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The bipanconnectivity and m-panconnectivity of the folded hypercube

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Abstract

The interconnection network considered in this paper is the folded hypercube that is an attractive variance of the well-known hypercube. The folded hypercube is superior to the hypercube in many criteria, such as diameter, connectivity and fault diameter. In this paper, we study the path embedding aspects, bipanconnectivity and m-panconnectivity, of the n-dimensional folded hypercube. A bipartite graph is bipanconnected if each pair of vertices x and y are joined by the bipanconnected paths that include a path of each length s satisfying $N - 1 \geq s \geq dist(x, y)$ and s-$dist(x, y)$ is even, where N is the number of vertices, and $dist(x, y)$ denotes the shortest distance between x and y. A graph is m-panconnected if each pair of vertices x and y are joined by the paths that include a path of each length ranging from m to $N - 1$. In this paper, we introduce a new graph called the Path-of-Ladders. By presenting algorithms to embed the Path-of-Ladders into the folded hypercube, we show that the n-dimensional folded hypercube is bipanconnected for n is an odd number. We also show that the n-dimensional folded hypercube is strictly $(n - 1)$-panconnected for n is an even number. That is, each pair of vertices are joined by the paths that include a path of each length ranging from $n - 1$ to $N - 1$; and the value $n - 1$ reaches the lower bound of the problem.

Keywords: Interconnection networks; Algorithms; Panconnectivity; Folded hypercubes

1. Introduction

In massively parallel MIMD systems, the interconnection network plays a crucial role in issues such as communication performance, hardware costs, potentialities for efficient algorithms and fault tolerant capabilities [19]. An interconnection network is usually represented by a graph where the vertices represent the nodes and the edges represent the links.

Various interconnection networks are proposed, thus the portability of an algorithm across these interconnection networks demonstrates considerable importance. That a host interconnection network can embed another guest interconnection network implies that the algorithms on the guest can be simulated on the host systematically. *Paths* and *cycles* are popular interconnection networks owing to their simple structures and low degree. Moreover, many

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parallel algorithms have been devised on them [21,23]. Many literatures have addressed how to embed cycles and paths into various interconnection networks [4,14].

A graph is *Hamiltonian* if it embeds a *Hamiltonian cycle* that contains each vertex exactly once [9]. In other words, that a graph is Hamiltonian implies that it embeds the maximal cycle. However, in the *resource-allocated systems*, each vertex may be allocated with or without a resource [5,10]. Thus, it makes sense to discuss how to join a specific pair of vertices with a *Hamiltonian path* in such systems. For example, let x and y be two vertices in a resource-allocated system, where the former and the latter are assigned with an input device and an output device, respectively. If we find a Hamiltonian path joining the pair of vertices, we can utilize the whole system to perform the systolic algorithm on a linear array [23]. No wonder that there are many researchers discussing the *Hamiltonian-connectivity* of various interconnection networks [8,24]. A graph is Hamiltonian-connected if there is a Hamiltonian path joining each pair of vertices.

On the other hand, to execute a parallel program efficiently, the size of the allocated cycle must accord with the problem size of the program [19]. Thus, many researchers study the problem of how to embed cycles of different sizes into an interconnection network. A graph is *pancyclic* if it embeds cycles of every length ranging from 3 to N, where N is the order of the graph [3]. A graph is *m-pancyclic* if it embeds cycles of every length ranging from m to N, where $3 \leq m \leq N$. The *bi-pancyclicity* is a restriction of the concept of pancyclicity to bipartite graphs. Note that a bipartite graph contains no odd cycle [9]. A bipartite graph is bipancyclic if it embeds cycles of every even length ranging from 4 to N [20]. In a *heterogeneous computing system*, each vertex and each edge may be assigned with distinct computing power and distinct bandwidth, respectively [29]. Thus, it is meaningful to extend the concept of the pancyclicity to the *vertex-pancyclicity* and the *edge-pancyclicity* [1,13,16]. A graph is vertex-pancyclic (edge-pancyclic) if each vertex (edge) lies on a cycle of every length ranging from 3 to N.

To join each pair of vertices with flexible lengths, the concept of the Hamiltonian-connectivity is also extended to the *panconnectivity*. A graph G is *panconnected* if every two distinct vertices x, y of G are joined by a path of every length ranging from the shortest distance between x and y to $N - 1$ [8]. A graph is *m-panconnected* if each pair of vertices x and y are joined by the m-panconnected paths that include a path of each length ranging from m to $N-1$. Clearly, every m_1-panconnected graph must be m_2-panconnected, where $N-1 \geq m_2 \geq m_1$. A graph is *strictly m-panconnected* if it is not $(m - 1)$-panconnected but m-panconnected; that is, the value m reaches the lower bound of the problem. By definition, every m-panconnected graph is Hamiltonian-connected, where $N - 1 \geq m \geq 1$. The *bipanconnectivity* is a restriction of the concept of the panconnectivity to bipartite graphs [20]. A graph G is bipartite if the vertex set of G can be partitioned into two vertex subsets V_1 and V_2 such that each edge of G joins one vertex in V_1 and the other in V_2. Clearly, if x and y reside in the same vertex subset of a bipartite graph, there exists no path of an odd length joining x and y. On the other hand, if x and y reside in distinct subsets, there exists no path of an even length joining x and y. Thus, a bipartite graph is said to be bipanconnected if there exists a path of length s joining an arbitrary pair of vertices x and y for each $dist(x, y) \leq s \leq N - 1$ and s-$dist(x, y)$ is even, where $dist(x, y)$ is the distance between x and y. These paths are called the *bipanconnected paths*.

The *n-dimensional hypercube*, denoted by the $H(n)$, is an attractive interconnection network in both theoretical interests and practical systems [25]. It gains many nice properties, such as high degree of regularity, symmetry, fault tolerance, simple routing and broadcasting [19]. Many mathematicians also regard it as an interesting graph and investigate its mathematical issues [7,18]. On the other hand, many practical computer systems, e.g. NCUBE family [22], the Cosmic Cube [26], iPSC, iPSC/2 [17], the Symult S-series [2] and the Connection machines [28], employ the hypercubes as the interconnection networks.

The *n-dimensional folded hypercube*, denoted by the $FH(n)$, is an interesting variance of the $H(n)$ [12]. The $FH(n)$ is constructed by appending a *complementary edge* to every pair of vertices with complementary addresses. Due to the complement edge, the folded hypercube is superior to the hypercube in many criteria. For example, the diameter of the n-dimensional hypercube is n, whereas the diameter of the n-dimensional folded hypercube is improved to be $\lceil n/2 \rceil$ [12]. The n-dimensional hypercube is n-regular and n-connected, whereas the n-dimensional folded hypercube is improved to be $(n + 1)$-regular and $(n + 1)$-connected [12]. Because it demonstrates many attractive properties, researchers have devoted themselves to various issues of the folded hypercube, such as embedding algorithms [11,15], fault tolerance [30] and robustness [27]. Xu and Ma have discussed how to embed cycles in the folded hypercube [31]. From the result of Xu and Ma, we can directly know that the n-dimensional folded hypercube is bipancyclic for n is an odd number and n-pancyclic for n is an even number. To the best of our knowledge, there is no literature addressing the panconnectivity of the folded hypercube. In this paper, we show that the n-dimensional folded hypercube is

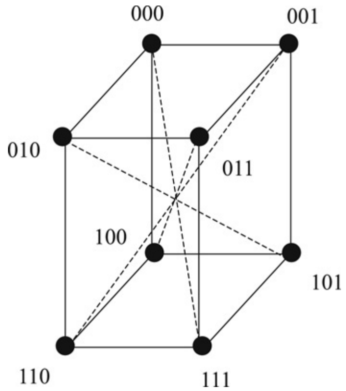

Fig. 1. The structure of the $FH(3)$.

bipanconnected for n is an odd number. We also show that the n-dimensional folded hypercube is strictly $(n-1)$-panconnected for n is an even number. That is, each pair of vertices are joined by the paths that include a path of each length ranging from $n-1$ to $N-1$; and the value $n-1$ reaches the lower bound of the problem.

The rest of this paper is organized as follows. In Section 2, we present some notations and background that will be used throughout this paper. By the algorithmic approach, we study the properties of the paths of the $FH(n)$ without and with complementary edge in Sections 3 and 4, respectively. In Section 5, we discuss the *bipanconnectivity* and the m-panconnectivity of the folded hypercube. Conclusions are given in Section 6.

2. Notations and background

For the definition of the hypercube, the *Cartesian product* of graphs is defined as follows:

Definition 1. Given two graphs $G = (V_G, E_G)$ and $F = (V_F, E_F)$, their Cartesian product, denoted by $G \times F$, is a graph (V_m, E_m), where $V_m = V_G \times V_F$ and $E_m = \{((x_1, y_1), (x_2, y_2)) | (x_1, y_1), (x_2, y_2) \in V_m$ and $((x_1 = x_2$ and $(y_1, y_2) \in E_F)$ or $(y_1 = y_2$ and $(x_1, x_2) \in E_G))\}$.

Definition 2. The $H(n)$ is defined recursively:

1. An $H(1)$ is a $K(2)$, where $K(2)$ is denoted for a complete graph with 2 vertices.
2. An $H(n)$ is $H(n-1) \times K(2)$ for $n \geq 2$.

