Topology and its Applications 156 (2009) 2812–2831

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Morphic cohomology and singular cohomology of motives over the complex numbers

Chenghao Chu

Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, USA

ARTICLE INFO

Article history:
Received 13 October 2008
Received in revised form 27 July 2009
Accepted 17 August 2009

MSC:
14F42
19E15
55N10

Keywords:
Motives
Morphic cohomology
Singular cohomology

ABSTRACT

Morphic cohomology and singular cohomology of motives over the complex numbers are defined via the triangulated category of motives. Regarding morphic cohomology as functors defined on the triangulated category of motives, natural transformations of morphic cohomology are studied.

1. Introduction

Let $\mathbb{C}$ denote the field of complex numbers. By *projective complex variety* we mean the set of zero locus of a family of homogeneous polynomials in some complex projective space $\mathbb{P}^n$. By *quasi-projective complex variety* or simply complex variety we mean a Zariski open subset of a projective variety. The *morphic cohomology* was first introduced and studied by Friedlander and Lawson in [8] for projective varieties and then generalized to normal quasi-projective varieties by Friedlander in [4]. The morphic cohomology of a complex variety X is bigraded and denoted as $L^qH^p(X)$, where p and q are integers satisfying $q, 2q-p \geqslant 0$. Roughly speaking, $L^qH^p(X)$ is the $(2q-p)$-th homotopy group of the space of q-cocycles over X and hence the theory of morphic cohomology is to study algebraic cycles using homotopy theory. Many properties of morphic cohomology have been proved for smooth complex varieties but not much for singular normal complex varieties.

Morphic cohomology has deep relations with many other cohomology theories such as singular cohomology and *motivic cohomology*. Recall from [16, §14] that the motivic cohomology of a complex variety X, denoted as $H^{p,q}(X)$ where q and p are integers with q nonnegative, can be represented in Voevodsky's triangulated category of motives over $\mathbb{C}$, denoted as *DM*. That is, there exists an object $\wp_{mot}(q)$ in *DM* such that

$$H^{p,q}(X) = Hom_{DM}\big(M(X), \wp_{mot}(q)[p-2q]\big),$$

where $M(X)$ denotes the motive of X. In this paper, we study the morphic cohomology from the motivic point of view, i.e., via the category *DM*. We construct objects $\wp_{mor}(q)$ and $\wp_{Sing}(q)$ in the category *DM* to represent morphic cohomology and

E-mail address: cchu@math.jhu.edu.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.08.017

singular cohomology of complex varieties. More precisely, for any smooth quasi-projective complex variety X, we prove in Theorem 5.1 and Theorem 6.9 that

$$L^qH^p(X) = Hom_{DM}\big(M(X), \wp_{mor}(q)[p-2q]\big),$$
$$H^p_{Sing}\big(X^{an}\big) = Hom_{DM}\big(M(X), \wp_{Sing}(q)[p-2q]\big).$$

The formula for singular cohomology remains valid if X is singular. As an application, we can define morphic cohomology and singular cohomology on motives, i.e., objects in DM. In particular, we re-define the morphic cohomology of a singular complex variety X to be

$$L^qH^p(X) = Hom_{DM}\big(M(X), \wp_{mor}(q)[p-2q]\big).$$

With this definition, many properties of morphic cohomology can be extended from smooth varieties to singular varieties. For example, we show in Theorem 6.12 that Friedlander's comparison result $L^qH^p(X) \cong H^p_{Sing}(X^{an})$, where X is smooth of pure dimension d and $q \geqslant d$, can be generalized to singular varieties. As a second application, we consider natural transformations of morphic cohomology of motives, where morphic cohomology is now regarded as functors on DM. Theorem 7.9 suggests that the *Friedlander–Mazur s-operation* (see [7]) is conjecturally the only nontrivial natural transformation of morphic cohomology of motives.

Here is an overview of the paper. We briefly recall in Section 2 the backgrounds, notations and techniques that are needed in this paper. Section 3 provides a general criteria for representing cohomology theories in the triangulated category of motives over a perfect field, which is used in Section 5 and Section 6 to represent morphic cohomology and singular cohomology of complex varieties. In Section 4, we apply the functor-on-curves topology (see [11]) on the category $Cor_{\mathbb{C}}$ to show that $Cor_{\mathbb{C}}$ is a topological category. This fact helps us to define the object representing morphic cohomology in Section 5. Section 7 provides a calculation of natural transformations of morphic cohomology of motives. We show that the s-operation is a natural transformation of morphic cohomology of motives. We also show that, if there is a natural transformation which is not generated by s, then there is a natural transformation, say τ, such that there exists for any natural number n a unique natural transformation τ_n satisfying $\tau = n\tau_n$.

2. Recollections and notations

2.1. Motivic cohomology

Throughout this subsection, letter K denotes a perfect field. The *category of finite correspondences over* K, denoted as Cor_K, has objects smooth quasi-projective varieties over K. For objects $X, Y \in Cor_K$, define $Hom_{Cor_K}(X,Y)$ to be the free abelian group on the *elementary finite correspondences* from X to Y, i.e., irreducible closed subvarieties of $X \times Y$ that are finite and surjective onto some irreducible component of X along the projection $X \times Y \longrightarrow X$. When K is understood, we simply write $Cor(X,Y)$ for $Hom_{Cor_K}(X,Y)$. For elementary finite correspondences $V \in Cor(X,Y)$ and $W \in Cor(Y,Z)$, the composition morphism $W \circ V$ is defined as the push-forward of the cycle theoretic intersection $(V \times Z) \bullet (X \times W)$ along the projection $X \times Y \times Z \longrightarrow X \times Z$. This extends by linearity to the definition of the composition map $Cor(X,Y) \times Cor(Y,Z) \longrightarrow Cor(X,Z)$. By identifying a morphism $f: X \longrightarrow Y$ as its graph in $X \times Y$, the usual category of smooth quasi-projective varieties over K, denoted as Sm/K, is a subcategory of Cor_K.

A *presheaf with transfers* is a contravariant functor from the category Cor_K to the category of abelian groups. A *Nisnevich sheaf with transfers* is a presheaf with transfer which is also a Nisnevich sheaf (see [17] for definition). Let $Sh_{Nis}(Cor_K)$ denote the category of Nisnevich sheaves with transfers over K. It is proved by Voevodsky that $Sh_{Nis}(Cor_K)$ is an abelian category with enough injective objects. Let $D^-(Sh_{Nis}(Cor_K))$ denote the derived category of $Sh_{Nis}(Cor_K)$ whose objects are cochain complexes bounded from above. We also refer cochain complexes simply as chain complexes.

Let $\mathbb{A}^s$ denote the s-dimensional affine space over K. A presheaf F on Sm/K is said to be *homotopy invariant* if $\pi^*: F(X) \longrightarrow F(X \times \mathbb{A}^1)$ is an isomorphism for any $X \in Sm/K$, where π is the projection $X \times \mathbb{A}^1 \longrightarrow X$. A chain complex of presheaves P^* is said to be *strictly homotopy invariant*, if all cohomology presheaves of P^* are homotopy invariant. A chain complex of Nisnevich sheaves F^* is said to be $\mathbb{A}^1$-*local*, if all the cohomology sheaves (with respect to the Nisnevich topology) of F^* are homotopy invariant. If P^* is a strictly homotopy invariant chain complex of presheaves with transfers, the Nisnevich sheafification of P^* is $\mathbb{A}^1$-local by [16, Theorems 22.1, 22.2].

Definition 2.1. ([16, Definition 14.1]) The triangulated category of motives over a perfect field K, denoted as $DM^{eff,-}_{Nis}(Cor_K)$, is defined to be the full subcategory of the derived category $D^-(Sh_{Nis}(Cor_K))$, which consists of only the $\mathbb{A}^1$-local objects.

Since we assume that K is perfect, $DM^{eff,-}_{Nis}(Cor_K)$ inherits the triangulated category structure from $D^-(Sh_{Nis}(Cor_K))$. We next construct a functor $C^*: D^-(Sh_{Nis}(Cor_K)) \longrightarrow DM^{eff,-}_{Nis}(Cor_K)$. For any nonnegative integer n, let Δ^n denote the algebraic n-simplex which is defined as the affine variety

$$\Delta^n = Spec\left(\frac{K[x_0, \dots, x_n]}{x_0 + \dots + x_n = 1}\right).$$

Exactly as in the topological situation, we have the boundary maps $\partial_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ and the degeneracy maps $\sigma_i^n : \Delta^{n+1} \longrightarrow \Delta^n$ where $0 \leqslant i \leqslant n$. In other words, we have a cosimplicial object $\Delta^\bullet = (\Delta^n, \partial_\bullet^n, \sigma_\bullet^n \mid n \geqslant 0)$ in the category Sm/K. If P is a presheaf of abelian groups on Sm/K, then there is a chain complex of presheaves bounded from above, denoted as C^*P, which sends $X \in Sm/K$ to the following chain complex

$$\cdots \longrightarrow P\big(X \times \Delta^2\big) \longrightarrow P\big(X \times \Delta^1\big) \longrightarrow P\big(X \times \Delta^0\big) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

In the above chain complex, the group $P(X \times \Delta^0)$ has degree 0 and the differential maps are defined as the alternating sums of the boundary maps. If F^* is a bounded above chain complex in $Sh_{Nis}(Cor_K)$, then there is a bi-complex $(s,t) \mapsto F^t(X \times \Delta^s)$, whose total complex is denoted as C^*F^*. It is easy to see that the chain complex C^*F^* is also bounded from above. As shown in [10, Lemma 4.1], C^*F^* is $\mathbb{A}^1$-local and hence an object in $DM_{Nis}^{eff,-}(Cor_K)$. Actually, C^* is right adjoint to the inclusion of $DM_{Nis}^{eff,-}(Cor_K)$ into $D^-(Sh_{Nis}(Cor_K))$. Let X be a not necessarily smooth quasi-projective variety over K. X may not be an object in Cor_K but we still have a well-defined presheaf with transfers sending $Y \in Cor_K$ to the free abelian group $Cor(Y,X)$ on elementary finite correspondences from Y to X. We denote this presheaf with transfers by h_X. It is shown in [16, Lemma 6.2] that h_X is a Nisnevich sheaf with transfers. So $M(X) = C^*h_X$ is an object in $DM_{Nis}^{eff,-}(Cor_K)$ and it is called the *motive* of X.

Definition 2.2. ([16, Definition 14.17, Theorem 15.2]) Let X be a quasi-projective variety over a perfect field K. For any integers p and q, define the motivic cohomology of X to be

$$H^{p,q}(X) = Hom_{DM_{Nis}^{eff,-}(Cor_K)}\big(M(X), M\big(\mathbb{P}^{q/q-1}\big)[p-2q]\big).$$

Here $M(\mathbb{P}^{q/q-1})$ is the cone of the obvious map $M(\mathbb{P}^{q-1}) \longrightarrow M(\mathbb{P}^q)$ in $DM_{Nis}^{eff,-}(Cor_K)$.

Remark 2.3. Usually, the objects of Cor_K are defined to be all smooth Noetherian separated schemes of finite type over K. Here we define Cor_K to contain only the smooth quasi-projective K-varieties. It is easy to show that this difference does not affect the category $DM_{Nis}^{eff,-}(Cor_K)$. That is, the category $DM_{Nis}^{eff,-}(Cor_K)$ constructed here is naturally equivalent to the usual one as defined, for example, in [16].

The motivic cohomology and the triangulated category of motives over a perfect field K with coefficients in a commutative ring R, denoted as $DM_{Nis}^{eff,-}(Cor_K, R)$, is defined in a similar way. For example, $DM_{Nis}^{eff,-}(Cor_K, \mathbb{Z}/n)$ is defined to be a full subcategory of $D^-(Sh_{Nis}(Cor_K), \mathbb{Z}/n)$, the derived category of Nisnevich sheaves with transfers of $\mathbb{Z}/n$-modules. Objects in $DM_{Nis}^{eff,-}(Cor_K, \mathbb{Z}/n)$ are $\mathbb{A}^1$-local objects in $D^-(Sh_{Nis}(Cor_K), \mathbb{Z}/n)$. Please see [16] for details about the discussion in this subsection.

2.2. The functor-on-curves topology

Let X and Y be quasi-projective complex varieties with X normal. Friedlander and Walker defined in [11] a topology on the set of morphisms from X to Y. For convenience, we call this topology the *functor-on-curves topology*. We need this topology to define the topological category structure on the category of finite correspondences over the complex numbers in Section 4.

In general, for any schemes A and B over the complex numbers $\mathbb{C}$, let $Hom(A,B)$ denote the set of morphisms from A to B over $Spec(\mathbb{C})$. Let X and Y be quasi-projective complex varieties and hence complex schemes, there is a contravariant functor of sets

$$\begin{aligned} \mathcal{M}or(X,Y) : (Sm/\mathbb{C})_{\leqslant 1} &\longrightarrow (Sets), \\ C &\mapsto Hom(X \times C, Y). \end{aligned}$$

Here $(Sm/\mathbb{C})_{\leqslant 1}$ is a category whose objects are spectra of those $\mathbb{C}$-algebras that have krull dimension at most one and can be realized as the localization of a finitely generated smooth integral algebra over $\mathbb{C}$. For example, the point $Spec(\mathbb{C})$, the affine line $Spec(\mathbb{C}[t])$ and its open subscheme $Spec(\mathbb{C}[t]_t)$ are in $(Sm/\mathbb{C})_{\leqslant 1}$ but the union of two points are not; the affine plane $X = Spec(\mathbb{C}[t_1,t_2])$ is not in $(Sm/\mathbb{C})_{\leqslant 1}$ but $Spec(\mathcal{O}_{X,\{x_1,\dots,x_n\}})$ is for any finite number of generic points $x_1,\dots,x_n$ of codimension one in X. Morphisms in $(Sm/\mathbb{C})_{\leqslant 1}$ are the usual morphisms of schemes over $\mathbb{C}$. Please see [11, §1] for more details and reasons for considering this category.

One useful property of the functor $\mathcal{M}or(X,Y)$ is that $\mathcal{M}or(X,Y)$ admits "proper, constructible presentation" which is defined as follows.

