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Theta function identities in the study of wavelets satisfying advanced differential equations

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ABSTRACT

The study of wavelets that satisfy the advanced differential equation $K'(t) = K(qt)$ is continued. The connections linking the theories of theta functions, wavelets, and advanced differential equations are further explored. A direct algebraic–analytic estimate is given for the maximal allowable translation parameter $\mathcal{N}(q)$ such that $b < \mathcal{N}(q)$ guarantees $\Lambda(0, q, b) \equiv \{(q^{m/2}/\sqrt{c_0})K(q^m t - nb) \mid m, n \in \mathbb{Z}\}$ is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$, where $\sqrt{c_0}$ is the $\mathcal{L}^2$ norm of K. For any $q > 1$ and any $b > 0$ we find conditions guaranteeing that $\Lambda(p, q, b) \equiv \{(q^{m/2}/\|K^{(p)}\|)K^{(p)}(q^m t - nb) \mid m, n \in \mathbb{Z}\}$ is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$ where $K^{(p)}$ denotes the pth derivative/antiderivative of K. The frames $\Lambda(p, q, b)$ become snug as either $p \to -\infty$ or $q \to \infty$, and their lower frame bounds $A(p, q, b) \to \infty$ as $q \to \infty$.

1. Introduction

We continue the study of the mother wavelet $K(t)$ defined for each $q > 1$ and $t \geqslant 0$ by

$$K(t) = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{e^{-q^k t}}{q^{k(k+1)/2}}, \tag{1}$$

where $K(t)$ in (1) satisfies the advanced differential equation

$$\frac{dK}{dt}(t) = K(qt). \tag{2}$$

Since $K(0^+) = K(\infty) = 0$, and since, by repeated use of the differential equation (2), it is clear that $K(t)$ is flat at $t = 0$ from the right, we set $K(t) = 0$ for $t < 0$ to obtain a smooth function on all of the reals. We set $\sqrt{c_0}$ to be the $\mathcal{L}^2$ norm of K over $\mathbb{R}$. We also observe that repeated applications of (2) yield

$$K^{(p)}(t) \equiv \frac{d^p K}{dt^p}(t) = q^{p(p-1)/2} K\left(q^p t\right) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^{(k+p)}t}}{q^{(k+p)(k-p+1)/2}}, \tag{3}$$

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and that (3) holds when $p < 0$ in which case we interpret $K^{(p)}$ as the $|p|$th antiderivative of K. For each $p \in \mathbb{Z}$ and $q > 1$, $K^{(p)}$ satisfies the advanced differential equation

$$\frac{dK^{(p)}}{dt}(t) = q^p K^{(p)}(qt). \tag{4}$$

As we will see later, the $K^{(p)}$ will be shown to generate wavelet frames for $\mathcal{L}^2(\mathbb{R})$.

Our study highlights the nexus between three seemingly distant areas of mathematics: theta functions, wavelets, and advanced-delayed differential equations. The link between these areas occurs via the fact that certain algebraic relations for theta functions correspond both to statements about advanced-delayed differential equations and to statements about properties of wavelets. We will utilize this link to interpret a class of results connecting these three areas. We will also exploit this link to provide direct algebraic–analytic estimates for translation parameters in obtaining frames.

In [7] we established the relation between $K(t)$ and the Jacobi theta function $\theta(\omega)$ which is defined for a given $q > 1$ by:

$$\theta(\omega) = \theta(q;\omega) \equiv \sum_{n=-\infty}^{\infty} \frac{\omega^n}{q^{n(n-1)/2}} = \mu_q \prod_{n=0}^{\infty}\left(1+\frac{\omega}{q^n}\right)\left(1+\frac{1}{\omega q^{n+1}}\right), \tag{5}$$

where μ_q is taken to be

$$\mu_q \equiv \prod_{n=0}^{\infty}\left(1-\frac{1}{q^{n+1}}\right).$$

We note here that the minimum value of $\theta(q^2;\omega^2)$ over $\omega \in \mathbb{R}\backslash\{0\}$ is

$$\nu_q \equiv \theta\left(q^2;1/q\right) = \sum_{n=-\infty}^{\infty}\frac{1}{q^{n^2}} = \mu_{q^2}\prod_{n=0}^{\infty}\left(1+\frac{1}{q^{2n+1}}\right)^2,$$

which is justified in Section 2.

The relation between $K(t)$ and $\theta(\omega)$ occurs via the Fourier transform [7]:

$$\hat{K}(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}K(t)\,dt = \frac{i(\mu_q)^3}{\sqrt{2\pi}\,\omega\theta(i\omega)}. \tag{6}$$

Eq. (6) establishes a foundation for linking a given algebraic identity on θ to its corresponding statement in the areas of wavelets and differential equations. Two such important algebraic identities on θ that we emphasize are:

$$\theta(q\omega) = q\omega\theta(\omega), \tag{7}$$

and

$$\theta(\omega) = \theta\left(1/(q\omega)\right). \tag{8}$$

We remark that the algebraic identity (7) is equivalent to the multiplicatively advanced differential equation (2) (under the assumption that (6) holds), and this in turn implies the wavelet statement that K has vanishing moments of all orders [7] through a repeated application of integration by parts. We point out that we do not pick arbitrary scale factors a in a frame formed from $K(a^m t - nb)$, for $m,n \in \mathbb{Z}$, because by picking a scale factor $a = q$ we have the natural identities (2), (7), (8) along with the vanishing of all moments. We further obtain $\mathcal{L}^2$ inner product relations such as $\langle K(t), K(q^{2n+1}t)\rangle = 0$ which hold when $a = q$. Further inner product computations reveal that $\langle K^{(p)}(q^m t - nb), t^k\rangle = 0$ giving vanishing of all moments for derivatives and antiderivatives of K. So in this sense q is the natural frequency associated to $K(t)$, and hence we only allow a frequency scale of $a = q$ throughout this work. We further note that as q varies, so does $K(t)$, in a non-linear manner.

Both identities (7) and (8) are key in providing direct algebraic–analytic estimates in studying the following frame condition for K,

$$0 < \inf_{1\leqslant|\omega|\leqslant q}\sum_{j\in\mathbb{Z}}\left(\left|\hat{K}\left(q^j\omega\right)\right|^2 - \sum_{k\in\mathbb{Z}\backslash\{0\}}\left|\hat{K}\left(q^j\omega\right)\hat{K}\left(q^j\omega+2\pi k/b\right)\right|\right), \tag{9}$$

where the algebraic identity (8) on θ gives us the surprising wavelet result that the term we call the “diagonal” term

$$G_0(\omega) \equiv \sum_{j\in\mathbb{Z}}\left|\hat{K}\left(q^j\omega\right)\right|^2$$

in (9) is a constant independent of ω. On the other hand, Eq. (7) under iterative application gives us direct algebraic–analytic bounds on the term we call the "off-diagonal" term

$$G_1(\omega) \equiv \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\hat{K}\left(q^j\omega\right)\hat{K}\left(q^j\omega + 2\pi k/b\right)\right|$$

without resorting to more commonly utilized bounds obtained by establishing decay rates on $\hat{K}(\omega)$. In tandem, careful deployment of the algebraic identities (7) and (8) allow us, in the wavelet arena, to generate wavelet frames with translation parameters b in (9) that are many orders of magnitude greater than those obtainable via traditional decay-rate determination on $\hat{K}(\omega)$ [3,4,7]. The spirit of these algebraic–analytic bounds is similar to the algebraic estimates used in [6].

Thus a first main result of this paper is to utilize properties of theta functions to establish an estimate for maximal allowable shift parameters in wavelet frames in Theorem 1, and a second main result is to find a wide class of frequency parameters q and translation parameters b for mother wavelets of form $K^{(p)}/\|K^{(p)}\|$ to generate a frame for $\mathcal{L}^2(\mathbb{R})$ in Theorem 4.

Theorem 1. *Let $2\pi/\sqrt{q} > b > 0$, and $\pi\sqrt{q} > b > 0$. Define*

$$F(q) = \left(1 + \sqrt{\frac{\pi \ln q}{2}}\right)\left(\frac{6}{q} + \frac{5}{q^2}\right) + \left(\frac{10}{q} + \frac{6}{q^{3/2}} + \frac{2}{q^2}\right) + \left(\left(1 + \sqrt{\frac{\pi \ln q}{2}}\right)\frac{5}{2q^2} + \frac{1}{q} + \frac{3}{2q^{3/2}} + \frac{1}{q^2}\right)\sqrt{\frac{2\pi}{\ln q}}. \tag{10}$$

Then for

$$\frac{2\pi\nu_q}{F(q)} > b > 0$$

we have $\Lambda(0,q,b) \equiv \{(q^{m/2}/\sqrt{c_0})K(q^m t - nb) \mid n, m \in \mathbb{Z}\}$ is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$.

Proof. Adding all the bounds in Propositions 5 and 6 in Section 3, and factoring out the common terms $\mu_q^4\mu_{q^2}/(2\pi)$, $b/(2\pi)$, and $1/\nu_q$, along with q^2, we have an upper bound for the off-diagonal term of

$$G_1(\omega) \equiv \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\hat{K}\left(q^j\omega\right)\hat{K}\left(q^j\omega + 2\pi k/b\right)\right| \leqslant \frac{\mu_q^4\mu_{q^2}}{2\pi} q^2 \frac{b}{2\pi}\frac{1}{\nu_q} F(q). \tag{11}$$

Utilizing Theorem 5 in Section 2, we explicitly compute the diagonal term as

$$G_0(\omega) \equiv \sum_{j\in\mathbb{Z}} \left|\hat{K}\left(q^j\omega\right)\right|^2 = \frac{\mu_q^4\mu_{q^2}}{2\pi} q^2. \tag{12}$$

Combining (11) and (12) we obtain

$$G_0(\omega) - G_1(\omega) \geqslant \frac{\mu_q^4\mu_{q^2}}{2\pi} q^2\left(1 - \frac{b}{2\pi\nu_q}F(q)\right) > 0 \quad \Longleftrightarrow \quad \frac{2\pi\nu_q}{F(q)} > b. \qquad \square \tag{13}$$

We remark that as q approaches infinity $2\pi\nu_q/F(q)$ grows, and the condition $2\pi/\sqrt{q} > b$ becomes the governing bound for large $q > q_0$, where $q_0 \approx 9.39033$ is the value of q with $2\pi\nu_{q_0}/F(q_0) = 2\pi/\sqrt{q_0}$. For $1 < q < q_0$ the bound $2\pi\nu_q/F(q)$ is the largest upper bound our methods can guarantee. Setting

$$\mathcal{N}(q) \equiv \min\left\{2\pi\nu_q/F(q), 2\pi/\sqrt{q}\right\}$$

gives the bounding curve $b = \mathcal{N}(q)$ in the (q,b) plane below which the functions $(q^{m/2}/\sqrt{c_0})K(q^m t - nb)$ generate a wavelet frame for $\mathcal{L}^2(\mathbb{R})$, as is illustrated in Fig. 1. Fig. 1 exhibits an apparent local minimum for $\mathcal{N}$ at $q_1 \approx 1.24667$ with $\mathcal{N}(q_1) \approx 0.44345$. So any choice of translation parameter b less than 0.44345 will allow for the ability of K to generate wavelet frames for $\mathcal{L}^2(\mathbb{R})$ for an arbitrary choice of q in the interval $(1, 200.75)$. The horizontal line $b = 1$ crosses $b = \mathcal{N}(q)$ at $q \approx 4.1374$ and at $q = (2\pi)^2$. Thus translation by integral multiples of $b = 1$ along with dilation by integral powers of q will give wavelet frames generated by K for q throughout the interval $(4.1374, (2\pi)^2)$. Although $(q,b) = (2,1)$ falls above $b = \mathcal{N}(q)$ and Theorem 1 cannot guarantee that $(2^{m/2}/\sqrt{c_0})K(2^m t - n)$ generates a wavelet frame for $\mathcal{L}^2(\mathbb{R})$, Theorem 4 and Corollary 1 will find a way around this to produce another wavelet, $K^{(-1)}$, generating a wavelet frame for $\mathcal{L}^2(\mathbb{R})$ when $(q,b) = (2,1)$.

A somewhat simpler, more algebraic version of Theorem 1 is obtained by estimating $2\pi\nu_q/F(q)$ from below.