That is, an $H(n)$ is a $K(2)^n$ which comprises 2^n vertices, each vertex x labelled by an n-bit number $v_n v_{n-1} \ldots v_2 v_1$. The vertex x is connected to another vertex y if and only if they differ by exactly one bit v_d, where $1 \leq d \leq n$; and (x, y) is called the *dimension d* edge of vertices x and y. The shortest distance of two vertices x and y in the $H(n)$ is $|x \oplus y|$, where $|\alpha|$ is denoted for the number of 1's in α and $\oplus$ is denoted for the *bitwise XOR operation* [25]. The vertex set of the $H(n)$ can be partitioned into two subsets $V_{\text{odd}}(H(n)) = \{x | x \in$ the vertex set of the $H(n)$ and $|x|$ is odd$\}$ and $V_{\text{even}}(H(n)) = \{x | x \in$ the vertex set of the $H(n)$ and $|x|$ is even$\}$. Clearly, each edge of the $H(n)$ joins a vertex in the $V_{\text{odd}}(H(n))$ and another vertex in the $V_{\text{even}}(H(n))$. Thus, the $H(n)$ is bipartite. A subcube can be represented by a string of n symbols over the set $\{0, 1, *\}$, where * is a "don't care" symbol. In this paper, the *outline graph* of an $H(n)$, denoted by the $OG(H(n))$, is to take each $v_n v_{n-1} \ldots v_2{}^*$ subnetwork as a supervertex. Owing to the recursive structure of an $H(n)$, if each $v_n v_{n-1} \ldots v_2{}^*$ subnetwork of an $H(n)$ is taken as a supervertex, the $H(n)$ will be transformed to an $H(n-1)$. We have the following proposition:

Proposition 1. *An $OG(H(n))$ is an $H(n-1)$.*

Definition 3. The $FH(n)$ is obtained by appending an edge $(x, \overline{x})$ to each pair of complementary vertices $x = v_n v_{n-1} \ldots v_2 v_1$ and $\overline{x} = \overline{v}_n \overline{v}_{n-1} \ldots \overline{v}_2 \overline{v}_1$ of the $H(n)$, where $\overline{\alpha}$ is denoted for the complement of α.

As illustrated in Fig. 1, the structure of the $FH(3)$ is shown. The vertex $\overline{x} = \overline{v}_n \overline{v}_{n-1} \ldots \overline{v}_2 \overline{v}_1$ is called the *complementary vertex* of the vertex $x = v_n v_{n-1} \ldots v_2 v_1$. Clearly, there are two classes of edges in the $FH(n)$.

Definition 4. A vertex x of the $H(n)$ and the $FH(n)$ is called odd (even) vertex if and only if $|x|$ is an odd (even) number.

Definition 5. The *hypercubic edges* (*h-edge*) of the $FH(n)$ are the edges of the original $H(n)$.

Definition 6. The *complementary edge* (*c-edge*) of the vertex $x = v_n v_{n-1} \ldots v_2 v_1$ is the edge $(x, \overline{x}) = (v_n v_{n-1} \ldots v_2 v_1, \overline{v}_n \overline{v}_{n-1} \ldots \overline{v}_2 \overline{v}_1)$ of the $FH(n)$.

Definition 7. A path containing a complementary edge is called a *complementary path* (*c-path* for short).

Definition 8. A path containing no complementary edge is called a *hypercubic path* (*h-path* for short).

Consider a pair of vertices x and y of the $FH(n)$. The shortest h-path joining x and y can be derived by correcting the bits that x differs from y hop by hop. Thus, we have

Proposition 2. *The length of the shortest h-path joining each pair of vertices x and y of the $FH(n)$ is $|x \oplus y|$.*

That a c-path contains two or more c-edges implies that it is not the shortest path; because it can be shortened by reducing a pair of c-edges. That is, the shortest c-path contains only one c-edge. By concatenating the edge $(x, \overline{x})$ and the shortest h-path joining $\overline{x}$ and y, the shortest c-path joining x and y can be derived. Thus, we have

Proposition 3. *The length of the shortest c-path joining each pair of vertices x and y of the $FH(n)$ is $1 + |\overline{x} \oplus y| = n + 1 - |x \oplus y|$.*

Combining Propositions 2 and 3, we have

Proposition 4. *The length of the shortest path joining x and y of the $FH(n)$ is $|x \oplus y|$ if $|x \oplus y| \le \lceil n/2 \rceil$ [12].*

Proposition 5. *The length of the shortest path joining x and y of the $FH(n)$ is $n + 1 - |x \oplus y|$ if $|x \oplus y| > \lceil n/2 \rceil$ [12].*

Lemma 1. *The $FH(n)$ is a bipartite graph if and only if n is an odd number.*

Proof. ($\leftarrow$) The vertex set of the $FH(n)$ can be partitioned into two subsets $V_{\text{odd}}(FH(n)) = \{x | x \in$ the vertex set of the $FH(n)$ and $|x|$ is odd$\}$ and $V_{\text{even}}(FH(n)) = \{x | x \in$ the vertex set of the $FH(n)$ and $|x|$ is even$\}$. Clearly, each h-edge of the $FH(n)$ joins a vertex in the $V_{\text{odd}}(FH(n))$ and another vertex in the $V_{\text{even}}(FH(n))$. By Definition 6, each c-edge joins a pair of vertices $(x, \overline{x})$. Clearly, $|x| + |\overline{x}| = n$. Since n is an odd number, one of them is in the $V_{\text{odd}}(FH(n))$ and the other is in the $V_{\text{even}}(FH(n))$.

($\rightarrow$) If there exists a bipartite graph $F(n)$ where n is an even number, there exists a shortest h-path p joining 0^n and 1^n with length n. Clearly, concatenating the path p and edge $(1^n, 0^n)$, an odd cycle of length $n+1$ can be derived; that contradicts the fact that a bipartite graph contains no odd cycle. $\square$

In this paper, the *outline graph* of an $FH(n)$, denoted by the $OG(FH(n))$, is to take each $v_n v_{n-1} \ldots v_2{}^*$ subnetwork as a supervertex. The supervertex $v_n v_{n-1} \ldots v_2{}^*$ is connected to the supervertex $u_n u_{n-1} \ldots u_2{}^*$ by an h-edge if and only if $v_n v_{n-1} \ldots v_2$ and $u_n u_{n-1} \ldots u_2$ differ by exactly one bit position; the supervertex $v_n v_{n-1} \ldots v_2{}^*$ is connected to the supervertex $u_n u_{n-1} \ldots u_2{}^*$ by a c-edge if and only if $u_n u_{n-1} \ldots u_2 = \overline{v}_n \overline{v}_{n-1} \ldots \overline{v}_2$. Thus, if each $v_n v_{n-1} \ldots v_2{}^*$ subnetwork of an $FH(n)$ is taken as a supervertex, the $FH(n)$ will be transformed to an $FH(n-1)$. We have the following proposition:

Proposition 6. *An $OG(FH(n))$ is an $FH(n-1)$.*

An embedding of a *guest graph* $G = (V_G, E_G)$ into a *host graph* $F = (V_F, E_F)$ is a mapping $\phi : G \to F$ comprising two mappings ϕ_V: $V_G \to V_F$ and ϕ_E: $E_G \to Ph(F)$, where the $Ph(F)$ denotes the set of paths in the graph F. The mapping ϕ_E maps each edge $(x_1, x_2) \in E_G$ to a path $p \in Ph(F)$ such that p joins $\phi_V(x_1)$ and $\phi_V(x_2)$. The *dilation* of an edge $e \in E_G$ under the embedding ϕ is the length of the path $\phi_E(e)$. The dilatation of the embedding ϕ is the maximal dilatation of all edges in G. In this paper, we only consider the embeddings with dilatation 1. The ratio $|V_F|/|V_G|$ is called the *expansion* of the embedding. Clearly, that an embedding is called expansion 1 implies that the guest graph and the host graph have the same number of vertices.

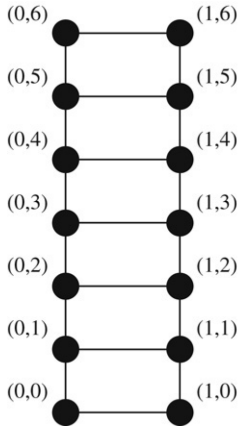

Fig. 2. The structure of an $LD(6)$.

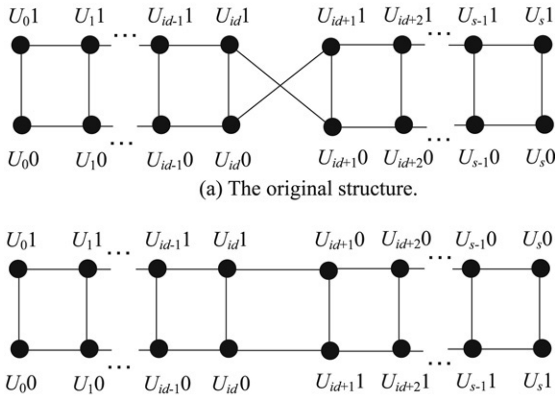

Fig. 3. Embedding the $L(s)$ into the $FH(n)$.