Definition 2.4. ([11, Definition 2.1]) Consider the data: $\mathcal{Y} = \coprod_{d=1}^{\infty} Y_d$, a disjoint union of complex projective varieties; $\mathcal{E} = \coprod_{d=1}^{\infty} E_d$, where each E_d is a constructible algebraic subset of Y_d; a "proper, constructible presentation" $R = \overline{R} \cap (\mathcal{E}^{\times 2})$, where $\overline{R} \subset \mathcal{Y}^{\times 2}$ is a closed equivalence relation such that $R = \overline{R} \cap (\mathcal{E} \times \mathcal{Y})$. Then we say that $(\mathcal{Y}, \mathcal{E}, R)$ is a proper, constructible presentation of a functor $F : (Sm/\mathbb{C})_{\leqslant 1} \longrightarrow (Sets)$ if F is the functor given by sending $C \in (Sm/\mathbb{C})_{\leqslant 1}$ to $Hom(C, \mathcal{E})/Hom(C, R)$. In general, if E is a constructible subset of a variety Y, we defined $Hom(X, E)$ to be the set of those morphisms from X to Y whose images land in E.

For any quasi-projective complex varieties X and Y, the proper, constructible presentation of the functor $\mathcal{M}or(X, Y)$ is given in the following proposition.

Proposition 2.5. *([11, Proposition 2.2]) Let X and Y be quasi-projective complex varieties and let X^w be the weak normalization of X. Assume that $X^w \subset \overline{X}^w$ and $Y \subset \overline{Y}$ are their projective closures. Then $\mathcal{M}or(X, Y) : (Sm/\mathbb{C})_{\leqslant 1} \longrightarrow (Sets)$ admits a proper, constructible presentation $(\mathcal{C}(\overline{X}^w \times \overline{Y}), \mathcal{E}_{0,1}(X^w, Y), R)$ defined as follows: $\mathcal{C}(\overline{X}^w \times \overline{Y})$ is the Chow variety of effective cycles in $\overline{X}^w \times \overline{Y}$ which have dimension equal to the dimension of X^w (locally); $\mathcal{E}_{0,1}(X^w, Y) \subset \mathcal{C}(\overline{X}^w \times \overline{Y})$ is the constructible subset of those cycles whose restriction to $X^w \times \overline{Y}$ are graphs of morphisms from X^w to Y; and R is the equivalence relation associated to the diagonal action of $\mathcal{C}((\overline{X}^w - X^w) \times \overline{Y})$, the subset of those cycles supported on $(\overline{X}^w - X^w) \times \overline{Y}$, on $(\mathcal{C}(\overline{X}^w \times \overline{Y}))^{\times 2}$.*

If $F : (Sm/\mathbb{C})_{\leqslant 1} \longrightarrow (Sets)$ admits a proper, constructible presentation $(\mathcal{Y}, \mathcal{E}, R)$, we denote the set of $\mathbb{C}$-points of $\mathcal{E}$ and R by $\mathcal{E}(\mathbb{C})$ and $R(\mathbb{C})$ respectively. Let $\mathcal{E}^{an}$ and R^{an} denote the associated analytic space of $\mathcal{E}(\mathbb{C})$ and $R(\mathbb{C})$, respectively. R^{an} is still a closed equivalence relation on $\mathcal{E}^{an}$. Let $(\mathcal{E}/R)^{an}$ denote the quotient topological space and we call its topology the **functor-on-curves topology**. Since when the functor F is $\mathcal{M}or(X, Y)$ where X is normal, the underlying set of $(\mathcal{E}/R)^{an}$ is exactly the set of morphisms $Hom(X, Y)$, so we obtain the functor-on-curves topology on the set $Hom(X, Y)$. The resulting topological space is denoted as $\mathcal{M}or(X, Y)$ as usual. The following theorem is useful when working with this topology. For more details and constructions on this functor-on-curves topology, please see [11].

Theorem 2.6. *([11, Theorem 2.3]) Let $F, F' : (Sm/\mathbb{C})_{\leqslant 1} \longrightarrow (Sets)$ be contravariant functors provided with proper, constructible presentations $(\mathcal{Y}, \mathcal{E}, R)$ and $(\mathcal{Y}', \mathcal{E}', R')$. Then a natural transformation $\psi : F \longrightarrow F'$ induces a continuous map $\psi^{an} : (\mathcal{E}/R)^{an} \longrightarrow (\mathcal{E}'/R')^{an}$.*

2.3. Morphic cohomology

All varieties in this subsection are complex varieties. If X is a complex variety, let X^{an} denote the topological space whose points are complex points of X and whose topology is the usual analytic topology.

Recall that if M is an abelian topological monoid, the *naive topological group completion* of M, denoted as M^+, is the quotient space of $M \times M$ with respect to the following equivalence relation $\backsim$:

$$(a_1, b_1) \backsim (a_2, b_2) \quad \text{in } M \times M \text{ if } a_1 + b_2 + c = b_1 + a_2 + c \text{ for some } c \in M.$$

In general, M^+ may not be a topological group. For the topological monoids considered in this paper, all their naive topological group completions are compactly generated Hausdorff topological groups as explained in [4, Theorem 1.5] or [14, Remark 2.2].

Let Y be a projective complex variety defined in some projective space $\mathbb{P}^n$. For $r \geqslant 0$, the *Chow monoid* of r cycles on Y is defined as the disjoint union $\mathcal{C}_r(Y) = \coprod_{d \geqslant 0} \mathcal{C}_{r,d}(Y)$, where $\mathcal{C}_{r,d}(Y)$ is the *Chow variety* of r-cycles of degree d on Y. Let X be a normal quasi-projective complex variety, then we have the functor-on-curves topology defined on $Hom(X, \mathcal{C}_{r,d}(Y))$ and the resulting topological space is denoted as $\mathcal{M}or(X, \mathcal{C}_{r,d}(Y))$. By Theorem 2.6 one can check that $\mathcal{M}or(X, \mathcal{C}_r(Y)) = \coprod_{d \geqslant 0} \mathcal{M}or(X, \mathcal{C}_{r,d}(Y))$ is a topological monoid whose addition is induced by the addition of cycles. It follows from definition that, if Y' is a closed subvariety of Y, $\mathcal{M}or(X, \mathcal{C}_r(Y'))$ is a closed submonoid of $\mathcal{M}or(X, \mathcal{C}_r(Y))$. Let $\mathcal{Z}^t(X)$ denote the naive topological group completion of the quotient topological monoid $\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^t))/\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^{t-1}))$. By [4, Proposition 2.2], one can equivalently define $\mathcal{Z}^t(X)$ as the quotient topological group of the naive topological group completions of $\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^t))$. That is

$$\mathcal{Z}^t(X) = \big(\mathcal{M}or\big(X, \mathcal{C}_0(\mathbb{P}^t)\big)/\mathcal{M}or\big(X, \mathcal{C}_0\big(\mathbb{P}^{t-1}\big)\big)\big)^+ = \mathcal{M}or\big(X, \mathcal{C}_0\big(\mathbb{P}^t\big)\big)^+/\mathcal{M}or\big(X, \mathcal{C}_0\big(\mathbb{P}^{t-1}\big)\big)^+.$$

Definition 2.7. ([4, Proposition 3.4]) The *morphic cohomology* of a normal quasi-projective complex variety X is defined as:

$$L^t H^{2t-i}(X) = \pi_i \mathcal{Z}^t(X), \quad t \geqslant 0 \text{ and } i \geqslant 0.$$

Remark 2.8. The topological group $\mathcal{Z}^t(X)$ defined here is the same topological group $\mathcal{Z}^t(X)$ defined in [4, Definition 2.4] by [4, Proposition 1.9] and [11, Proposition 2.4]. So the above definition of morphic cohomology is consistent with the original definition [4, Proposition 3.4, (3.4.1)].

By [4, Proposition 6.3], morphic cohomology is homotopy invariant on smooth quasi-projective complex varieties. That is, for a smooth quasi-projective complex variety X, the natural map $L^qH^p(X) \xrightarrow{\cong} L^qH^p(X \times \mathbb{A}^1)$ is an isomorphism. Morphic cohomology also satisfies the Mayer–Vietoris property for smooth quasi-projective complex varieties. That is, if $X = U \cup V$ is an open covering of a smooth quasi-projective complex variety X, there is a natural long exact sequence

$$\cdots \longrightarrow L^qH^p(X) \longrightarrow L^qH^p(U) \oplus L^qH^p(V) \longrightarrow L^qH^p(U \cap V) \longrightarrow L^qH^{p+1}(X) \longrightarrow \cdots.$$

This fact is well known to experts. For lacking of a suitable reference, we sketch a proof as follows. We recall the definition of *Lawson homology* on quasi-projective complex varieties from [15]. The localization sequence of Lawson homology [6,14] implies that Lawson homology satisfies the Mayer–Vietoris property. Now we apply the *Duality Theorem* [4] to get the desired Mayer–Vietoris sequence for morphic cohomology from the corresponding Mayer–Vietoris sequence for Lawson homology.

By [3, Proposition 3.4], there is a natural transformation $L^qH^p(X) \longrightarrow H^p_{Sing}(X^{an})$, where X is a normal quasi-projective complex variety. If X is smooth projective and $q \geqslant Dim(X)$, this natural transformation is an isomorphism by [9, Theorems 5.6, 5.8]. When X is smooth quasi-projective and $q \geqslant Dim(X)$, this map is still an isomorphism which follows from the compatibility of the long exact sequence of [4, Corollary 6.2] and the similar long exact sequence of singular cohomology.

Now we briefly review the construction of the s-operation on morphic cohomology, which is a dual version of the s-operation on Lawson homology introduced by Friedlander and Mazur in [7]. Let $[\underline{x}] = [x_0, x_1, \ldots, x_n]$ and $[\underline{y}] = [y_0, y_1, \ldots, y_m]$ be homogeneous coordinates in the projective space $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively. We use $[\underline{x}, \underline{y}]$ as homogeneous coordinates for $\mathbb{P}^{n+m+1}$. Let $W \subset \mathbb{P}^n$ be a closed subvariety defined by homogeneous polynomials $f_1(\underline{x}), \ldots, f_s(\underline{x})$ and $V \subset \mathbb{P}^m$ be a closed subvariety defined by homogeneous polynomials $g_1(\underline{y}), \ldots, g_t(\underline{y})$, then the *algebraic join* of W and V, denoted as $W \# V$, is the closed subvariety in $\mathbb{P}^{n+m+1}$ defined by the polynomials

$$f_1(\underline{x}), f_2(\underline{x}), \ldots, f_s(\underline{x}), g_1(\underline{y}), g_2(\underline{y}), \ldots, g_t(\underline{y}).$$

For normal quasi-projective complex varieties X and Y, [7, Section 3.5] defines a biadditive continuous map

$$\begin{aligned} \mathcal{M}or\big(X, \mathcal{C}_0(\mathbb{P}^n)\big) \times \mathcal{M}or\big(Y, \mathcal{C}_0(\mathbb{P}^m)\big) &\xrightarrow{\#} \mathcal{M}or\big(X \times Y, \mathcal{C}_1(\mathbb{P}^{m+n+1})\big), \\ (f, g) &\mapsto f \# g, \end{aligned}$$

where the map $f \# g$ sends $(x \times y) \in X \times Y$ to $\sum n_i m_j V_i \# W_j$ if $f(x) = \sum n_i V_i$ and $g(y) = \sum m_j W_j$. This further induces a continuous biadditive map on their naive group completions. If Y is a point and $m = 0$, then $\mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^0))^+ = \mathbb{Z} \cdot \mathbb{P}^0$ is the free abelian group on the point $\mathbb{P}^0$. In this case, the restriction of # to $\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^n))^+ \times \mathbb{P}^0$ is called the *algebraic suspension* map and is denoted as

$$\Sigma\!\!\!/ : \mathcal{M}or\big(X, \mathcal{C}_0(\mathbb{P}^n)\big)^+ \longrightarrow \mathcal{M}or\big(X, \mathcal{C}_1(\mathbb{P}^{n+1})\big)^+.$$

The Algebraic Suspension Theorem for Cocycles [8, Theorem 3.3] implies that $\Sigma\!\!\!/$ is a homotopy equivalence.

Now we consider the following diagram of continuous maps of topological spaces

$$\mathcal{M}or\big(X, \mathcal{C}_0(\mathbb{P}^n)\big)^+ \times \mathcal{M}or\big(pt, \mathcal{C}_0(\mathbb{P}^1)\big)^+ \xrightarrow{\#} \mathcal{M}or\big(X, \mathcal{C}_1(\mathbb{P}^{n+2})\big)^+ \xleftarrow{\Sigma\!\!\!/} \mathcal{M}or\big(X, \mathcal{C}_0(\mathbb{P}^{n+1})\big)^+.$$

Clearly, it induces a diagram on the quotient spaces

$$\mathcal{Z}^n(X) \times \mathcal{M}or\big(pt, \mathcal{C}_0(\mathbb{P}^1)\big)^+ \xrightarrow{\#} \mathcal{M}or\big(X, \mathcal{C}_1(\mathbb{P}^{n+2})\big)^+ / \mathcal{M}or\big(X, \mathcal{C}_1(\mathbb{P}^{n+1})\big)^+ \xleftarrow{\Sigma\!\!\!/} \mathcal{Z}^{n+1}(X). \tag{1}$$

The map $\Sigma\!\!\!/$ on the right-hand side of diagram (1) is still a homotopy equivalence. Since $\mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1))^+ = \mathcal{C}_0(\mathbb{P}^1)^+ = (\coprod_{d \geqslant 0} S^d(\mathbb{P}^1)^{an})^+$ where $S^d(-)$ is the d-fold symmetric product of a topological space, by the Dold–Thom Theorem [1] we see that $\pi_2 \mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1))^+$ is the singular homology group $H_2((\mathbb{P}^1)^{an})$ which is the free abelian group $\mathbb{Z}$. So by taking homotopy groups of the spaces in (1), we obtain the following commutative diagram.

$$\begin{array}{ccc} \pi_i \mathcal{Z}^n(X) \otimes \pi_2 \mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1))^+ & \longrightarrow & \pi_{i+2} \mathcal{Z}^{n+1}(X) \\ \downarrow \cong & & \| \\ L^nH^{2n-i}(X) & \xrightarrow{s} & L^{n+1}H^{2n-i}(X) \end{array}$$

The induced bottom map is by definition the s-operation. Please see [4,7,9] for more details.

