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Elementary proofs concerning results about functions on the n-sphere

E.R. Verheul

Fakulteit Wiskunde en Informatica, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, Netherlands

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Abstract

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A notion of sign map is introduced for functions on the n-sphere. There are three kinds of signs: $-$, 0, $+$. It is shown that if two sign maps are homotopic then they have the same sign. From this theorem many classical results on functions of the n-sphere can be derived in a fairly elementary fashion.

Keywords: Brouwer fixed point theorem, fundamental theorem of algebra, hairy ball theorem, homotopy, sign of a map, simplicial approximation.

AMS (MOS) Subj. Class.: 30C15, 55M20, 55M25.

Introduction

In [3] Hirsch gave an elementary proof of the nonretractibility of the cell on its boundary. In this proof he approximated an assumed retraction of the cell onto the boundary by a simplicial retraction of the cell into its boundary, and he obtained a contradiction via the construction of an "arc" of simplices. The idea behind the proof is beautifully simple and what is more, one can see geometricly what is going on. Alas the existence of such a simplicial retraction is not quite an elementary result, and so this proof involves even more preparation as, for example, a proof using Sperners lemma.

In [2] Dugundji defines the degree of a map between n-spheres and proves that homotopic maps have the same degree. This is very short and elementary, and provides more information than Brouwer's fixed point theorem or any of its relatives.

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Inspecting the proof of Dugundji one sees that the degree of a map occurs actually as the degree of a "proper vertex map" depending on a triangulation of the sphere. This triangulation may be different at the time of definition and at the time a homotopy with some other function is considered. When Dugundji attempts to prove that the degree does not depend on the simplicial presentation of the sphere, he supposes things easier than they really are, and his argument is in error. By the use of special "simplicial approximations" it seems possible to repair this proof, but the arguments become more elaborate and non-elementary.

Our aim is not to repair Dugundji's proof. We shall restrict ourselves to consider the sign of the degree (without referring to the notion of degree). We are then able to derive many of the classical consequences. In our proofs we will develop a new method which appeals to the same geometric intuition as Hirsch's construction. This self-contained method avoids the existence of special simplicial approximations, and is more widely of use than the original method of Hirsch.

On the class of (continuous) functions of the n-sphere into $\mathbb{R}^{n+1}\backslash\{0\}$ we will introduce the notion *sign map*; we consider three kinds of sign, negative, zero and positive. Roughly, a positive (respectively negative) sign map turns "small" n-simplices into n-simplices without (respectively with) changing of the orientation. A zero sign map sends n-simplices into something of lower dimension. This notion behaves very well and it turns out that surprisingly many functions are sign maps. For example a polynominal in $\mathbb{C}$ is a sign map, when restricted to a sufficiently large sphere. All kinds of "normal" functions like linear maps and constant maps are sign maps too. Our main theorem states that if sign maps are homotopic they have the same sign. Among the corollaries are: the fixed point theorem of Brouwer, the fundamental theorem of algebra and the "nonzero vectorfield" theorem. It should be noted that contrary to the Brouwer fixed point theorem, there are not many "elementary" proofs of the "nonzero vectorfield" theorem.

In [1] Dodson gave a proof of the fundamental theorem of algebra. Dodson mentioned some other results that his method could give too, without actually proving them. It turns out that these results can be easily proved with our methods.

1. Properties and examples of sign maps

For $q>1$ let $S_R^{q-1} := \{x\in\mathbb{R}^q \mid \|x\| = R\}$ be the $(q-1)$-sphere of radius R. We usually think of $\|\cdot\|$ as the standard norm, but occasionally we allow it to be another norm.

Definition. Let $s\in\{-1,0,+1\}$. A map $h: S_R^{q-1}\to\mathbb{R}^q\backslash\{0\}$ is called a *sign map* (*of sign* s) if there is a $\delta>0$ with the following property:

If $\{x_0,\dots,x_{q-1}\}$ is a linearly independent set of diameter less than δ, then

$$\operatorname{sign}\det(x_0,x_1,\dots,x_{q-1})\cdot\operatorname{sign}\det(h(x_0),h(x_1),\dots,h(x_{q-1})) = s.$$

Observe that for a nonzero sign map the explicit demand that the image set avoids zero is superfluous.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^q$. Let π be the *radial projection*,

$$\pi:\mathbb{R}^q\backslash\{0\}\to S_R^{q-1},\quad x\to\frac{x\cdot R}{\|x\|}.$$

We make a few simple observations which are stated here for later reference.

Example 1.1. Let R_1 be a positive number.

(a) If $f:S_R^{q-1}\to\mathbb{R}^q\backslash\{0\}$ is a sign map, then $\pi\circ f$ is a sign map with the same sign as f.

(b) Let $\|\cdot\|_1$ be another norm on $\mathbb{R}^q$. Then the projection of the sphere $\{x\in\mathbb{R}^q\mid \|x\|_1=R_1\}$ onto the sphere S_R^1, i.e., the restriction of π to $\{x\in\mathbb{R}^q\mid\|x\|_1=R_1\}$, is a + sign map.

Example 1.2. Let $g:S_R^{q-1}\to\mathbb{R}^q$ and $h:S_R^{q-1}\to S_R^{q-1}$ be sign maps. If $\operatorname{sign}(h)$ is nonzero, or if g is the restriction to S_R^1 of a regular linear map, then the composition $g\circ h$ is also a sign map, with sign $\operatorname{sign}(g)\cdot\operatorname{sign}(h)$.

In the remainder of this section we shall give some important examples of sign maps.

Example 1.3. Let $c\in\mathbb{R}^q$ be different from the origin. Then the constant map $c:S_R^{q-1}\to\mathbb{R}^q$, $x\to c$, is a sign map of sign 0.

Example 1.4. A nonsingular linear map $A:\mathbb{R}^q\to\mathbb{R}^q$, restricted to a $(q-1)$-sphere, is a sign map with sign equal to $\operatorname{sign}\det(A)$.

In what follows we see $\mathbb{C}$ as $\mathbb{R}^2$, the norm shall be the standard (Euclidean) norm. Recall there is a continuous function $\arg:\mathbb{C}\backslash\{0\}\to\mathbb{R} \bmod 2\pi$ defined by

$$\arg(z)=\mu\ \Leftrightarrow\ |z|\cdot e^{i\mu}=z.$$

We also have the following equality for all $z_1,z_2\in\mathbb{C}\backslash\{0\}$,

$$\det(z_1,z_2)=\operatorname{Im}(\bar z_1\cdot z_2)=|z_1|\cdot|z_2|\cdot\sin(\arg(z_2)-\arg(z_1)).$$

Let $a,b\in\mathbb{C}$, and let n be an integer.