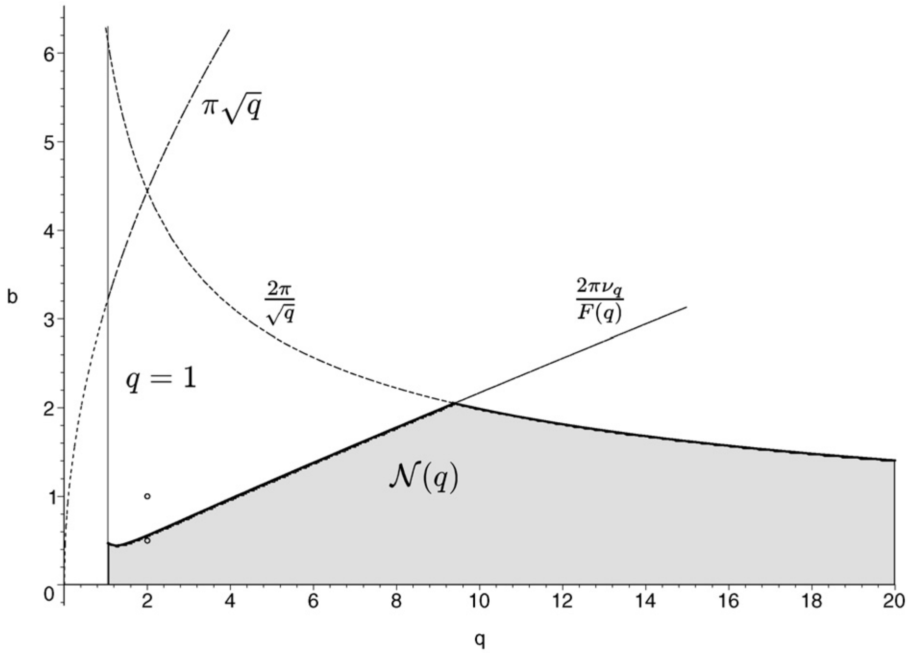

Fig. 1. The dark curve $b = \mathcal{N}(q)$ represents the maximal translation shift parameter; the points $(2, 1)$ and $(2, 0.5)$ are plotted for reference; the gray region represents the allowable (q, b) for which $(q^{m/2}/\sqrt{c_0})K(q^m t - nb)$ generate a wavelet frame for $\mathcal{L}^2(\mathbb{R})$.

Theorem 2. *Let $2\pi/\sqrt{q} > b > 0$, $\pi\sqrt{q} > b > 0$, and*

$$\frac{2\pi(q-1+\sqrt{1+2/\ln q})}{11\sqrt{\pi\ln q/2}+37+6\sqrt{2\pi/\ln q}} > b > 0.$$

Then $\Lambda(0, q, b) \equiv \{(q^{m/2}/\sqrt{c_0})K(q^m t - nb) \mid n, m \in \mathbb{Z}\}$ is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$.

Proof. In (10) of Theorem 1, replace each $1/q^p$ term in $F(q)$ by $1/q$ and estimate $29 + 5\pi/2$ from above by 37 in order to obtain a bound from above,

$$F(q) < (1/q)(11\sqrt{\pi\ln q/2}+37+6\sqrt{2\pi/\ln q}), \tag{14}$$

and replace each $1/q^p$ term in $F(q)$ by $1/q^2$ and estimate $29 + 5\pi/2$ from below by 36 to obtain a bound from below,

$$\left(1/q^2\right)(11\sqrt{\pi\ln q/2}+36+6\sqrt{2\pi/\ln q}) < F(q). \tag{15}$$

By (34) of Lemma 1 in Section 2, we have

$$1+\sqrt{\pi/\ln q} > \nu_q > 1+(1/q)(\sqrt{1+2/\ln q}-1). \tag{16}$$

By (14), (15), and (16) we have

$$\frac{2\pi q^2(1+\sqrt{\frac{\pi}{\ln q}})}{11\sqrt{\frac{\pi\ln q}{2}}+36+6\sqrt{\frac{2\pi}{\ln q}}} > \frac{2\pi\nu_q}{F(q)} > \frac{2\pi(q-1+\sqrt{1+\frac{2}{\ln q}})}{11\sqrt{\frac{\pi\ln q}{2}}+37+6\sqrt{\frac{2\pi}{\ln q}}}. \tag{17}$$

From (13) of Theorem 1, we have a wavelet frame provided

$$\frac{2\pi\nu_q}{F(q)} > b > 0.$$

Thus if $b > 0$ is less than the rightmost expression in (17) we have a wavelet frame. This yields Theorem 2. □

Remark. The leftmost expression in (17) can easily be shown to be less than $\pi\sqrt{q}$ for $q \in [1, 2]$, and, since $2\pi/\sqrt{q} < \pi\sqrt{q}$ for $q \in (2, \infty)$, we have $\mathcal{N}(q) < \pi\sqrt{q}$ on $[1, \infty)$. So we only need assume $0 < b < \mathcal{N}(q)$ in Theorems 1 and 2, and the assumption that $b < \pi\sqrt{q}$ is superfluous there (even though it arose in a natural way in Proposition 6 and its supporting propositions). On the other hand, the rightmost expression in (17) is clearly positive for all q in the interval $(1,\infty)$ and has a limit of $\sqrt{\pi}/3$ as $q \to 1^+$. We conclude that: $2\pi\nu_q/F(q)$ is then positive on the interval $(1,q_0)$; that $\mathcal{N}(q)$ remains positive on $(1,\infty)$; and that as $q \to 1^+$ we can take reasonably large translation parameters of order at least $\sqrt{\pi}/3$ while still generating wavelet frames for $\mathcal{L}^2(\mathbb{R})$.

We take the lower frame bound of our frame $\Lambda(0,q,b)$ to be

$$A(0,q,b) \equiv \inf\left\{ \frac{2\pi}{bc_0}\big(G_0(\omega) - G_1(\omega)\big) \,\Big|\, \omega \in [1,q] \right\},$$

and the upper frame bound of our frame $\Lambda(0,q,b)$ to be

$$B(0,q,b) \equiv \sup\left\{ \frac{2\pi}{bc_0}\big(G_0(\omega) + G_1(\omega)\big) \,\Big|\, \omega \in [1,q] \right\}.$$

A consequence of the estimates obtained in proving the above results is the following:

Theorem 3. *Assume $0 < b < \mathcal{N}(q)$. Then the lower frame bound $A(0,q,b)$ for $\Lambda(0,q,b)$ and the upper frame bound $B(0,q,b)$ for $\Lambda(0,q,b)$ satisfy*

$$\lim_{q\to\infty} \frac{B(0,q,b)}{A(0,q,b)} = 1.$$

Thus as q grows $\Lambda(0,q,b) = \{(q^{m/2}/\sqrt{c_0})K(q^m t - nb) \mid n,m \in \mathbb{Z}\}$ becomes snug [5].

Proof. We have, by (11) and (12),

$$\begin{aligned} A(0,q,b) &= \inf_{|\omega|\in[1,q]} \frac{2\pi}{bc_0} \sum_{j\in\mathbb{Z}} \left(\left|\hat{K}(q^j\omega)\right|^2 - \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\hat{K}(q^j\omega)\hat{K}(q^j\omega + 2\pi k/b)\right| \right) \\ &\geqslant \frac{2\pi}{bc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi} q^2 \left(1 - \frac{b}{2\pi\nu_q}F(q)\right) \\ &\geqslant \frac{2\pi}{bc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi} q^2 \left(1 - \frac{\sqrt{q}}{2}F(q)\right), \end{aligned} \tag{18}$$

and

$$\begin{aligned} B(0,q,b) &= \sup_{|\omega|\in[1,q]} \frac{2\pi}{bc_0} \sum_{j\in\mathbb{Z}} \left(\left|\hat{K}(q^j\omega)\right|^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\hat{K}(q^j\omega)\hat{K}(q^j\omega + 2\pi k/b)\right| \right) \\ &\leqslant \frac{2\pi}{bc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi} q^2 \left(1 + \frac{b}{2\pi\nu_q}F(q)\right) \\ &\leqslant \frac{2\pi}{bc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi} q^2 \left(1 + \frac{\sqrt{q}}{2}F(q)\right). \end{aligned} \tag{19}$$

Where (18) and (19) follow from the hypothesis that $b < \mathcal{N}(q) < \pi\sqrt{q}$ and the fact that $1/\nu_q < 1$. Thus

$$1 \leqslant \frac{B(0,q,b)}{A(0,q,b)} \leqslant \frac{1 + (\sqrt{q}/2)F(q)}{1 - (\sqrt{q}/2)F(q)},$$

and since $(\sqrt{q}/2)F(q) \to 0$ as $q \to \infty$, the ratio $B(0,q,b)/A(0,q,b) \to 1$. □

We remark that as q varies, so does our mother-wavelet, $K(t)$, which depends on q. A snug frame, as in [5], satisfies that the ratio of frame bounds B/A is close to one, making invertibility efficient. The frames generated by K for large q are snug. Also, since c_0 grows with order at most q^1 as q approaches ∞, then $A(0,q,b)$ also approaches ∞. Thus there is increasing clarity of signal representation with increasing q, as in [2,3].

We next harness Theorem 1 to obtain a wide versatility in choice of frequency coefficient and translation parameter. Before proceeding, we have the first of a pair of preliminary observations.

Proposition 1. *For all $q > 1$, for all $b > 0$, and for all $p,m,n \in \mathbb{Z}$*

$$\frac{q^{m/2}}{\|K^{(p)}\|} K^{(p)}\big(q^m t - nb\big) = \frac{q^{(m+p)/2}}{\|K^{(0)}\|} K^{(0)}\big(q^{(m+p)}t - n\big(bq^p\big)\big), \tag{20}$$

where $K^{(p)}$ is the pth derivative (or $|p|$th antiderivative when $p < 0$) of K, and the norm is the $\mathcal{L}^2$ norm.

Proof. First notice that

$$\|K^{(p)}\|^2 = \int_{-\infty}^{\infty} \left(K^{(p)}(t)\right)^2 dt = \int_{-\infty}^{\infty} \left(q^{p(p-1)/2} K\left(tq^p\right)\right)^2 dt$$
$$= q^{p(p-1)} \int_{-\infty}^{\infty} \left(K(u)\right)^2 q^{-p}\, du = q^{p(p-2)} \|K\|^2 = q^{p(p-2)} c_0, \tag{21}$$

where (3) was utilized in (21). Now we have

$$\frac{q^{m/2}}{\|K^{(p)}\|} K^{(p)}\left(q^m t - nb\right) = \frac{q^{m/2}}{q^{p(p-2)/2}\|K\|} q^{p(p-1)/2} K\left(\left(q^m t - nb\right)q^p\right)$$
$$= \frac{q^{m/2}}{\|K\|} q^{p/2} K\left(q^{(m+p)} t - n\left(bq^p\right)\right), \tag{22}$$

which gives the proposition upon noting that (21) and (3) were used to obtain (22). □

Next we observe

Proposition 2. *The Fourier transform of $K^{(p)}$ is given by*

$$\widehat{K^{(p)}}(\omega) = q^{p(p-3)/2} \hat{K}\left(q^{-p}\omega\right). \tag{23}$$

Proof. By (3) we have

$$\widehat{K^{(p)}}(\omega) = q^{p(p-1)/2} \int_{-\infty}^{\infty} e^{-it\omega} K\left(q^p t\right) dt$$
$$= q^{p(p-1)/2} \int_{-\infty}^{\infty} e^{-iuq^{-p}\omega} K(u) q^{-p}\, du$$
$$= q^{p(p-3)/2} \int_{-\infty}^{\infty} e^{-iuq^{-p}\omega} K(u)\, du = q^{p(p-3)/2} \hat{K}\left(q^{-p}\omega\right),$$

where we have relied on the change of variables $u = q^p t$. □

We now can prove the second main result of the paper.