3. Strictly homotopy invariant pseudo-flasque presheaf with transfers

Let K be a perfect field and let Sm/K denote the category of smooth separated Noetherian scheme of finite type over K as usual. As before, we also use the term "chain complex" to mean a cochain complex bounded from above. In particular, a chain complex may have differential maps of degree $+1$.

Recall from [5] that a chain complex of presheaves P^* is said to be *pseudo-flasque* if for any $U \in Sm/K$ and open covering $U = U_1 \cup U_2$, we have a distinguished triangle of chain complexes of abelian groups

$$P^*(U) \longrightarrow P^*(U_1) \oplus P^*(U_2) \longrightarrow P^*(U_1 \cap U_2).$$

We need the following theorem from [5] concerning the cohomology presheaves of a pseudo-flasque chain complex of presheaves. For our purpose we only state it for a smooth scheme X.

Theorem 3.1. *(Friedlander [5, Theorem 3.1]) Let $X \in Sm/K$. Let $\mathcal{HP}_X$ denote the homotopy category of the category $\mathcal{P}_X$ of chain complexes of presheaves of abelian groups on the small Zariski site Zar_X. Denote $\mathcal{D}_X$ the localization of $\mathcal{P}_X$ with respect to the thick subcategory of those $P \in \mathcal{P}_X$ with the property that every stalk of P is acyclic. Denote $\mathbf{Z}_X$ the presheaf sending Y to the free abelian group $\mathbb{Z}(Hom(Y, X))$. For any $P \in \mathcal{P}_X$ and $i \in \mathbb{Z}$, we have*

$$H_i\big(P(X)\big) = Hom_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], P\big).$$

If P is pseudo-flasque, then for any $i \in \mathbb{Z}$, the natural map

$$Hom_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], P\big) \longrightarrow Hom_{\mathcal{D}_X}\big(\mathbf{Z}_X[i], P\big)$$

is an isomorphism.

Note that the chain complexes in [5] are bounded from below and the differential maps have degree -1. Such a chain complex P_* can be naturally identified with a bounded above cochain complex P^* by setting $P^i = P_{-i}$ and using the same differential maps. It is obvious that $H^i(P^*) = H_{-i}(P_*)$. The following lemma follows easily from Theorem 3.1.

Lemma 3.2. *Let $X \in Sm/K$. If P^* is a pseudo-flasque chain complex of presheaves on Sm/K or Zar_X which is bounded from above, there is a natural isomorphism*

$$H^i\big(P^*(X)\big) = \mathbb{H}^i\big(X, P^*_{Zar}\big),$$

where P^*_{Zar} is the chain complex of Zariski sheaves associated to P^*.

Proof. Let P_* denote the corresponding chain complex which has differential maps of degree -1, then P_* satisfies the condition of Theorem 3.1. So we have

$$H^i\big(P^*(X)\big) \overset{(1)}{=} H_{-i}\big(P_*(X)\big) \overset{(2)}{=} Hom_{\mathcal{D}_X}\big(\mathbf{Z}_X[-i], P_*\big) \overset{(3)}{=} Hom_{\mathcal{D}_X}\big(\mathbf{Z}_X, P_*[i]\big) \overset{(4)}{=} \mathbb{H}^i\big(X, P^*_{Zar}\big).$$

The isomorphisms at (1) and (3) are trivial. (2) is guaranteed by Theorem 3.1. Let I_* be an injective resolution of the associated complex of sheaves $(P_*)_{Zar}$. Then I^* is also an injective resolution of P^*_{Zar}. So we have the following computations.

$$Hom_{\mathcal{D}_X}\big(\mathbf{Z}_X, P_*[i]\big) = H_{-i}\big(I_*(X)\big) = H^i\big(I^*(X)\big) = \mathbb{H}^i\big(X, P^*_{Zar}\big).$$

This proves the isomorphism at (4). □

The following theorem of Voevodsky plays an important role to represent motivic cohomology in the category $DM^{eff,-}_{Nis}(Cor_K)$ (see Section 2.1). Please see [16, Proposition 13.10] for details.

Theorem 3.3 *(Voevodsky). Let P^* be a bounded above chain complex of Nisnevich sheaves with transfers, whose cohomology sheaves are homotopy invariant. Then the natural map $\mathbb{H}^n_{Zar}(X, P^*) \longrightarrow \mathbb{H}^n_{Nis}(X, P^*)$ is an isomorphism for any $X \in Sm/K$ where K is a perfect field.*

Since we are working with presheaves, we need the following generalized version of Theorem 3.3.

Lemma 3.4. *Let P^* be a bounded above chain complex of presheaves with transfers. If P^* is strictly homotopy invariant, i.e., all cohomology presheaves of P^* are homotopy invariant, then the natural map*

$$\mathbb{H}^n_{Zar}\big(X, P^*_{Zar}\big) \longrightarrow \mathbb{H}^n_{Nis}\big(X, P^*_{Nis}\big)$$

is an isomorphism for any $X \in Sm/K$, where P^*_{Zar} and P^*_{Nis} are the sheafifications of P^* in the corresponding topology and K is a perfect field.

Proof. Let $Dim(X) = d$, then both the Zariski and the Nisnevich cohomology dimension of X are less than or equal to d, i.e., $H^{>d}_{Zar}(X, F) \cong H^{>d}_{Nis}(X, G) = 0$ for any Zariski sheaf F and Nisnevich sheaf G. So the following two hypercohomology spectral sequences both converge.

$$E_2^{p,q} = H^p_{Zar}\big(X, \big(H^q P^*\big)_{Zar}\big) \implies \mathbb{H}^{p+q}_{Zar}\big(X, P^*_{Zar}\big),$$
$$E_2^{p,q} = H^p_{Nis}\big(X, \big(H^q P^*\big)_{Nis}\big) \implies \mathbb{H}^{p+q}_{Nis}\big(X, P^*_{Nis}\big).$$

Here we used the fact that $(H^p P^*)_{Zar} = (H^p(P^*_{Zar}))_{Zar}$ and $(H^p P^*)_{Nis} = (H^p(P^*_{Nis}))_{Nis}$, which is true because sheafification in Zariski topology and Nisnevich topology are exact functors.

There is a natural morphism from the first spectral sequence to the second one, which is induced by the Nisnevich sheafification functor. By assumption, the presheaf with transfers $H^q P^*$ is homotopy invariant, which implies $(H^q P^*)_{Zar} = (H^q P^*)_{Nis}$ as presheaves with transfers by [16, Theorems 22.1, 22.2]. So the natural map $H^p_{Zar}(X, (H^q P^*)_{Zar}) \longrightarrow H^p_{Nis}(X, (H^q P^*)_{Nis})$ is an isomorphism by Theorem 3.3. So the natural map $\mathbb{H}^n_{Zar}(X, P^*_{Zar}) \longrightarrow \mathbb{H}^n_{Nis}(X, P^*_{Nis})$ is also an isomorphism. □

We are now ready to prove the main result of this section.

Theorem 3.5. *Let K be a perfect field. Let P^* be a chain complex of presheaves with transfers which is bounded from above, strictly homotopy invariant and pseudo-flasque. For any smooth scheme X over K, there exist natural isomorphisms*

$$H^n\big(P^*(X)\big) \overset{(1)}{\cong} \mathbb{H}^n_{Zar}\big(X, P^*_{Zar}\big) \overset{(2)}{\cong} \mathbb{H}^n_{Nis}\big(X, P^*_{Nis}\big) \overset{(3)}{\cong} \mathbb{H}^n_{Zar}\big(X, P^*_{Nis}\big)$$
$$\overset{(4)}{\cong} Hom_{D^-}\big(h_X, P^*_{Nis}[n]\big) \overset{(5)}{\cong} Hom_{DM^{eff,-}_{Nis}(Cor_K)}\big(M(X), P^*_{Nis}[n]\big).$$

Here $D^- = D^-(Sh_{Nis}(Cor_K))$ denotes the derived category of bounded above chain complexes of Nisnevich sheaves with transfers.

Proof. (1) and (2) follow from Lemmas 3.2 and 3.4. Since P^* is strictly homotopy invariant, all Nisnevich cohomology sheaves of P^*_{Nis} are also homotopy invariant thanks to [16, Theorems 22.1, 22.2]. So P^*_{Nis} is $\mathbb{A}^1$-local and hence (3) follows from [16, Proposition 13.10], (4) and (5) follow from [16, Proposition 14.16]. □

4. Topological category structure on $Cor_{\mathbb{C}}$

All varieties in this section are defined over the complex numbers $\mathbb{C}$. Let $Cor_{\mathbb{C}}$ denote the category of finite correspondences over $\mathbb{C}$. For any $X, Y \in Cor_{\mathbb{C}}$, recall that morphisms from X to Y in $Cor_{\mathbb{C}}$ are called finite correspondences which are formal sums of the elementary finite correspondences with integral coefficients. If all the coefficients are nonnegative, the finite correspondence is called *effective finite correspondence*. Let $Eff^d(X, Y)$ denote the set of effective finite correspondences of degree d over X and let $Eff(X, Y)$ denote the monoid $\coprod_{d \geqslant 0} Eff^d(X, Y)$.

If X is irreducible, the "graph construction" of [3, Theorem 1.4] gives a natural identification between the set $Eff^d(X, Y)$ and the set $Hom(X, S^d Y)$, where $S^d Y$ is the d-fold symmetric product of Y. A cycle $V \in Eff^d(X, Y)$ is identified with the morphism sending $x \in X$ to the cycle theoretic intersection $(x \times Y) \bullet V$ in $X \times Y$. This identification defines a natural isomorphism of discrete monoids

$$\tau_{X,Y} : Eff(X, Y) \xrightarrow{\cong} Hom\bigg(X, \coprod_{d \geqslant 0} S^d Y\bigg).$$

As recalled in Section 2.2, we have the functor-on-curves topology on $Hom(X, \coprod_{d \geqslant 0} S^d Y) = \coprod_{d \geqslant 0} Hom(X, S^d Y)$ and the resulting space is denoted as $\mathcal{M}or(X, \coprod_{d \geqslant 0} S^d Y)$. We give $Eff(X, Y)$ the topology induced from the isomorphism $\tau_{X,Y}$. It is clear that $Eff(X, Y)$ and $\mathcal{M}or(X, \coprod_{d \geqslant 0} S^d Y)$ are now topological monoids naturally identified with each other by the homeomorphism $\tau_{X,Y}$.

If $X = X_1 \cup X_2$ where X_1 and X_2 are unions of irreducible components of X, then $X_1 \cap X_2$ is empty because X is smooth. So $Eff(X, Y) = Eff(X_1, Y) \times Eff(X_2, Y)$ and we give $Eff(X, Y)$ the product topology. Finally, we define the topology on $Cor(X, Y)$ to be the quotient topology with respect to the obvious surjective map

$$Eff(X, Y) \times Eff(X, Y) \longrightarrow Cor(X, Y); \qquad (V_1, V_2) \mapsto V_1 - V_2.$$

Now $Cor(X, Y)$ is the naive topological group completion of the topological monoid $Eff(X, Y)$. For convenience, we still call the topologies on $Eff(X, Y)$ and $Cor(X, Y)$ the functor-on-curves topology.

If Y is projective, $\coprod_{d \geqslant 0} S^d Y = \mathcal{C}_0(Y)$ is the Chow monoid of zero cycles on Y. So by the natural isomorphism $\tau_{X,Y}$ we have $Eff(X, Y) \xrightarrow{\cong} \mathcal{M}or(X, \mathcal{C}_0(Y))$ and hence $Cor(X, Y) \xrightarrow{\cong} \mathcal{M}or(X, \mathcal{C}_0(Y))^+$. Let $Cor(X, \mathbb{P}^{r/r-1})$ denote the quotient topological group $Cor(X, \mathbb{P}^r)/Cor(X, \mathbb{P}^{r-1})$. Obviously, we have

$$Cor(X, \mathbb{P}^{r/r-1}) = Cor(X, \mathbb{P}^r)/Cor(X, \mathbb{P}^{r-1}) \cong \mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^r))^+/\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^{r-1}))^+ = \mathcal{Z}^r(X).$$

The topological group $\mathcal{Z}^r(X)$ is recalled in Section 2.3 and its homotopy groups are the morphic cohomologies of X. So we have the following formula.

$$\boxed{L^r H^{2r-i}(X) = \pi_i \mathcal{Z}^r(X) = \pi_i Cor(X, \mathbb{P}^{r/r-1}), \quad r \geqslant 0 \text{ and } i \geqslant 0.}$$

With the functor-on-curves topology defined on the *Hom*-sets in $Cor_{\mathbb{C}}$, we show in the remaining part of this section that $Cor_{\mathbb{C}}$ is a topological category.

Lemma 4.1. *For any quasi-projective complex varieties X and Y with X being normal, there exists a natural continuous map*

$$\begin{aligned} S^d : \mathcal{M}or(X, Y) &\longrightarrow \mathcal{M}or(S^d X, S^d Y), \\ f &\mapsto S^d f : [x_1, \dots, x_d] \mapsto [f(x_1), \dots, f(x_d)], \end{aligned}$$

where $[x_1, \dots, x_d] \in S^d X$ denotes the image of $(x_1, \dots, x_d) \in X^{\times d}$.