Example 1.5. *The power map $\mathbb{C}\to\mathbb{C}$, $z\to z^n$ restricted to S_R^1, for $R>0$ fixed, is a sign map with sign equal to* $\operatorname{sign}(n)$.

Example 1.6. *If $|a|\neq|b|$, then the map $\mathbb{C}\to\mathbb{C}$, $z\to az^n+b\bar z^n$ restricted to s_R^1, for $R>0$ fixed, is a sign map with sign equal to* $\operatorname{sign}(n(|a|-|b|))$.

Example 1.7. *Let $p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$ be a polynomial in $\mathbb{C}$ of degree $n\geq 1$. Then the following hold.*[1]

(a) *The restriction of p to S_R^1 is a + sign map, provided*

$$\operatorname{Re}[z\cdot p'(z)\cdot\overline{p(z)}]>0\quad \textit{for every } z\in S_R^1. \tag{1}$$

[1] The following conditions on R are of special interest, for instance in Theorem 3.8. If one is only interested in the *existence* of a $R>0$ such that p restricted to S_R^1 is a + sign map, then one can find a much easier proof.

(b) *The lefthand side of* (1) *is minorized by*

$$\sum_{k=1}^{n} k|a_k|^2R^{2k} - \sum_{0\leq k<l\leq n} (k+l)|a_la_k| \cdot R^{k+l}. \tag{2}$$

(c) *Suppose* $R>0$ *is such that formula* (2) *takes a positive value. Then the restriction of* p *to* S^1_R *is a* $+$ *sign map.*

Proofs. For a proof of the statements in Examples 1.5, 1.6 and 1.7, we first make the following observation. Let $f: S^1_R \to \mathbb{C}$ be a continuous function, and let $s \in \{-1, 1\}$. Clearly, f is a sign map of sign s if there is a $\delta > 0$ such that for $z, w \in S^1_R$ with $|z-w|<\delta$ and $z\neq w$ the following equality holds,

$$\operatorname{sign}(\operatorname{Im}(\overline{f(z)}\cdot f(w))\cdot \operatorname{Im}(\bar{z}\cdot w)^{-1}) = s.$$

Now put $w = z\cdot e^{i\mu}$ with $\mu\in(-\pi, \pi]$, which implies that $\operatorname{Im}(\bar{z}\cdot w) = R^2\cdot \sin(\mu)$. Since the restriction of arg to S^1_R is a continuous open function this is equivalent with the existence of a $\delta>0$ such that

$$0<|\mu|<\delta \Rightarrow \operatorname{sign}(\operatorname{Im}(\overline{f(z)}\cdot f(z\cdot e^{i\mu}))\cdot \sin(\mu)^{-1}) = s \quad \text{for every } z\in S^1_R. \tag{3}$$

First, we will show Example 1.6 (from which 1.5 trivially follows). Let f be such as described in Example 1.6, and let $R>0$. If $n=0$, then Example 1.6 follows from Example 1.3. So we may assume that $n\neq 0$. Let $z\in S^1_R$ and $\mu\in(-\pi, \pi]$. A straightforward calculation shows that $\operatorname{Im}(\overline{f(z)}\cdot f(z\cdot e^{i\mu}))$ equals $R^{2n}\cdot(|a|^2-|b|^2)\cdot \sin(n\cdot\mu)$. So if we take any δ with $0<\delta<|2\cdot\pi/n|$, then we are in the situation of property (3), with $s=\operatorname{sign}(n(|a|-|b|))$.

Next, we will give a proof of 1.7. We will first show 1.7(b). To this end, let $R>0$. Direct verification yields the following equality for $z\in S^1_R$,

$$\operatorname{Re}[z\cdot p'(z)\cdot\overline{p(z)}] = \sum_{k=1}^{n} k|a_k|^2R^{2k} + \sum_{0\leq k<l\leq n}(k+l)\operatorname{Re}[a_k\bar{a}_l\cdot z^k\cdot \bar{z}^l]. \tag{4}$$

From this the desired inequality

$$\operatorname{Re}[z\cdot p'(z)\cdot\overline{p(z)}] \geq \sum_{k=1}^{n} k|a_k|^2R^{2k} - \sum_{0\leq k<l\leq n}(k+l)|a_la_k|\cdot R^{k+l},$$

easily follows.

For a proof of Example 1.7(a) (from which 1.7(c) trivially follows), let $R>0$ be such that (1) holds. Compactness of the sphere S^1_R together with an continuity argument shows the existence of a $\theta>0$ such that[2] $\operatorname{Re}[z\cdot p'(z)\cdot\overline{p(z)}]\geq\theta$ for all $z\in S^1_R$. For $z\in S^1_R$ and $0\leq k, l\leq n$ we write

$$T_{k,l}(z) = \bar{a}_k\bar{z}^k a_l z^l.$$

[2] If we are in the situation of (c), then we do not need to use the compactness of S^1_R, as formula (2) provides us with such a θ.

Observe that $T_{k,k}(z)=|a_k|^2\cdot R^{2k}$. Using this notation, formula (4) yields

$$\mathrm{Re}[z\cdot p'(z)\cdot \overline{p(z)}]=\sum_{k=1}^{n} k\cdot T_{k,k}(z)+\sum_{0\leq k<l\leq n}(k+l)\cdot \mathrm{Re}[T_{k,l}(z)]. \tag{5}$$

We aim at an application of formula (3). To this end, let $\mu\in(-\pi,\pi]$. Consider the following equalities, which can be deduced by direct computations:

$$\begin{aligned}\mathrm{Im}[\overline{p(z)}\cdot p(z\cdot \mathrm{e}^{\mathrm{i}\mu})]&=\sum_{k=1}^{n}\mathrm{Im}(T_{k,k}(z)\cdot \mathrm{e}^{\mathrm{i}k\mu})+\sum_{0\leq k,l\leq n;k\neq l}\mathrm{Im}(T_{k,l}(z)\cdot \mathrm{e}^{\mathrm{i}l\mu})\\ &=\sum_{k=1}^{n}T_{k,k}(z)\cdot\sin(k\mu)+\sum_{0\leq k<l\leq n;k\neq l}\mathrm{Im}(T_{k,l}(z)\\ &\quad\cdot \mathrm{e}^{il\mu}+T_{l,k}(z)\cdot \mathrm{e}^{ik\mu})\\ &=\sum_{k=1}^{n}T_{k,k}(z)\cdot\sin(k\mu)+\sum_{0\leq k<l\leq n;k\neq l}\mathrm{Im}(T_{k,l}(z)\\ &\quad\cdot \mathrm{e}^{il\mu}+\overline{T_{k,l}(z)}\cdot \mathrm{e}^{ik\mu}).\end{aligned} \tag{6}$$

Now let $0\leq k<l\leq n$. One can easily verify the following equality.