Theorem 4. *Set $\mathcal{N}(q) = \min\{(2\pi\nu_q)/F(q), 2\pi/\sqrt{q}\}$. For any $q > 1$ and any $b > 0$*

$$\Lambda(p,q,b) \equiv \left\{ \frac{q^{m/2}}{\|K^{(p)}\|} K^{(p)}\left(q^m t - nb\right) \;\middle|\; m, n \in \mathbb{Z} \right\}$$

is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$ $\forall p \leqslant p_0 \equiv p_0(q,b) \equiv \sup\{p \in \mathbb{Z} \mid bq^p < \mathcal{N}(q)\}$. Furthermore,

$$\Lambda(p,q,b) = \Lambda\left(0,q,bq^p\right) \quad \forall p \in \mathbb{Z}.$$

For $p \leqslant p_0$, and letting $A(p,q,b)$ and $B(p,q,b)$ be the lower and upper frame bounds for $\Lambda(p,q,b)$ as in (24) and (26) below, we have

$$A(p,q,b) = A\left(0,q,bq^p\right) \quad \text{and} \quad B(p,q,b) = B\left(0,q,bq^p\right).$$

For $p \leqslant p_0$ the frames $\Lambda(p,q,b)$ become snug as either $p \to -\infty$ or as $q \to \infty$, that is

$$\lim_{p \to -\infty} \frac{B(p,q,b)}{A(p,q,b)} = 1 = \lim_{q \to \infty} \frac{B(p,q,b)}{A(p,q,b)}.$$

Proof. By (20) in Proposition 1, we immediately obtain that $\Lambda(p,q,b)=\Lambda(0,q,bq^p)$, since the functions in these two sets are equal. By Theorem 1 we know that $\Lambda(0,q,bq^p)$ is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$ if the translation term bq^p satisfies $bq^p<\mathcal{N}(q)$, which by definition of p_0 holds for all $p\leqslant p_0$. Thus $\Lambda(p,q,b)$ is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$ for all $p\leqslant p_0$. Since the functions in each frame are the same, their frame bounds are equal, as we next verify directly. We have,

$$\begin{aligned}
&A(p,q,b) \qquad (24)\\
&\equiv \inf_{|\omega|\in[1,q]} \frac{2\pi}{b\|K^{(p)}\|^2}\sum_{j\in\mathbb{Z}}\left(\left|\widehat{K^{(p)}}\left(q^j\omega\right)\right|^2-\sum_{k\neq 0}\left|\widehat{K^{(p)}}\left(q^j\omega\right)\widehat{K^{(p)}}\left(q^j\omega+\frac{2\pi k}{b}\right)\right|\right)\\
&= \inf_{|\omega|\in[1,q]} \frac{2\pi q^{p(p-3)}}{bq^{p(p-2)}c_0}\sum_{j\in\mathbb{Z}}\left(\left|\hat{K}\left(q^{j-p}\omega\right)\right|^2-\sum_{k\neq 0}\left|\hat{K}\left(q^{j-p}\omega\right)\hat{K}\left(q^{j-p}\omega+\frac{2\pi kq^{-p}}{b}\right)\right|\right)\\
&= \inf_{|\omega|\in[1,q]} \frac{2\pi}{bq^pc_0}\sum_{J\in\mathbb{Z}}\left(\left|\hat{K}\left(q^J\omega\right)\right|^2-\sum_{k\neq 0}\left|\hat{K}\left(q^J\omega\right)\hat{K}\left(q^J\omega+2\pi k/\left(bq^p\right)\right)\right|\right)\\
&\equiv A\left(0,q,bq^p\right)\\
&\geqslant \frac{2\pi}{bq^pc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi}q^2\left(1-\frac{bq^p}{2\pi\nu_q}F(q)\right)\\
&\geqslant \frac{2\pi}{bq^pc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi}q^2\left(1-\frac{\sqrt{q}}{2}F(q)\right), \qquad (25)
\end{aligned}$$

where (21) from Proposition 1, as well as (23) from Proposition 2, allow us to convert from $K^{(p)}$ to K, and from $\widehat{K^{(p)}}$ to $\hat{K}$, respectively. A reindexing from $j-p$ to J leads us to equality with $A(0,q,bq^p)$. At this point the estimates from Theorem 1 lead us to (25).

The computation for $B(p,q,b)$ is similar, except for taking supremum and adding the off-diagonal term. It again leads to equality of upper frame bounds,

$$\begin{aligned}
&B(p,q,b) \qquad (26)\\
&\equiv \sup_{|\omega|\in[1,q]} \frac{2\pi}{b\|K^{(p)}\|^2}\sum_{j\in\mathbb{Z}}\left(\left|\widehat{K^{(p)}}\left(q^j\omega\right)\right|^2+\sum_{k\neq 0}\left|\widehat{K^{(p)}}\left(q^j\omega\right)\widehat{K^{(p)}}\left(q^j\omega+\frac{2\pi k}{b}\right)\right|\right)\\
&= \sup_{|\omega|\in[1,q]} \frac{2\pi}{bq^pc_0}\sum_{J\in\mathbb{Z}}\left(\left|\hat{K}\left(q^j\omega\right)\right|^2+\sum_{k\neq 0}\left|\hat{K}\left(q^j\omega\right)\hat{K}\left(q^j\omega+2\pi k/\left(bq^p\right)\right)\right|\right)\\
&\equiv B\left(0,q,bq^p\right)\\
&\leqslant \frac{2\pi}{bq^pc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi}q^2\left(1+\frac{bq^p}{2\pi\nu_q}F(q)\right)\\
&\leqslant \frac{2\pi}{bq^pc_0}\frac{\mu_q^4\mu_{q^2}}{2\pi}q^2\left(1+\frac{\sqrt{q}}{2}F(q)\right). \qquad (27)
\end{aligned}$$

Here (25) and (27) follow from the facts that $bq^p<\mathcal{N}(q)<\pi\sqrt{q}$ for $p\leqslant p_0$ and that $1/\nu_q<1$. Thus

$$1\leqslant\frac{B(p,q,b)}{A(p,q,b)}\leqslant\frac{1+bq^pF(q)/(2\pi\nu_q)}{1-bq^pF(q)/(2\pi\nu_q)}\leqslant\frac{1+(\sqrt{q}/2)F(q)}{1-(\sqrt{q}/2)F(q)},$$

and since $(bq^pF(q))/(2\pi\nu_q)\to 0$ as $p\to-\infty$, and since $(\sqrt{q}/2)F(q)\to 0$ as $q\to\infty$, the ratio $B(p,q,b)/A(p,q,b)\to 1$ in either case.

Finally, since c_0 grows with order at most q^1 as $q\to\infty$, then (25) gives that $A(p,q,b)\to\infty$ for $p\leqslant\min\{0,p_0\}$. Thus there is increasing clarity of signal representation with increasing q for all $p\leqslant\min\{0,p_0\}$, as per [2,3]. Similarly as $p\to-\infty$, by (25) $A(p,q,b)\to\infty$, and we have increasing clarity in this case as well. □

Corollary 1. *We have that* $(q,b)=(2,1)$ *are frequency and translation parameters for a wavelet frame generated by* $K^{(-1)}$. *That is*

$$\Lambda(-1,2,1)=\left\{\frac{2^{m/2}}{\|K^{(-1)}\|}K^{(-1)}\left(2^mt-n\right)\,\middle|\,m,n\in\mathbb{Z}\right\}=\Lambda\left(0,2,2^{-1}\right)$$

is a wavelet frame for $\mathcal{L}^2(\mathbb{R})$, where

$$K^{(-1)}(t)=\sum_{k=-\infty}^{\infty}(-1)^k\frac{e^{-q^{(k-1)}t}}{q^{(k-1)(k+2)/2}}\quad\textit{and}\quad\frac{dK^{(-1)}}{dt}(t)=q^{-1}K^{(-1)}(qt).\tag{28}$$

Proof. Since $2^{-1}<\mathcal{N}(2)=(2\pi\nu_2)/F(2)\approx 0.55723$, Theorem 4 gives the result, after noting that (28) follows from (3) and (4). □

Remark. Each function in each frame $\Lambda(p,q,b)$ has all moments vanishing, as can be seen by converting the function to a multiple of K and changing variables when integrating against polynomials. Also there is an algebraic version of Theorem 4 that relies on the lower bound (17). If we set

$$L(q)\equiv 2\pi\left(q-1+\sqrt{1+2/\ln q}\right)\left(11\sqrt{\pi\ln q/2}+37+6\sqrt{2\pi/\ln q}\right)^{-1},$$

$\tilde{\mathcal{N}}(q)\equiv\min\{2\pi/\sqrt{q},L(q)\}$, and $\tilde{p}_0\equiv\tilde{p}_0(q,b)\equiv\sup\{p\in\mathbb{Z}\mid bq^p<\tilde{\mathcal{N}}(q)\}$, then, for $p\leqslant\tilde{p}_0$, the $\Lambda(p,q,b)$ are wavelet frames generating $\mathcal{L}^2(\mathbb{R})$ with snugness properties as in Theorem 4.

2. Relevant properties of the Jacobi theta function

Our analysis depends on properties of the Jacobi theta function, as defined in (5). We first prove identity (8) on θ:

Proposition 3. *$\theta(q;1/(q\omega))=\theta(q;\omega)$.*

Proof. We have

$$\begin{aligned}\theta\big(q;1/(q\omega)\big)&=\mu_q\prod_{n=0}^{\infty}\big(1+\big\{1/(q\omega)\big\}/q^n\big)\big(1+1/\big(\big\{1/(q\omega)\big\}q^{n+1}\big)\big)\\&=\mu_q\prod_{n=0}^{\infty}\big(1+1/\big(q^{n+1}\omega\big)\big)\big(1+\omega/\big(q^n\big)\big)\\&=\theta(q;\omega).\qquad\square\end{aligned}$$

Proposition 4. *$\theta(q;q\omega)=q\omega\theta(q;\omega)$.*

Proof. We have

$$\begin{aligned}\theta(q;q\omega)&=\mu_q\prod_{n=0}^{\infty}\big(1+(q\omega)/q^n\big)\big(1+1/\big((q\omega)q^{n+1}\big)\big)\\&=\mu_q\prod_{n=0}^{\infty}\big(1+\omega/q^{n-1}\big)\big(1+1/\big(\omega q^{n+2}\big)\big)\\&=(1+q\omega)\big(1+1/(q\omega)\big)^{-1}\mu_q\prod_{n=0}^{\infty}\big(1+\omega/q^n\big)\big(1+1/\big(\omega q^{n+1}\big)\big)\\&=q\omega\theta(q;\omega).\qquad\square\end{aligned}$$

Successive iterations of Proposition 4 give that for $n\geqslant 0$

$$\theta\big(q;q^n\omega\big)=q^{n(n+1)/2}\omega^n\theta(q;\omega),\tag{29}$$

whence $\theta(q;\omega)=\theta(q;q(\omega/q))=q(\omega/q)\theta(q;\omega/q)$ gives $\omega^{-1}\theta(q;\omega)=\theta(q;\omega/q)$ which under iterations gives that (29) holds for all negative n and thus for all $n\in\mathbb{Z}$.