Proof. Following [11], we identify $\mathcal{M}or(X, Y)$ and $\mathcal{M}or(S^d X, S^d Y)$ as functors of sets on the category $(Sm_{\mathbb{C}})_{\leqslant 1}$ of smooth quasi-projective varieties with dimension not greater than 1, which send $C \in (Sm_{\mathbb{C}})_{\leqslant 1}$ to the sets $Hom(X \times C, Y)$ and $Hom(S^d X \times C, S^d Y)$ respectively. Note that the value of these two functors on the point $Spec(\mathbb{C})$ are exactly the underlying sets of the spaces $\mathcal{M}or(X, Y)$ and $\mathcal{M}or(S^d X, S^d Y)$, because X is normal and hence $S^d X$ is also normal. We need to construct a natural transformation

$$\tau : \mathcal{M}or(X, Y) \longrightarrow \mathcal{M}or(S^d X, S^d Y),$$

whose restriction to the point $Spec\,\mathbb{C}$ is the map S^d in the lemma. Once τ is constructed, the continuity of S^d follows from [11, Theorem 2.3].

We construct a natural transformation τ in the following way. For any $f \in Hom(X \times C, Y)$, there is a composition of morphisms

$$\Theta : C \times X^{\times d} \xrightarrow{\Delta_C \times Id} C^{\times d} \times X^{\times d} \xrightarrow{f^{\times d}} Y^{\times d} \xrightarrow{\pi} S^d Y,$$

where Δ_C is the diagonal morphism and π is the projection that sends $(y_1, \dots, y_d)$ to $[y_1, \dots, y_d]$. Since Θ is invariant under the action of Σ_d on $X^{\times d}$, it induces a natural map $\widetilde{\Theta} : C \times S^d X \longrightarrow S^d Y$ by the universal property of symmetric product and fiber product. Define $\tau(f) = \widetilde{\Theta}$. It is a routine calculation to check that τ is a required natural transformation. □

There is a morphism $Tr : S^c S^d X \longrightarrow S^{cd} X$ which sends the point $[[x_{1,1}, \dots, x_{1,d}], \dots, [x_{c,1}, \dots, x_{c,d}]] \in S^c S^d X$ to the point $[x_{1,1}, \dots, x_{1,d}, \dots, x_{c,1}, \dots, x_{c,d}] \in S^{cd}(X)$. Following [11], we call the morphism Tr the *trace morphism*.

Lemma 4.2. *For any quasi-projective complex varieties Y and X with X being normal, the trace morphism induces a natural continuous map*

$$Tr : \mathcal{M}or(X, S^c S^d Y) \longrightarrow \mathcal{M}or(X, S^{cd} Y).$$

Proof. The proof is similar to the proof of Lemma 4.1. The natural transformation between the functors $\mathcal{M}or(X, S^c S^d Y)$ and $\mathcal{M}or(X, S^{cd} Y)$ is defined as composition with the trace morphism Tr. □

Remark 4.3. If X and Y are projective so that we can talk about their Chow varieties, the above two lemmas are special cases of [11, Propositions 1.4, 1.5].

Proposition 4.4. *For any quasi-projective complex varieties X, Y and Z with X and Y being normal, the following map is continuous.*

$$\begin{aligned} \mathcal{M}or\Big(X, \coprod_{c \geqslant 0} S^c Y\Big) \times \mathcal{M}or\Big(Y, \coprod_{d \geqslant 0} S^d Z\Big) &\xrightarrow{\Phi} \mathcal{M}or\Big(X, \coprod_{e \geqslant 0} S^e Z\Big), \\ \big(X \xrightarrow{f} S^c Y, Y \xrightarrow{g} S^d Z\big) &\mapsto \big(Tr \circ S^c g \circ f : X \longrightarrow S^{cd} Z\big). \end{aligned}$$

Proof. Without loss of generality, we can assume that both X and Y are connected. Obviously we have $\mathcal{M}or(X, \coprod_{c \geqslant 0} S^c Y) \times \mathcal{M}or(Y, \coprod_{d \geqslant 0} S^d Z) = \coprod_{c,d \geqslant 0} \mathcal{M}or(X, S^c Y) \times \mathcal{M}or(Y, S^d Z)$, so it is enough to show that the restriction $\Phi_{c,d} = \Phi|_{\mathcal{M}or(X, S^c Y) \times \mathcal{M}or(Y, S^d Z)}$ is continuous for any $c, d \geqslant 0$. One can check that $\Phi_{c,d}$ fits into the following commutative diagram.

$$\begin{array}{ccc} \mathcal{M}or(X,S^cY)\times\mathcal{M}or(Y,S^dZ) & \xrightarrow{\Phi_{c,d}} & \mathcal{M}or(X,S^{cd}Z) \\ \big\downarrow{\scriptstyle Id\times S^c} & & \big\uparrow{\scriptstyle Tr} \\ \mathcal{M}or(X,S^cY)\times\mathcal{M}or(S^cY,S^cS^dZ) & \xrightarrow{Comp.} & \mathcal{M}or(X,S^cS^dZ) \end{array}$$

The left vertical map $Id \times S^c$ is continuous by Lemma 4.1. The bottom horizontal map *Comp.* is the composition map which is continuous by [11, Proposition 1.7]. The right vertical map *Tr* is continuous by Lemma 4.2. So their composition, the top horizontal map $\Phi_{c,d}$, is also continuous. □

Remark 4.5. If Y and Z are projective varieties, the above proposition is a special case of [11, Proposition 1.7] where $r = 0$.

If we forget the topologies, the map Φ in Proposition 4.4 defines a biadditive morphism of the underlying discrete monoids

$$\Phi : Hom\Big(X, \coprod_{c\geqslant 0} S^cY\Big) \times Hom\Big(Y, \coprod_{d\geqslant 0} S^dZ\Big) \longrightarrow Hom\Big(X, \coprod_{e\geqslant 0} S^eZ\Big).$$

As we mentioned before, these monoids are naturally identified with the monoids of effective finite correspondences. We next show that the map Φ is the composition map in the category $Cor_{\mathbb{C}}$ under these identifications. We need the following fact.

Lemma 4.6. *Let X, Y and Z be smooth quasi-projective varieties, let $x \in X$ and $y \in Y$ be two points. Assume that $W \subset Y \times Z$ is an irreducible closed subvariety which intersects $y \times Z$ properly. Then*

$$(x \times y \times Z) \bullet (x \times W) = (x \times y \times Z) \bullet (X \times W),$$

where the first intersection is in $x \times Y \times Z$ and the second is in $X \times Y \times Z$. The resulting cycles are compared as cycles on $X \times Y \times Z$.

Proof. It is easy to see that the intersections are proper. If X is projective, the projection π from $X \times Y \times Z$ to $x \times Y \times Z$ is proper. The Projection Formula implies

$$(x \times y \times Z) \bullet (x \times W) = \big(\pi(x \times y \times Z)\big) \bullet (x \times W) = \pi\big((x \times y \times Z) \bullet (X \times W)\big) = (x \times y \times Z) \bullet (X \times W).$$

The last equality is valid because the intersection are points defined over $\mathbb{C}$, so the field extensions involved are trivial.

If X is quasi-projective, Hironaka's resolution of singularities [13] implies that X admits a smooth projective closure $\overline{X}$. So we have the following calculations.

$$(x \times y \times Z) \bullet (x \times W) = (x \times y \times Z) \bullet (\overline{X} \times W) = (x \times y \times Z) \bullet (X \times W).$$

The first equality holds because $\overline{X}$ is projective. The second equality holds because the intersections equal set theoretically and the corresponding multiplicities are the same since X is open in $\overline{X}$ and intersection multiplicities only depend on local rings at the intersection. □

Lemma 4.7. *Assume X and $\overline{Y}$ are smooth quasi-projective varieties and Y is a locally closed subvariety of $\overline{Y}$. If V is a finite correspondence from X to Y, V is also a finite correspondence from X to $\overline{Y}$.*

Proof. It is enough to assume that V is an elementary finite correspondence. Let f denote the composition $V \longrightarrow X \times Y \longrightarrow X \times \overline{Y}$, let g denote the finite map $V \longrightarrow X$ and let h denote the projection $X \times \overline{Y} \longrightarrow X$. Because $g = h \circ f$ is finite and h is separated, f is proper. This shows that V is also closed in $X \times \overline{Y}$. So V is a finite correspondence from X to $\overline{Y}$. □

Proposition 4.8. *For smooth quasi-projective varieties X, Y and Z, there is a commutative diagram of discrete monoids.*

$$\begin{array}{ccc} Eff(X,Y)\times Eff(Y,Z) & \xrightarrow{Comp.} & Eff(X,Z) \\ {\scriptstyle(\tau_{X,Y},}\big\downarrow{\scriptstyle\tau_{Y,Z})} & & \big\downarrow{\scriptstyle\tau_{X,Z}} \\ \mathcal{M}or(X,\coprod_{c\geqslant 0}S^cY)\times\mathcal{M}or(Y,\coprod_{d\geqslant 0}S^dZ) & \xrightarrow{\Phi} & \mathcal{M}or(X,\coprod_{e\geqslant 0}S^eZ) \end{array}$$

Proof. To simplify notations, we regard an effective cycle also as the corresponding morphism. Without loss of generality, we assume X, Y and Z are irreducible. It is enough to check that, for any $V \subset X \times Y$ being irreducible and finite surjective over X of degree c and $W \subset Y \times Z$ being irreducible and finite surjective over Y of degree d, we have $W \circ V = \Phi(V, W)$

as morphisms from X to $S^{cd}Z$. Choose $x \in X$ and denote $V(x) = (x \times Y) \bullet V = \sum n_i(x \times y_i)$. If we identify $x \times Y$ as Y, then we also have $V(x) = \sum n_i y_i$.

It is easy to calculate $\Phi(V, W)(x)$. By the definition of Φ in Proposition 4.4, we have

$$\Phi(V, W)(x) = Tr \circ S^c W \circ V(x) = Tr \circ S^c W\Big(\sum n_i y_i\Big) = \sum n_i W(y_i) = \sum n_i\big((y_i \times Z) \bullet W\big).$$

Next we calculate $(W \circ V)(x)$. If Y is projective, intersection theory (see [11]) gives the following calculation. Here π denotes the projection $X \times Y \times Z \longrightarrow X \times Z$.

$$\begin{aligned}
(W \circ V)(x) &= (x \times Z) \bullet \pi\big((V \times Z) \bullet (X \times W)\big) \\
&= \pi\big((x \times Y \times Z \bullet V \times Z) \bullet (X \times W)\big) \quad \text{by the Projection Formula} \\
&= \pi\big(\big((x \times Y \bullet V) \times Z\big) \bullet (X \times W)\big) \\
&= \pi\Big(\Big(\sum n_i(x \times y_i) \times Z\Big) \bullet (X \times W)\Big) \quad \text{since } x \times Y \bullet V = \sum n_i(x \times y_i) \\
&= \sum n_i \pi\big((x \times y_i \times Z) \bullet (X \times W)\big) \\
&= \sum n_i \pi\big((x \times y_i \times Z) \bullet (x \times W)\big) \quad \text{by Lemma 4.6} \\
&= \sum n_i \pi\big(x \times (y_i \times Z \bullet W)\big) \\
&= \sum n_i\big((y_i \times Z) \bullet W\big) \quad \text{since the intersection are } \mathbb{C}\text{-points.}
\end{aligned}$$

This shows $(W \circ V)(x) = Tr \circ S^c W \circ V(x)$ for any $x \in X$. So it completes the prove when Y is projective. If Y is only quasi-projective, let $\overline{Y}$ be a smooth projective closure of Y as given by the resolution of singularities [13]. Lemma 4.7 implies that $V \subset X \times Y \subset X \times \overline{Y}$ is also a finite correspondence from X to $\overline{Y}$. Denote $\overline{W}$ the closure of $W \subset \overline{Y} \times Z$. $\overline{W}$ may not be a finite correspondence, but we still have $V \times Z \cap X \times W = V \times Z \cap X \times \overline{W}$ for the same reason as in the proof of Lemma 4.7. So $V \times Z \bullet X \times \overline{W}$ is also a proper intersection and equal to $V \times Z \bullet X \times W$ since intersection multiplicity only depends on local rings. Similarly, $x \times y_i \times Z \bullet x \times \overline{W} = x \times y_i \times Z \bullet x \times W$. Now we have the following calculation, where π denotes the projections from $X \times Y \times Z$ or $X \times \overline{Y} \times Z$ to $X \times Z$.

$$\begin{aligned}
(W \circ V)(x) &= (x \times Z) \bullet \pi\big((V \times Z) \bullet (X \times W)\big) \\
&= (x \times Z) \bullet \pi\big((V \times Z) \bullet (X \times \overline{W})\big) \\
&= \sum n_i \pi\big((x \times y_i \times Z) \bullet (x \times \overline{W})\big) \quad \text{as calculated in the projective case} \\
&= \sum n_i \pi\big((x \times y_i \times Z) \bullet (x \times W)\big) \\
&= \sum n_i\big((y_i \times Z) \bullet W\big).
\end{aligned}$$

This shows that $(W \circ V)(x) = Tr \circ S^c W \circ V(x)$ for any $x \in X$, when Y is quasi-projective. The proof is complete. □

Theorem 4.9. *With the functor-on-curves topology, $Cor_{\mathbb{C}}$ is a topological category. That is, for any smooth quasi-projective varieties X, Y and Z, the composition in $Cor_{\mathbb{C}}$*

$$Comp. : Cor(X, Y) \times Cor(Y, Z) \longrightarrow Cor(X, Z)$$

is a continuous biadditive map of topological groups.