$$\begin{aligned}&\mathrm{Im}(T_{k,l}(z)\cdot \mathrm{e}^{\mathrm{i}l\mu})+\mathrm{Im}(\overline{T_{k,l}(z)}\cdot \mathrm{e}^{\mathrm{i}k\mu})\\ &\quad=\mathrm{Re}(T_{k,l}(z))(\sin(l\mu)+\sin(k\mu))+\mathrm{Im}(T_{k,l}(z))(\cos(l\mu)-\cos(k\mu)).\end{aligned} \tag{7}$$

The following formulae are standard:

$$\lim_{\mu\to 0}\sin(\mu)^{-1}\cdot(\sin(l\mu)+\sin(k\mu))=k+l,$$

$$\lim_{\mu\to 0}\sin(\mu)^{-1}\cdot(\cos(l\mu)-\cos(k\mu))=0.$$

In view of these formulae and formula (7), we can uniformly (with respect to z) approximate the term $(k+l)\cdot\mathrm{Re}(T_{k,l}(z))$ of (5) by $\sin(\mu)^{-1}\cdot[\mathrm{Im}(T_{k,l}(z)\cdot \mathrm{e}^{\mathrm{i}l\mu})+\mathrm{Im}(\overline{T_{k,l}(z)}\cdot \mathrm{e}^{\mathrm{i}k\mu})]$ (observe that all $T_{k,l}$ are bounded). Similarly, we can uniformly approximate $k\cdot T_{k,k}(z)$ by $\sin(\mu)^{-1}\cdot T_{k,k}(z)\cdot\sin(k\mu)$. Hence by (6) and (5) we can uniformly approximate $\mathrm{Re}[z\cdot p'(z)\cdot\overline{p(z)}]$ by $\sin(\mu)^{-1}\cdot\mathrm{Im}[\overline{p(z)}\cdot p(z\cdot \mathrm{e}^{\mathrm{i}\mu})]$. That is, we can find a $\delta>0$ such that for all $0<|\mu|<\delta$ and $z\in S^1_R$

$$|\mathrm{Re}[z\cdot p'(z)\cdot\overline{p(z)}]-\mathrm{Im}[\overline{p(z)}\cdot p(z\cdot \mathrm{e}^{\mathrm{i}\mu})]\cdot\sin(\mu)^{-1}|<\tfrac{1}{2}\theta.$$

So we conclude that for all $0<|\mu|<\delta$ and $z\in S^1_R$ the inequality $\mathrm{Im}[\overline{p(z)}\cdot p(z\cdot \mathrm{e}^{\mathrm{i}\mu})]\cdot\sin(\mu)^{-1}|>0$, holds. By formula (3) this implies that p restricted to S^1_R is a sign map of sign +. □

2. The main theorem and its proof

Our main result is:

Theorem 2.1. *If two sign maps are homotopic, then they have the same sign.*

This is a "strong" result as the corollaries in Section 3 will show.

Prior to the proof of this theorem we shall show a crucial lemma. First we introduce some notation. The (closed) simplex σ spanned by an affinely independent set $\{a_0, \ldots, a_n\}$ will be denoted by $\langle a_0, \ldots, a_n\rangle$. The interior σ^0 of σ is the set

$$\left\{b \,\middle|\, b = \sum_{i=0}^{n} \mu_i a_i, \mu_i > 0, \sum_{i=0}^{n} \mu_i = 1\right\}.$$

Note that such simplices are unordered. If the vertices are ordered by $a_0 < a_1 < \cdots < a_n$, then the resulting *ordered* simplex will be denoted by $(a_0, \ldots, a_n)$.

Definition. Let E be a q-simplex in $\mathbb{R}^q$, B be a $(q-1)$-face of E and v the only vertex of E not in B. The affine hull of B gives a hyperplane P in $\mathbb{R}^q$, so $\mathbb{R}^q$ is the disjoint union of this hyperplane P and two open halfspaces. Because v can not be in P it is in one of the open halfspaces, say H. This halfspace will be called the *good side* of B in E. Now take an (ordered) $(q-1)$-simplex $(u_0, u_1, \ldots, u_{q-1})$ in B and any y at the good side of B in E (for example v) and define the *sign* of $(u_0, u_1, \ldots, u_{q-1})$ in E, denoted by $\mathrm{sign}((u_0, u_1, \ldots, u_{q-1}), E)$, as the sign of $\det(u_0 - y, u_1 - y, u_2 - y, \ldots, u_{q-1} - y)$. As $\{u_0 - y, u_1 - y, u_2 - y, \ldots, u_{q-1} - y\}$ is a basis of $\mathbb{R}^q$ for every y in H, that determinant is nonzero. Furthermore, as "taking a determinant" is a continuous operation, and as H is connected, it is clear that the sign is independent of the choice of y in H.

Lemma 2.2 (Key-lemma). *Let A be a convex set with affine dimension $q-1$ in $\mathbb{R}^q$, E a q-simplex in $\mathbb{R}^q$ and B a $(q-1)$-face of E. If $f: E \to A$ is an affine map such that the affine dimension of $f(B)$ equals $q-1$, then the following are true.*

(1) *There is a function $g: B^0 \to \partial E \backslash B$ such that if $b \in B^0$, then $g(b)$ is the unique point c in $\partial E \backslash B$ with $f(c) = f(b)$.*

Let C be a $(q-1)$-face of E and let $e \in B^0$.

(2) *If $g(e) \in C \backslash C^0$ and V is a neighborhood of e in B^0, then there is a point e^* in V with $g(e^*) \in C^0$.*

Let $g(e) \in C^0$.

(3) (a) *There are convex open sets U (containing e) and V (containing $g(e)$) in B^0 respectively C^0 such that $g|_U: U \to V$ is an affine isomorphism.*

(b) *If $\langle u_0, u_1, \ldots, u_{q-1}\rangle$ is a $(q-1)$-simplex in U, then $\langle g(u_0), \ldots, g(u_{q-1})\rangle$ is a $(q-1)$-simplex in V.*

(c) $\mathrm{sign}((u_0, \ldots, u_{q-1}), E)$ *and* $\mathrm{sign}((g(u_0), \ldots, g(u_{q-1})), E)$ *are opposite signs.*

Proof. Suppose $E = \langle a_0, \ldots, a_q\rangle$. Without loss of generality, $B = \langle a_0, \ldots, a_{q-1}\rangle$, $a_0 \in C$ and $a_0 = 0 = f(a_0) \in A$.