An immediate consequence of Proposition 3 is the following key result in our study, the constancy of the diagonal term in the frame condition (9):

Theorem 5. *The diagonal is a constant independent of ω:*

$$G_0(\omega)=\sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|^2=\frac{\mu_q^4\mu_{q^2}q^2}{2\pi}\quad\forall\omega\in\mathbb{R}\backslash\{0\}.$$

Proof. Eq. (6) gives, upon observing that the conjugate of $\theta(i\omega)$ is $\theta(-i\omega)$, the identity

$$\left|\hat{K}(\omega)\right|^2 = \frac{\mu_q^6}{2\pi\omega^2\theta(-i\omega)\theta(i\omega)} = \frac{\mu_q^4\mu_{q^2}}{2\pi\omega^2\theta(q^2;\omega^2)}, \tag{30}$$

which follows from the fact that

$$\frac{\mu_{q^2}}{\mu_q^2}\theta(i\omega)\theta(-i\omega) = \mu_{q^2}\prod_{n=0}^{\infty}\left(1+\frac{\omega^2}{q^{2n}}\right)\left(1+\frac{1}{\omega^2 q^{2n+2}}\right)$$
$$= \theta\left(q^2;\omega^2\right) = \sum_{n=-\infty}^{\infty} q^{-n(n-1)}\omega^{2n}. \tag{31}$$

Thus, utilizing (30), letting $\kappa_q \equiv (\mu_q^4\mu_{q^2})/(2\pi)$, and relying on (29) in the first row of (32) below we have

$$\sum_{j\in\mathbb{Z}}\left|\hat{K}\left(q^j\omega\right)\right|^2 = \sum_{j\in\mathbb{Z}}\frac{\kappa_q}{q^{2j}\omega^2\theta(q^2;(q^{2j}\omega^2))} = \sum_{j\in\mathbb{Z}}\frac{\kappa_q}{q^{2j}\omega^2 q^{j(j+1)}\omega^{2j}\theta(q^2;\omega^2)}$$
$$= \frac{\kappa_q}{\omega^2\theta(q^2;\omega^2)}\sum_{j\in\mathbb{Z}}\frac{(q^{-2}\omega^{-2})^j}{q^{j(j+1)}} = \frac{\kappa_q q^2\omega^2}{\omega^2\theta(q^2;\omega^2)}\sum_{J\in\mathbb{Z}}\frac{(q^{-2}\omega^{-2})^J}{q^{(J-1)(J)}}$$
$$= \frac{\kappa_q q^2}{\theta(q^2;\omega^2)}\theta\left(q^2;1/\left(q^2\omega^2\right)\right) = \kappa_q q^2, \tag{32}$$

where we have reindexed to $J = j+1$ in the second row, then utilized the summation expression (31) for θ in proceeding from the second to the third row, and finally relied on Proposition 3 for the last equality. This gives that the diagonal is a constant independent of $\omega \in \mathbb{R}\backslash\{0\}$ and yields the theorem. □

Because of (30), it will be useful to find the minimum value of $\theta(q^2;\omega^2)$, so we differentiate, to obtain after simplification,

$$\frac{d\theta(q^2;\omega^2)}{d\omega} = \frac{2}{\omega}\sum_{k=1}^{\infty}\frac{k((q\omega^2)^{2k}-1)}{\omega^{2k}q^{k(k+1)}}.$$

Thus, solving $(q\omega^2)^{2k} - 1 = 0$, we find that $\theta(q^2;\omega^2)$ is increasing for $\omega > 1/\sqrt{q}$, and it is decreasing in the range $0 < \omega < 1/\sqrt{q}$. The minimum value, by symmetry about the origin, occurs at $\omega = \pm 1/\sqrt{q}$, and is

$$\theta\left(q^2;q^{-1}\right) = \sum_{n\in\mathbb{Z}}\frac{1}{q^{n^2}} \equiv \nu_q \geqslant 1. \tag{33}$$

We also observe that $\theta(q^2;0) = \theta(q^2;\pm\infty) = +\infty$.

We can more sharply estimate ν_q from above and below with the following lemma.

Lemma 1. *ν_q is bounded above and below by*

$$1+\sqrt{\pi/\ln q} \geqslant \nu_q \geqslant 1 + (1/q)(\sqrt{1+2/\ln q}-1). \tag{34}$$

Proof. We bound from below by noting that

$$\nu_q = \theta\left(q^2;1/q\right) = \sum_{k\in\mathbb{Z}}\frac{1}{q^{k^2}} = 1 + 2\sum_{k\geqslant 1}\frac{1}{q^{k^2}} \geqslant 1 + 2\int_1^{\infty} e^{-(\ln q)x^2}\,dx \tag{35}$$

$$= 1 + \frac{2}{\sqrt{\ln q}}\int_{\sqrt{\ln q}}^{\infty} e^{-u^2}\,du \geqslant 1 + \frac{2}{\sqrt{\ln q}}\frac{e^{-(\sqrt{\ln q})^2}}{\sqrt{\ln q}+\sqrt{\ln q + 2}} \tag{36}$$

$$= 1 + \frac{2}{q\sqrt{\ln q}}\frac{\sqrt{\ln q+2}-\sqrt{\ln q}}{2} = 1 + \frac{1}{q}(\sqrt{1+2/\ln q}-1), \tag{37}$$

where we have: compared the sum with the corresponding integral in (35); changed variables and relied on the bound (51) in (36); rationalized the rightmost denominator of (36) with the conjugate $\sqrt{\ln q+2}-\sqrt{\ln q}$ to obtain (37) and then simplified.

We bound from above with

$$\nu_q = \theta\big(q^2; 1/q\big) = \sum_{k\in\mathbb{Z}} \frac{1}{q^{k^2}} = 1 + 2\sum_{k\geqslant 1} \frac{1}{q^{k^2}} \leqslant 1 + 2\int_0^\infty e^{-(\ln q)x^2}\,dx$$
$$= 1 + \frac{1}{\sqrt{\ln q}} \int_{-\infty}^{\infty} e^{-u^2}\,du = 1 + \sqrt{\pi/\ln q}. \qquad \square$$

We also record a very useful estimate:

Lemma 2. *For* $1 \leqslant \omega \leqslant q$,

$$\frac{\omega^p}{\sqrt{\theta(q^2;\omega^2)}} \leqslant \begin{cases} q^{p-1}/\sqrt{\nu_q} & \text{if } p > 1, \\ 1/\sqrt{\nu_q} & \text{if } p \leqslant 1. \end{cases} \tag{38}$$

Proof. By relying first on (29) with $n = -1$, and then on (33) we have

$$\frac{\omega^p}{\sqrt{\theta(q^2;\omega^2)}} = \frac{\omega^{p-1}}{\sqrt{(\omega^2)^{-1}\theta(q^2;\omega^2)}} = \frac{\omega^{p-1}}{\sqrt{\theta(q^2;\omega^2/q^2)}}$$
$$\leqslant \frac{\omega^{p-1}}{\sqrt{\theta(q^2;1/q)}} = \frac{\omega^{p-1}}{\sqrt{\nu_q}}.$$

The result now follows after bounding from above by letting $\omega = q$ if $p > 1$ or $\omega = 1$ if $p \leqslant 1$. $\square$

Finally, we observe that the maximal value obtained by $|\hat{K}(\omega)|$ is $(\sqrt{\kappa_q}q)/\sqrt{\nu_q}$, when $\omega = \pm q^{-3/2}$. From (30) we have

$$\big|\hat{K}(\omega)\big|^2 = \frac{\mu_q^4 \mu_{q^2}}{2\pi\omega^2\theta(q^2;\omega^2)} = \frac{\kappa_q}{q^{-2}(q^2\omega^2)\theta(q^2;\omega^2)}$$
$$= \frac{\kappa_q q^2}{\theta(q^2;q^2\omega^2)} \leqslant \frac{\kappa_q q^2}{\theta(q^2;1/q)} = \frac{\kappa_q q^2}{\nu_q}, \tag{39}$$

where we have relied on (29) with $n = 1$ to move the $q^2\omega^2$ term inside the θ function, and then on (33) for the inequality. We note, also by (33), that the maximal value is attained when $q^2\omega^2 = 1/q$ or when $\omega = \pm q^{-3/2}$.

3. Bounding the off-diagonal term $G_1(\omega)$

Having explicitly determined the diagonal term to be $(\mu_q^4 \mu_{q^2} q^2)/(2\pi)$ in the frame condition (9), we turn our sights on using theta function identities to obtain tight estimates for the off-diagonal term

$$G_1(\omega) = \sum_j \sum_{k\neq 0} \big|\hat{K}\big(q^j\omega\big)\big|\big|\hat{K}\big(q^j\omega + 2\pi k/b\big)\big|$$
$$= \frac{\mu_q^4\mu_{q^2}}{2\pi} \sum_j \sum_{k\neq 0} \frac{1}{|q^j\omega|\sqrt{\theta(q^2;(q^j\omega)^2)}} \frac{1}{|q^j\omega + 2\pi k/b|\sqrt{\theta(q^2;(q^j\omega + 2\pi k/b)^2)}},$$

for $1 \leqslant |\omega| \leqslant q$, where we have relied on (30). By symmetry of $|\hat{K}(\omega)|$ about the origin, we restrict ourself without loss of generality to estimates over $1 \leqslant \omega \leqslant q$. For conciseness we define:

$$\kappa_q \equiv \frac{\mu_q^4\mu_{q^2}}{2\pi}.$$

We further define for each fixed j, q, ω, and b

$$k_0 \equiv k_0(j) \equiv k_0(j,q,\omega,b) \equiv \inf\big\{k < 0 \;\big|\; 1/\sqrt{q} < q^j\omega + 2\pi k/b\big\},$$
$$k_1 \equiv k_1(j) \equiv k_1(j,q,\omega,b) \equiv \sup\big\{k < 0 \;\big|\; q^j\omega + 2\pi k/b < -1/\sqrt{q}\big\},$$

where we take $k_0 = -1$ in the case that $\{k < 0 \mid 1/\sqrt{q} < q^j\omega + 2\pi k/b\} = \emptyset$, and where we write $k_0(j)$ and $k_1(j)$ when we wish to emphasize dependence of k_0 and k_1 on j. The purpose of such a k_1 and k_0 is to mark the last translate of $q^j\omega$ by a multiple of $2\pi/b$ before reaching $\pm 1/\sqrt{q}$ (the optimal points for $\theta(q^2;\omega^2)$) in the second factor of the off-diagonal term.

We will subdivide estimating the off-diagonal term, under appropriate restrictions on b, into four cases. To do so it will be convenient to define a partial sum for G_1 as:

$$\widetilde{G}_1(\omega;\, j\in\mathcal{A};\, k\in\mathcal{B}) \equiv \sum_{j\in\mathcal{A}}\bigl|\hat{K}\bigl(q^j\omega\bigr)\bigr|\sum_{k\in\mathcal{B}}\bigl|\hat{K}\bigl(q^j\omega+2\pi k/b\bigr)\bigr|.$$

The first three cases are handled with:

Proposition 5. *For $1\leqslant\omega\leqslant q$, $0<b<2\pi/\sqrt{q}$, and*

Case 1. *$j\in\mathbb{Z}$ and $k>0$:*

$$\begin{aligned}\widetilde{G}_1(\omega;\, j\in\mathbb{Z};\, k>0) &= \sum_{j\in\mathbb{Z}}\bigl|\hat{K}\bigl(q^j\omega\bigr)\bigr|\sum_{k>0}\bigl|\hat{K}\bigl(q^j\omega+2\pi k/b\bigr)\bigr|\\ &\leqslant \frac{b}{2\pi}\frac{\kappa_q}{\nu_q}\left(3q+2+\sqrt{\frac{2\pi}{\ln q}}\right)\left(1+\sqrt{\frac{\pi\ln q}{2}}\right).\end{aligned}$$

Case 2. *$j\in\mathbb{Z}$ and $k<k_1(j)$:*

$$\begin{aligned}\widetilde{G}_1\bigl(\omega;\, j\in\mathbb{Z};\, k<k_1(j)\bigr) &= \sum_{j\in\mathbb{Z}}\bigl|\hat{K}\bigl(q^j\omega\bigr)\bigr|\sum_{k<k_1(j)}\bigl|\hat{K}\bigl(q^j\omega+2\pi k/b\bigr)\bigr|\\ &\leqslant \frac{b}{2\pi}\frac{\kappa_q}{\nu_q}\left(3q+2+\sqrt{\frac{2\pi}{\ln q}}\right)\left(1+\sqrt{\frac{\pi\ln q}{2}}\right).\end{aligned}$$

Case 3. *$j\in\mathbb{Z}$ and $k_0(j)<k<0$:*

$$\begin{aligned}\widetilde{G}_1\bigl(\omega;\, j\in\mathbb{Z};\, k_0(j)<k<0\bigr) &= \sum_{j\in\mathbb{Z}}\bigl|\hat{K}\bigl(q^j\omega\bigr)\bigr|\sum_{k_0(j)<k<0}\bigl|\hat{K}\bigl(q^j\omega+2\pi k/b\bigr)\bigr|\\ &\leqslant \frac{b}{2\pi}\frac{\kappa_q}{\nu_q}\left(1+\frac{1}{2}\sqrt{\frac{2\pi}{\ln q}}\right)\left(1+\sqrt{\frac{\pi\ln q}{2}}\right).\end{aligned}$$

Each of Cases 1, 2, 3 is a "tail" case bounded with similar methods. The final case is for special k values:

Proposition 6. *For $1\leqslant\omega\leqslant q$, $0<b<2\pi/\sqrt{q}$, $0<b<\pi\sqrt{q}$, and*