Proof. In the commutative diagram of Proposition 4.8, both vertical maps are homeomorphisms. Φ is continuous by Proposition 4.4. So the top horizontal map

$$Comp. : Eff(X, Y) \times Eff(Y, Z) \longrightarrow Eff(X, Z)$$

is also continuous. The map in the theorem is induced by this map. So it is continuous and biadditive. □

Corollary 4.10. *Let $q \geqslant 0$ be an integer and let X be a smooth quasi-projective complex variety. Let $Cor(X, \mathbb{P}^{q/q-1})$ denote the quotient topological group $Cor(X, \mathbb{P}^q)/Cor(X, \mathbb{P}^{q-1})$. There is a presheaf with transfers of abelian topological groups*

$$\begin{aligned}
Cor\big(-, \mathbb{P}^{q/q-1}\big) : Cor_{\mathbb{C}} &\longrightarrow \text{Abelian Topological Groups}, \\
X &\mapsto Cor\big(X, \mathbb{P}^{q/q-1}\big) = \mathcal{Z}^q(X).
\end{aligned}$$

5. Morphic cohomology of motives over $\mathbb{C}$

We denote the triangulated category of motives over $\mathbb{C}$ by *DM*. By Corollary 4.10, we have a presheaf with transfers of abelian topological groups sending any $X \in Cor_{\mathbb{C}}$ to the topological group $Cor(X, \mathbb{P}^{q/q-1})$ for any $q \geqslant 0$. The topology on $Cor(X, \mathbb{P}^{q/q-1})$ is defined in Section 4 in such a way that we have

$$L^q H^{2q-i}(X) = \pi_i Cor\big(X, \mathbb{P}^{q/q-1}\big).$$

Let $P^*(q)$ denote the chain complex of presheaves with transfers such that $P^i(q)(X) = Sing_{-i} Cor(X, \mathbb{P}^{q/q-1})$ for $i \leqslant 0$ and 0 for $i > 0$, where $Sing_n(-)$ is the usual set of continuous maps from the usual topological n-simplex to a topological space. The differential maps of $P^*(q)$ are defined in the obvious manor. Let $\wp_{mor}(q) = P^*(q)_{Nis}$ denote the Nisnevich sheafification of $P^*(q)$.

Theorem 5.1. *$\wp_{mor}(q)$ is $\mathbb{A}^1$-local and there is a natural isomorphism*

$$L^q H^p(X) = Hom_{DM}\big(M(X), \wp_{mor}(q)[p-2q]\big)$$

for any smooth quasi-projective complex variety X and integers p and q with q and $2q - p$ nonnegative.

Proof. It follows from the definitions that $L^q H^p(X) = H^{p-2q}(P^*(q)(X))$. Since morphic cohomology is homotopy invariant and satisfies the Mayer–Vietoris property, $P^*(q)$ is strictly homotopy invariant and pseudo-flasque. By Theorem 3.5 and its proof, $\wp_{mor}(q)$ is $\mathbb{A}^1$-local and we have

$$L^q H^p(X) = H^{p-2q}\big(P^*(X)\big) = Hom_{DM}\big(M(X), \wp_{mor}(q)[p-2q]\big). \qquad \square$$

Definition 5.2. For any $K^* \in DM$ and integers p and q with q nonnegative, define the (q, p)-th morphic cohomology of K^* with integral coefficients as

$$L^q H^p(K^*) = Hom_{DM}\big(K^*, \wp_{mor}(q)[p-2q]\big).$$

In particular, the (q, p)-th morphic cohomology of any quasi-projective complex variety X is defined as $L^q H^p(X) = L^q H^p(M(X))$.

Remark 5.3. When X is smooth, Theorem 5.1 implies that the definition of morphic cohomology of X given by Definition 5.2 agrees with the usual definition as given in [4]. If X is singular but normal, we do not know whether or not these two definitions give the same morphic cohomology of X.

The category *DM* admits several good properties. One can use the distinguished triangles as listed in [16, §14] to re-obtain many properties about morphic cohomology. Moreover, as an advantage of Definition 5.2 of the morphic cohomology on singular varieties, we can extend existing properties of morphic cohomology from smooth complex varieties to singular varieties. For example, we prove a vanishing result of morphic cohomology on singular quasi-projective varieties, assuming the Friedlander–Mazur Conjecture which claims that $L^q H^p(X) = 0$ for any smooth quasi-projective variety X and negative integer p.

Proposition 5.4. *If the Friedlander–Mazur Conjecture is true, then $L^q H^p(X) = 0$ for any quasi-projective complex variety X and integers $p < 0$ and $q \geqslant 0$.*

Proof. By the Friedlander–Mazur Conjecture, $H^n(P^*(q)) = L^q H^{n+2q} = 0$ as a presheaf with transfers when $n < -2q$. This implies that $(H^n(\wp_{mor}(q)))_{Nis} = (H^n(P^*(q)))_{Nis} = 0$ where $n < -2q$. So the chain complex $\wp_{mor}(q)$ is exact at degrees less than $-2q$. So, up to quasi-isomorphisms, $\wp_{mor}(q)[n]$ has non-zero terms only at positive degrees when $n < -2q$. The chain complex h_X (see Section 2.1) has non-zero terms only at degree zero. In this case, it is a standard fact about derived categories that $Hom_{D^-}(h_X, \wp_{mor}(q)[n]) = 0$ for $n < -2q$, where D^- is the derived category of bounded above chain complexes of Nisnevich sheaves with transfers. So

$$L^q H^{n+2q}(X) = Hom_{DM}\big(M(X), \wp_{mor}(q)[n]\big) = Hom_{D^-}\big(h_X, \wp_{mor}(q)[n]\big) = 0,$$

where $n < -2q$. The middle isomorphism is proved in [16, Proposition 14.16]. This proves that $L^q H^p(X) = 0$ when $p < 0$. $\square$

If Z is a closed smooth subvariety of X of codimension c, there is a Gysin triangle in *DM* (see [16, §14])

$$M(X - Z) \longrightarrow M(X) \longrightarrow M(Z)(c)[2c] \longrightarrow M(X - Z)[1].$$

To use this triangle to calculate morphic cohomology, we need to know the group $L^q H^p(M(Z)(c)[2c])$.

Proposition 5.5. *Let Z be a smooth quasi-projective complex variety and c be a natural number, then there exists a natural isomorphism*

$$L^qH^p\big(M(Z)(c)[2c]\big) \xrightarrow{\cong} L^{q-c}H^{p-2c}(Z).$$

Proof. One may prove this isomorphism by decoding the construction of Gysin map in motivic cohomology explained in [16] and the Gysin map in morphic cohomology explained in [4]. Here we give a formal prove.

Let O denote the origin of the c-dimensional affine plane $\mathbb{A}^c$. Let $\mathbb{A}^{c,*}$ denote the open subvariety $\mathbb{A}^c - O$. Regard $Z = Z \times O$ as a codimension c closed smooth subvariety of $Z \times \mathbb{A}^c$. Denote $j : \mathbb{A}^{c,*} \longrightarrow \mathbb{A}^c$ the open embedding. By the Gysin triangle we have the Gysin long exact sequence

$$\cdots \longrightarrow L^qH^p\big(M(Z)(c)[2c]\big) \longrightarrow L^qH^p\big(Z \times \mathbb{A}^c\big) \xrightarrow{j^*} L^qH^p\big(Z \times \mathbb{A}^{c,*}\big) \longrightarrow L^qH^{p+1}\big(M(Z)(c)[2c]\big) \longrightarrow \cdots.$$

Let P be a fixed point in $\mathbb{A}^{c,*}$. We have the following diagram.

$$\begin{array}{ccc} Z \times P & \underset{\pi}{\overset{i_2}{\rightleftarrows}} & Z \times \mathbb{A}^c \\ & {\scriptstyle i_1}\searrow \quad \nearrow{\scriptstyle j} & \\ & Z \times \mathbb{A}^{c,*} & \end{array}$$

where i_1, i_2 and j are obvious embeddings and π is induced by the projection from $\mathbb{A}^c$ to the point P. Taking morphic cohomology we see that $(\pi^* \circ i_1^*) \circ j^* = \pi^* \circ (j \circ i_1)^* = \pi^* \circ i_2^*$ which is an isomorphism since morphic cohomology is $\mathbb{A}^1$-local on smooth varieties. So the map j^* is splitting injective and the Gysin long exact sequence breaks into short exact sequences

$$0 \longrightarrow L^qH^p\big(Z \times \mathbb{A}^c\big) \xrightarrow{j^*} L^qH^p\big(Z \times \mathbb{A}^{c,*}\big) \longrightarrow L^qH^{p+1}\big(M(Z)(c)[2c]\big) \longrightarrow 0.$$

Similarly, the Gysin long exact sequence of [4, Corollary 6.2] also breaks into short exact sequences

$$0 \longrightarrow L^qH^p\big(Z \times \mathbb{A}^c\big) \xrightarrow{j^*} L^qH^p\big(Z \times \mathbb{A}^{c,*}\big) \longrightarrow L^{q-c}H^{p+1-2c}(Z) \longrightarrow 0.$$

Being the quotient group of the same group homomorphism, $L^qH^p(M(Z)(c)[2c])$ and $L^{q-c}H^{p-2c}(Z)$ are naturally isomorphic. □

6. Singular cohomology of motives over $\mathbb{C}$

For any $X \in Cor_{\mathbb{C}}$, it is well know that the usual singular cohomology (with integral coefficients) of the underlying analytic space X^{an} can be defined as

$$H^{2q-i}_{Sing}\big(X^{an}\big) = \pi_0 Cont\big(X^{an}, K(\mathbb{Z}, 2q-i)\big) = \pi_i Cont\big(X^{an}, K(\mathbb{Z}, 2q)\big),$$

where $Cont(-,-)$ is the mapping space with the compact-open topology and $K(\mathbb{Z}, 2q)$ is the Eilenberg–Maclane space. It follows from the Dold–Thom Theorem [1] that $H_i((\mathbb{P}^q)^{an}) = \pi_i(\coprod_{d\geqslant 0}(S^d\mathbb{P}^q)^{an})^+$. So we can choose a model of $K(\mathbb{Z}, 2q)$ as the quotient topological group $(\coprod_{d\geqslant 0}(S^d\mathbb{P}^q)^{an})^+/(\coprod_{d\geqslant 0}(S^d\mathbb{P}^{q-1})^{an})^+$.

Remark 6.1. With the compact-open topology, we do not know if

$$Cont\big((-)^{an}, K(\mathbb{Z}, 2q)\big) : X \mapsto Cont\big(X^{an}, K(\mathbb{Z}, 2q)\big)$$

is a functor from $Cor_{\mathbb{C}}$ to the category of abelian topological groups. The problem is that the compact-open topology does not behave well with some constructions such as compositions of continuous maps.

Recall from [19] the definition of compactly generated topology, the category $\mathfrak{CG}$ of compactly generated topological spaces and the retraction functor k sending a Hausdorff space X to its associated compactly generated space $k(X) \in \mathfrak{CG}$ and a map f to itself. The functor k is right adjoin to the inclusion of $\mathfrak{CG}$ into the category of Hausdorff spaces, which means

$$Cont(X, Y) = Cont\big(X, k(Y)\big)$$

as sets if $X \in \mathfrak{CG}$ and Y is Hausdorff. In particular, $k(X)$ and X have the same singular chain complexes. For any X and Y in $\mathfrak{CG}$, let $X \times_{\mathfrak{CG}} Y$ denote the categorical product in $\mathfrak{CG}$. Note that in [19], $X \times_{\mathfrak{CG}} Y$ is denoted as $X \times Y$, which we reserve here for the usual product in the category of topological spaces.

Definition 6.2. G is a topological group in $\mathfrak{CG}$ means:

1. G is a group and also a compactly generated topological space;
2. The inverse map $Inv : G \longrightarrow G$ and the multiplication map $Mul : G \times_{\mathfrak{CG}} G \longrightarrow G$ are continuous.

For example, if G is a Hausdorff topological group, then kG is a topological group in $\mathfrak{CG}$. The converse is not necessarily true, because $G \times_{\mathfrak{CG}} G \longrightarrow G \times G$ is not necessarily a homeomorphism. As explained in [14, Remark 2.2], the topological group $K(\mathbb{Z}, 2q)$ is compactly generated, so it is a topological group in $\mathfrak{CG}$. Let X be a space and G be a topological group, the mapping space $Cont(X, G)$ is not necessarily a topological group in $\mathfrak{CG}$ because the space $Cont(X, G)$ may not be an object in $\mathfrak{CG}$.

Lemma 6.3. *If $X \in \mathfrak{CG}$ and G is a topological group in $\mathfrak{CG}$, $kCont(X, G)$ is a topological group in $\mathfrak{CG}$.*

Proof. By [19, Theorem 5.9], the map $Inv : kCont(X, G) \longrightarrow kCont(X, G)$ is continuous because it is induced by the inverse map of G. The multiplication map $Mul : kCont(X, G) \times_{\mathfrak{CG}} kCont(X, G) \longrightarrow kCont(X, G)$ is continuous because it can be decomposed as

$$kCont(X, G) \times_{\mathfrak{CG}} kCont(X, G) = kCont(X, G \times_{\mathfrak{CG}} G) \longrightarrow kCont(X, G).$$

The first map is a natural homeomorphism by [19, Theorem 5.4]. The second map is continuous by [19, Theorem 5.9], because it is induced by the multiplication map of G. □

Corollary 6.4. *For any $X \in Cor_{\mathbb{C}}$, $kCont(X^{an}, K(\mathbb{Z}, 2q))$ is a topological group in $\mathfrak{CG}$.*

Corollary 6.5. *If $X \in \mathfrak{CG}$ and G is a topological group in $\mathfrak{CG}$, the set $Cont(X, G)$ is a group.*

Let G be a topological group in $\mathfrak{CG}$. G may not be a topological group, but the multiplication with a fixed element is still a homeomorphism of G according to the following proposition.

Proposition 6.6. *If G is a topological group in $\mathfrak{CG}$ and g is an element in G, the map $Mul_g : G \longrightarrow G$ sending $g' \in G$ to $g'g$ and the map ${}_gMul : G \longrightarrow G$ sending $g' \in G$ to gg' are continuous.*

Proof. Let $(Id, g) : G \longrightarrow G \times_{\mathfrak{CG}} G$ be the unique continuous map defined by the pair of continuous maps $(G \xrightarrow{Id} G, G \xrightarrow{g} g)$ in $\mathfrak{CG}$. We see that $Mul_g = Mul \circ (Id, g)$ is continuous. Similarly ${}_gMul$ is also continuous. □

Lemma 6.7. *Let G be an abelian topological group whose topology is compactly generated. For any $Y \in \mathfrak{CG}$, there exists a continuous map*

$$\begin{aligned} S^d : kCont(Y, G) &\longrightarrow kCont\big(S^d Y, G\big), \\ f &\mapsto \left(S^d f : [y_1, \ldots, y_d] \mapsto \sum_{i=1}^{d} f(x_i)\right). \end{aligned}$$

Proof. By [19, 2.6], Y is in $\mathfrak{CG}$ implies $S^d Y$ (with the quotient topology) is also in $\mathfrak{CG}$. The map S^d in the lemma is obviously well defined. We first show that $S^d : kCont(Y, G) \longrightarrow Cont(S^d Y, G)$ is continuous.