Let $\mathrm{hull}(A)$ denote the *linear* hull of A. We can consider f as a restriction of a linear map $F: \mathbb{R}^q \to \mathrm{hull}(A)$. Now let $b \in B^0$, say, $b = \sum_{i=0}^{q} \mu_i a_i$ with $\mu_i > 0$ if $i < q$, $\mu_q = 0$ and $\sum_{i=0}^{q} \mu_i = 1$. Because A, B and $f(B)$ have affine dimension $(q-1)$ and $0 \in A, B, f(B)$ it follows that $\mathrm{hull}(A)$, $\mathrm{hull}(B)$, $F(\mathrm{hull}(B))$ are $(q-1)$-dimensional in $\mathbb{R}^q$. Therefore F is a linear bijection of $\mathrm{hull}(B)$ onto $\mathrm{hull}(A)$ and the kernel of F in $\mathbb{R}^q$ is one-dimensional. Let k be a nontrivial element from the kernel, as

$k \notin \text{hull}(B)$ we can write $k = \sum_{i=0}^{q} \theta_i a_i$ with $\sum_{i=0}^{q} \theta_i = 0$ and $\theta_q \neq 0$. We may assume that $\theta_q > 0$. An element of $f^{-1}(f(b)) \cap \partial E \backslash B$ takes the form

$$\sum_{i=0}^{q} (\mu_i + \lambda\theta_i) a_i$$

where $\mu_i + \lambda\theta_i \geq 0$ for all i, $\mu_q + \lambda\theta_q > 0$ and there is a $i < q$ such that $\mu_i + \lambda\theta_i = 0$. It follows that

$$\lambda = \min\{-\mu_i/\theta_i \mid \theta_i < 0\},$$

existence and unicity therefore follow directly. This proves (1).

For a proof of (2) and (3)(a), (b), first note that if $g(e) \in C$ then k cannot be in hull(C). Since k is parallel to $g(e) - e$ and $e \in B^0$. Consequently F restricted to hull(C) is a linear isomorphism. This yields an isomorphism $G: \text{hull}(B) \to \text{hull}(C)$

$$G := F|_{\text{hull}(C)}^{-1} F|_{\text{hull}(B)},$$

extending $g|_{g^{-1}(C)}$. In the situation of (2), we find that $G(V) \cap C$ is a neighborhood of $g(e)$ in C. Because the interior of C is dense in C one can find a c in $G(V) \cap C^0$ and consequently there is a e^* in V with

$$g(e^*) = G(e^*) = c \in C^0.$$

In the situation of (3)(a), the set $U := G^{-1}(C^0) \cap B^0$ is clearly convex open in B^0 and contains e. Then $V := G(U) = g(U)$ is a convex open set in C^0 containing $g(e)$ and g restricted to U is an affine isomorphism. This proves (3)(a) which immediately implies (3)(b).

As the contribution of a_q to k, θ_q, is positive and $a_0 = 0 \in B \cap C$, it follows that k is at the good side of hull(B) in E. We have already remarked that k is not in hull(C), we have in fact:

Claim. *k is at the good side of* hull(C) *in E.*

Proof. Assume that $-k$ is not at the good side of hull(C) in E, then k must be at the good side of hull(C) in E. But as e is at the good side of hull(C) in E and as $a_0 = 0 \in C$ this would mean that $e + \lambda k$ would be at the good side of hull(C) in E for all $\lambda > 0$ too, but this would mean that $g(e)$ is not in hull$(C) \supset C$, a contradiction.

As $u_0 \in \text{hull}(B)$ and $g(u_0) \in \text{hull}(C)$ we find that $u_0 + k$ and $g(u_0) - k$ are also at the good side of hull(B) respectively hull(C). Now for all $\lambda_0, \lambda_1, \ldots, \lambda_{q-1} \in \mathbb{R}$ we have

$$\begin{aligned}
&\det(u_0 - (u_0 + k), u_1 - (u_0 + k), u_2 - (u_0 + k), \ldots, u_{q-1} - (u_0 + k)) \\
&\quad = \det(-k, u_1 - (u_0 + k), u_2 - (u_0 + k), \ldots, u_{q-1} - (u_0 + k)) \\
&\quad = -\det(k, u_1 - (u_0 + k), u_2 - (u_0 + k), \ldots, u_{q-1} - (u_0 + k)) \\
&\quad = -\det(k, u_1 - (u_0 + k) - (\lambda_1 - 2)k, u_2 - (u_0 + k) - (\lambda_2 - 2)k, \ldots, u_{q-1} \\
&\qquad - (u_0 + k) - (\lambda_{q-1} - 2)k) \\
&\quad = -\det(u_0 + \lambda_0 k - (u_0 + \lambda_0 k - k), u_1 - (u_0 + \lambda_1 k - k), u_2 \\
&\qquad - (u_0 + \lambda_2 k - k), \ldots, u_{q-1} - (u_0 + \lambda_{q-1} k - k)).
\end{aligned}$$

The way g was constructed gives that, for certain $\lambda_0, \ldots, \lambda_{q-1}$ the last determinant equals

$$-\det(g(u_0)-(g(u_0)-k), g(u_1)-(g(u_0)-k), g(u_2)$$
$$-(g(u_0)-k), \ldots, g(u_{q-1})-(g(u_0)-k)).$$

This proves (3)(c). □

For the sake of brevity we say that (f, B, A) has the key-property if it has the properties of Lemma 2.2. So A has to be a convex set with affine dimension $q-1$ in $\mathbb{R}^q$, E a q-simplex in $\mathbb{R}^q$, B a $(q-1)$-face of E and $f: E \to A$ has to be an affine map such that the affine dimension of $f(B)$ equals $q-1$.

Remark. If (f, B, A) has the key-property and C is as in part (3) of Lemma 2.2, then (f, C, A) has the key-property too, as it follows directly from (3)(a).

Let $M^q := [-1, 1]^q \backslash (-\frac{1}{2}, \frac{1}{2})^q$ a q-dimensional annulus in $\mathbb{R}^q$. $\partial[-\frac{1}{2}, \frac{1}{2}]^q$ and $\partial[-1, 1]^q$ are respectively called the inner and outer boundary of M^q. Denote the outer boundary of M^q by S^{q-1}.

Proof of Theorem 2.1. In view of Examples 1.1 and 1.2 we only have to prove the theorem for one particular norm and one sphere, and only for sign maps with an image in that sphere. There is a norm on $\mathbb{R}^q$ such that its unit sphere is the outer boundary S^{q-1} of M^q. So assume that $f, g: S^{q-1} \to S^{q-1}$ are homotopic sign maps with different sign, say: f is not a 0 sign map. We may assume that $f((1, 0, \ldots, 0)) = (1, 0, \ldots, 0)$. To this end, let p denote the projection from S^{q-1} onto the Euclidean $(q-1)$ sphere of radius one (cf. Example 1.1). If H is a homotopy connecting f and g then $p \circ H \circ p^{-1}$ is a homotopy connecting the sign maps $p \circ f \circ p^{-1}$ and $p \circ g \circ p^{-1}$ with different signs and $\operatorname{sign}(p \circ f \circ p^{-1}) \neq 0$. As a rotation on the Euclidean sphere is a + sign function, we can use such a function to obtain the desired situation (cf. Example 1.2).