A path of length s is denoted by a $P(s)$; and a cycle of length s is denoted by a $C(s)$. A ladder of length s, denoted by an $L(s)$, is a $P(s) \times K(2)$. Each vertex of an $L(s)$ is labelled by (b_0, b_1), where $b_0 = 0$ or $b_0 = 1$, and $0 \le b_1 \le s$. Each edge $((0, b_1), (1, b_1))$ is called a *rung* of the ladder $L(s)$, where $0 \le b_1 \le s$. Specifically, it is called the b_1th rung. The 0th rung is called the *bottom rung* of the ladder. The two paths $((0, 0), (0, 1), \dots, (0, s))$ and $((1, 0), (1, 1), \dots, (1, s))$ are called the *bands* of the $L(s)$. Specifically, the former is called the 0th band and the latter is called the 1st band. As shown in Fig. 2, the structure of an $L(6)$ is illustrated. Clearly, a path of length $2l + 1$, $((0, 0), (0, 1), \dots, (0, l), (1, l), \dots, (1, 1), (1, 0))$, can be embedded in an $L(s)$, where $0 \le l \le s$.

Proposition 7. *There exists a path of each odd length ranging from* 1 *to* $2s + 1$ *joining* $(0, 0)$ *and* $(1, 0)$ *in an* $L(s)$.

In the supervertex $U^* = v_n v_{n-1} \dots v_2{*}$, the vertex $v_n v_{n-1} \dots v_2 v_1$ is said to be the v_1 vertex of the U^*. By the structure of the $H(n)$, vertex v_1 of U^* and vertex v_1 of W^* are connected if and only if U^* and W^* are connected in the $OG(H(n))$. Clearly, if the $OG(H(n))$ embeds a path of the supervertices with length s, $(U_0{*}, U_1{*}, U_2{*}, \dots, U_s{*})$, the $H(n)$ embeds the structure of $P(s) \times K(2)$ that is an $L(s)$. Likewise, if the $OG(FH(n))$ embeds a h-path of the supervertices with length s, $(U_0{*}, U_1{*}, U_2{*}, \dots, U_s{*})$, the $FH(n)$ also embeds the structure of an $L(s)$. On the other hand, if the $OG(FH(n))$ embeds a path of the supervertices with length s, $(U_0{*}, U_1{*}, U_2{*}, \dots, U_{id}{*}, U_{id+1}{*}, \dots, U_s{*})$, where $(U_{id}{*}, U_{id+1}{*})$ is a c-edge, the $FH(n)$ embeds the structure illustrated in Fig. 3(a) that can be redrawn in Fig. 3(b); that is, it is also an $L(s)$. Similarly, that a $P(s)$ of the $OG(FH(n))$ contains multiple c-edges also implies that the $FH(n)$ embeds an $L(s)$. Thus, we have

Proposition 8. *If the* $OG(FH(n))$ *embeds a* $P(s)$*, the* $FH(n)$ *embeds an* $L(s)$*.*

Definition 9. An *automorphism* of a graph $G = (V, E)$ is a permutation σ on V such that for each (x, y) in E if and only if $(\sigma(x), \sigma(y))$ is also in E [19].

Definition 10. A graph $G = (V, E)$ is *vertex transitive* if for each pair of vertices x, y in V, there exists an automorphism of G that maps x to y [19].

Definition 11. A graph $G = (V, E)$ is *edge transitive* if for each pair of edges (x_1, y_1) and (x_2, y_2) in E, there exists an automorphism of G that maps (x_1, y_1) to (x_2, y_2) [19].

Informally, a vertex(edge) transitive graph looks the same when viewed from each vertex(edge). A graph possesses vertex transitivity or edge transitivity implies that it is a symmetric graph to a high degree. In fact, the $FH(n)$ is vertex transitive. Let $x = u_n u_{n-1} \dots u_2 u_1$ and $y = v_n v_{n-1} \dots v_2 v_1$ be two arbitrary vertices of the $FH(n)$, we can relabel each vertex $w_n w_{n-1} \dots w_2 w_1$ as $(w_n \oplus u_n \oplus v_n)|\ (w_{n-1} \oplus u_{n-1} \oplus v_{n-1})| \dots |\ (w_2 \oplus u_2 \oplus v_2)|\ (w_1 \oplus u_1 \oplus v_1)$ to map $u_n u_{n-1} \dots u_2 u_1$ to $v_n v_{n-1} \dots v_2 v_1$, where $\alpha|\beta$ is denoted for the concatenation of α and β. It is easy to see that the mapping is an automorphism on the $FH(n)$; and we have

Proposition 9. *The $FH(n)$ is vertex transitive.*

From the above relabelling, the vertex $\overline{u}_n \overline{u}_{n-1} \dots \overline{u}_2 \overline{u}_1$ will be mapped onto $(\overline{u}_n \oplus u_n \oplus v_n)|\ (\overline{u}_{n-1} \oplus u_{n-1} \oplus v_{n-1})| \dots |\ (\overline{u}_2 \oplus u_2 \oplus v_2)|\ (\overline{u}_1 \oplus u_1 \oplus v_1) = \overline{v}_n \overline{v}_{n-1} \dots \overline{v}_2 \overline{v}_1$. In other words, it maps the complementary edge of x, the edge $(x, \overline{x})$, to the complementary edge of y, the edge $(y, \overline{y})$. Thus, we have

Proposition 10. *For any pair of complementary edges $(x, \overline{x})$ and $(y, \overline{y})$ of the $FH(n)$, there exists an automorphism σ on the $FH(n)$, such that $(\sigma(x), \sigma(\overline{x})) = (y, \overline{y})$.*

For each permutation π on $\{1, 2, \dots, n\}$, we can relabel each vertex $w_n w_{n-1} \dots w_2 w_1$ of the $FH(n)$ as $w_{\pi(n)} w_{\pi(n-1)} \dots w_{\pi(2)} w_{\pi(1)}$. Clearly, the mapping is an automorphism on the $FH(n)$. Combining the Proposition 9, we know

Proposition 11. *For any pair of h-edges (x_1, x_2) and (y_1, y_2) of the $FH(n)$, there exists an automorphism σ on the $FH(n)$, such that $(\sigma(x_1), \sigma(x_2)) = (y_1, y_2)$.*

A *binary Gray code* of length n is an ordered sequence of 2^n n-bit code words. It is a permutation from 0 to $2^n - 1$ such that successive code words differ by exactly one bit. A *reflected Gray code* with length n, denoted by $RG(n)$, and the rth code word in the reflected Gray code, denoted by $g(n, r)$ are defined as follows [6]:

$$RG(1) = \{0, 1\}$$

and let

$$RG(k) = \{g(k, 0), g(k, 1), \dots, g(k, 2^k - 1)\}$$

then

$$RG(k + 1) = \{0g(k, 0), 0g(k, 1), \dots, 0g(k, 2^k - 1), 1g(k, 2^k - 1), \dots, 1g(k, 1), 1g(k, 0)\}.$$

For example, a $RG(2) = \{00, 01, 11, 10\}$ and a $RG(3) = \{000, 001, 011, 010, 110, 111, 101, 100\}$. A $RG(n)$ can be represented by its *transition sequence* which is the ordered list of the bit positions that change as it proceeds from one code word to the next one. For example, a $RG(3)$ can be represented by $(1, 2, 1, 3, 1, 2, 1)$. We also use the transition sequence to represent a path in the $H(n)$ and the $FH(n)$. The complementary edge in the transition sequence of the $FH(n)$ is denoted as c.

Definition 12. A Path-of-Ladder is a graph unified by a *bone path BP* and *sl* ladders $LD(0), LD(1), \dots, LD(sl - 1)$ with $BR(0), BR(1), \dots, BR(sl - 1)$ as the bottom rungs, respectively, such that each $BR(i)$ is contained in the BC where $0 \le i \le sl - 1$ and $BR(0), BR(1), \dots, BR(sl - 1)$ disjoint each other.

As illustrated in Fig. 4, the structure of a path-of-ladders graph is shown, where $(x_0, x_1, x_2, x_3, x_4, x_5)$ is the bone path and (x_0, x_1), (x_2, x_3) and (x_4, x_5) are the $BR(0)$, $BR(1)$ and $BR(2)$, respectively.

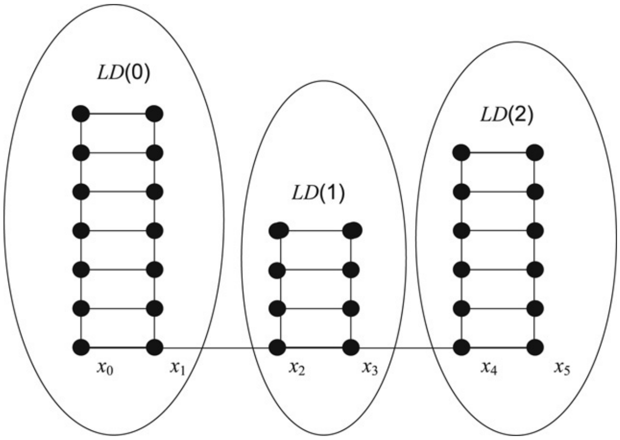

Fig. 4. The structure of a Path-of-Ladders graph.

3. The *h*-paths

In this section, we study the h-path of the folded hypercube by the algorithmic approach. To investigate the h-path of the folded hypercube, the discussion about how to embed a ladder into the folded hypercube with expansion 1 is required.