Case 4. *$j\in\mathbb{Z}$ and $k_1(j)\leqslant k\leqslant k_0(j)$:*

$$\begin{aligned}&\widetilde{G}_1\bigl(\omega;\, j\in\mathbb{Z};\, k_1(j)\leqslant k\leqslant k_0(j)\bigr)\\ &\quad= \sum_{j\in\mathbb{Z}}\bigl|\hat{K}\bigl(q^j\omega\bigr)\bigr|\sum_{k_1(j)\leqslant k\leqslant k_0(j)}\bigl|\hat{K}\bigl(q^j\omega+2\pi k/b\bigr)\bigr|\\ &\quad\leqslant \frac{b}{2\pi}\frac{\kappa_q}{\nu_q}\bigl(10q+6\sqrt{q}+2+\bigl\{q+(3/2)\sqrt{q}+1\bigr\}\sqrt{2\pi/\ln q}\bigr).\end{aligned}$$

4. Bounding the tail Cases 1, 2, 3

4.1. Preliminaries

We denote the greatest integer function of a real number r by $\lfloor r\rfloor$, and let $0\leqslant\epsilon<1$ denote the difference between a real number and its corresponding greatest integer $r=\lfloor r\rfloor+\epsilon$. For $E,k,b>0$ we have

$$\begin{aligned}E+2\pi k/b &= q^{\log_q(E+2\pi k/b)} = q^{-1/2+\{1/2+\log_q(E+2\pi k/b)\}}\\ &= q^{-1/2+\lfloor 1/2+\log_q(E+2\pi k/b)\rfloor+\epsilon} = q^{-1/2+a+\epsilon}\end{aligned}\tag{40}$$

where for conciseness we take $a\equiv\lfloor 1/2+\log_q(E+2\pi k/b)\rfloor$ in (40) and throughout this section. This gives us

$$\begin{aligned}\theta\big(q^2;(E+2\pi k/b)^2\big) &= \theta\big(q^2;q^{2(a-1/2+\epsilon)}\big) = \theta\big(q^2;q^{2a}q^{(-1+2\epsilon)}\big)\\ &= q^{a(a+1)}\big(q^{(-1+2\epsilon)}\big)^a\theta\big(q^2;q^{-1+2\epsilon}\big) &(41)\\ &\geqslant q^{a^2}\theta\big(q^2;q^{-1}\big) = q^{\lfloor 1/2+\log_q(E+2\pi k/b)\rfloor^2}\nu_q &(42)\\ &\geqslant \nu_q q^{(-1/2+\log_q(E+2\pi k/b))^2} &(43)\\ &= \nu_q q^{1/4}(E+2\pi k/b)^{\{\log_q(E+2\pi k/b)-1\}} &(44)\end{aligned}$$

where we have used the algebraic identity (29) to obtain (41), the fact that $\theta(q^2;w^2)$ has the minimum value of $\theta(q^2;q^{-1})=\nu_q$ to obtain (42), and the fact that $(\lfloor r\rfloor)^2 \geqslant (r-1)^2$ for $r-1>0$ to obtain (43) where we must now assume the added constraint that $-1/2+\log_q(E+2\pi k/b)>0$ which will always hold if $2\pi/\sqrt{q}>b$. The point here is to represent $E+2\pi k/b$ as an integral power of q times a term $q^{-1/2+\epsilon}$ that is as close as possible to the minimum point $q^{-1/2}$ of $\theta(q^2;\omega^2)$ and then harness the power of (29).

Proposition 7. *For $E>0$ and $(2\pi/\sqrt{q})>b>0$ we have*

$$\begin{aligned}&\sum_{k>0}\big|\hat{K}(E+2\pi k/b)\big|\\ &\quad\leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}q^{-1/8}\sum_{k>0}(E+2\pi k/b)^{-1/2\{\log_q(E+2\pi k/b)+1\}} &(45)\\ &\quad\leqslant \frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right). &(46)\end{aligned}$$

Proof. We have

$$\begin{aligned}\big|\hat{K}(E+2\pi k/b)\big| &= \frac{\sqrt{\kappa_q}}{|E+2\pi k/b|\sqrt{\theta(q^2;(E+2\pi k/b)^2)}} &(47)\\ &\leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}q^{-1/8}(E+2\pi k/b)^{-1/2\{\log_q(E+2\pi k/b)+1\}}, &(48)\end{aligned}$$

where (30) gives (47), and the bound (44) implies (48) upon adding exponents. We then obtain (45) by summing over $k>0$.

The bound (46) follows by first comparing the sum (45) to the corresponding integral. For conciseness below, we let $\tau\equiv\{\ln(E+2\pi/b)-\ln\sqrt{q}\}/(\sqrt{2\ln q})$ in (50) through (52).

$$\begin{aligned}&\sum_{k>0}(E+2\pi k/b)^{-1/2\{\log_q(E+2\pi k/b)+1\}}\\ &\quad\leqslant (E+2\pi/b)^{-1/2\{\log_q(E+2\pi/b)+1\}}+\int_1^\infty (E+2\pi x/b)^{-1/2\{\log_q(E+2\pi x/b)+1\}}\,dx\\ &\quad= (E+2\pi/b)^{-1/2\{\log_q(E+2\pi/b)+1\}}+\frac{b}{2\pi}\int_{E+2\pi/b}^\infty (v)^{-1/2\{\log_q(v)+1\}}\,dv &(49)\\ &\quad= (E+2\pi/b)^{-1/2\{\log_q(E+2\pi/b)+1\}}+\frac{bq^{1/8}\sqrt{2\ln q}}{2\pi}\int_\tau^\infty e^{-u^2}\,du, &(50)\end{aligned}$$

where we have made the change of variables $v=E+2\pi x/b$ in (49) and then $u=(\ln v-\ln\sqrt{q})/(\sqrt{2\ln q})$ in (50). Then, by applying the rightmost bound in (51) (see [1]) for $x\geqslant 0$

$$\frac{e^{-x^2}}{x+\sqrt{x^2+2}} \leqslant \int_x^\infty e^{-u^2}\,du \leqslant \frac{e^{-x^2}}{x+\sqrt{x^2+4/\pi}} \leqslant \frac{\sqrt{\pi}}{2}e^{-x^2} \tag{51}$$

to the integral in (50), we obtain

$$\begin{aligned}\int_\tau^\infty e^{-u^2}\,du &\leqslant \frac{\sqrt{\pi}}{2}e^{-\tau^2}\\ &= \frac{\sqrt{\pi}}{2}\left(\frac{Eb+2\pi}{b\sqrt{q}}\right)^{-(1/2)\{\log_q((Eb+2\pi)/(b\sqrt{q}))\}}\\ &= \frac{\sqrt{\pi}}{2}\left(E+\frac{2\pi}{b}\right)^{-(1/2)\{\log_q(E+2\pi/b)-1/2\}}\left(\frac{1}{\sqrt{q}}\right)^{-(1/2)\{\log_q(E+2\pi/b)-1/2\}}\\ &= \frac{\sqrt{\pi}}{2}\left(E+\frac{2\pi}{b}\right)^{-(1/2)\log_q(E+2\pi/b)+1/2} q^{-1/8}. \end{aligned} \tag{52}$$

Applying (52) to (50) we obtain

$$\begin{aligned}&\sum_{k>0}(E+2\pi k/b)^{-1/2\{\log_q(E+2\pi k/b)+1\}}\\ &\quad\leqslant \left(E+\frac{2\pi}{b}\right)^{-1/2\{\log_q(E+\frac{2\pi}{b})+1\}}+\frac{b}{2\pi}\sqrt{\frac{\pi\ln q}{2}}\left(E+\frac{2\pi}{b}\right)^{-(1/2)\log_q(E+\frac{2\pi}{b})+1/2}\\ &\quad= \left(E+\frac{2\pi}{b}\right)^{-(1/2)\log_q(E+\frac{2\pi}{b})+1/2}\left(\left(E+\frac{2\pi}{b}\right)^{-1}+\frac{b}{2\pi}\sqrt{\frac{\pi\ln q}{2}}\right)\\ &\quad= \left(E+\frac{2\pi}{b}\right)^{-(1/2)\log_q(E+\frac{2\pi}{b})+1/2}\frac{b}{2\pi}\left(\left(\frac{Eb}{2\pi}+1\right)^{-1}+\sqrt{\frac{\pi\ln q}{2}}\right)\\ &\quad\leqslant \frac{b}{2\pi}\left(E+\frac{2\pi}{b}\right)^{-(1/2)\log_q(E+\frac{2\pi}{b})+1/2}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right)\\ &\quad\leqslant \frac{b}{2\pi}q^{1/8}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right), \end{aligned} \tag{53}$$

with the last inequality in (53) holding by the fact that

$$f(x)=x^{-(1/2)\log_q(x)+1/2}$$

attains a maximum value of $q^{1/8}$ at $x=\sqrt{q}$. Applying (53) to (45) gives (46) and the proposition. □

4.2. Further bounds

Proposition 8. *For $q>1$ and $\omega\in[1,q]$,*

$$\sum_{j\in\mathbb{Z}}\left|\hat{K}\left(q^j\omega\right)\right| = \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;\omega^2)}}q\theta(\omega)\leqslant\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(3q+2+\sqrt{\frac{2\pi}{\ln q}}\right).$$

Proof. We obtain the equality by observing

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left|\hat{K}\left(q^j\omega\right)\right| &= \sum_j\frac{\sqrt{\kappa_q}}{q^j\omega\sqrt{\theta(q^2;(q^j\omega)^2)}} = \frac{\sqrt{\kappa_q}}{\omega\sqrt{\theta(q^2;\omega^2)}}\sum_j\frac{1}{q^jq^{j(j+1)/2}\omega^j}\\ &= \frac{\sqrt{\kappa_q}q(q\omega)^{-1}}{\sqrt{\theta(q^2;\omega^2)}}\sum_j\frac{(q\omega)^{-j}}{q^{j(j+1)/2}} = \frac{\sqrt{\kappa_q}q(q\omega)^{-1}}{\sqrt{\theta(q^2;\omega^2)}}\theta(q\omega) = \frac{\sqrt{\kappa_q}q\theta(\omega)}{\sqrt{\theta(q^2;\omega^2)}}, \end{aligned}$$

where (29) with $n=-1$ was used to obtain the last equality. For the inequality we have

$$\begin{aligned}&\frac{\sqrt{\kappa_q}}{\omega\sqrt{\theta(q^2;\omega^2)}}\sum_j\frac{1}{q^jq^{j(j+1)/2}\omega^j}\\ &\quad= \frac{\sqrt{\kappa_q}}{\omega\sqrt{\theta(q^2;\omega^2)}}\sum_j e^{-(1/2)\ln q\{j^2+j(3+2\log_q(\omega))\}}\end{aligned}$$

$$= \frac{\sqrt{\kappa_q} q^{9/8+(3/2)\log_q(\omega)+(1/2)(\log_q(\omega))^2}}{\omega\sqrt{\theta(q^2;\omega^2)}} \sum_j e^{-(1/2)\ln q\{j+3/2+\log_q(\omega)\}^2} \tag{54}$$

$$= \frac{\sqrt{\kappa_q} q^{9/8+(1/2)\log_q(\omega)+(1/2)(\log_q(\omega))^2}}{\sqrt{\theta(q^2;\omega^2)}} \sum_j e^{-(1/2)\ln q\{j+3/2+\log_q(\omega)\}^2} \tag{55}$$

where (54) comes from a completion of squares, and (55) comes from canceling the ω in the denominator. We next split the summation in (55) into three cases $j \leqslant -4$, $-3 \leqslant j \leqslant -1$, $j \geqslant 0$, and then rely on the following bound (56) for $\alpha \geqslant 0$,

$$\sum_{j\geqslant 0} e^{-(1/2)\ln q(j+\alpha)^2} \leqslant q^{-(\alpha^2/2)}\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right), \tag{56}$$

to obtain for the $j \leqslant -4$ case

$$\begin{aligned}
\sum_{j\leqslant -4} e^{-(1/2)\ln q\{j+3/2+\log_q(\omega)\}^2} &= \sum_{J\leqslant 0} e^{-(1/2)\ln q\{J-5/2+\log_q(\omega)\}^2} \\
&= \sum_{J\leqslant 0} e^{-(1/2)\ln q\{J-3/2+(-1+\log_q(\omega))\}^2} = \sum_{L\geqslant 0} e^{-(1/2)\ln q\{L+3/2+(1-\log_q(\omega))\}^2} \\
&\leqslant q^{-(1/2)\{1-\log_q(\omega)\}^2} \sum_{L\geqslant 0} e^{-(1/2)\ln q\{L+3/2\}^2} \\
&= q^{-(1/2)+\log_q(\omega)-(1/2)\{\log_q(\omega)\}^2} q^{-9/8}\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right),
\end{aligned} \tag{57}$$

where the reindexings $J = j+4$ and $L = -J$ were used. The $j \geqslant 0$ case yields

$$\begin{aligned}
\sum_{j\geqslant 0} e^{-(1/2)\ln q\{j+3/2+\log_q(\omega)\}^2} &\leqslant \sum_{j\geqslant 0} e^{-(1/2)\ln q\{j+3/2\}^2-(1/2)\ln q\{\log_q\omega\}^2} \\
&\leqslant q^{-(1/2)\{\log_q\omega\}^2} q^{-9/8}\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right),
\end{aligned} \tag{58}$$

where we have used (56) with $\alpha = 3/2$ to obtain the last inequality. Finally the $-3 \leqslant j \leqslant -1$ case gives

$$\begin{aligned}
&\sum_{-3\leqslant j\leqslant -1} e^{-(1/2)\ln q\{j+3/2+\log_q(\omega)\}^2} \\
&\quad = q^{-9/8+(3/2)\log_q\omega-(1/2)(\log_q\omega)^2} + q^{-1/8+(1/2)\log_q\omega-(1/2)(\log_q\omega)^2} \\
&\qquad + q^{-1/8-(1/2)\log_q\omega-(1/2)(\log_q\omega)^2}.
\end{aligned} \tag{59}$$