Let $\delta_1 > 0$ (for f) and $\delta_2 > 0$ (for g) be as in the definition of a sign map. Now a homotopy connecting f and g induces a function $F: M^q \to S^{q-1}$, which equals f on the outer boundary and sends x in the inner boundary to $g(2x)$.

Let A be the face $\{x \in S^{q-1} \mid x_1 = 1\}$ of M^q, $b := (1, 0, \ldots, 0) \in A^0$ and $\alpha := d(b, S^{q-1} \backslash A)$ (in which d is the Euclidean norm on $\mathbb{R}^q$). Choose $\delta > 0$ such that: $\forall x, y \in M^q$: $d(x, y) < \delta \to d(F(x), F(y)) < \frac{1}{2}\alpha$. Take a triangulation D of M^q with $\operatorname{mesh}(D) < \delta, \delta_1, \frac{1}{2}\delta_2$ which has b as a vertex.

Let F^* be the "affine approximation" of F on D (i.e. $F^*(v) = F(v)$ for vertices in D; extend affinely on simplices in D). Note that $F^*(b) = b$ as b is a vertex of D and that in general the image of F^* is not in S^{q-1}. It can be easily seen that F^* has the following property on the inner and outer boundary.

(I) If B is a $(q-1)$-simplex of D contained in the outer boundary (respectively inner boundary) and $\{x_0, \ldots, x_{q-1}\}$ are independent (affine and linear dependence are equivalent here) in B, then

$$\operatorname{sign}\det(x_0, \ldots, x_{q-1}) \cdot \operatorname{sign}\det(F^*(x_0), \ldots, F^*(x_{q-1}))$$
$$= \operatorname{sign}(f) \text{ (respectively } \operatorname{sign}(g)).$$

Let B be a $(q-1)$-simplex of D contained in the outer boundary with $b \in B$ and E the unique q-simplex in D which has B as a face. Note that B must be contained in A.

Now consider sequences $(B_i, E_i, e_i)_{i=0}^{i=m}$ with B_i a $(q-1)$-face of E_i, E_i a q-simplex in D, $B_0 = B$, $E_0 = E$ with the following properties (see Fig. 1):

(1) $E_i \cap E_{i-1} = B_i \; \forall i > 0$;

(2) $\forall i \leq m \; e_i \in B_i^0$: $F^*(e_i) = F^*(e_0)$;

(3) all e_i are different.

As D is finite and as such a sequence exists, at least for $m = 0$ (take any $e_0 \in B_0^0$) there is a sequence of maximal length $(B_i, E_i, e_i)_{i=0}^{i=n}$.

Now let e be a vertex of E_i, we have:

$$d(F^*(e), b) \leq d(F^*(e), F^*(e_i)) + d(F^*(e_i), b)$$
$$= d(F^*(e), F^*(e_i)) + d(F^*(e_0), F^*(b)) < \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha = \alpha.$$

Therefore $F^*(E_i) \subset A$. By (I) and as $\operatorname{sign}(f) \neq 0$ we conclude that $(F^*|_{E_0}, B_0, A)$ has the key-property. As $e_1 \in B_1^0$ the remark following the key-lemma gives that $(F^*|_{E_0}, B_1, A)$ also has the key-property. Because $F(E_1) \subset A$ it then follows that $(F^*|_{E_1}, B_1, A)$ has the key-property. As all $e_i \in B_i^0$ and $F(E_i) \subset A$ we can repeat this argument and we have:

$(F^*|_{E_i}, B_i, A)$ *has the key-property* $\forall i \leq n$.

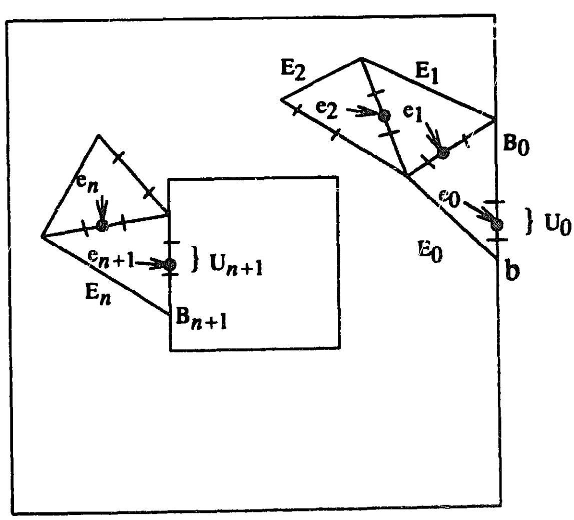

Fig. 1.

Now let g_i for $i=0,1,\ldots,n$ be the functions $B_i^0 \to \partial E_i \backslash B_i$ as described in the key-lemma. According to (3)(a) of the key-lemma there are convex open sets $U_i \subset B_i^\circ$ containing e_i for $i=0,1,\ldots,n$ such that for $i<n$ g_i restricted to U_i is an affine isomorphism on U_{i+1}. Furthermore according to (1) of the key-lemma there is a $(q-1)$-face $B:=B_{n+1}$ in E_n different from B_n in which an element $e:=g_n(e_n)$ with $F^*(e_n)=F^*(e)$. This e need not be in B_{n+1}°, but according to (2) of the key-lemma combined with (3)(a) there are points e_n^*, e_{n+1}^* and convex open sets U_n^*, U_{n+1}^* with $e_n^* \in U_n^* \subset U_n$ and $e_{n+1}^* \in U_{n+1}^* \subset B_{n+1}^\circ$ such that $e_{n+1}^* := g_n(e_n^*)$ and g_n restricted to U_n^* is an affine isomorphism onto U_{n+1}^*. For $i<n$ define points e_i^* and convex open $U_i^* \subset U_i$ by taking pre-images under the g_i (so $e_{n-1}^* := g_{n-1}^{-1}(e_n^*)$ and $U_{n-1}^* := g_{n-1}^{-1}(U_n^*)$ etc.). So we have convex open sets U_i^* and points $e_i^* \in U_i^* \subset B_i$ for $i=0,1,\ldots,n+1$ with $e_{i+1}^* = g_i(e_i^*)$ and such that the g_i restricted to U_i^* are affine isomorphisms onto U_{i+1}^*. Note that all e_i^* remain different for all $i \leqslant n$ and that at least $e_{n+1}^* \neq e_n^*$. Take a $(q-1)$-simplex $(a_0,\ldots,a_{q-1})$ in U_0^*. Let $\sigma_1 := (g_0(a_0),\ldots,g_0(a_{q-1}))$ be the pointwise image. According to (3)(b) of the key-lemma this is also an (ordered) $(q-1)$-simplex in B_1. Define $\sigma_i \subset U_i^*$ for $i=1,2,\ldots,n+1$ analogously. Denote σ_{n+1} by $(b_0,\ldots,b_{q-1})$. By construction we have that