Lemma 2. *An $L(s)$ is embedded into an $FH(n)$, where $s = 2^{n-1} - 1$.*

Proof. We map each vertex $(0, i)$ of the $L(s)$ to the vertex $g(n-1, i)0$ of the $FH(n)$; and each vertex $(1, i)$ of the $L(s)$ to the vertex $g(n-1, i)1$ of the $FH(n)$, where $0 \le i \le 2^{n-1} - 1$. By the definition of the Gray codes, $g(n-1, i)0$ and $g(n-1, i+1)0$ is connected; and $g(n-1, i)1$ and $g(n-1, i+1)1$ is also connected. Moreover, $g(n-1, i)0$ and $g(n-1, i)1$ is connected. This completes the proof. □

Clearly, the mapping of Lemma 2 includes no c-edges. According to Proposition 11, we know

Corollary 3. *An $L(2^{n-1} - 1)$ is embedded into the $FH(n)$ by an arbitrary h-edge of the $FH(n)$ as the bottom rung of the ladder.*

First, we consider two vertices x and y of the $FH(n)$, where $|x \oplus y| = 2a + 1$. Without loss of generality, we assume that $x = 0^n$ and $y = 0^{n-2a-1}1^{2a+1}$. Clearly, a path $(U_0 = x, U_1, \ldots, U_{2a+1} = y)$ can be derived by the transition sequence $(1, 2a+1, 2, 2a, \ldots, a+3, a, a+2, a+1)$. The $FH(n)$ can be decomposed into the subcube $(^*)^{n-2a-1}0(^*)^{2a}$ that contains the edge (U_0, U_1), the subcube $(^*)^{n-2a-1}10(^*)^{2a-1}$ that contains the edge (U_2, U_3), the subcube $(^*)^{n-2a-1}110(^*)^{2a-2}$ that contains the edge $(U_4, U_5), \ldots,$ the subcube $(^*)^{n-2a-1}(1)^{a-1}0(^*)^{a+1}$ that contains the edge (U_{2a-2}, U_{2a-1}) and the subcube $(^*)^{n-2a-1}(1)^a(^*)^{a+1}$ that contains the edge (U_{2a}, U_{2a+1}). From Corollary 3, we know that we can embed an $L(2^{n-k-2} - 1)$ with the edge (U_{2k}, U_{2k+1}) as the bottom rung in each corresponding subcube $(^*)^{n-2a-1}(1)^k0(^*)^{2a-k}$ for each $0 \le k < a$; and an $L(2^{n-k-1} - 1)$ with the edge (U_{2k}, U_{2k+1}) as the bottom rung in the corresponding subcube $(^*)^{n-2a-1}(1)^k(^*)^{k+1}$ for $k = a$. By applying the path joining $x = 0^n$ and $y = 0^{n-2a-1}1^{2a+1}$ with transition sequence $(1, 2a+1, 2, 2a, \ldots, a+3, a, a+2, a+1)$ in the $FH(n)$ as the bone path and concatenating the above ladders with the bone path, the Path-of-Ladders graph of 2^n vertices and with the bone path joining x and y, denoted by the $POL(x, y, 2^n)$, is derived.

Algorithm A_1. /* Generating the $POL(0^n, 0^{n-2a-1}1^{2a+1}, 2^n)$ with the bone path starting from 0^n along the transition sequence $(1, 2a+1, 2, 2a, \ldots, a+3, a, a+2, a+1)$ */
Input: a, n;
Output: Generate a ladder $L(2^{n-k-2} - 1)$ with the kth bottom rung $BT(k) = (0^{n-2a-1}1^k0^{2a-2k+1}1^k, 0^{n-2a-1}1^k0^{2a-2k}1^{k+1})$ in the kth subcube $SC(k) = (^*)^{n-2a-1}(1)^k0(^*)^{2a-k}$ for each $0 \le k < a$; and generate the kth ladder $LD(k)$ as an $L(2^{n-k} - 1)$ with the kth bottom rung $BT(k) = (0^{n-2a-1}1^k01^k, 0^{n-2a-1}1^{2k+1})$ in the kth subcube $SC(k) = (^*)^{n-2a-1}(1)^k(^*)^{k+1}$ for $k = a$. And concatenate the bottom rungs $BT(k)$, for each

$0 \leq k \leq a$, by the bone path starting from $x = 0^n$ to $y = 0^{n-2a-1}1^{2a+1}$ along the transition sequence $(1, 2a+1, 2, 2a, \ldots, a+3, a, a+2, a+1)$ to generate the $POL(0^n, 0^{n-2a-1}1^{2a+1}, 2^n)$.

Initialization: $k = 0$;

For ($k = 0$; $k < a$; $k = k+1$)

$BT(k) = (0^{n-2a-1}1^k0^{2a-2k+1}1^k, 0^{n-2a-1}1^k0^{2a-2k}1^{k+1})$.

$SC(k) = (^*)^{n-2a-1}(1)^k0(^*)^{2a-k}$.

Generate the kth ladder $LD(k)$ as an $L(2^{n-k-2}-1)$ with the bottom rung $BT(k)$ in the subcube $SC(k)$ by the reflected grey codes and the symmetrical properties of the folded hypercube.

/* If ($k \geq 1$) Connect the pair of vertices $0^{n-2a-1}1^k0^{2a-2k+1}1^k$ and $0^{n-2a-1}1^{k-1}0^{2a-2k+2}1^k$.

Endif */

Loop /* for loop */

$BT(a) = (0^{n-2a-1}1^a01^a, 0^{n-2a-1}1^{2a+1})$.

$SC(a) = (^*)^{n-2a-1}(1)^a(^*)^{a+1}$.

Generate the ath ladder $LD(a)$ as an $L(2^{n-a-1}-1)$ with the bottom $BT(a)$ in the subcube $SC(a)$ by the reflected grey codes and the symmetrical properties of the folded hypercube.

Concatenate the bottom rungs $BT(k)$, for each $0 \leq k \leq a$, by the path starting from $x = 0^n$ to $y = 0^{n-2a-1}1^{2a+1}$ with the transition sequence $(1, 2a+1, 2, 2a, \ldots, a+3, a, a+2, a+1)$ to generate the $POL(0^n, 0^{n-2a-1}1^{2a+1}, 2^n)$.

/* Connect the pair of vertices $0^{n-2a-1}1^a01^a$ and $0^{n-2a-1}1^{a-1}0^21^a$. */

/* End of Algorithm A_1*/

For example, let $a = 3, n = 9$. The subcube $SC(0) =$ **0******, and the bottom rung $BT(0) = (000000000, 000000001)$. Clearly, the $BT(0)$ resides in the $SC(0)$. The $SC(1) =$ **10*****, and the $BT(1) = (001000001, 001000011)$. The $SC(2) =$ **110****, and the $BT(2) = (001100011, 001100111)$. The $SC(3) =$ **111****, and the $BT(3) = (001110111, 001111111)$. Clearly, each bottom rung $BT(k)$ resides in corresponding subcube $SC(k)$, where $0 \leq k \leq 3$. Moreover, the vertex set of $SC(0) \cup SC(1) \cup SC(2) \cup SC(3)$ is equal to the vertex set of the $FH(9)$. By Corollary 3, each $LD(k)$ with the $BT(k)$ as the bottom rung is embedded into each subcube $SC(k)$ with expansion 1 for each $0 \leq k \leq 3$. In other words, the $LD(0)$ that is an $L(127)$, the $LD(1)$ that is an $L(63)$, the $LD(2)$ that is an $L(31)$ and the $LD(3)$ that is an $L(31)$, can be generated, respectively. Recall that in fact each $BT(k)$ is a pair of successive vertices in the path joining 0^n and $0^{n-2a-1}1^{2a+1}$. Thus, we can concatenate each $BT(k)$, where $0 \leq k < 3$, by the path joining 000000000 and 001111111 with the transition sequence $(1, 7, 2, 6, 3, 5, 4)$ to generate the $POL(0^9, 0^21^7, 512)$ as illustrated in Fig. 5.

By Proposition 7, the $BT(k)$ can be replaced by a path $Pt(k)$ of length $2l_k+1$ for each $0 \leq k \leq a$, the length of the path $(Pt(0), Pt(1), \ldots, Pt(a))$ will be $a + (2l_0+1) + (2l_1+1) + \cdots + (2l_a+1) = 2a+1+2(l_0+l_1+\cdots+l_a)$. Thus, for an arbitrary odd number $2L+1$, where $2a+1 \leq 2L+1 \leq 2^n-1$, if a set of $l_0, l_1, \ldots$, and l_a is chosen as $l_0+l_1+\cdots+l_a = L-a$, the length of the path $(Pt(0), Pt(1), \ldots, Pt(a))$ will be $2L+1$.

Algorithm A_2. /* By the $POL(0^n, 0^{n-2a-1}1^{2a+1}, 2^n)$, generating a path joining $x = 0^n$ and $y = 0^{n-2a-1}1^{2a+1}$ of each odd length ranging from $|x \oplus y|$ to 2^n-1, where $|x \oplus y|$ is odd. */

For an arbitrary odd length $2L+1$, where $|x \oplus y| \leq 2L+1 \leq 2^n-1$, a path of length $2L+1$ joining x and y can be generated as follows:

Step 1. By algorithm A_1, generate a series of ladders $LD(0), LD(1), \ldots, LD(a)$ and concatenate the bottom rungs $BT(0), BT(1), \ldots, BT(a)$ to construct the $POL(0^n, 0^{n-2a-1}1^{2a+1}, 2^n)$.