Let K be a compact subset of $S^d Y$ and U be an open subset of G. Let $V(K, U)$ be the set of maps $g \in Cont(S^d Y, G)$ such that $g(K) \subset U$. Let $f \in kCont(Y, G)$ be an arbitrary element in $(S^d)^{-1}(V(K, U))$, we must find an open neighborhood (in the topology of $kCont(Y, G)$) of f which is included in $(S^d)^{-1}(V(K, U))$.

Let $\pi : Y^{\times d} \longrightarrow S^d Y$ be the natural projection. By [18, Lemma 6.3], K is compact in $S^d(Y)$ implies that $\pi^{-1}(K)$ is also compact in $Y^{\times d}$. Let $K' = \bigcup_{i=1}^{d} \pi_i(\pi^{-1} K)$ where π_i is the natural projection from $Y^{\times d}$ to the i-th copy of Y. It is easy to see that K' is compact in Y and $y \in K'$ if and only if y appears as a component of a point in K.

By our assumption, $S^d f(K)$ is compact and it is contained in the open subset U of G. For any point $a \in S^d f(K)$, there exists an open neighborhood U_a of 0 of G such that $a + U_a \subset U$. It is well known that for this open set U_a, there exists an open subset V_a containing 0 of G such that $V_a + V_a \subset U_a$. So for any $a' \in S^d f(K) \cap (a + V_a)$, we have $a' + V_a \subset a + V_a + V_a \subset a + U_a \subset U$. Since $S^d f(K)$ is compact, we can find finitely many such points a_j and their corresponding neighborhoods V_{a_j} of 0 in G such that

1. $\bigcup (a_j + V_{a_j})$ is an open covering of $S^d f(K)$;
2. For any $b \in S^d f(K)$, if $b \in a_j + V_{a_j}$ then $b + V_{a_j} \subset U$.

Let V be the intersection of these finitely many open sets V_{a_j}, then V is an open neighborhood of 0 in G and $S^d f(K) + V \subset U$. Let U' be an even smaller neighbor hood of 0 in G such that the summation of any d elements in U' is in V. The existence of U' is a general fact about topological groups.

Let $V(K', U')$ be the set of maps $h \in Cont(Y, G)$ such that $h(K') \subset U'$. By definition, $V(K', U')$ is open in $Cont(Y, G)$ and hence in $kCont(Y, G)$. G is obviously a topological group in $\mathfrak{CG}$, so $kCont(Y, G)$ is a topological group in $\mathfrak{CG}$ by Lemma 6.3. In particular, addition by f is a homeomorphism of $kCont(Y, G)$ by Proposition 6.6. So $f + V(K', U')$ is an open subset of $kCont(Y, G)$ and $f = f + 0$ belongs to $f + V(K', U')$ where 0 is the constant map with image $0 \in U'$. If $h \in f + V(K', U')$, then $h(x) - f(x) \in U'$ for any $x \in K'$. So, for any $[x_1, \ldots, x_d] \in K$, we have $S^d h([x_1, \ldots, x_d]) = \sum_{i=1}^d h(x_i) = \sum_{i=1}^d f(x_i) + \sum_{i=1}^d (h(x_i) - f(x_i)) \subset S^d f(K) + \sum_{i=1}^d U' \subset S^d f(K) + V \subset U$. This proves that the open set $f + V(K', U')$ is a subset of $(S^d)^{-1}(V(K, U))$. So $S^d : kCont(Y, G) \longrightarrow Cont(S^d Y, G)$ is continuous. Applying the functor k, we obtain the continuity of the map in the lemma $S^d : kCont(Y, G) = kkCont(Y, G) \longrightarrow kCont(S^d Y, G)$. □

When X is irreducible, an effective finite correspondence $f \in Eff(X, Y)$ can be identified with an algebraic map $X \longrightarrow S^d Y$ for some integer d and hence a continuous map in the analytic topology. There is an induced map of sets

$$f^* : Cont\big(Y^{an}, K(\mathbb{Z}, 2q)\big) \longrightarrow Cont\big(X^{an}, K(\mathbb{Z}, 2q)\big),$$

which sends a continuous map $\alpha : Y^{an} \longrightarrow K(\mathbb{Z}, 2q)$ to the composition

$$f^*(\alpha) : X^{an} \xrightarrow{f} S^d Y^{an} \xrightarrow{S^d \alpha} S^d K(\mathbb{Z}, 2q) \xrightarrow{\Sigma_d} K(\mathbb{Z}, 2q).$$

Note that f is continuous in the analytic topology; $S^d \alpha$ is continuous because the d-fold symmetric product S^d is a functor from the category of topological spaces to itself; Σ_d is induced from the continuous map $K(\mathbb{Z}, 2q)^{\times d} \xrightarrow{+} K(\mathbb{Z}, 2q)$ sending $(a_1, \ldots, a_d)$ to $a_1 + \cdots + a_d$, so Σ_d is continuous. This shows that $f^*(\alpha)$ is continuous. So f^* is well defined.

Theorem 6.8. *The following functor is a presheaf with transfers of abelian topological groups in* $\mathfrak{CG}$.

$$\begin{aligned} Cont\big((-)^{an}, K(\mathbb{Z}, q)\big) : Cor_{\mathbb{C}} &\longrightarrow \textit{Abelian Topological Groups in } \mathfrak{CG}, \\ X &\mapsto kCont\big(X^{an}, K(\mathbb{Z}, q)\big), \\ f &\mapsto f^*. \end{aligned}$$

Proof. A straight forward calculation shows that, for any $f \in Cor(X, Y)$ and $g \in Cor(Y, Z)$ in $Cor_{\mathbb{C}}$, f^* and g^* are morphisms in the category of abelian groups and $(g \circ f)^* = f^* \circ g^*$. It only remains to show that f^* is continuous. Since for any $f' \in Cor(X, Y)$ we have $(f + f')^* = f^* + f'^*$ by definition and the set of continuous maps from $kCont(Y^{an}, K(\mathbb{Z}, q))$ to $kCont(X^{an}, K(\mathbb{Z}, q))$ forms a group by Corollary 6.5, we can assume that f is an effective finite correspondence. If f corresponds to a morphism $X \longrightarrow S^d Y$, then f^* decomposes as a composition

$$kCont\big(Y^{an}, K(\mathbb{Z}, q)\big) \longrightarrow kCont\big(S^d Y^{an}, K(\mathbb{Z}, q)\big) \longrightarrow kCont\big(X^{an}, K(\mathbb{Z}, q)\big).$$

The first map in the diagram is continuous by Lemma 6.7 and the second map is continuous by [19, 5.9], so f^* is continuous. □

For any $X \in Cor_{\mathbb{C}}$, we set

$$F^i(q)(X) = Sing_{-i} Cont\big(X^{an}, K(\mathbb{Z}, 2q)\big) = Sing_{-i} kCont\big(X^{an}, K(\mathbb{Z}, 2q)\big)$$

for $i \leqslant 0$ and we set $F^i(q)(X) = 0$ for $i > 0$. Theorem 6.8 together with Corollary 6.5 implies that $F^*(q)$ is a chain complex of presheaves with transfers under the obvious differential maps. Since $H^p(F^*(q))(X) = H^{-p}_{Sing}(X^{an})$ for all integer p, $F^*(q)$ is strictly homotopy invariant and pseudo-flasque. Denote $\wp_{Sing}(q) = F^*(q)_{Nis}$, then $\wp_{Sing}(q)$ is $\mathbb{A}^1$-local. By Theorem 3.5 we have

$$H^p_{Sing}\big(X^{an}\big) = Hom_{DM}\big(M(X), \wp_{Sing}(q)[p - 2q]\big)$$

for any $X \in Cor_{\mathbb{C}}$. We next generalize this formula to singular varieties.

To continue we need to recall the notion of *abstract blow-up* which is defined on schemes over arbitrary base field. See, for example, [16, Definition 12.21] for details. In our case, an abstract blow-up of a complex variety X is the following diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\ i\ } & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{\ i\ } & X \end{array}$$

where $p : X' \longrightarrow X$ is a proper map; Z is a proper closed subvariety of X and called the center of the blow-up; $Z' = p^{-1}(Z) = Z \times_X X'$; p induces an isomorphism from $(X' - Z')$ to $(X - Z)$.

Given an abstract blow-up $p : X' \longrightarrow X$ with center Z and $Z' = Z \times_X X'$, there is an abstract blow-up distinguished triangle in DM (see [16, Theorem 13.26 and §14]):

$$M(Z') \longrightarrow M(X') \oplus M(Z) \longrightarrow M(X) \longrightarrow M(Z')[1].$$

So there is an abstract blow-up sequence for the functor $Hom_{DM}(M(-), \wp_{Sing}(q)[p-2q])$ on complex varieties. For singular cohomology, the similar blow-up exact sequence is well known.

Theorem 6.9. *For any quasi-projective complex variety X,*

$$H^p_{Sing}\big(X^{an}\big) = Hom_{DM}\big(M(X), \wp_{Sing}(q)[p-2q]\big).$$

Proof. Both $H^p_{Sing}((-)^{an})$ and $Hom_{DM}(M(-), \wp_{Sing}(q)[p-2q])$ admit abstract blow-up long exact sequences and there is a natural transformation $H^p_{Sing}((-)^{an}) \longrightarrow Hom_{DM}(M(-), \wp_{Sing}(q)[p-2q])$ which is an isomorphism on smooth quasi-projective complex varieties. An induction on the dimension of X finishes the proof. Since we will use the same argument in the following theorem, details are skipped. □

Definition 6.10. For any $K^* \in DM$ and integers p and q with q nonnegative, define the p-th singular cohomology of K^* with integral coefficients as

$$H^p_{Sing}(K^*) = Hom_{DM}\big(K^*, \wp_{Sing}(q)[p-2q]\big).$$

From the constructions we see that $\wp_{Sing}(q)[2] = \wp_{Sing}(q+1)$ in the category DM, so changing the parameter q does not change the cohomology ring $H^*_{Sing}(X^{an})$. So the parameter q does not appear in the singular cohomology of motives just as it should be.

Using the model $(\coprod_{d\geqslant 0}(S^d\mathbb{P}^q)^{an})^+/(\coprod_{d\geqslant 0}(S^d\mathbb{P}^{q-1})^{an})^+$ for $K(\mathbb{Z}, 2q)$, we recall from [4, Proposition 3.4] that there is a continuous group homomorphism

$$\tau_q : Cor\big(X, \mathbb{P}^{q/q-1}\big) = \mathcal{M}or\bigg(X, \coprod_{d\geqslant 0} S^d\mathbb{P}^q\bigg)^+ \Big/ \mathcal{M}or\bigg(X, \coprod_{d\geqslant 0} S^d\mathbb{P}^{q-1}\bigg)^+ \longrightarrow Cont\big(X^{an}, K(\mathbb{Z}, 2q)\big),$$

which is induced by identifying an algebraic map to the continuous map which it defines in the analytic topology. It is easy to check that τ_q is compatible with the transfer maps defined for morphic cohomology and singular cohomology in Proposition 4.4 and Theorem 6.8, respectively. So we obtain a natural map $\tau_q : \wp_{mor}(q) \longrightarrow \wp_{Sing}(q)$ and hence a natural transformation of δ-functors

$$\tau_q : L^qH^*(-) \longrightarrow H^*_{Sing}\big((-)^{an}\big),$$

where both $L^qH^p(-)$ and $H^p_{Sing}((-)^{an})$ are now defined on the triangulated category of motives DM.

Remark 6.11. To show that the chain complex $Sing_*Cont(X^{an}, K(\mathbb{Z}, 2q))$ is a chain complex of presheaves with transfers, we identify it as $Sing_*kCont(X^{an}, K(\mathbb{Z}, 2q))$ because the associated compactly generated topology behaves better. The reason why we still use $Cont(X^{an}, K(\mathbb{Z}, 2q))$, not $kCont(X^{an}, K(\mathbb{Z}, 2q))$, to define $\wp_{Sing}(q)$ is the convenience in the above definition of the natural transformation τ_q.

We define the dimension of a quasi-projective variety X as

$$Dim(X) = Max\big\{Dim(X_i) \bigm| X_i \text{ is an irreducible component of } X\big\}.$$

As we mentioned in Section 2.3, when X is smooth of dimension d, there exists a natural isomorphism $L^qH^p(X) \xrightarrow{\cong} H^p_{Sing}(X^{an})$ for $q \geqslant d$ and for all p. This isomorphism is induced by the map $\wp_{mor}(q) \xrightarrow{\tau_q} \wp_{Sing}(q)$ as one can see from the natural isomorphisms in Theorem 3.5.

Theorem 6.12. *Let X be a quasi-projective complex variety of dimension d. The natural map*

$$\tau_q : L^qH^p(X) \longrightarrow H^p_{Sing}\big(X^{an}\big)$$

is an isomorphism for all integers p and q with $q \geqslant d$.

Proof. We proof by induction on d. The case $d = 0$ is trivial. Assume the statement is true for all quasi-projective varieties of dimension less than or equal to d. Consider a quasi-projective variety X of dimension $d + 1$. By resolution of singularities [13], there is an abstract blow-up square

$$\begin{array}{ccc} Z' & \xrightarrow{i} & X' \\ \downarrow{p} & & \downarrow{p} \\ Z & \xrightarrow{i} & X \end{array}$$

where Z is the locus of singularities of X and X' is smooth of dimension $Dim(X') = Dim(X) = d + 1$. Note that $Dim(Z) < Dim(X)$.

If Z' contains an irreducible component, say Z'', such that the dimension of Z'' is $d + 1$, then Z'' must also be an irreducible component of X'. Since X' is smooth, Z'' is actually a connected component of X'. Since $p(Z'') \subset Z$, we can throw away Z'' and the remaining pair $(Z' - Z'', X' - Z'')$ is still an abstract blow-up of X with center Z. This shows that we can require the dimension of Z' is less than or equal to d.