(II) $\quad (F^*(a_0), F^*(a_1),\ldots,F^*(a_{q-1})) = (F^*(b_0), F^*(b_1),\ldots,F^*(b_{q-1}))$

(equality as *ordered* simplices). (3)(c) of the key-lemma states that $\operatorname{sign}(\sigma_0, E_0)$ and $\operatorname{sign}(\sigma_1, E_0)$ are opposite. As E_0 and E_1 lie on opposite sides of B_1 we conclude that $\operatorname{sign}(\sigma_0, E_0)$ and $\operatorname{sign}(\sigma_1, E_1)$ are equal. Using this argument repeatedly it follows that for $i=0,1,\ldots,n$ $\operatorname{sign}(\sigma_0, E_0)$ and $\operatorname{sign}(\sigma_{i+1}, E_i)$ are opposite, in particular

(III) $\quad \operatorname{sign}(\sigma_0, E_0)$ and $\operatorname{sign}(\sigma_{n+1}, E_n)$ are opposite.

Now we claim that B_{n+1}, and hence also σ_{n+1}, has to be in either the outer- or inner boundary. This would finish the proof. Indeed, if σ_{n+1} is in the outer boundary then the origin is at the good side of σ_{n+1} so (III) implies that $\det(a_0, a_1,\ldots,a_{q-1})$ and $\det(b_0, b_1,\ldots,b_{q-1})$ have different sign. This, together with (II) contradicts (I).

If σ_{n+1} is the inner boundary then the origin is not at the good side of σ_{n+1} so (III) implies that $\det(a_0, a_1,\ldots,a_{q-1})$ and $\det(b_0, b_1,\ldots,b_{q-1})$ have the same sign. This, together with (II), also contradicts (I), and the theorem is proven.

So let us prove the claim. If B_{n+1} is not in the inner or outer boundary we can define E_{n+1} as the unique q-simplex in D *different* from E_n such that $E_n \cap E_{n+1} = B_{n+1}$. We show that the sequence $(B_i, E_i, e_i^*)_{(i=0,1,\ldots,n+1)}$ also has properties (1), (2), (3) contradicting the maximality of the original sequence.

(1) and (2) follow by construction.

As for (3), we only have to show that $e_{n+1}^* = e_i^*$ implies $i=n+1$. Assume $i \neq n+1$. By the construction of e_{n+1}^* it follows that $i<n$. Now $e_i^* \in B_i^0 \cap B_{n+1}^0$ so as D is a triangulation we have $B_i = B_{n+1}$. It turns out that E_n, E_{i-1} and E_i have one $(q-1)$-simplex as a common face, namely B_{n+1}, we conclude from the fact that D is a triangulation of an annulus in Euclidean space that $E_n = E_{i-1}$ or $E_n = E_i$ or $E_i = E_{i-1}$.

The first case (1) of the key-lemma gives $e_{i-1}^* = e_n^*$, a contradiction. The other cases give similar contradictions. □

3. Consequences of the main theorem

Most of the following corollaries are classical.

Corollary 3.1. *A sign map $S_R^{q-1} \to \mathbb{R}^q \setminus \{0\}$ of nonzero sign is not nullhomotopic.*

Corollary 3.2 (Brouwer). *The identity $S_R^{q-1} \to S_R^{q-1}$ is not nullhomotopic.*

Corollary 3.3. *If the restrictions to S_R^{q-1} of two nonsingular linear maps $\mathbb{R}^q \to \mathbb{R}^q$ to S^{q-1} are homotopic, then the determinants of these maps have the same sign.*

Corollary 3.4. *In an even-dimensional sphere the identity and the antipodal map $(x \to -x)$ are not homotopic.*

Corollary 3.5. *The nth power map $S_R^1 \to S_R^1$, $z \to z^n$ is not nullhomotopic for $n \neq 0$.*

Corollary 3.6. *The map $S_R^1 \to \mathbb{C} \setminus \{0\}$, $z \to az^n + b\bar{z}^n$ is not nullhomotopic if $|a| \neq |b|$ and $n \neq 0$.*

Corollary 3.7. *A polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ in $\mathbb{C}$ of degree $n \geqslant 1$ has a root in $\mathbb{C}$.*

Proofs. Corollaries 3.1 to 3.6 directly follow from the main theorem and the examples in Section 1.

For a proof of Corollary 3.7: According to Example 1.7 there is a $R > 0$ such that p restricted to S_R^1 is a + sign map. That restriction can therefore not be nullhomotopic by Corollary 3.2(1). Then p must have roots in $\{z \in \mathbb{C} \,|\, |z| \leqslant R\}$. If not then $H(x, t) := p(t \cdot x)$ would define a nullhomotopy, a contradiction. □

Note that formula (2) gives a way to effectively find a radius for a circle inside of which must be a zero. In general the estimated circle need not contain *all* zeros of f. For example one can easily deduce that for $(2z-1)(z+9)$ the radius $\frac{3}{4}$ satisfies the condition 1.7(c), whereas -9 is a zero outside this circle. This yields an important difference with the work of Dodson in [1], who proved that a polynomial as in Corollary 3.7 must have a zero within the circle with radius $R = \max\{1, |a_n|^{-1}(a_{n-1}| + |a_{n-2}| + \cdots + |a_0|)\}$. This bound can be obtained directly. Indeed as

$$\left| \sum_{j=0}^{n} a_j z^j \right| \geqslant |a_n z^n| - \sum_{j=0}^{n-1} |a_j z^j| = |z|^{n-1} \left(|a_n| |z| - \sum_{j=0}^{n-1} |a_j| |z|^{j-n+1} \right)$$

and $R \geqslant 1$ we have for $|z| > R$ that first of all

$$|p(z)| > |a_n|\left(R - \sum_{i=0}^{n-1} |a^{i-n+1}|\right).$$

Furthermore as $R \geqslant |a_n|^{-1}(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|)$ the right side of the last inequality is non-negative. Hence *all* zeros of f have to be within the circle with radius R.

By Example 1.7 we obtain that a complex polynominal p restricted to S^1_R is a + sign map if we take $R > 0$ large enough. The following theorem gives more specific information.