Step 2. In each $LD(k)$, for each $0 \leq k \leq a$, choose a path $Pt(k)$ of length $2l_k+1$ joining the two vertices of $BT(k)$ in $LD(k)$ such that $l_0+l_1+\cdots+l_a = L-a$.

Step 3. Concatenate $Pt(0), Pt(1), \ldots Pt(a)$ to derive the path of length $2(l_0+l_1+\cdots+l_a)+a+1+a = 2L+1$ joining x and y.

/* End of algorithm A_2 */

For example, let $a = 3, n = 9$. We explain how to use the Algorithm A_2 to join the pair of vertices $x = 000000000$ and $y = 001111111$ in the $FH(9)$ with a path of length 477. Because $2L+1 = 477$, $L = 238$. As stated above, a series

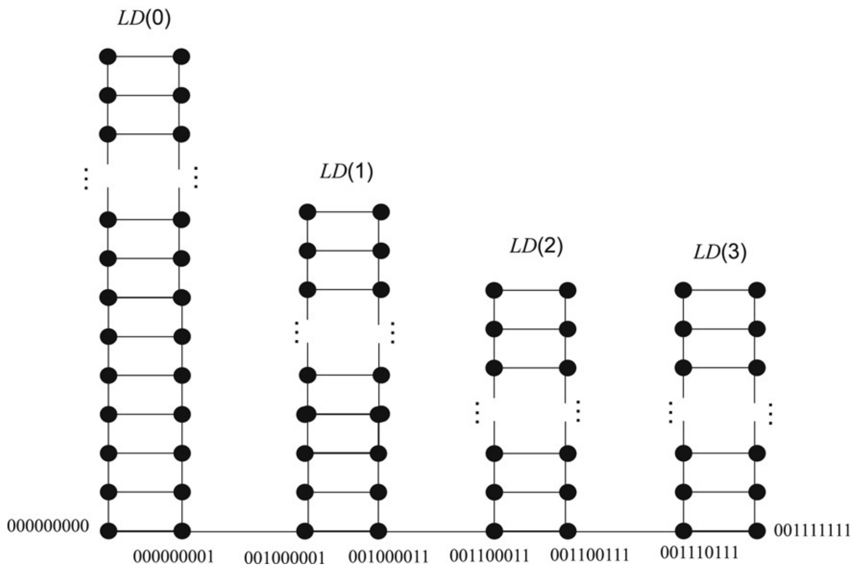

Fig. 5. Connecting $LD(0)$, $LD(1)$, $LD(2)$, $LD(3)$ by the path joining 0^9 and 0^21^7 with the transition sequence (1, 7, 2, 6, 3, 5, 4).

of ladders, the $LD(0)$ that is an $L(127)$, the $LD(1)$ that is an $L(63)$, the $LD(2)$ that is an $L(31)$ and the $LD(3)$ that is an $L(31)$, can be generated and unified as a $POL(0^9, 0^21^7, 512)$ by algorithm A_1. In each $LD(k)$, for each $0 \leq k \leq 3$, generate a path $Pt(k)$ of length $2l_k + 1$ joining the two vertices of $BT(k)$ such that $l_0 + l_1 + \cdots + l_a = L - a$. That is, l_0, l_1, l_2, and l_3 are chosen such that $l_0 + l_1 + l_2 + l_3 = L - a = 238 - 3 = 235$, where $1 \leq l_0 \leq 2^{n-2} - 1 = 127$, $1 \leq l_1 \leq 2^{n-3} - 1 = 63$, $1 \leq l_2 \leq 2^{n-4} - 1 = 31$, $1 \leq l_3 \leq 2^{n-4} - 1 = 31$. Let the set of l_0, l_1, l_2, and l_3 is chosen as 127, 63, 31 and 14, respectively. Concatenating $Pt(0)$, $Pt(1)$, $Pt(2)$ and $Pt(3)$, the path joining $x = 000000000$ and $y = 001111111$ with length $255 + 127 + 63 + 29 + 3 = 477$ is generated. Clearly, in Algorithm A_1 and Algorithm A_2, the $POL(0^n, 0^{n-2a-1}1^{2a+1}, 2^n)$ includes no c-edges. Thus, we state

Lemma 4. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a+1$, there exist h-paths joining x and y of each odd length ranging from $2a + 1$ to $2^n - 1$.*

Then, we consider the case of x and y, where $|x \oplus y| = 2a$. Without loss of generality, we assume that $x = 0^n$ and $y = 0^{n-2a}1^{2a}$. Recall that $OG(FH(n))$ is an $FH(n-1)$. Clearly, by Algorithm A_2 and Proposition 8, we can generate a path joining $0^{n-1}*$ and $0^{n-2a}1^{2a-1}*$ by a path of supervertices, $(0^{n-1}* = U_0{}^*, U_1{}^*, \ldots, U_{2h-1}{}^* = 0^{n-2a}1^{2a-1}*)$, of each odd length $2h - 1$ ranging from $2a - 1$ to $2^{n-1} - 1$ in the $OG(FH(n))$. Clearly, the $C(4)$, $(U_{2j}0, U_{2j}1, U_{2j+1}1, U_{2j+1}0)$, can be regarded as the j-th ladder $LD(j)$ of length 1 and with $(U_{2j}0, U_{2j+1}0)$ as the bottom for each $0 \leq j \leq h-1$, as illustrated in Fig. 6. In each $LD(j)$, where $0 \leq j \leq h-2$, we can choose a path $Pt(j)$ as $(U_{2j}0, U_{2j+1}0)$ or $(U_{2j}0, U_{2j}1, U_{2j+1}1, U_{2j+1}0)$. In the $LD(h-1)$, a path $Pt(h-1) = (U_{2h-2}0, U_{2h-2}1, U_{2h-1}1)$ is chosen. Thus, concatenating the $Pt(0), Pt(1), \ldots, Pt(h-1)$, a path joining 0^n and $0^{n-2a}1^{2a}$ with even length ranging from $2h$ to $4h-2$ can be generated. Because $2h - 1$ is ranging from $2a - 1$ to $2^{n-1} - 1$, $a \leq h \leq 2^{n-2}$. The paths joining 0^n and $0^{n-2a}1^{2a}$ contain

$$
\begin{aligned}
&\{P(s) | 2a \leq s \leq 4a - 2, s \text{ is even}\} \quad (\text{for } h = a)\\
&\cup \{P(s) | 2a + 2 \leq s \leq 4a + 2, s \text{ is even}\} \quad (\text{for } h = a + 1)\\
&\ldots,\\
&\cup \{P(s) | 2jr \leq s \leq 4jr - 2, s \text{ is even}\} \quad (\text{for } h = jr)\\
&\cup \{P(s) | 2jr + 2 \leq s \leq 4jr + 2, s \text{ is even}\} \quad (\text{for } h = jr + 1)\\
&\ldots,\\
&\cup \{P(s) | 2^{n-1} - 2 \leq s \leq 2^n - 6, s \text{ is even}\} \quad (\text{for } h = 2^{n-2} - 1)\\
&\cup \{P(s) | 2^{n-1} \leq s \leq 2^n - 2, s \text{ is even}\} \quad (\text{for } h = 2^{n-2})
\end{aligned}
$$

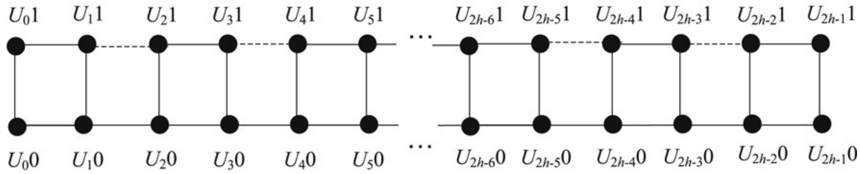

Fig. 6. Each $C(4)$, $(U_{2j}0, U_{2j+1}1, U_{2j+1}0, U_{2j}1)$, is regarded as an $L(1)$ with $(U_{2j}0, U_{2j+1}0)$ as the bottom, where $0 \leq j \leq h-1$.

Clearly, $4jr-2$ is always greater than or equal to $2jr+2$ for $jr \geq 2$. We investigate the case that $a = 1$ and $jr = a$. For $jr = 1$, the paths join 0^n and $0^{n-2a}1^{2a}$ contain $\{P(s)|2 \leq s \leq 2, s \text{ is even}\}$; and for $jr = 2$, the paths join 0^n and $0^{n-2a}1^{2a}$ contain $\{P(s)|4 \leq s \leq 6, s \text{ is even}\}$. Thus, the paths joins 0^n and $0^{n-2a}1^{2a}$ contains $\{P(s)|2a \leq s \leq 2^n-2, s \text{ is even}\}$.