The results (57), (58), and (59) combine with (55) to give, after canceling the common $(1/2)(\log_q\omega)^2$ terms in the exponents,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \left|\hat{K}\left(q^j\omega\right)\right| \\
&\quad \leqslant \frac{\sqrt{\kappa_q} q^{9/8+(1/2)\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}}\left(q^{-9/8}\left(1+q^{-(1/2)+\log_q(\omega)}\right)\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right)\right) \\
&\qquad + \frac{\sqrt{\kappa_q} q^{9/8+(1/2)\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}}\left(q^{-9/8+(3/2)\log_q\omega} + q^{-1/8+(1/2)\log_q\omega}\right) \\
&\qquad + \frac{\sqrt{\kappa_q} q^{9/8+(1/2)\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}} q^{-1/8-(1/2)\log_q\omega}
\end{aligned} \tag{60}$$

$$\begin{aligned}
&\quad = \sqrt{\kappa_q}\left(\frac{q^{(1/2)\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}} + \frac{q^{-(1/2)+(3/2)\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}}\right)\left(\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right)\right) \\
&\qquad + \frac{\sqrt{\kappa_q} q^{2\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}} + \frac{\sqrt{\kappa_q} q^{1+\log_q(\omega)}}{\sqrt{\theta(q^2;\omega^2)}} + \frac{\sqrt{\kappa_q} q^1}{\sqrt{\theta(q^2;\omega^2)}}
\end{aligned} \tag{61}$$

$$\begin{aligned}&\leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\big(1+q^{-(1/2)+(1/2)}\big)\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right)+\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\big(q^1+q^1+q^1\big) \qquad (62)\\ &= \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(\left(2+\sqrt{\frac{2\pi}{\ln q}}\right)+3q\right),\end{aligned}$$

which gives the proposition after noting that we have applied estimate (38) to each term in (61) to obtain (62). □

We will need the following corollary, similar to Proposition 8, when we do not sum j over all integers, but only over $j \geqslant 0$.

Corollary 2. *For $q > 1$ and $\omega \in [1, q]$*

$$\sum_{j\geqslant 0}\big|\hat{K}\big(q^j\omega\big)\big| \leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\frac{1}{2}\left(2+\sqrt{\frac{2\pi}{\ln q}}\right).$$

Proof. This is the $j \geqslant 0$ case in Proposition 8, where we use only (58) inserted into the first term of (60). □

4.3. Bounding the tail Cases 1, 2, 3

We next provide the proof of Proposition 5.

Proof of Proposition 5.

Case 1. The proof follows immediately by first applying Proposition 7 with E taken to be $q^j\omega$, and then applying Proposition 8:

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|\sum_{k>0}\big|\hat{K}\big(q^j\omega+2\pi k/b\big)\big| &= \widetilde{G}_1(\omega;\, j\in\mathbb{Z};\, k>0)\\ &\leqslant \sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|\frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right)\\ &\leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(3q+2+\sqrt{\frac{2\pi}{\ln q}}\right)\frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right).\end{aligned}$$

Case 2. Here we rely on the symmetry about the origin of $|\hat{K}(\omega)|$ before an application of Proposition 7, with E taken to be $-q^j\omega - 2\pi k_1(j)/b$, and then an application of Proposition 8 to obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|\sum_{k<k_1(j)}\big|\hat{K}\big(q^j\omega+2\pi k/b\big)\big| &= \widetilde{G}_1\big(\omega;\, j\in\mathbb{Z};\, k<k_1(j)\big)\\ &= \sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|\sum_{k<k_1(j)}\big|\hat{K}\big(\big\{-q^j\omega-2\pi k_1(j)/b\big\}+2\pi\big\{k_1(j)-k\big\}/b\big)\big|\\ &= \sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|\sum_{0<L}\big|\hat{K}\big(\big\{-q^j\omega-2\pi k_1(j)/b\big\}+2\pi\{L\}/b\big)\big| \qquad (63)\\ &\leqslant \sum_{j\in\mathbb{Z}}\big|\hat{K}\big(q^j\omega\big)\big|\frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right)\\ &\leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(3q+2+\sqrt{\frac{2\pi}{\ln q}}\right)\frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1+\sqrt{\frac{\pi\ln q}{2}}\right),\end{aligned}$$

where the reindexing was taken to be $L = k_1(j) - k$ in (63) for each fixed j.

Case 3. To be non-vacuous, the condition that $k_0(j) < k < 0$ implies that $k_0(j) < -1$ and this in turn implies $1/\sqrt{q} < q^j\omega + 2\pi k_0(j)/b < q^j\omega - 2\pi/b$ which restricts j to

$$\begin{aligned}
j &> -\log_q \omega + \log_q\big(-2\pi k_0(j)/b + 1/\sqrt{q}\big) \equiv N_0 \\
&> -\log_q \omega + \log_q(2\pi/b + 1/\sqrt{q}) \\
&> -\log_q \omega + \log_q(\sqrt{q} + 1/\sqrt{q}) \\
&> -\log_q \omega + 1/2 + \log_q(1 + 1/q) > -1/2,
\end{aligned}$$

or more simply $j > N_0 \geqslant 0$. Thus instead of relying on a sum for $j \in \mathbb{Z}$ we now utilize a sum for $j > N_0$, and later compare it to the sum over $j \geqslant 0$ to obtain

$$\begin{aligned}
\widetilde{G}_1\big(\omega; j \in \mathbb{Z}; k_0(j) < k < 0\big) &= \widetilde{G}_1\big(\omega; j > N_0; k_0(j) < k < 0\big) \\
&= \sum_{j>N_0} \big|\hat{K}\big(q^j\omega\big)\big| \sum_{k_0(j)<k<0} \big|\hat{K}\big(q^j\omega + 2\pi k/b\big)\big| \\
&\leqslant \sum_{j>N_0} \big|\hat{K}\big(q^j\omega\big)\big| \sum_{k_0(j)<k} \big|\hat{K}\big(\big\{q^j\omega + 2\pi k_0(j)/b\big\} + 2\pi\big\{k - k_0(j)\big\}/b\big)\big| \qquad (64) \\
&= \sum_{j>N_0} \big|\hat{K}\big(q^j\omega\big)\big| \sum_{0<L} \big|\hat{K}\big(\big\{q^j\omega + 2\pi k_0(j)/b\big\} + 2\pi L/b\big)\big| \qquad (65) \\
&\leqslant \sum_{j>N_0} \big|\hat{K}\big(q^j\omega\big)\big| \frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1 + \sqrt{\frac{\pi \ln q}{2}}\right) \qquad (66) \\
&\leqslant \sum_{j\geqslant 0} \big|\hat{K}\big(q^j\omega\big)\big| \frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1 + \sqrt{\frac{\pi \ln q}{2}}\right) \qquad (67) \\
&\leqslant \frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\frac{1}{2}\left(2 + \sqrt{\frac{2\pi}{\ln q}}\right)\frac{b}{2\pi}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}\left(1 + \sqrt{\frac{\pi \ln q}{2}}\right), \qquad (68)
\end{aligned}$$

where we abandoned the restriction that $k < 0$ and re-expressed the argument in terms of $k_0(j)$ in (64), reindexed by $L = k - k_0(j)$ for each fixed j to obtain (65), then Proposition 7 was applied to (65) with $E = \{q^j\omega + 2\pi k_0(j)/b\}$ to obtain (66), extended the summation to $j \geqslant 0$ in (67), and finally Corollary 2 was applied to (66) to obtain (68). $\square$

5. Bounds for special *k* values: Case 4

Henceforth, we assume that $2\pi/b > 2/\sqrt{q}$ (or equivalently $\pi\sqrt{q} > b$), which is already implied by our assumption $2\pi/\sqrt{q} > b$ when $q \geqslant 2$. One purpose of this assumption is to ensure that $q^j\omega - 2\pi k/b \in [-1/\sqrt{q}, 1/\sqrt{q}\,]$ holds for at most one value of k.

Now, Case 4, where $j \in \mathbb{Z}$ and $k_1(j) \leqslant k \leqslant k_0(j)$, further divides into 4 subcases determined by the behavior of j:

Case (4a). $k_1(j) \leqslant k \leqslant k_0(j)$ and $0 < q^j\omega < 1/\sqrt{q}$.
Case (4b). $k_1(j) \leqslant k \leqslant k_0(j)$ and $1/\sqrt{q} \leqslant q^j\omega < 2\pi/b - 1/\sqrt{q}$.
Case (4c). $k_1(j) \leqslant k \leqslant k_0(j)$ and $2\pi/b - 1/\sqrt{q} \leqslant q^j\omega \leqslant 2\pi/b + 1/\sqrt{q}$.
Case (4d). $k_1(j) \leqslant k \leqslant k_0(j)$ and $2\pi/b + 1/\sqrt{q} < q^j\omega$.

Remark. The cases are expressed above as a partition of the positive reals. However they actually describe the behavior of $q^j\omega - 2\pi/b$ relative to the interval $[-1/\sqrt{q}, 1/\sqrt{q}\,]$, and they help describe $k_1(j)$ and $k_0(j)$. For instance, (4a) gives that $q^j\omega - 2\pi/b < -1/\sqrt{q}$ and j tends to be negative; (4b) gives that $q^j\omega - 2\pi/b < -1/\sqrt{q}$ and j tends to be non-negative; (4c) gives that $-1/\sqrt{q} \leqslant q^j\omega - 2\pi/b \leqslant 1/\sqrt{q}$; and (4d) gives that $1/\sqrt{q} < q^j\omega - 2\pi/b$. These subcases impose conditions on $k_1(j)$ and $k_0(j)$, and the statements about j, while collectively are comprehensive, individually impose restrictions on j after taking logarithms. We repeat the cases from this perspective:

Case (4a). $k_1(j) = -1 \leqslant k \leqslant -1 = k_0(j)$ and $j < -\log_q \omega - 1/2 \equiv N_1$.
Case (4b). $k_1(j) = -1 \leqslant k \leqslant -1 = k_0(j)$ and $N_1 \equiv -\log_q \omega - 1/2 \leqslant j < -\log_q \omega + \log_q(2\pi/b - 1/\sqrt{q}) \equiv N_2$.
Case (4c). $k_1(j) = -2 \leqslant k \leqslant -1 = k_0(j)$ and $N_2 \equiv -\log_q \omega + \log_q(\frac{2\pi}{b} - \frac{1}{\sqrt{q}}) \leqslant j \leqslant -\log_q \omega + \log_q(2\pi/b + 1/\sqrt{q}) \equiv N_3$.
Case (4d). $k_1(j) \leqslant k \leqslant k_0(j)$ and $N_3 \equiv -\log_q \omega + \log_q(2\pi/b + 1/\sqrt{q}) < j$.