Theorem 3.8. *Let p be a polynominal of degree $n \geqslant 1$ and d a zero of p of maximal modulus $|d|$. Then the restriction of p to S^1_R is a + sign map for every $R > |d|$.*

Proof. Without loss of generality, we may assume that the coefficient of the highest power of p equals 1. By "the fundamental theorem of algebra" (cf. Corollary 3.7) we can write p as

$$p(z) := (z - d_1)(z - d_2) \cdots (z - d_n).$$

Where $d_1, \ldots, d_n$ are all zeros of p. We aim at the application of Example 1.7(a) so we have to prove that for every $R > |d|$ and z with $|z| = R$ we have

$$\mathrm{Re}[z \cdot p'(z) \cdot \overline{p(z)}] > 0.$$

Using the above form of p one can easily see that

$$\mathrm{Re}[z \cdot p'(z) \cdot \overline{p(z)}] = \sum_{i=1}^{n} \mathrm{Re}[z \cdot \overline{(z - d_i)}] \cdot \prod_{j \neq i} |z - d_j|^2. \qquad (*)$$

Now for z with $|z| = R$ one can easily show that $\mathrm{Re}(z \cdot \overline{(z - d_i)}) \geqslant R(R - |d_i|) > 0$. Applying this inequality to the right-hand side of $(*)$ finishes the proof. □

Let p be a polynominal in $\mathbb{C}$. The previous theorem shows that the sign of $\mathrm{Re}[z \cdot p'(z) \cdot \overline{p(z)}]$ gives information about the position of the zeros of p. In fact, if one takes a "fine" collection of points in a large ball of the complex plane and marks the points were $\mathrm{Re}[z \cdot p'(z) \cdot \overline{p(z)}]$ is negative, then one gets a good idea of the position of the zeros of p. I do not know whether one can construct an algorithm with this procedure.

To obtain further corollaries to the results in Corollaries 3.1 to 3.6, we need the following proposition.

Proposition 3.9. *Let $g : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ and let $f : \mathbb{C} \to \mathbb{C}$ satisfy*

$$\lim_{|z| \to \infty} f(z) \cdot g(z) = c \neq 0.$$

Then the following are true.

(a) *There is a* $L>0$ *such that* $f(z)\neq 0$ *for* $|z|\geqslant L$.

(b) *The restrictions of* f *and* $1/g$ *to* S^1_L *are homotopic.*

Suppose $R>0$ *is such that* $g|_{S^1_R}$ *is not nullhomotopic. Then,*

(c) f *must have a zero in* $\{z\in\mathbb{C}\,|\,|z|<L\}$. *Also*

(d) *if* $\lim_{|z|\to\infty} g(z)=0$, *then* f *is surjective.*

Proof. Part (a) is clear. For part (b) we only have to prove that $f_{S^1_L}\cdot g_{S^1_L}$ is homotopic with the constant function c. Indeed, multiplying this homotopy with $(c\cdot g_{S^1_L})^{-1}$ finishes the proof. Define $H: S^1_L\times I\to\mathbb{C}\setminus\{0\}$

$$H(z,t)=\begin{cases} c, & \text{if } t=0,\\ f\left(\dfrac{z}{t}\right)\cdot g\left(\dfrac{z}{t}\right), & \text{if } 0<t\leqslant 1.\end{cases}$$

H is easily to be seen continuous at $(z,0)$, and is a homotopy as desired.

For part (c), first note that the fact that $g|_{S^1_R}$ is not nullhomotopic easily implies that $g|_{S^1_L}$ is not nullhomotopic. Consequently, the function $(1/g)|_{S^1_L}$ is not nullhomotopic either. Now by part (b) we conclude that $f|_{S^1_L}$ is not nullhomotopic. If f has no zeros in $\{z\in\mathbb{C}\,|\,|z|<L\}$ then $F(x,t):=f(tx)$ would be a homotopy connecting $f|_{S^1_L}$ and the constant $f(0)$.

(d) follows directly from (c) by looking at $f(z)-a$ for any $a\in\mathbb{C}$. □

A different formulation of the previous proposition is as follows.

Proposition 3.9′. *Let* $g:\mathbb{C}\setminus\{0\}\to\mathbb{C}\setminus\{0\}$ *and let* $f:\mathbb{C}\to\mathbb{C}$ *satisfy*

$$\lim_{|z|\to\infty}\frac{f(z)}{g(z)}=c\neq 0.$$

Then the following are true.

(a) *There is a* $L>0$ *such that* $f(z)\neq 0$ *for* $|z|\geqslant L$.

(b) $f|_{S^1_L}$ *is homotopic to* $g|_{S^1_L}$.

Suppose $R>0$ *is such that* $g|_{S^1_R}$ *is not nullhomotopic. Then,*

(c) f *must have a zero in* $\{z\in\mathbb{C}\,|\,|z|<L\}$. *Also*

(d) if $\lim_{|z|\to\infty} 1/g(z)=0$, *then* f *is surjective.*

Corollary 3.10 (to Proposition 3.9). *If a polynominal* p *of degree* $n\geqslant 1$ *has no zeros on the circle with radius* R, *then* $p|_{S^1_R}$ *is homotopic to* $z^k|_{S^1_R}$, *where* k *is the number of zeros in the circle with radius* R, *counting multiplicities.*

Proof. First assume that p has no roots in the circle with radius R. Then $p|_{S^1_R}$ is clearly nullhomotopic and the corollary is clear. Secondly, let $d_1, d_2, \ldots, d_k$ be all zeros of norm less than R (counting multiplicities). Now define g by, $g(z):=\prod_{j=1}^k (z-d_j)$. By Proposition 3.9′(b) we conclude that $g|_{S^1_R}$ is homotopic to $z^k|_{S^1_R}$. So we only have to prove that $g|_{S^1_R}$ is homotopic to $p|_{S^1_R}$, which is equivalent with $(p/g)|_{S^1_R}$ being nullhomotopic. This however, is trivial as the polynominal $p\cdot g$ has no zeros in the circle with radius R. □

One can easily modify Proposition 3.9 and Corollary 3.10 to spheres with other points than zero as a centre. For Corollary 3.10 this yields: if there are k zeros in the sphere around a point a, counting multiplicities, and no zeros on the sphere. Then the polynominal restricted to the sphere is homotopic with $(z-a)^k$ restricted to the sphere.

Corollary 3.11 (to Corollary 3.6 and Propositions 3.9 and 3.9′). *Let $|a|\neq|b|$ and $f:\mathbb{C}\to\mathbb{C}$.*

(a) *Suppose,*

$$\lim_{|z|\to\infty} f(z)\cdot(az^n+b\bar{z}^n) \text{ exists and is nonzero.}$$

Then f is surjective if $n<0$ and f has a zero if $n>0$.

(b) *Suppose,*

$$\lim_{|z|\to\infty} \frac{f(z)}{az^n+b\bar{z}^n} \text{ exists and is nonzero.}$$

Then f is surjective if $n>0$ and f has a zero if $n<0$.