For $q \geqslant d + 1$, we have the following commutative diagram of abstract blow-up sequences.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & L^qH^p(X) & \longrightarrow & L^qH^p(X') \oplus L^qH^p(Z) & \longrightarrow & L^qH^p(Z') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^p_{Sing}(X^{an}) & \longrightarrow & H^p_{Sing}(X'^{an}) \oplus H^p_{Sing}(Z^{an}) & \longrightarrow & H^p_{Sing}(Z'^{an}) & \longrightarrow & \cdots \end{array}$$

The second and the third vertical arrows in the diagram are isomorphisms by the induction assumption and the smooth case, so the first vertical arrow is also an isomorphism by the five-lemma. □

The morphic cohomology of the motive $M(Z)(c)[2c]$ is calculated in Proposition 5.5. We have a similar result for the singular cohomology of the motive $M(Z)(c)[2c]$. The proof is the same as in Proposition 5.5 and hence skipped.

Proposition 6.13. *For any smooth complex variety Z and natural number c, there exists a natural isomorphism $H^p_{Sing}(M(Z)(c)[2c]) \xrightarrow{\cong} H^{p-2c}_{Sing}(Z^{an})$.*

7. Natural transformations of morphic cohomology of motives

7.1. The s-operation on morphic cohomology of motives

By Definition 5.2, the morphic cohomology is defined and represented in the triangulated category of motives DM. By the Yoneda Lemma, the group of natural transformations from L^qH^p to $L^{q+i}H^{p+j}$ is given by the group

$$\begin{aligned} &Hom_{DM}\big(\wp_{mor}(q)[p-2q], \wp_{mor}(q+i)\big[(p+j)-2(q+i)\big]\big) \\ &\quad = Hom_{DM}\big(\wp_{mor}(q), \wp_{mor}(q+i)\big[(2q+j)-2(q+i)\big]\big) \\ &\quad = L^{q+i}H^{2q+j}\big(\wp_{mor}(q)\big). \end{aligned}$$

If $f \in L^{q+i}H^{2q+j}(\wp_{mor}(q))$, we denote $f_* : L^qH^p \longrightarrow L^{q+i}H^{p+j}$ for the induced natural transformation. We want to calculate the group $L^{q+i}H^{2q+j}(\wp_{mor}(q))$ and hence understand the natural transformations of morphic cohomology of motives.

In this subsection, we show that the s-operation studied in [7,8] is natural with respect to the transfer maps, so the composition $s^i = s \circ s \circ \cdots \circ s$ (i terms) is an element in $L^{q+i}H^{2q}(\wp_{mor}(q))$ for any $i \geqslant 0$. We recall from Section 2.3 the topological monoid $\mathcal{M}or(X, \mathcal{C}_r(Y))$ where X is a smooth quasi-projective complex variety and Y is a projective complex variety.

Lemma 7.1. *For any integer $r \geqslant 0$ and smooth projective variety Y, the presheaf*

$$\begin{aligned} \mathcal{M}or\big(-, \mathcal{C}_r(Y)\big)^+ : Cor_{\mathbb{C}} &\longrightarrow Ab, \\ X &\mapsto \mathcal{M}or\big(X, \mathcal{C}_r(Y)\big)^+ \end{aligned}$$

is a presheaf with transfers of abelian topological groups.

Proof. Let X and Z be smooth quasi-projective complex varieties. We claim that there is a biadditive continuous map

$$\mathcal{M}or\Big(X, \coprod_{c\geqslant 0} S^c Z\Big) \times \mathcal{M}or\big(Z, \mathcal{C}_r(Y)\big) \xrightarrow{\Phi} \mathcal{M}or\big(X, \mathcal{C}_r(Y)\big),$$

$$(f, g) \mapsto \Big(x \mapsto \sum_{z\in f(x)} g(z)\Big).$$

The continuity of Φ can be proved in the same way as the proof of Proposition 4.4. Notice that if Z is projective, this is only a special case of [11, Proposition 1.7].

Taking naive group completion and restricting Φ to $\mathcal{M}or(Z, \mathcal{C}_r(Y))^+ \times \{V\}$ for any $V \in Cor(X, Z) = \mathcal{M}or(X, \coprod_{c\geqslant 0} S^c Z)^+$, we obtain the continuous group homomorphism

$$V^* : \mathcal{M}or\big(Z, \mathcal{C}_r(Y)\big)^+ \longrightarrow \mathcal{M}or\big(X, \mathcal{C}_r(Y)\big)^+.$$

Using the definition of Φ, a direct computation shows that $V^* \circ W^* = (W \circ V)^*$ for any $W \in Cor(Z, Z')$ in $Cor_{\mathbb{C}}$. □

Lemma 7.2. *In the construction of the s-operation, the join map # is a biadditive natural transformation between presheaves with transfers of topological groups for any nonnegative integer r,*

$$\mathcal{M}or\big(-, \mathcal{C}_r(Y)\big)^+ \times \mathcal{M}or\big(pt, \mathcal{C}_0(\mathbb{P}^1)\big)^+ \xrightarrow{\#} \mathcal{M}or\big(-, \mathcal{C}_{r+1}(Y \# \mathbb{P}^1)\big)^+;$$

the suspension map $\Sigma\!\!\!/$ is a natural transformation of presheaves with transfers of topological groups,

$$\mathcal{M}or\big(-, \mathcal{C}_0(Y)\big)^+ \xrightarrow{\Sigma\!\!\!/} \mathcal{M}or\big(-, \mathcal{C}_1(\Sigma\!\!\!/ Y)\big)^+.$$

Proof. All functors here are presheaves with transfers of abelian topological groups by Lemma 7.1. As reviewed in Section 2.3, everything is clear except the compatibility of # and $\Sigma\!\!\!/$ with transfer maps. Since $\Sigma\!\!\!/$ is the restriction of # in a special case, we only need to show that # is compatible with transfer maps. It is enough to check that the following diagram commutes for any elementary finite correspondence $V \in Cor(X, Z)$.

$$\begin{array}{ccc}
\mathcal{M}or(Z, \mathcal{C}_r(Y)) \times \mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1)) & \xrightarrow{\#} & \mathcal{M}or(Z, \mathcal{C}_{r+1}(Y \# \mathbb{P}^1)) \\
\big\downarrow{\scriptstyle V^*\times Id} & & \big\downarrow{\scriptstyle V^*} \\
\mathcal{M}or(X, \mathcal{C}_r(Y)) \times \mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1)) & \xrightarrow{\#} & \mathcal{M}or(X, \mathcal{C}_{r+1}(Y \# \mathbb{P}^1))
\end{array}$$

For any $(f, \sum n_i p_i) \in \mathcal{M}or(Z, \mathcal{C}_r(Y)) \times \mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1))$, a straight forward calculation shows that both $V^*(\#(f, \sum n_i p_i))$ and $\#(V^* \times Id(f, \sum n_i p_i))$ are the same morphism sending $x \in X$ to the element $\sum_{z\in V(x)} f(z) \# (\sum n_i p_i)$ in $\mathcal{C}_{r+1}(Y \# \mathbb{P}^1)$. So $\# \circ (V^* \times Id) = V^* \circ \#$ and the proof is complete. □

We are now ready to prove the main result of this subsection.

Theorem 7.3. $s \in L^{q+1}H^{2q}(\wp_{mor}(q))$. *That is, the usual s-operation is compatible with transfer maps.*

Proof. The proof is to write down the construction of the s-operation and check that the natural transformation at each step is compatible with transfer maps. By Lemma 7.2, the diagram (1) in Section 2.3 defines transformations of presheaves with transfers of topological spaces

$$\mathcal{Z}^q(-) \times \mathcal{Z}_0\big(\mathbb{P}^1\big) \xrightarrow{\#} \mathcal{M}or\big(-, \mathcal{C}_1\big(\mathbb{P}^{q+2}\big)\big)^+ / \mathcal{M}or\big(-, \mathcal{C}_1\big(\mathbb{P}^{q+1}\big)\big)^+ \xleftarrow{\Sigma\!\!\!/} \mathcal{Z}^{q+1}(-), \tag{2}$$

where $\mathcal{Z}^q(-)$ is the functor sending $X \in Cor_{\mathbb{C}}$ to the topological group $\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^q))^+ / \mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^{q-1}))^+$ and $\mathcal{Z}_0(\mathbb{P}^1)$ denotes the topological group $\mathcal{M}or(pt, \mathcal{C}_0(\mathbb{P}^1))^+$. Taking the singular chain complexes $Sing_*(-)$, we obtain natural transformations between contravariant functors from $Cor_{\mathbb{C}}$ to the category of simplicial sets

$$Sing_*\big(\mathcal{Z}^q(-) \times \mathcal{Z}_0\big(\mathbb{P}^1\big)\big) \xrightarrow{\#} Sing_*\big(\mathcal{M}or\big(-, \mathcal{C}_1\big(\mathbb{P}^{q+2}\big)\big)^+ / \mathcal{M}or\big(-, \mathcal{C}_1\big(\mathbb{P}^{q+1}\big)\big)^+\big) \xleftarrow{\Sigma\!\!\!/} Sing_*\big(\mathcal{Z}^{q+1}(-)\big). \tag{3}$$

The map $\Sigma\!\!\!/$ in (3) is a natural transformation of contravariant functors from $Cor_{\mathbb{C}}$ to $Ch(Ab)$, the category of chain complexes of abelian groups. The map # in (3) is not a group homomorphism, but it is biadditive and $\mathbb{Z}$-balanced in the sense that $(a \# nb) = (na \# b)$ for any integer n and elements a and b. So the map # in (3) induces a natural transformation

$$d\big(Sing_* \mathcal{Z}^q(-) \otimes Sing_* \mathcal{Z}_0\big(\mathbb{P}^1\big)\big) \xrightarrow{\#} Sing_*\big(\mathcal{M}or\big(-, \mathcal{C}_1\big(\mathbb{P}^{q+2}\big)\big)^+ / \mathcal{M}or\big(-, \mathcal{C}_1\big(\mathbb{P}^{q+1}\big)\big)^+\big), \tag{4}$$

where $d(Sing_* \mathcal{Z}^q(-) \otimes Sing_* \mathcal{Z}_0(\mathbb{P}^1))$ is the diagonal complex of the bisimplicial group

$$(s,t) \mapsto Sing_s \mathcal{Z}^q(-) \otimes Sing_t \mathcal{Z}_0(\mathbb{P}^1).$$

The map # in (4) is a natural transformation between contravariant functors from $Cor_{\mathbb{C}}$ to $Ch(Ab)$ by the universal property of tensor product. The generalized Eilenberg–Zilber Theorem [2] gives a natural quasi-isomorphism

$$d\big(Sing_* \mathcal{Z}^q(-) \otimes Sing_* \mathcal{Z}_0(\mathbb{P}^1)\big) \xrightarrow{EZ} Tot\big(Sing_* \mathcal{Z}^q(-) \otimes Sing_* \mathcal{Z}^1(pt)\big). \tag{5}$$

Let $s \in Sing_2 \mathcal{Z}_0(\mathbb{P}^1)$ be a generator of the group $\pi_2 Sing_* \mathcal{Z}_0(\mathbb{P}^1) = \mathbb{Z}$. The inclusion

$$Sing_* \mathcal{Z}^q(-) \cong Sing_* \mathcal{Z}^q(-) \otimes (\mathbb{Z} \cdot s) \longrightarrow Tot\big(Sing_* \mathcal{Z}^q(-) \otimes Sing_* \mathcal{Z}^1(pt)\big)$$

gives a natural transformation of contravariant functors from $Cor_{\mathbb{C}}$ to $Ch(Ab)$:

$$Sing_* \mathcal{Z}^q(-)[2] \xrightarrow{i} Tot\big(Sing_* \mathcal{Z}^q(-) \otimes Sing_* \mathcal{Z}^1(pt)\big). \tag{6}$$

Put (3), (4), (5) and (6) together and after sheafification, we obtain an element $s \in L^{q+1}H^{2q}(\wp_{mor}(q))$ in the derived category of Nisnevich sheaves with transfers

$$s = (\not{\Sigma})^{-1} \circ \# \circ (EZ)^{-1} \circ i : \wp_{mor}(q)[2] \longrightarrow \wp_{mor}(q+1).$$

Since the construction described above is the same construction of the s-operation as described in Section 2.3, the induced operation s_* on smooth varieties is the same as the usual s-operation. □

It follows from the theorem that s^i is an element in $L^{q+i}H^{2q}(\wp_{mor}(q))$ for any $i \geqslant 0$. We will need the following fact about the element s^i in the next subsection. Recall that an element g in an abelian group G is called torsion if $ng = 0$ for some natural number n and g is called n-divisible if $g \neq 0$ and there exists some element g' such that $g = ng'$.

Lemma 7.4. *For any $i \geqslant 0$, $s^i \in L^{q+i}H^{2q}(\wp_{mor}(q))$ is not torsion or n-divisible for any natural number n.*

Proof. It is shown in [8] that the morphic cohomology ring $L^*H^*(Spec(\mathbb{C}))$ is the polynomial ring $\mathbb{Z}[s]$ and the operation s^i acts as multiplication by s^i. So the operation s^i is not torsion or n-divisible for any n. □

Since $\tau_q = \tau_{q+1}s$ (see [8, Theorem 5.2]), Theorem 6.12 immediately implies the following result.

Theorem 7.5. *Let X be a quasi-projective complex variety and q be an integer such that all irreducible components of X have dimension less than or equal to q, then the s-operation is an isomorphism for all p.*

$$s : L^qH^p(X) \xrightarrow{\cong} L^{q+1}H^p(X).$$

7.2. Natural transformations of morphic cohomology of motives

We continue to calculate the group $L^{q+i}H^{2q+j}(\wp_{mor}(q)) = Hom_{DM}(\wp_{mor}(q), \wp_{mor}(q+i)[j-2i])$. In fact, the calculation is much easier with finite coefficients.