Proposition 9. *In Case* (4a) *we have for* $q > 1$, $\omega \in [1, q]$, $2\pi/\sqrt{q} > b > 0$, *and* $\pi\sqrt{q} > b > 0$

$$\widetilde{G}_1(\omega; j < N_1; k = -1) \leqslant \frac{b}{2\pi}\frac{\kappa_q}{\nu_q}\left(6q + 2 + \sqrt{\frac{2\pi}{\ln q}}\right).$$

Proof. The condition $j < N_1$ gives that either (i) $j \leqslant -2$ or (ii) $j = -1$ and $1 \leqslant \omega < \sqrt{q}$. For (i) we obtain

$$\frac{q^j \omega b}{2\pi} \leqslant \frac{q^j q b}{2\pi} \leqslant \frac{q^j q\sqrt{q}}{2} \leqslant \frac{q^{-1/2}}{2} \leqslant \frac{1}{2}.$$

For (ii) we obtain

$$\frac{q^j \omega b}{2\pi} \leqslant \frac{q^j \sqrt{q} b}{2\pi} \leqslant \frac{q^{-1}\sqrt{q}\sqrt{q}}{2} = \frac{1}{2},$$

which gives in either case that

$$1 - \frac{q^j \omega b}{2\pi} \geqslant 1 - 1/2 = 1/2 \quad \text{or} \quad \frac{1}{1 - q^j \omega b/(2\pi)} \leqslant 2. \tag{69}$$

Thus

$$\begin{aligned}
\widetilde{G}_1(\omega; j < N_1; k = -1) &= \sum_{j<N_1} \left|\hat{K}\left(q^j\omega\right)\right| \sum_{k=-1} \left|\hat{K}\left(q^j\omega + 2\pi k/b\right)\right| \\
&= \sum_{j<N_1} \left|\hat{K}\left(q^j\omega\right)\right| \frac{\sqrt{\kappa_q}}{|q^j\omega - 2\pi/b|\sqrt{\theta(q^2; (q^j\omega - 2\pi/b)^2)}} \\
&= \sum_{j<N_1} \left|\hat{K}\left(q^j\omega\right)\right| \frac{b}{2\pi} \frac{\sqrt{\kappa_q}}{|q^j\omega b/(2\pi) - 1|\sqrt{\theta(q^2; (q^j\omega - 2\pi/b)^2)}} \\
&\leqslant \sum_{j<N_1} \left|\hat{K}\left(q^j\omega\right)\right| \frac{b}{2\pi} \frac{2\sqrt{\kappa_q}}{\sqrt{\theta(q^2; 1/q)}} \qquad (70) \\
&\leqslant \sum_{j\leqslant -1} \left|\hat{K}\left(q^j\omega\right)\right| \frac{b}{2\pi} \frac{2\sqrt{\kappa_q}}{\sqrt{\nu_q}} \\
&\leqslant \frac{b}{2\pi} 2\frac{\kappa_q}{\nu_q}\left(3q + \frac{1}{2}\left(2 + \sqrt{\frac{2\pi}{\ln q}}\right)\right), \qquad (71)
\end{aligned}$$

where we have used (69) to obtain (70), along with Proposition 8 with the cases $j \leqslant -4$ and $j = -3, -2, -1$ to obtain (71). □

Proposition 10. *In Case* (4b) *we have for* $q > 1$, $\omega \in [1, q]$, $2\pi/\sqrt{q} > b > 0$, *and* $\pi\sqrt{q} > b > 0$

$$\widetilde{G}_1(\omega; N_1 \leqslant j < N_2; k = -1) \leqslant \frac{b}{2\pi}\frac{2\sqrt{q}\,\kappa_q}{\sqrt{\nu_q}}\left(\frac{1}{\sqrt{\theta(q^2; \omega^2)}} \sum_{N_1 \leqslant j < N_2} \frac{1}{q^{j(j+1)/2}\omega^j}\right). \tag{72}$$

Proof. The condition $N_1 \leqslant j < N_2$ gives that either (i) $j = -1$ and $\sqrt{q} \leqslant \omega \leqslant q$ or (ii) $0 \leqslant j < -\log_q \omega + \log_q(2\pi/b - 1/\sqrt{q})$. These conditions on j imply

$$q^j\omega - \frac{2\pi}{b} < \frac{-1}{\sqrt{q}} \iff \frac{b}{2\pi} < \frac{1}{q^j\omega + 1/\sqrt{q}} \iff \frac{q^j\omega b}{2\pi} < \frac{q^j\omega}{q^j\omega + 1/\sqrt{q}}. \tag{73}$$

Whence,

$$1 - \frac{q^j\omega b}{2\pi} > \frac{1/\sqrt{q}}{q^j\omega + 1/\sqrt{q}} = \frac{1}{q^{j+1/2}\omega + 1}, \tag{74}$$

or

$$\frac{1}{1 - q^j\omega b/(2\pi)} < q^{j+1/2}\omega + 1. \tag{75}$$

We have reached a stage parallel to (69) in Case (4a), however, unlike that case, we do not obtain a bound corresponding to the upper bound of 2 in Case (4a). Since j becomes positive, the right-hand side of (75) can be quite large for small values of b. Thus we incorporate another factor from our summand in (72) to obtain the following bound:

$$\frac{1}{q^j\omega}\frac{1}{(1-q^j\omega b/(2\pi))} < \frac{q^{j+1/2}\omega+1}{q^j\omega} = \sqrt{q}+\frac{1}{q^j\omega} \leqslant \sqrt{q}+\sqrt{q}, \tag{76}$$

where the last inequality on $1/q^j\omega$ is obtained in the maximal case (i) of (4b) with $j=-1$ and $\omega=\sqrt{q}$.

We are now set to obtain our bound (72):

$$\begin{aligned}
&\sum_{N_1\leqslant j<N_2}\left|\hat{K}\left(q^j\omega\right)\right|\sum_{k=-1}\left|\hat{K}\left(q^j\omega+2\pi k/b\right)\right| \\
&\leqslant \sum_{N_1\leqslant j<N_2}\frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega)^2)}}\frac{1}{q^j\omega\frac{2\pi}{b}|q^j\omega b/(2\pi)-1|}\frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega-\frac{2\pi}{b})^2)}} \\
&\leqslant \frac{b}{2\pi}\kappa_q\sum_{N_1\leqslant j<N_2}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}\frac{2\sqrt{q}}{1}\frac{1}{\sqrt{\theta(q^2;1/q)}} \\
&= \frac{b}{2\pi}\frac{2\sqrt{q}\kappa_q}{\sqrt{\nu_q}}\left(\frac{1}{\sqrt{\theta(q^2;\omega^2)}}\sum_{N_1\leqslant j<N_2}\frac{1}{q^{j(j+1)/2}\omega^j}\right),
\end{aligned} \tag{77}$$

where (77) was obtained from (76). □

Proposition 11. *In Case* (4c) *we have for* $q>1$, $\omega\in[1,q]$, $2\pi/\sqrt{q}>b>0$, *and* $\pi\sqrt{q}>b>0$

$$\widetilde{G}_1(\omega;N_2\leqslant j\leqslant N_3;k=-1,-2) \leqslant \frac{b}{2\pi}\frac{\kappa_q}{\sqrt{\nu_q}}(2q+2\sqrt{q})\left(\frac{1}{\sqrt{\theta(q^2;\omega^2)}}\sum_{N_2\leqslant j\leqslant N_3}\frac{1}{q^{j(j+1)/2}\omega^j}\right). \tag{78}$$

Proof. The condition $N_2\leqslant j\leqslant N_3$ gives that

$$1/\sqrt{q} < -1/\sqrt{q}+2\pi/b < q^j\omega < 1/\sqrt{q}+2\pi/b$$

which yields the bound

$$\frac{1}{q^j\omega} < \frac{1}{-1/\sqrt{q}+2\pi/b} = \frac{b}{2\pi}\frac{1}{1-b/(2\pi\sqrt{q})} < \frac{b}{2\pi}2, \tag{79}$$

where the last inequality follows from the hypothesis $b<\pi\sqrt{q}$ and the fact that

$$\frac{2}{\sqrt{q}} < \frac{2\pi}{b} \quad\Longleftrightarrow\quad \frac{b}{2\pi\sqrt{q}} < \frac{1}{2}. \tag{80}$$

Furthermore, we have that $q^j\omega-4\pi/b < 1/\sqrt{q}-2\pi/b < -1/\sqrt{q}$ whence

$$\frac{1}{|q^j\omega-4\pi/b|} < \frac{1}{-1/\sqrt{q}+2\pi/b} = \frac{b}{2\pi}\frac{1}{(1-b/(2\pi\sqrt{q}))} < \frac{b}{2\pi}2. \tag{81}$$

We now obtain the estimate

$$\begin{aligned}
&\sum_{N_2\leqslant j\leqslant N_3}\left|\hat{K}\left(q^j\omega\right)\right|\sum_{k=-1,-2}\left|\hat{K}\left(q^j\omega+2\pi k/b\right)\right| \\
&= \sum_{N_2\leqslant j\leqslant N_3}\left|\hat{K}\left(q^j\omega\right)\right|\left(\left|\hat{K}\left(q^j\omega-\frac{2\pi}{b}\right)\right|+\frac{\sqrt{\kappa_q}}{|q^j\omega-\frac{4\pi}{b}|\sqrt{\theta(q^2;(q^j\omega-\frac{4\pi}{b})^2)}}\right) \\
&\leqslant \sum_{N_2\leqslant j\leqslant N_3}\frac{\sqrt{\kappa_q}}{q^j\omega\sqrt{\theta(q^2;(q^j\omega)^2)}}\left(\frac{\sqrt{\kappa_q}q}{\sqrt{\theta(q^2;1/q)}}+\frac{b}{2\pi}\frac{2\sqrt{\kappa_q}}{\sqrt{\theta(q^2;1/q)}}\right) \qquad (82)\\
&\leqslant \frac{\kappa_q}{\sqrt{\nu_q}}\sum_{N_2\leqslant j\leqslant N_3}\frac{1}{q^j\omega}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}\left(q+\frac{b}{2\pi}2\right) \qquad (83)
\end{aligned}$$

$$\leqslant \frac{\kappa_q}{\sqrt{\nu_q}} \sum_{N_2 \leqslant j \leqslant N_3} \frac{b}{2\pi} 2 \frac{1}{q^{j(j+1)/2}\omega^j \sqrt{\theta(q^2;\omega^2)}} (q+\sqrt{q}) \tag{84}$$

$$\leqslant \frac{b}{2\pi} \frac{\kappa_q}{\sqrt{\nu_q}} (2q + 2\sqrt{q}) \left(\frac{1}{\sqrt{\theta(q^2;\omega^2)}} \sum_{N_2 \leqslant j \leqslant N_3} \frac{1}{q^{j(j+1)/2}\omega^j} \right),$$

where (39) and (81) were used to obtain (82), (29) was used to obtain (83), (79) and (80) were used to obtain (84). □

Proposition 12. *In Case* (4d) *we have for* $q > 1$, $\omega \in [1, q]$, $2\pi/\sqrt{q} > b > 0$, *and* $\pi\sqrt{q} > b > 0$

$$\widetilde{G}_1\big(\omega; N_3 < j; k_1(j) \leqslant k \leqslant k_0(j)\big) \leqslant \frac{b}{2\pi} \frac{\kappa_q}{\sqrt{\nu_q}} (2q + 3\sqrt{q}) \left(\frac{1}{\sqrt{\theta(q^2;\omega^2)}} \sum_{N_3 < j} \frac{1}{q^{j(j+1)/2}\omega^j} \right). \tag{85}$$

Proof. The condition $N_3 < j$ gives analogues of (73), (74), and (75) when $k = k_1(j), k_0(j)$. For instance, when $k = k_1(j)$ we have

$$q^j\omega + \frac{2\pi k_1(j)}{b} < \frac{-1}{\sqrt{q}} \iff \frac{-b}{2\pi k_1(j)} < \frac{1}{q^j\omega + 1/\sqrt{q}}$$

$$\iff \frac{-q^j\omega b}{2\pi k_1(j)} < \frac{q^j\omega}{q^j\omega + 1/\sqrt{q}}.$$

Whence,

$$1 + \frac{q^j\omega b}{2\pi k_1(j)} > \frac{1/\sqrt{q}}{q^j\omega + 1/\sqrt{q}} = \frac{1}{q^{j+1/2}\omega + 1},$$

or

$$\frac{1}{1 + q^j\omega b/(2\pi k_1(j))} < q^{j+1/2}\omega + 1. \tag{86}$$

Thus (86) gives

$$\frac{1}{q^j\omega} \frac{1}{(1 + q^j\omega b/(2\pi k_1(j)))} < \frac{q^{j+1/2}\omega + 1}{q^j\omega} = \sqrt{q} + \frac{1}{q^j\omega} < \sqrt{q} + \sqrt{q}, \tag{87}$$

where the last inequality on $1/(q^j\omega)$ follows since $1/\sqrt{q} < q^j\omega$.