Algorithm A_3. /* Generating a path joining $x = 0^n$ and $y = 0^{n-2a}1^{2a}$ of an $FH(n)$ with an arbitrary even length ranging from $|x \oplus y|$ to 2^n-2 */
For an arbitrary even length $2L$, where $|x \oplus y| = 2a \leq 2L \leq 2^n-2$, a path of length $2L$ joining x and y can be generated as follows:

Case 1: $2L < 2^{n-1}$.
Step 1. By Algorithm A_2, generating the path of supervertices, $(0^{n-1}* = U_0*, U_1*, \ldots, U_{2L-1}* = 0^{n-2a}1^{2a-1}*)$, in the $OG(FH(n))$.
Step 2. From the above path of supervertices in the $OG(FH(n))$, the path $(U_0 0, U_1 0, \ldots, U_{2L-1}0, U_{2L-1}1)$ of length $2L$ in the $FH(n)$ can be derived.
Case 2: $2L \geq 2^{n-1}$.
Step 1. By Algorithm A_2, generating a path of supervertices, $(0^{n-1}* = U_0*, U_1*, \ldots, U_{N/2-1}* = 0^{n-2a}1^{2a-1}*)$, in the $OG(FH(n))$, where $N = 2^n$.
Step 2. For each $0 \leq j < (2L-2^{n-1})/2 = L-2^{n-2}$, generate the path $Pt(j) = (U_{2j}0, U_{2j}1, U_{2j+1}1, U_{2j+1}0)$.
Step 3. For each $(2L-2^{n-1})/2 \leq j \leq 2^{n-2}-2$, generate the path $Pt(j) = (U_{2j}0, U_{2j+1}0)$.
Step 4. For $j = 2^{n-2}-1$, generate the path $Pt(j) = (U_{2j}0, U_{2j+1}0, U_{2j+1}1)$.
Step 5. Concatenating the $Pt(0)$, $Pt(1)$, …, $Pt(2^{n-1}-1)$, a path joining 0^n and $0^{n-2a}1^{2a}$ with length $2L$ can be generated.

/* End of algorithm A_3 */

For example, let $a = 4$, $n = 10$. We explain how to use the algorithm A_3 to join the pair of vertices $x = 0000000000$ and $y = 0011111111$ in the $FH(10)$ with length 156 and 680, respectively. Because $156 < 2^9$, Case 1 is applied. By the Step 1, we use the Algorithm A_2 to generate a path ($U_0* = 000000000*$, U_1*, U_2*, …, $U_{155}* = 001111111*$) in the $OG(FH(10))$ (i.e., an $FH(9)$). The path $(U_0 0, U_1 0, \ldots, U_{155}0, U_{155}1)$ can be derived for joining 0000000000 and 0011111111 with length 156.

Because $680 \geq 2^9$, Case 2 is applied and $L = 340$ here. By the Step 1, we use the Algorithm A_2 to generate the path ($U_0* = 000000000*$, U_1*, U_2*, …, $U_{511}* = 001111111*$) in the $OG(FH(10))$ (i.e., an $FH(9)$). By the Step 2, for each $0 \leq j < L-2^{n-2} = 340-256 = 84$, generate the path $Pt(j) = (U_{2j}0, U_{2j}1, U_{2j+1}1, U_{2j+1}0)$. By the Step 3, for each $84 \leq j \leq 2^{n-2}-2 = 254$, generate the path $Pt(j) = (U_{2j}0, U_{2j+1}0)$. By the Step 4, for $j = 2^{n-2}-1$, generate the path $Pt(j) = (U_{2j}0, U_{2j+1}0, U_{2j+1}1)$. Thus, the total length of the path that concatenates $Pt(0)$, $Pt(1)$, …, $Pt(82)$ and $Pt(83)$ is $3 \times 84 + 83 = 335$. The total length of the path that concatenates $Pt(84)$, $Pt(85)$, …, $Pt(253)$ and $Pt(254)$ is $171 + 170 = 341$. Clearly, the length of $Pt(255) = (U_{510}0, U_{511}1, U_{511}1)$ is 2. One edge concatenates $Pt(83)$ and $Pt(84)$, one edge concatenates $Pt(254)$ and $Pt(255)$; thus, the total length of the path that concatenates $Pt(0)$, $Pt(1)$, …, $Pt(253)$, $Pt(254)$ and $Pt(255)$ is $335 + 341 + 2 + 2 = 680$.

Lemma 5. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$, there exist h-paths joining x and y of each even length ranging from $2a$ to 2^n-2.*

4. The c-paths

In this section, we study the c-paths of the folded hypercube by the algorithmic approach. First, we consider a pair of vertices x and $\overline{x}$ of an $FH(n)$. Clearly, if and only if x and y have the complementary addresses, the length of the shortest c-path joining x and y of the $FH(n)$ is 1. Without loss of generality, we assume that $x = 0^n$ and $y = 1^n$.

Lemma 6. *An $L(s)$ is embedded into an $FH(n)$ with $(0^n, 1^n)$ as the bottom rung, where $s = 2^{n-1} - 1$.*

Proof. We map each vertex $(0, i)$ of the $L(s)$ to the vertex $g(n-1, i)0$ of the $FH(n)$; and each vertex $(1, i)$ of the $L(s)$ to the complementary vertex of $g(n-1, i)0$ of the $FH(n)$, where $0 \le i \le 2^{n-1} - 1$. By the definition of the Gray codes, $g(n-1, i)0$ and $g(n-1, i+1)0$ are connected; and the complementary vertex of $g(n-1, i)0$ and the complementary vertex of $g(n-1, i+1)0$ are also connected for each $0 \le i \le 2^{n-1} - 2$. Moreover, each $g(n-1, i)0$ and the complementary vertex of the $g(n-1, i)0$ are connected. This completes the proof. □

From the above lemma, Propositions 7 and 10, we have

Lemma 7. *For each pair of complementary vertices x and $\overline{x}$ of the $FH(n)$, there exist paths joining x and $\overline{x}$ of each odd length ranging from 1 to $2^n - 1$.*

Now, we consider each pair of vertices x and y where $|x \oplus y|$ is an even number $2a$ and n is an even number $2f$, where $a < f$. According to the symmetrical properties of the $FH(n)$, without loss of generality, we assume that $x = 0^n = 0^{2f}$ and $y = 0^{n-2a}1^{2a} = 0^{2f-2a}1^{2a}$. By the transition sequence $(2f, c, 2a+1, 2f-1, 2a+2, 2f-2, \ldots, f+a+1, f+a)$, a path $(x = 0^{2f}, 10^{2f-1}, 01^{2f-1}, 01^{2f-2a-2}01^{2a}, 0^21^{2f-2a-3}01^{2a}, 0^21^{2f-2a-4}0^21^{2a}, \ldots, 0^{f-a-1}1^20^{f-a-1}1^{2a}, 0^{f-a}10^{f-a-1}1^{2a}, 0^{2f-2a}1^{2a} = y)$ can be derived. Clearly, the edge $(0^{2f}, 10^{2f-1})$ resides in the subcube $(*)^{2f-1}0$. By Corollary 3, we can generate the 0th ladder $LD(0)$ with an $L(2^{2f-2} - 1)$ and the edge $(0^{2f}, 10^{2f-1})$ as the bottom rung in the subcube $(*)^{2f-1}0$. According to the symmetrical properties of the $FH(n)$, we can modify Algorithm A_1 slightly by relabelling all the vertices and edges of $(*)^{2f-1}1$ to generate a $POL(01^{2f-1}, 0^{2f-2a}1^{2a}, 2^{2f-1})$. Unifying the $LD(0)$ and the $POL(01^{2f-1}, 0^{2f-2a}1^{2a}, 2^{2f-1})$ by the c-edge $(10^{2f-1}, 01^{2f-1})$, we can generate a $POL(0^{2f}, 0^{2f-2a}1^{2a}, 2^{2f})$. We can modify algorithm A_2 slightly to handle the $POL(0^{2f}, 0^{2f-2a}1^{2a}, 2^{2f})$, the c-paths of each odd length ranging from $2f - 2a + 1$ to $2^n - 1$ to join the two vertices 0^{2f} and $0^{2f-2a}1^{2a}$ can be derived. According to the symmetrical properties and combining Lemma 7, we have

Lemma 8. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$ and n is an even number $2f$, where $a \le f$, there exist c-paths joining x and y of each odd length ranging from $2f - 2a + 1$ to $2^n - 1$.*

Likewise, we can derive similar result for the case that $|x \oplus y|$ is an odd number $2a + 1$ and n is an odd number $2f + 1$.

Lemma 9. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a + 1$ and n is an odd number $2f + 1$, where $a \le f$, there exist c-paths joining x and y of each odd length ranging from $2f - 2a + 1$ to $2^n - 1$.*