By the Universal Coefficient Theorem from commutative algebra, it is easy to see that the complex $\wp_{mor}(q)/n = \wp_{mor}(q) \otimes \mathbb{Z}/n$ is $\mathbb{A}^1$-local and hence an object in the category $DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)$. The category $DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)$ is constructed in a similar way as of DM in Section 2.1. Objects in $DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)$ are bounded above $\mathbb{A}^1$-local chain complexes of Nisnevich sheaves with transfers of $\mathbb{Z}/n$-modules. Please see [16] for more details. The same argument in the proof of Theorem 5.1 shows that $\wp_{mor}(q)[p-2q]/n$ represents the (q,p)-th morphic cohomology with coefficients in $\mathbb{Z}/n$ on smooth quasi-projective varieties.

Definition 7.6. For any $K^* \in DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)$ and integers p and q with $q \geqslant 0$, the (q,p)-th morphic cohomology of K^* with coefficients in $\mathbb{Z}/n$ is defined as

$$L^qH^p(K^*, \mathbb{Z}/n) = Hom_{DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)}\big(K^*, \big(\wp_{mor}(q)/n\big)[p-2q]\big).$$

In particular, the (q,p)-th morphic cohomology of a quasi-projective complex variety X with coefficients in $\mathbb{Z}/n$ is defined as $L^qH^p(X, \mathbb{Z}/n) = L^qH^p(M(X)/n, \mathbb{Z}/n)$.

As usual, let $M(\mathbb{P}^{q/q-1})$ denote the cone of the natural map $M(\mathbb{P}^{q-1}) \longrightarrow M(\mathbb{P}^q)$ in the triangulated category DM. It follows from [20, §9] that the motivic cohomology and morphic cohomology coincide on smooth quasi-projective varieties X with finite coefficients, i.e., the natural map

$$H^p\big(\big(M\big(\mathbb{P}^{q/q-1}\big)/n\big)(X)\big) \longrightarrow H^p\big(\big(\wp_{mor}(q)/n\big)(X)\big)$$

is an isomorphism for any integer p and nonnegative integer q. So the natural map $M(\mathbb{P}^{q/q-1})/n \xrightarrow{\cong} \wp_{mor}(q)/n$ is an isomorphism in $DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)$.

Lemma 7.7.

$$L^{q+i}H^{2q+j}\big(\wp_{mor}(q)/n, \mathbb{Z}/n\big) = \begin{cases} \mathbb{Z}/n & \text{if } i \geqslant 0 \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $M(\mathbb{P}^{q/q-1})/n \cong \wp_{mor}(q)/n$, we have

$$L^{q+i}H^{2q+j}\big(\wp_{mor}(q)/n, \mathbb{Z}/n\big) = L^{q+i}H^{2q+j}\big(M\big(\mathbb{P}^{q/q-1}\big)/n, \mathbb{Z}/n\big).$$

To calculate the group on the right-hand side, we consider the following long exact sequence of morphic cohomology and singular cohomology associated to the distinguished triangle $M(\mathbb{P}^{q-1})/n \longrightarrow M(\mathbb{P}^q)/n \longrightarrow M(\mathbb{P}^{q/q-1})/n \longrightarrow M(\mathbb{P}^{q-1})/n[1]$.

$$\begin{array}{ccccccc} \cdots \longrightarrow & L^sH^n(M(\mathbb{P}^{q/q-1})/n, \mathbb{Z}/n) & \longrightarrow & L^sH^n(\mathbb{P}^q, \mathbb{Z}/n) & \longrightarrow & L^sH^n(\mathbb{P}^{q-1}, \mathbb{Z}/n) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^n_{Sing}(M(\mathbb{P}^{q/q-1})/n, \mathbb{Z}/n) & \longrightarrow & H^n_{Sing}(\mathbb{P}^q, \mathbb{Z}/n) & \longrightarrow & H^n_{Sing}(\mathbb{P}^{q-1}, \mathbb{Z}/n) & \longrightarrow \cdots \end{array}$$

In the diagram, the vertical maps are given by the natural transformation τ_* as discussed in Section 6. The horizontal maps are induced by the standard embedding $\mathbb{P}^{q-1} \longrightarrow \mathbb{P}^q$. If $s = q + i \geqslant q$, all vertical maps are isomorphisms and the result follows easily. If $s = q + i < q$, the result follows from the fact that the vertical maps are isomorphisms for $n \leqslant 2s$ and the fact that $L^sH^n(X) = 0$ for any $n > 2s$ on a smooth variety X. □

Since $\mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n$ is a resolution by representable Nisnevich sheaves in the category $Sh_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z})$, it follows from the definition of the tensor structure $\overset{tr}{\otimes}$ in DM (see [16, §8]) that we have a natural short exact sequence

$$0 \longrightarrow Hom_{DM}(M^*, N^*)/n \longrightarrow Hom_{DM}\left(M^*, N^* \overset{tr}{\bigotimes} \mathbb{Z}/n\right) \longrightarrow Tor_1\big(Hom_{DM}\big(M^*, N^*[1]\big), \mathbb{Z}/n\big) \longrightarrow 0.$$

To compute the middle term, we recall from [12, Lemma 4] the natural isomorphism

$$Hom_{DM}\left(M^*, N^* \overset{tr}{\bigotimes} \mathbb{Z}/n\right) \xrightarrow{\cong} Hom_{DM^{eff,-}_{Nis}(Cor_{\mathbb{C}}, \mathbb{Z}/n)}(M^*/n, N^*/n).$$

When applied to the case $N^* = \wp_{mor}(q+i)[j-2i]$, these two results together give the following fact.

Lemma 7.8. *For any $M^* \in DM$, there is a natural short exact sequence*

$$0 \longrightarrow L^{q+i}H^{2q+j}(M^*)/n \longrightarrow L^{q+i}H^{2q+j}(M^*, \mathbb{Z}/n) \longrightarrow Tor_1\big(L^{q+i}H^{2q+j+1}(M^*), \mathbb{Z}/n\big) \longrightarrow 0.$$

We are now ready to prove the main result of this section.

Theorem 7.9.

$$L^{q+i}H^{2q+j}\big(\wp_{mor}(q)\big) = \begin{cases} \mathbb{Z} \oplus V_{i,q} & \text{if } i \geqslant 0 \text{ and } j = 0, \\ W_{i,j,q} & \text{otherwise.} \end{cases}$$

Here $V_{i,q}$ and $W_{i,j,q}$ are some vector spaces over the rational numbers $\mathbb{Q}$.

Proof. By Lemma 7.8 we have the following short exact sequence

$$0 \longrightarrow L^{q+i}H^{2q+j}\big(\wp_{mor}(q)\big)/n \longrightarrow L^{q+i}H^{2q+j}\big(\wp_{mor}(q), \mathbb{Z}/n\big) \longrightarrow Tor_1\big(L^{q+i}H^{2q+j+1}\big(\wp_{mor}(q)\big), \mathbb{Z}/n\big) \longrightarrow 0.$$

If $i < 0$ or $j \neq 0$, we have $L^{q+i}H^{2q+j}(\wp_{mor}(q), \mathbb{Z}/n) = 0$ by Lemma 7.7. So

$$L^{q+i}H^{2q+j}\big(\wp_{mor}(q)\big)/n = Tor_1\big(L^{q+i}H^{2q+j+1}\big(\wp_{mor}(q)\big), \mathbb{Z}/n\big) = 0.$$

If $i \geqslant 0$ and $j = 0$, we have $L^{q+i}H^{2q}(\wp_{mor}(q), \mathbb{Z}/n) \cong \mathbb{Z}/n$ by Lemma 7.7. By Lemma 7.4, $L^{q+i}H^{2q}(\wp_{mor}(q))$ contains an element s^i which is not n-divisible for any n. So the group $L^{q+i}H^{2q}(\wp_{mor}(q))/n$ contains a copy of $\mathbb{Z}/n$ as a subgroup, which enforces the second injective map in the short exact sequence to be an isomorphism. So we have

$$L^{q+i}H^{2q}\big(\wp_{mor}(q)\big)/n = \mathbb{Z}/n \quad \text{and} \quad Tor_1\big(L^{q+i}H^{2q+1}\big(\wp_{mor}(q)\big), \mathbb{Z}/n\big) = 0.$$

This calculation immediately implies that $L^{q+i}H^{2q+j}(\wp_{mor}(q))$ is a $\mathbb{Q}$-vector space if $i < 0$ or $j \neq 0$. For $i \geqslant 0$ and $j = 0$, the group $L^{q+i}H^{2q}(\wp_{mor}(q))/n$ is isomorphic to $\mathbb{Z}/n$ for any natural number n and the group $L^{q+i}H^{2q}(\wp_{mor}(q))$ is torsion free. The obvious map $\lambda : M(\mathbb{P}^{q/q-1}) \longrightarrow \wp_{mor}(q)$ induces a map $\lambda^* : L^{q+i}H^{2q}(\wp_{mor}(q)) \longrightarrow L^{q+i}H^{2q}(M(\mathbb{P}^{q/q-1})) \cong \mathbb{Z}$ and a natural map between the short exact sequences of Lemma 7.8, for which the first 3 terms are written down in the following diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & L^{q+i}H^{2q}(\wp_{mor}(q))/n & \xrightarrow{\cong} & L^{q+i}H^{2q}(\wp_{mor}(q), \mathbb{Z}/n) \\ & & \downarrow{\scriptstyle \lambda^*\otimes 1_n} & & \downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & L^{q+i}H^{2q}(M(\mathbb{P}^{q/q-1}))/n & \xrightarrow{\cong} & L^{q+i}H^{2q}(M(\mathbb{P}^{q/q-1}), \mathbb{Z}/n) \end{array}$$

The three isomorphisms in the diagram have already been explained. It follows that the left vertical map $\lambda^* \otimes 1_n$ is an isomorphism for any n. So $\lambda^* : L^{q+i}H^{2q}(\wp_{mor}(q)) \longrightarrow \mathbb{Z}$ is onto. Denote the kernel of λ^* as $V_{i,q}$, then $L^{q+i}H^{2q}(\wp_{mor}(q)) = \mathbb{Z} \oplus V_{i,q}$. Clearly $V_{i,q}$ is torsion free and divisible, so it is a $\mathbb{Q}$-vector space. □

Remark 7.10. It follows from the proof that s^i is a generator of the direct summand $\mathbb{Z}$ of $L^{q+i}H^{2q}(\wp_{mor}(q))$ where $i \geqslant 0$. An element t in the vector spaces $W_{i,j,q}$ or $V_{i,q}$ has the following property: for any integer n, there exists a unique natural transformation t_n such that $t = nt_n$. Conjecturally, there should be no such non-zero natural transformation t since we do not see any $\mathbb{Q}$-space in known morphic cohomology groups with integral coefficients.

The natural transformations of morphic cohomology of motives considered in this section are special (non-stable) morphic cohomology operations. Namely, these natural transformations are those operations which are compatible with the transfer maps.

Acknowledgements

This paper is based on my PhD thesis. I would like to thank my supervisor, Eric Friedlander, for his guidance and many helps. I would also like to thank Christian Haesemeyer for his helpful discussions.

I thank the referee for many useful and important comments and suggestions.

References

[1] A. Dold, R. Thom, Quasifaserungen und unendliche symmetrische produkte, Ann. of Math. (2) 67 (1956) 230–281.
[2] A. Dold, D. Puppe, Homologie nicht-additiver Funktoren; Anwendungen, Ann. Inst. Fourier (Grenoble) 11 (1961) 201–312.
[3] E. Friedlander, Algebraic cycles, Chow varieties, and Lawson homology, Compos. Math. 77 (1) (1991) 55–93.
[4] E. Friedlander, Algebraic cocyles on normal, quasi-projective varieties, Compos. Math. 110 (2) (1998) 127–162.
[5] E. Friedlander, Bloch–Ogus properties for topological cycle theory, Ann. Sci. Ecole Norm. Sup. (4) 33 (2000) 57–79.
[6] E. Friedlander, O. Gabber, Cycle spaces and intersection theory, in: Topological Methods in Modern Mathematics, Stony Brook, NY, 1991, pp. 325–370.
[7] E. Friedlander, B. Mazur, Filtrations on the homology of algebraic quasi-projective varieties, Mem. Amer. Math. Soc. 110 (529) (1994).
[8] E. Friedlander, H. Lawson, A theory of algebraic cocycles, Ann. of Math. 136 (1992) 361–428.
[9] E. Friedlander, H. Lawson, Duality relating spaces of algebraic cocycles and cycles, Topology 36 (1997) 533–565.
[10] E. Friedlander, V. Voevodsky, Bivariant cycle cohomology, in: Cycles, Transfer, and Motivic Homology Theories, in: Ann. of Math. Stud., vol. 143, 2000.
[11] E. Friedlander, M. Walker, Function spaces and continuous algebraic pairings for quasi-projective varieties, Compos. Math. 125 (1) (2001) 69–110.
[12] C. Hasemeyer, J. Hornbostel, Motives and etale motives with finite coefficients, K-Theory 34 (2005) 195–207.
[13] H. Hironaka, Resolution of singularities of an algebraic quasi-projective variety over a field of characteristic zero, Ann. of Math. 79 (1964), I: 109–203, II: 205–326; Ann. of Math. 129 (1989) 253–291.
[14] P. Lima-Filho, Completions and fibrations for topological monoids, Trans. Amer. Math. Soc. 340 (1) (1993) 127–147.
[15] P. Lima-Filho, Lawson homology for quasiprojective varieties, Compos. Math. 84 (1) (1992) 1–23.
[16] C. Mazza, V. Voevodsky, C. Weibel, Lectures on Motivic Cohomology, Clay Math. Monogr., vol. 2, 2006.
[17] Y. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory, in: Algebraic K-Theory: Connections with Geometry and Topology, Kluwer, 1989, pp. 241–342.
[18] E. Spanier, Infinite symmetric products, function spaces, and duality, Ann. of Math. (2) 69 (1959) 142–198.
[19] N. Steenrod, A convenient category of topological spaces, Michigan Math. J. 14 (1967) 133–152.
[20] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic quasi-projective varieties, Invent. Math. 123 (1996) 61–94.