When $k = k_0(j)$ we have

$$q^j\omega + \frac{2\pi k_0(j)}{b} > \frac{1}{\sqrt{q}} \iff \frac{-b}{2\pi k_0(j)} > \frac{1}{q^j\omega - 1/\sqrt{q}}$$

$$\iff \frac{-q^j\omega b}{2\pi k_0(j)} > \frac{q^j\omega}{q^j\omega - 1/\sqrt{q}}.$$

Whence,

$$1 + \frac{q^j\omega b}{2\pi k_0(j)} < \frac{-1/\sqrt{q}}{q^j\omega - 1/\sqrt{q}} = \frac{-1}{q^{j+1/2}\omega - 1},$$

or

$$\frac{1}{|1 + q^j\omega b/(2\pi k_0(j))|} < q^{j+1/2}\omega - 1. \tag{88}$$

Thus (88) gives

$$\frac{1}{q^j\omega} \frac{1}{|1 + q^j\omega b/(2\pi k_0(j))|} < \frac{q^{j+1/2}\omega - 1}{q^j\omega} = \sqrt{q} - \frac{1}{q^j\omega} < \sqrt{q}. \tag{89}$$

So in the $k = k_1(j), k_0(j)$ cases we have

$$
\begin{aligned}
&\sum_{N_3<j} \left|\hat{K}\left(q^j\omega\right)\right| \sum_{k=k_1(j),k_0(j)} \left|\hat{K}\left(q^j\omega + 2\pi k/b\right)\right| \\
&\quad = \sum_j \sum_{k=k_1,k_0} \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega)^2)}} \frac{1}{q^j\omega|q^j\omega+2\pi k/b|} \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega+2\pi k/b)^2)}} \\
&\quad \leqslant \sum_{N_3<j} \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega)^2)}} \frac{b}{2\pi}(2\sqrt{q}+\sqrt{q}) \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;1/q)}} && (90)\\
&\quad \leqslant \frac{b}{2\pi}\frac{\kappa_q}{\sqrt{\nu_q}} 3\sqrt{q} \sum_{N_3<j} \frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}, && (91)
\end{aligned}
$$

where (87) and (89) yield (90), and (29) and (33) give (91).

Finally, we handle the case that $k_1(j) < k < k_0(j)$, where

$$-1/\sqrt{q} < q^j\omega + 2\pi k/b < 1/\sqrt{q}.$$

By (80), there is at most one such k for each j. Then by (39)

$$\left|\hat{K}\left(q^j\omega + 2\pi k/b\right)\right| \leqslant \sqrt{\kappa_q}q/\sqrt{\theta\left(q^2;1/q\right)} = \sqrt{\kappa_q}q/\sqrt{\nu_q}. \tag{92}$$

Furthermore,

$$\frac{1}{q^j\omega} < \frac{1}{-2\pi k/b - 1/\sqrt{q}} = \frac{b}{2\pi|k|}\frac{1}{(1+b/(2\pi k\sqrt{q}))} < \frac{b/(2\pi)}{(1+b/(2\pi k\sqrt{q}))}$$

and

$$0 < \frac{-b}{2\pi\sqrt{q}k} < \frac{b}{2\pi\sqrt{q}} < \frac{1}{2} \quad\Longrightarrow\quad \frac{1}{1+b/(2\pi\sqrt{q}k)} < 2$$

combine to give

$$\frac{1}{q^j\omega} < \frac{b}{2\pi}2. \tag{93}$$

Hence

$$
\begin{aligned}
&\sum_{N_3<j} \left|\hat{K}\left(q^j\omega\right)\right| \sum_{k_1(j)<k<k_0(j)} \left|\hat{K}\left(q^j\omega + 2\pi k/b\right)\right| && (94)\\
&\quad \leqslant \sum_{N_3<j}\left(\sum_{k_1(j)<k<k_0(j)} \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega)^2)}}\frac{1}{q^j\omega}\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}q\right) && (95)\\
&\quad \leqslant \sum_{N_3<j} \frac{\sqrt{\kappa_q}}{\sqrt{\theta(q^2;(q^j\omega)^2)}}\frac{b}{2\pi}2\frac{\sqrt{\kappa_q}}{\sqrt{\nu_q}}q && (96)\\
&\quad \leqslant \frac{b}{2\pi}\frac{\kappa_q}{\sqrt{\nu_q}}2q\sum_{N_3<j}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}, && (97)
\end{aligned}
$$

where we have used (92) to bound the right factor of (94) to obtain (95), and then employed (93) to obtain (96). Adding (91) and (97) gives the proposition. □

Lemma 3. *For $q > 1$ and $\omega \in [1, q]$ we have*

$$\sum_{-1\leqslant j}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}} \leqslant \frac{1}{\sqrt{\nu_q}}\left(2+\frac{1}{2}\sqrt{2\pi/\ln q}\right). \tag{98}$$

Proof. We have

$$\sum_{-1\leqslant j}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}$$

$$=\sum_{-1\leqslant j}\frac{e^{-(1/2)\ln q\{j^2+j(1+2\log_q\omega)\}}}{\sqrt{\theta(q^2;\omega^2)}}$$

$$=\frac{e^{(1/2)\ln q(1/2+\log_q\omega)^2}}{\sqrt{\theta(q^2;\omega^2)}}\left(e^{-(1/2)\ln q\{-1/2+\log_q\omega\}^2}\right)+\frac{e^{(1/2)\ln q(1/2+\log_q\omega)^2}}{\sqrt{\theta(q^2;\omega^2)}}\left(\sum_{0\leqslant j}e^{-(1/2)\ln q\{j+1/2+\log_q\omega\}^2}\right) \tag{99}$$

$$=\frac{q^{1/8+(1/2)\log_q\omega+(1/2)(\log_q\omega)^2}}{\sqrt{\theta(q^2;\omega^2)}}\left(q^{-1/8+(1/2)\log_q\omega-(1/2)(\log_q\omega)^2}\right)$$

$$+\frac{q^{1/8+(1/2)\log_q\omega+(1/2)(\log_q\omega)^2}}{\sqrt{\theta(q^2;\omega^2)}}\left(\sum_{0\leqslant j}e^{-(1/2)\ln q\{j+1/2+\log_q\omega\}^2}\right) \tag{100}$$

$$\leqslant\frac{q^{\log_q\omega}}{\sqrt{\theta(q^2;\omega^2)}}+\frac{q^{1/8+(1/2)\log_q\omega+(1/2)(\log_q\omega)^2}}{\sqrt{\theta(q^2;\omega^2)}}\sum_{0\leqslant j}e^{-(1/2)\ln q\{(j+1/2)^2+(\log_q\omega)^2\}} \tag{101}$$

$$\leqslant\frac{1}{\sqrt{\nu_q}}+\frac{q^{1/8+(1/2)\log_q\omega}}{\sqrt{\theta(q^2;\omega^2)}}\sum_{0\leqslant j}e^{-(1/2)\ln q(j+1/2)^2} \tag{102}$$

$$\leqslant\frac{1}{\sqrt{\nu_q}}+\frac{q^{1/8+(1/2)\log_q\omega}}{\sqrt{\theta(q^2;\omega^2)}}q^{-1/8}\frac{1}{2}\left(2+\sqrt{2\pi/\ln q}\right) \tag{103}$$

$$=\frac{1}{\sqrt{\nu_q}}+\frac{q^{(1/2)\log_q\omega}}{\sqrt{\theta(q^2;\omega^2)}}\frac{1}{2}\left(2+\sqrt{2\pi/\ln q}\right) \tag{104}$$

$$\leqslant\frac{1}{\sqrt{\nu_q}}+\frac{1}{\sqrt{\nu_q}}\frac{1}{2}\left(2+\sqrt{2\pi/\ln q}\right), \tag{105}$$

where we have: completed the squares and separated the $j=-1$ term from the $j\geqslant 0$ sum to obtain (99); re-expressed terms with a base q in (100); canceled like terms in the exponents and dropped the $-\ln q(j+1/2)(\log_q\omega)$ from the exponent in the $j\geqslant 0$ sum to obtain (101); applied the useful estimate (38) of Lemma 2 on the first term to obtain (102); applied estimate (56) to bound the $j\geqslant 0$ sum in obtaining (103); canceled like terms in exponents to obtain (104); and again applied (38) of Lemma 2 for (105), yielding the lemma. □

Proposition 13. *For $q>1$, $\omega\in[1,q]$, $2\pi/\sqrt{q}>b>0$, and $\pi\sqrt{q}>b>0$ we have*

$$\widetilde{G}_1\big(\omega;N_1\leqslant j;k_1(j)\leqslant j\leqslant k_0(j)\big)\leqslant\frac{b}{2\pi}\frac{\kappa_q}{\nu_q}\left(2q+3\sqrt{q}\right)\left(2+\frac{1}{2}\sqrt{2\pi/\ln q}\right).$$

Proof. Noticing that $\max\{2\sqrt{q},2q+2\sqrt{q},2q+3\sqrt{q}\}=2q+3\sqrt{q}$ and then bounding $2\sqrt{q}$ by $2q+3\sqrt{q}$ in (72), and $2q+2\sqrt{q}$ by $2q+3\sqrt{q}$ in (78), and by adding the resulting analogues of (72) and (78) to (85), we obtain

$$\sum_{N_1\leqslant j}^{\infty}\left|\hat{K}\big(q^j\omega\big)\right|\sum_{k_1(j)\leqslant k\leqslant k_0(j)}\left|\hat{K}\big(q^j\omega+2\pi k/b\big)\right|$$

$$\leqslant\frac{b}{2\pi}\frac{\kappa_q}{\sqrt{\nu_q}}\left(2q+3\sqrt{q}\right)\left(\sum_{N_1\leqslant j}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}\right) \tag{106}$$

$$\leqslant\frac{b}{2\pi}\frac{\kappa_q}{\sqrt{\nu_q}}\left(2q+3\sqrt{q}\right)\left(\sum_{-1\leqslant j}\frac{1}{q^{j(j+1)/2}\omega^j\sqrt{\theta(q^2;\omega^2)}}\right) \tag{107}$$

$$\leqslant\frac{b}{2\pi}\frac{\kappa_q}{\sqrt{\nu_q}}\left(2q+3\sqrt{q}\right)\frac{1}{\sqrt{\nu_q}}\left(2+\frac{1}{2}\sqrt{2\pi/\ln q}\right), \tag{108}$$

where we have factored out a maximum $2q+3\sqrt{q}$ and combined all sums for (106), proceeded from the sum over $N_1\leqslant j$ to the sum over the possibly slightly larger index $-1\leqslant j$ for (107), and used (98) in Lemma 3 to obtain (108) and hence the proposition. □

Proof of Proposition 6. Add the bounds in Propositions 9 and 13. □

In summary, we have been able to show the efficacy of larger translation parameters in the generation of wavelet frames for $\mathcal{L}^2(\mathbb{R})$. The driving force for this improvement is the use of theta function identities in obtaining an exact calculation of G_0 and in obtaining accurate estimates for G_1. This allows us to establish a threshold $b = \mathcal{N}(q)$ below which the parameters (q, b) allow K to generate wavelet frames. Similarly for $bq^p < \mathcal{N}(q)$ the parameters (q, b) allow $K^{(p)}$ to generate wavelet frames. Every function in our frames has many interesting properties, including the fact that each has all moments vanishing and each satisfies an advanced differential equation. For large q and for very negative p our frames become snug which will impact efficiency in invertibility.

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