Then, we consider each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a + 1$ and n is an even number $2f$, where $a < f$. According to the symmetrical properties of the $FH(n)$, without loss of generality, we assume that $x = 0^n = 0^{2f}$ and $y = 0^{n-2a-2}1^{2a+1}0 = 0^{2f-2a-2}1^{2a+1}0$. Recall that the $OG(FH(n))$ is an $FH(n-1)$. According to Lemma 9, there exists a c-path, $(0^{2f-1*} = U_0*, U_1*, \ldots, U_{2i}*, U_{2i+1}*, \ldots, U_{2h}*, U_{2h+1}* = 0^{2f-2a-2}1^{2a+1}*)$, of each odd length $2h + 1$ ranging from $n - 2a - 1 = 2f - 2a - 1$ to $2^{n-1} - 1$ in the $OG(FH(n))$, where $(U_{2i}*, U_{2i+1}*)$ is the c-edge. From Proposition 8, we know that there exists the corresponding $L(2h + 1)$ in the $FH(n)$ for each $2h + 1$ ranging from $2f - 2a - 1$ to $2^{n-1} - 1$. Clearly, for each j, the four vertices $U_{2j}0, U_{2j}1, U_{2j+1}1, U_{2j+1}0$ can form a $C(4)$ that can be regarded as the jth ladder $LD(j)$ with length 1; where $(U_{2j}0, U_{2j+1}0)$ as the bottom rung for $0 \le j < i$, $(U_{2j}0, U_{2j+1}1)$ as the bottom rung for $j = i$, $(U_{2j}1, U_{2j+1}1)$ as the bottom rung for $i < j \le h$. The structure is illustrated in Fig. 7. In each $LD(j)$, we can choose a path $Pt(j)$ as $(U_{2j}0, U_{2j+1}0)$ or $(U_{2j}0, U_{2j}1, U_{2j+1}1, U_{2j+1}0)$ where $0 \le j < i$; in each $LD(j)$, we can choose a path $Pt(j)$ as $(U_{2j}0, U_{2j+1}1)$ or $(U_{2j}0, U_{2j}1, U_{2j+1}0, U_{2j+1}1)$ where $j = i$; in each $LD(j)$, we can choose a path $Pt(j)$ as $(U_{2j}1, U_{2j+1}1)$ or $(U_{2j}1, U_{2j}0, U_{2j+1}0, U_{2j+1}1)$ where $i < j < h$; and in the $LD(h)$, the path $Pt(h) = (U_{2h}1, U_{2h}0, U_{2h+1}0)$ is chosen. Concatenating the $Pt(0), Pt(1), \ldots, Pt(h)$, a path joining $x = 0^n = 0^{2f}$ and $y = 0^{n-2a-2}1^{2a+1}0 = 0^{2f-2a-2}1^{2a+1}0$

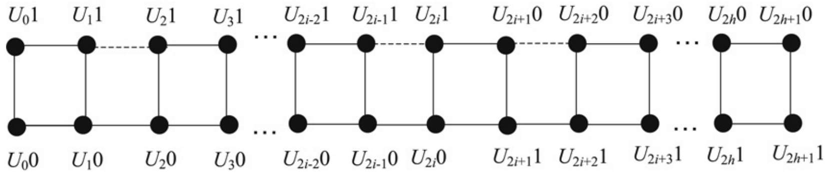

Fig. 7. Each $(U_{2j}{}^*, U_{2j+1}{}^*)$ is as an $L(1)$ with $(U_{2j}0, U_{2j+1}0)$ as the bottom rung, for $0 \le j \le i-1$, with $(U_{2j}0, U_{2j+1}1)$ as the bottom rung for $j = i$ and with $(U_{2j}1, U_{2j+1}1)$ as the bottom rung for $i < j < h$.

with length ranging from $2h+2$ to $4h+2$ can be generated. Because $2h+1$ is ranging from $2f-2a-1$ to $2^{n-1}-1$, the value of h is ranging from $f-a-1$ to $2^{n-2}-1$. The paths joining $0^n = 0^{2f}$ and $0^{n-2a-2}1^{2a+1}0 = 0^{2f-2a-2}1^{2a+1}0$ contains

$$\begin{aligned}
&\{P(s) | 2f-2a \le s \le 4f-4a-2, s \text{ is even}\} \quad (\text{for } h = f-a-1)\\
&\cup \{P(s) | 2f-2a+2 \le s \le 4f-4a+2, s \text{ is even}\} \quad (\text{for } h = f-a)\\
&\ldots,\\
&\cup \{P(s) | 2jr+2 \le s \le 4jr+2, s \text{ is even}\} \quad (\text{for } h = jr)\\
&\cup \{P(s) | 2jr+4 \le s \le 4jr+6, s \text{ is even}\} \quad (\text{for } h = jr+1)\\
&\ldots,\\
&\cup \{P(s) | 2^{n-1}-2 \le s \le 2^n-6, s \text{ is even}\} \quad (\text{for } h = 2^{n-2}-2)\\
&\cup \{P(s) | 2^{n-1} \le s \le 2^n-2, s \text{ is even}\} \quad (\text{for } h = 2^{n-2}-1)
\end{aligned}$$

Clearly, $4jr+2$ is always greater than or equal to $2jr+4$ for $jr \ge 1$. We investigate the case that $f-a=1$ and $jr = f-a-1 = 0$. For $jr = 0$, the paths joining 0^{2f} and $0^{2f-2a-2}1^{2a+1}0$ contain $\{P(s) | 2 \le s \le 2, s \text{ is even}\}$; and for $jr = 1$, the paths joining 0^{2f} and $0^{2f-2a-2}1^{2a+1}0$ contain $\{P(s) | 4 \le s \le 6, s \text{ is even}\}$. Thus, the paths joining 0^{2f} and $0^{2f-2a-2}1^{2a+1}0$ contain $\{P(s) | 2f-2a \le s \le 2^n-2, s \text{ is even}\}$. Thus, we have

Lemma 10. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a+1$ and n is an even number $2f$, where $a < f$, there exist c-paths joining x and y of each even length ranging from $2f-2a$ to $2^{2f}-2 = 2^n-2$.*

Likewise, we can derive the following lemma:

Lemma 11. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$ and n is an odd number $2f+1$, where $a \le f$, there exist c-paths joining x and y of each even length ranging from $2f-2a+2$ to $2^{2f}-2 = 2^n-2$.*

5. The bipanconnectivity and the m-panconnectivity

In this section, we study the bipanconnectivity and the m-panconnectivity of the $FH(n)$ by the above discussions about the h-paths and the c-paths of the $FH(n)$. By Lemma 1, we know that an $FH(n)$ is a bipartite graph if n is an odd number $2f+1$. For $1 \le 2a+1 \le \lceil (2f+1)/2 \rceil = f+1$, we can derive that $2f+2 \ge 4a+2$, and $2f-2a+1 \ge 2a+1$. Combining Lemmas 4 and 9, we know

Lemma 12. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a+1$, n is an odd number $2f+1$ and $1 \le 2a+1 \le \lceil (2f+1)/2 \rceil = f+1$, there exist paths joining x and y of each odd length ranging from $2a+1$ to 2^n-1.*

For $2f+1 \ge 2a+1 > \lceil (2f+1)/2 \rceil = f+1$, we can derive that $4a > 2f$ and $2a+1 > 2f-2a+1$. Combining Lemmas 4 and 9, we know

Lemma 13. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a+1$, n is an odd number $2f+1$ and $2f+1 \ge 2a+1 > \lceil (2f+1)/2 \rceil = f+1$, there exist paths joining x and y of each odd length ranging from $2f-2a+1$ to 2^n-1.*

For $2 \leq 2a \leq \lceil (2f+1)/2 \rceil = f+1$, we can derive that $2f+2 \geq 4a$ and $2f-2a+2 \geq 2a$. Combining Lemmas 5 and 11, we have

Lemma 14. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$, n is an odd number $2f+1$ and $2 \leq 2a \leq \lceil (2f+1)/2 \rceil = f+1$, there exist paths joining x and y of each even length ranging from $2a$ to $2^n - 2$.*

For $2f+1 > 2a > \lceil (2f+1)/2 \rceil = f+1$, we can derive that $4a > 2f+2$ and $2a > 2f-2a+2$. Combining Lemmas 5 and 11, we have

Lemma 15. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$, n is an odd number $2f+1$ and $2f+1 > 2a > \lceil (2f+1)/2 \rceil = f+1$, there exist paths joining x and y of each even length ranging from $2f-2a+2$ to $2^n - 2$.*

By the above four lemmas, Propositions 4 and 5, we have

Theorem 16. *The $FH(n)$ is bipanconnected for n is an odd number.*

By Lemma 1, we know that the $FH(n)$ is not a bipartite graph if n is an even number. Combining Lemmas 5 and 8, we know

Lemma 17. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$ and n is an even number $2f$, where $2 \leq 2a \leq 2f$, there exist paths joining x and y of each length $\in \{2a, 2a+2, \ldots, 2^n-2\} \cup \{2f-2a+1, 2f-2a+3, \ldots, 2^n-1\}$.*

For $2a \leq f$, $2f-2a+1$ is greater than $2a$; $\{2a, 2a+2, \ldots, 2^n-2\} \cup \{2f-2a+1, 2f-2a+3, \ldots, 2^n-1\} = \{2a, 2a+2, \ldots, 2f-2a-2, 2f-2a, 2f-2a+1, 2f-2a+2, 2f-2a+3, \ldots, 2^n-1\}$. Thus, we have

Corollary 18. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$ and n is an even number $2f$, where $2 \leq 2a \leq f$, there exist paths joining x and y of each length $\in \{2a, 2a+2, \ldots, 2f-2a-2, 2f-2a, 2f-2a+1, 2f-2a+2, 2f-2a+3, \ldots, 2^n-1\}$.*

Clearly, the maximum value of $2f-2a$ is $2f-2$ as $a=1$, for each $2 \leq 2a \leq f$. Similarly, for $f < 2a \leq 2f$, we can derive that $2f-2a$ is less than $2a$. Because both of $2f-2a$ and $2a$ are even, $2f-2a+1$ is also less than $2a$; $\{2a, 2a+2, \ldots, 2^n-2\} \cup \{2f-2a+1, 2f-2a+3, \ldots, 2^n-1\} = \{2f-2a+1, 2f-2a+3, \ldots, 2a-3, 2a-1, 2a, 2a+1, 2a+2, \ldots, 2^n-1\}$. Thus, we have

