the ring of algebraic integers of $\mathbb{Q}(\alpha_1)$ let $[\theta]$ denote the ideal generated by θ in that ring. Then $([\alpha_1/k], [\alpha_2/k]) = [1]$.

Let u_n be the nth term of a non-degenerate recurrence sequence as in (14), with $t = 3$, $f_2(n)$ a non-zero constant, λ_2 say, and $|\alpha_1| = |\alpha_2| = |\alpha_3|$. Thus, by the above remarks,

$$\begin{aligned} u_n &= \lambda_1\alpha_1^n + \lambda_2\alpha_2^n + f_3(n)\alpha_3^n \\ &= k^n(\lambda_1\gamma_1^n + \lambda_2\gamma_2^n + f_3(n)\gamma_3^n), \end{aligned}$$

where $\gamma_i = \alpha_i/k$ for $i = 1, 2, 3$ and $([\gamma_1], [\gamma_2]) = [1]$ in the ring of algebraic integers of $\mathbb{Q}(\gamma_1)$. Let us put

$$w_n = \lambda_1\gamma_1^n + \lambda_2\gamma_2^n + f_3(n)\gamma_3^n,$$

for $n = 0, 1, 2, \ldots$. Notice that $(w_n)_{n=0}^{\infty}$ is a non-degenerate linear recurrence sequence with $|\gamma_1| = |\gamma_2| = |\gamma_3|$ and as before

$$\gamma_1\gamma_2 = \gamma_3^2. \tag{78}$$

Put $K = \mathbb{Q}(\gamma_1)$. Let S be the set of non-Archimedean valuations on K, normalized as in (22), for which $\|\gamma_2\|_v < 1$. Each prime ideal $\mathfrak{p}$ dividing $[\gamma_2]$ also divides $[\gamma_3]$ by (78) and does not divide $[\gamma_1]$ since $([\gamma_1], [\gamma_2]) = [1]$. Therefore, by (78),

$$\|\gamma_2\|_v = \|\gamma_3\|_v^2, \qquad \text{for} \quad v \in S,$$

and so,

$$1 = \|\gamma_1\|_v > \|\gamma_3\|_v > \|\gamma_2\|_v, \qquad \text{for} \quad v \in S.$$

Thus $\prod_{v\in S}(\|\gamma_1\|_v/\|\gamma_3\|_v) = \prod_{v\in S}\|\gamma_2\|_v^{-1/2}$, and by the product formula (22), $\prod_{v\in S}\|\gamma_2\|_v^{-1/2} = \prod_{v\in T}\|\gamma_2\|_v^{1/2}$, where T denotes the set of normalized Archimedean valuations on K. T consists of a single element v_0 and $\|\gamma_2\|_{v_0} = |\gamma_2|^2$. Therefore

$$\prod_{v\in S}(\|\gamma_1\|_v/\|\gamma_3\|_v) = |\gamma_2|. \tag{79}$$

Denote the maximum of P, k, and 2 by P_0. We now apply Theorem 6 with $v_1, \ldots, v_r$ the valuations in S. By (79) there are only finitely many integers s, x, q, and n with $sx^q = w_n$, $s \neq 0$, and the greatest prime factor of s at most P_0, $n \geqslant 0$, and $q > 2$. Similarly putting $2^3 w_n = z_n$ for $n = 0, 1, 2, \ldots$, we see by Theorem 6 that there are only finitely many integers s and n with $s \neq 0$ and the greatest prime factor of s at most P_0, $n \geqslant 0$, and $s \cdot 2^3 = z_n$ or equivalently $s = w_n$.

Suppose that there are infinitely many integer quadruples (s, x, q, n) with $s \neq 0$ and the greatest prime factor of s at most P_0, $x > 1$, $n \geqslant 0$, $q > 2$, and $sx^q = u_n$. Then either there exist infinitely many integer quadruples (s, x, q, n) with $s \neq 0$ and the greatest prime factor of s at most P_0, $n \geqslant 0$, $q > 2$, and $sx^q = u_n$ or there exist infinitely many integer pairs (s, n) with $s \neq 0$ and the greatest prime factor of s at most P_0, $n \geqslant 0$, and $s = u_n$. Recall that w_n is an integer and that $u_n = k^n w_n$ for $n = 0, 1, 2, \ldots$. Thus in the former case there are infinitely many integer quadruples (s, x, q, n) with $s \neq 0$ and the greatest prime factor of s at most P_0, $n \geqslant 0$, $q > 2$, and $sx^q = w_n$. By the preceding paragraph there are only finitely many such quadruples. In the latter case there are infinitely many integer pairs (s, n) with $s \neq 0$ and the greatest prime factor of s at most P_0, $n \geqslant 0$, and $s = w_n$. Again, by the preceding paragraph, this is not possible. Therefore the above supposition is false and this establishes our result.

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References

1. A. Baker, Bounds for the solutions of the hyperelliptic equation, *Proc. Cambridge Philos. Soc.* **65** (1969), 439–444.
2. A. Baker, A sharpening of the bounds for linear forms in logarithms II, *Acta Arith.* **24** (1973), 33–36.
3. A. Baker, "Transcendental Number Theory," Cambridge Univ. Press, Cambridge, 1975.
4. B. Brindza, On S-integral solutions of the equation $y^m = f(x)$, *Acta Math. Hungar.* **44** (1984), 133–139.
5. P. Kiss, Differences of the terms of linear recurrences, *Studia Sci. Math. Hungar.*, to appear.
6. S. Lang, "Fundamentals of Diophantine Geometry," Springer-Verlag, New York, 1983.
7. D. J. Lewis, Diophantine equations: p-adic methods, *in* "Studies in Number Theory 6" (W. J. Leveque, Ed.), M.A.A., Prentice-Hall, Englewood Cliffs, 1969.
8. K. Mahler, Eine arithmetische Eigenschaft der Taylor-Koeffizienten rationaler Funktionen, *Proc. Akad. Wetensch. Amsterdam* **38** (1935), 50–60.

9. L. J. MORDELL, "Diophantine Equations," Academic Press, New York/London, 1969.
10. I. NEMES AND A. PETHÖ, Polynomial values in linear recurrences, *Publ. Math.* **31** (1984), 229–233.
11. I. NEMES AND A. PETHÖ, Polynomial values in linear recurrences II, *J. Number Theory*, to appear.
12. A. PETHÖ, Perfect powers in second order linear recurrences, *J. Number Theory* **15** (1982), 5–13.
13. A. J. VAN DER POORTEN, Linear forms in logarithms in the p-adic case in Transcendence theory: Advances and applications (A. Baker and D. W. Masser, Eds.), Academic Press, New York/London, 1977.
14. W. M. SCHMIDT, Diophantine approximation, *in* "Lecture Notes in Mathematics," Vol. 785, Springer-Verlag, Berlin, 1980.
15. T. N. SHOREY AND C. L. STEWART, On the Diophantine equation $ax^{2t}+bx^ty+cy^2=d$ and pure powers in recurrence sequences, *Math. Scand.* **52** (1983), 24–36.
16. C. L. STEWART, "On Some Diophantine Equations and Related Linear Recurrence Sequences," Sém. de Théorie des Nombres, Paris, 1980-1981, Birkhauser, Boston, 1982.