In [1] only part (b) for $n>0$ was mentioned. This corollary gives another proof of the fundamental theorem of algebra. Using Corollary 3.11(b) one can deduce more theorems concerning polynominals. For example, a "harmonic" polynominal:

$$a_nz^n+\cdots+a_1z^1+a_0+\cdots+a_{-1}\bar{z}^1+\cdots+a_{-n}\bar{z}^n$$

with $|a_{-n}|\neq|a_n|$ has a zero in $\mathbb{C}$.

Corollary 3.12 (to Corollary 3.3). *Let σ be a permutation of n elements and let*

$$r:S^{n-1}\to S^{n-1},\quad r(x_1,x_2,\ldots,x_n):=(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}).$$

If σ is an odd permutation, then

$$\forall f:S^{n-1}\to S^{n-1}\ \exists x\in S^{n-1}:\quad f(x)=x \quad or \quad f(x)=r(x).$$

Proof. Suppose not. From Corollary 3.3 it follows that the identity cannot be homotopic to r. However the function $H:S^{n-1}\times I\to S^{n-1}$ defined by

$$H(x,t)=\begin{cases}((1-2t)x-2t\cdot f(x))\cdot|(1-2t)x-2t\cdot f(x)|^{-1} & \text{if } 0\leq t\leq\frac{1}{2}\\ ((2t-1)\cdot r(x)-(2-2t)\cdot f(x))\cdot|(2t-1)r(x)-(2-2t)\cdot f(x)|^{-1} & \text{if } \frac{1}{2}\leq t\leq 1\end{cases}$$

is such a homotopy. □

The following corollary is classical.

Corollary 3.13 (to Corollary 3.4). *Let $\langle\cdot,\cdot\rangle$ be the standard inner product in $\mathbb{R}^{n+1}$. Then the following are equivalent.*

(a) *n is even.*

(b) *The identity and the antipodal map of S^n are not homotopic.*

(c) *There is no continuous tangent vectorfield on S^n with nonzero vectors.*

(d) $\forall f:S^n\to S^n\ \exists x\in S^n: f(x)=x$ *or* $f(x)=-x$.

Proof. (a)$\Rightarrow$(b) This is Corollary 3.4.

(b)$\Rightarrow$(c) If there were such a vectorfield then $F(x,t):=x\cos\pi t+v(x)\sin\pi t$ would be a homotopy between the identity and the antipodal map.

(c)$\Rightarrow$(d) Suppose there is a map f not satisfying (d). Then

$$v(x):=(f(x)-\langle x,f(x)\rangle x)\cdot|f(x)-\langle x,f(x)\rangle x|^{-1}$$

defines a continuous nonzero vectorfield.

(d)$\Rightarrow$(c) If v is a vectorfield as described in (c), then it cannot satisfy (d) because $\langle \pm x, x\rangle$ is nonzero.

(c)$\Rightarrow$(a) If n is odd, say $n=2m+1$, then

$$v:S^n\to S^n,\quad v(x_1,x_2,\ldots,x_{2m+2})=(-x_2,x_1,-x_4,x_3,\ldots,-x_{2m+2},x_{2m+1})$$

defines a map not satisfying (c). $\square$

Part (c) of the above corollary is known as the "hairy ball theorem".

From Corollary 3.13 it follows that the identity and the antipode map are homotopic on a odd dimensional sphere. On the Euclidean unit circle we can take for example the following homotopy:

$$H_1:S^1\times I\to\mathbb{R}^2\backslash\{0\},\quad H_1(x_1,x_2,t):=\begin{pmatrix}\cos\pi t & \sin\pi t\\ -\sin\pi t & \cos\pi t\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

For a general odd dimensional sphere the desired homotopy is given by

$$H_{2m+1}:S^{2m+1}\times I\to\mathbb{R}^{2m+2}\backslash\{0\},$$

$$H_{2m+1}((x_1,x_2,\ldots,x_{2m+2}),t)$$
$$=(H_1((x_1,x_2),t),H_1((x_3,x_4),t),\ldots,H_1((x_{2m+1},x_{2m+2}),t)).$$

Notice that for a fixed $t\in I$, $H^{2m+1}(x,t)$ is the restriction to S^{2m+1} of a linear map $\mathbb{R}^{2m+2}\to\mathbb{R}^{2m+2}$. This linear map is selfadjoint only if $t=0$ or $t=1$. This is not a coincidence as the following theorem shows.

Theorem 3.14. *There is no homotopy $H:S^{q-1}\times I\to\mathbb{R}^q\backslash\{0\}$ between the identity and the antipode map such that, for each t, $H(-,t)$ is the restriction to S^{q-1} of a selfadjoint linear map $\mathbb{R}^q\to\mathbb{R}^q$.*

Proof. We need the following: If $A:\mathbb{R}^n\to\mathbb{R}^n$ is a selfadjoint map and $\sup_{x\in S^{n-1}}\langle A(x),x\rangle=0$, then A is singular. Indeed, as the above supremum is in fact a maximum we can find a $w\in S^{n-1}$ such that $\langle A(w),w\rangle=0$. If $c:=A(w)$, then for every $\lambda>0$ we have $\langle A(w+\lambda c),w+\lambda c\rangle\leq 0$ and hence that $\langle A(w),A(w)\rangle\leq -\frac{1}{2}\lambda\langle A(c),c\rangle$. This implies that $A(w)=0$. Now assume that H is a homotopy as in the theorem, and that $H_0=\mathrm{id}$. Define

$$p:I\to\mathbb{R},\quad t\mapsto\sup_{x\in S^{n-1}}\langle H(x,t),x\rangle$$

Now p is continuous and as $p(0)=1$ and $p(1)=-1$, connectivity of I gives a $s \in I$ for which $p(s)=0$. Now H_s is a restriction to S^{n-1} of a selfadjoint map A but then the above gives that A is singular so there is a $x \in S^{n-1}$ such that $A(x)=H(x,s)=0$, contradiction. □

Note that the above theorem gives a proof that every selfadjoint map has an eigenvalue. Indeed if not then

$$H: S^{q-1} \times [-1,1] \to \mathbb{R}^q \setminus \{0\}, \quad H(x,t) := (1-|t|)A + t \cdot \mathrm{id}$$

would give a selfadjoint homotopy connecting the identity and the antipodal map.

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References

[1] M.M. Dodson, A Brouwer type coincidence theorem and the fundamental theorem of algebra, Canad. Math. Bull. 27 (4) (1984) 478–480.

[2] J. Dugundji, Topology (Allyn and Bacon, Boston, MA, 1960) 335–340.

[3] M. Hirsch, A proof of the nonretractibility of a cell into its boundary, Proc. Amer. Math. Soc. 14 (1963) 364.