Corollary 19. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an even number $2a$ and n is an even number $2f$, where $f < 2a \leq 2f$, there exist paths joining x and y of each length $\in \{2f-2a+1, 2f-2a+3, \ldots, 2a-3, 2a-1, 2a, 2a+1, 2a+2, \ldots, 2^n-1\}$.*

Clearly, the maximum value of $2a-1$ is $2f-1$ as $a=f$, for each $f < 2a \leq 2f$. Combining Lemmas 4 and 10, we know

Lemma 20. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a+1$ and n is an even number $2f$, where $1 \leq 2a+1 \leq 2f$, there exist paths joining x and y of each length $\in \{2a+1, 2a+3, \ldots, 2^n-1\} \cup \{2f-2a, 2f-2a+2, 2f-2a+4, \ldots, 2^n-2\}$.*

For $1 \leq 2a+1 \leq f$, $2f-2a$ is greater than or equal to $2a+1$. Thus, $\{2a+1, 2a+3, \ldots, 2^n-1\} \cup \{2f-2a, 2f-2a+2, \ldots, 2^n-2\} = \{2a+1, 2a+3, \ldots, 2f-2a-3, 2f-2a-1, 2f-2a, 2f-2a+1, 2f-2a+2, \ldots, 2^n-1\}$. We have

Corollary 21. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a+1$ and n is an even number $2f$, where $1 \leq 2a+1 \leq f$, there exist paths joining x and y of each length $\in \{2a+1, 2a+3, \ldots, 2f-2a-3, 2f-2a-1, 2f-2a, 2f-2a+1, 2f-2a+2, \ldots, 2^n-1\}$.*

Clearly, the maximum value of $2f - 2a - 1$ is $2f - 1$ as $a = 0$, for each $1 \le 2a + 1 \le f$. Similarly, for $f < 2a + 1 < 2f$, we can derive that $2f - 2a$ is less than $2a + 2$. Because both of $2f - 2a$ and $2a + 2$ are even, $2f - 2a$ is also less than $2a + 1$. Thus, $\{2a + 1, 2a + 3, \ldots, 2^n - 1\} \cup \{2f - 2a, 2f - 2a + 2, \ldots, 2^n - 2\} = \{2f - 2a, 2f - 2a + 2, \ldots, 2a - 2, 2a, 2a + 1, 2a + 2, \ldots, 2^n - 1\}$. Thus, we have

Corollary 22. *For each pair of vertices x and y of the $FH(n)$, where $|x \oplus y|$ is an odd number $2a + 1$ and n is an even number $2f$, where $f < 2a + 1 < 2f$, there exist paths joining x and y of each length $\in \{2f - 2a, 2f - 2a + 2, \ldots, 2a - 2, 2a, 2a + 1, 2a + 2, \ldots, 2^n - 1\}$.*

Clearly, the maximum value of $2a$ is $2f - 2$ as $a = f - 1$, for each $f < 2a + 1 < 2f$. According to Corollaries 18, 19, 21 and 22, we have

Lemma 23. *The $HF(n)$ is $(n - 1)$-panconnected for n is an even number.*

We show the following lemma by contradiction:

Lemma 24. *The $HF(n)$ is not $(n - 2)$-panconnected for n is an even number.*

Proof. Suppose that the $HF(n)$ is $(n-2)$-panconnected. There should exist a path of length $n-2$ joining 0^n and $0^{n-1}1$. Because 0^n is an even vertex, and $0^{n-1}1$ is an odd vertex, there exists no h-path of even length joining them. However, by Proposition 3, the length of the shortest c-path joining 0^n and $0^{n-1}1$ is $n + 1 - |x \oplus y| = n$. Contradiction. □

By the above two lemmas, we have

Theorem 25. *The $HF(n)$ is strictly $(n - 1)$-panconnected for n is an even number.*

Recall that if an interconnection network is m-panconnected where $1 \le m \le N - 1$, it must be Hamiltonian-connected. Thus, we have

Corollary 26. *The $HF(n)$ is Hamiltonian-connected for n is an even number.*

6. Conclusions

In this paper, we present algorithms to generate the h-paths and c-paths joining an arbitrary pair of vertices in the n-dimensional folded hypercube. From the properties of these paths, we show that the n-dimensional folded hypercube is bipanconnected for n is an odd number. We also show that the n-dimensional folded hypercube is strictly $(n - 1)$-panconnected for n is an even number. That is, each pair of vertices are joined by the paths that include a path of each length ranging from $n - 1$ to $N - 1$; and the value $n - 1$ reaches the lower bound of the problem. The work will help the engineers to develop corresponding applications on the multiprocessor systems that employ the folded hypercubes as the interconnection networks. It will also help a further investigation on the folded hypercube. For example, to find a fault-tolerant algorithm to generate the bipanconnected paths and m-panconnected paths on the folded hypercube appears interesting.

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References

[1] B. Alspach, D. Hare, Edge-pancyclic block-intersection graphs, Discrete Math. 97 (1–3) (1991) 17–24.
[2] Ametek Corporation, Ametek Series 2010 Brochures.
[3] T. Araki, Y. Shibata, Pancyclicity of recursive circulant graphs, Inform. Process. Lett. 81 (4) (2002) 187–190.
[4] A. Auletta, A. Rescigno, V. Scarano, Embedding graphs onto the supercube, IEEE Trans. Comput. 44 (4) (1995) 593–597.
[5] M.M. Bae, Resource placement in torus-based networks, IEEE Trans. Comput. 46 (10) (1997) 1083–1092.
[6] J.R. Bitner, G.E. Ehrlich, M. Reingold, Efficient generation of the binary reflected Gray code and its applications, Commun. ACM 19 (9) (1976) 517–521.
[7] L.S. Chandran, T. Kavitha, The treewidth and pathwidth of hypercubes, Discrete Math. 306 (3) (2006) 359–365.

[8] J.M. Chang, J.S. Yang, J.S. Yang, Y.L. Wang, Y. Cheng, Panconnectivity, fault-tolerant hamiltonicity and hamiltonian-connectivity in alternating group graphs, Networks 44 (4) (2004) 302–310.
[9] G. Chartrand, O.R. Oellermann, Applied and Algorithmic Graph Theory, McGraw-Hill, New York, 1993.
[10] H.L. Chen, N.F. Tzeng, Efficient resource placement in hypercubes using multiple-adjacency codes, IEEE Trans. Comput. 43 (1) (1994) 23–33.
[11] S.A. Choudum, R.U. Nandini, Complete binary trees in folded and enhanced cubes, Networks 43 (4) (2004) 266–272.
[12] A. El-Amawy, S. Latifi, Properties and performance of folded hypercubes, IEEE Trans. Parallel Distrib. Syst. 2 (1991) 31–42.
[13] J. Fan, X. Lin, X. Jia, Node-pancyclicity and edge-pancyclicity of crossed cubes, Inform. Process. Lett. 93 (3) (2005) 133–138.
[14] J.-F. Fang, J.-Y. Hsiao, C.-Y. Tang, Embedding cycles and meshes onto incomplete hypercubes, Internat. J. Comput. Math. 75 (1) (2000) 1–19.
[15] V. Heun, E.W. Mayr, Efficient embeddings into hypercube-like topologies, The Computer Journal 46 (6) (2003) 632–644.
[16] A. Hobbs, The square of a block is vertex pancyclic, J. Combin. Theory Ser. B 20 (1) (1976) 1–4.
[17] Intel Corporation, The iPSC Data Sheet, Beaverton, Ore., 1985.
[18] M. Kobeissi, M. Mollard, Disjoint cycles and spanning graphs of hypercubes, Discrete Math. 288 (1–3) (2004) 73–87.
[19] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes, Mogran Kaufmann, California, 1992.
[20] T.K. Li, C.H. Tsai, J.J.M. Tan, L.H. Hsu, Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes, Inform. Process. Lett. 87 (2003) 107–110.
[21] Y.C. Lin, On balancing sorting on a linear array, IEEE Trans. Parallel Distrib. Syst. 4 (5) (1993) 566–571.
[22] NCUBE Corporation, NCUBE Handbook, Beaverton, Ore., 1986.
[23] D.R. O'Hallaron, Uniform approach for solving some classical problems on a linear array, IEEE Trans. Parallel Distrib. Syst. 2 (2) (1991) 236–241.
[24] C.D. Park, K.Y. Chwa, Hamiltonian properties on the class of hypercube-like networks, Inform. Process. Lett. 91 (1) (2004) 11–17.
[25] Y. Saad, M.H. Schultz, Topological properties of hypercubes, IEEE Trans. Comput. 37 (7) (1988) 867–872.
[26] C.L. Seitz, The cosmic cube, Commun. ACM 28 (1) (1985) 22–33.
[27] E. Simó, J.L.A. Yebra, The vulnerability of the diameter of folded n-cubes, Discrete Math. 174 (1997) 317–322.
[28] L.W. Tucker, G.G. Robertson, Architecture and applications of the connection machine, Computer 21 (8) (1988) 26–38.
[29] B. Ucar, C. Aykanat, K. Kaya, M. Ikinci, Task, Assignment in heterogeneous computing systems, J. Parallel Distrib. Comput. 66 (1) (2006) 32–46.
[30] D. Wang, Embedding Hamiltonian cycles into folded hypercubes with faulty links, J. Parallel Distrib. Comput. 61 (2001) 545–564.
[31] J.M. Xu, M. Ma, Cycles in folded hypercubes, Appl. Math. Lett. 19 (2006) 140–145.