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A nonlocal model with strain-based damage

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To my daughter Angela.

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ABSTRACT

A thermodynamically consistent formulation of nonlocal damage in the framework of the internal variable theories of inelastic behaviours of associative type is presented. The damage behaviour is defined in the strain space and the effective stress turns out to be additively splitted in the actual stress and in the nonlocal counterpart of the relaxation stress related to damage phenomena. An important advantage of models with strain-based loading functions and explicit damage evolution laws is that the stress corresponding to a given strain can be evaluated directly without any need for solving a nonlinear system of equations. A mixed nonlocal variational formulation in the complete set of state variables is presented and is specialized to a mixed two-field variational formulation. Hence a finite element procedure for the analysis of the elastic model with nonlocal damage is established on the basis of the proposed two-field variational formulation. Two examples concerning a one-dimensional bar in simple tension and a two-dimensional notched plate are addressed. No mesh dependence or boundary effects are apparent.

1. Introduction

The realistic modelling of inelastic behaviour of ductile or quasi-brittle materials is essential for the solution of numerous boundary-value problems occurring in various engineering fields. For example, microscopic defects and cracks cause reduction in strength of materials and shorten the life time of engineering structures. Therefore, a main issue in engineering applications is to provide realistic information on strain, stress and damage distributions within elements of such materials.

In order to describe the gradual internal deterioration of solids, within the general framework of continuum thermodynamics of irreversible processes, several continuum damage models have been proposed which are either phenomenologically based or micromechanically motivated.

Classical local theories can provide reasonable predictions of the conditions at which strain localization may occur, see e.g. Lemaitre (1996), Chambon et al. (2000) and Brünig (2003a,b). An important issue in such phenomenological constitutive models is the appropriate choice of the physical nature of mechanical variables describing the damage state of materials. For example, scalar valued damage variables have been proposed by Lemaitre (1985), Tai and Yang (1986), Alves et al. (2000) and Celentano and Chaboche (2007). However, isotropic damage models predict lower strength of materials compared to the theories of anisotropic damage (Chow and Wang, 1987, 1988).

Thus, second-order damage tensors have been introduced in order to describe anisotropic damage phenomena (see e.g. Kachanov, 1980; Ju, 1990; Baste and Audoin, 1991; Voyiadjis and Kattan, 1992; Bruhns and Schiesse, 1996; Steinmann and Carol, 1998; Brünig, 2004, 2008; Menzel et al., 2005).

In quasi-brittle materials such as concrete, rocks, soils, tough ceramics and composites, the degradation of the material elastic properties is associated with damage which induces a remarkable overall strength reduction when a certain damage threshold is attained. Failure is then preceded by a gradual development of a nonlinear fracture process zone and by localization of strain. The failure processes characterized by strain localization into shear bands provide an example in which classical continuum theories reach the limit of their range of application.

In fact, simple materials in the sense of Truesdell and Noll (1965) do not contain information about the size of the localization zone which tends to become infinitely thin in the continuum approach or takes on the size of the smallest finite element in the FE-approach.

In order to overcome the above-mentioned limits of classical continuum mechanics, many remedies have been proposed.

The first example of generalized continuum theories is provided by Cosserat, or micropolar, continuum proposed by Cosserat and Cosserat (1909). In such media each material particle is considered as a rigid body endowed with three additional degrees of freedom (microrotations). Further contributions to the development and generalization of micropolar continuum theories has been provided by Toupin (1962) and Eringen (1966). However, Cosserat continua found their applications in the analysis of failure

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problems as a method to regularize the balance equations in presence of strain localization only more later, see e.g. de Borst (1991), Steinmann (1994) and Ehlers and Volk (1998).

A natural generalization of micropolar continua is provided by continua with microstructure proposed by Toupin (1964) and Mindlin (1964) and also referred to as micromorphic continua (Eringen, 1968; Capriz, 1989; Eringen, 1998; Vardoulakis and Sulem, 1995; Forest and Sievert, 2003). Micromorphic continuum theories embedded micropolar continua as a special case since each particle of the body is regarded as being itself a deformable continuum subject to affine (homogeneous) deformations.

Nonlocal regularization methods introduce the characteristic length as an additional material parameter which describes the nonlocal micro-interactions produced in media suffering microdecohesion or damage processes.

Nonlocal theories are defined either in a strong form (integral-type), see e.g. Eringen (1981), Pijaudier-Cabot and Bažant (1987), Borino et al. (2003), Jirásek and Rolshoven (2003), Poh and Swaddiwudhipong (2009) or in a weak form (spatial higher gradients), i.e. Aifantis (1984, 1992), de Borst and Mühlhaus (1992), de Borst and Pamin (1996), Fleck and Hutchinson (2001), Liebe and Steinmann (2001), Gurtin (2003, 2004), Gurtin and Anand (2005), Samala et al. (2008), Jirásek and Rolshoven (2009a,b).

Nonlocal approaches in the softening regime, induced by damage or by strain-softening plasticity, regularize the inelastic dissipative model and re-establish a well-posed boundary value problem. Accordingly the variables that need to be regularized are the ones related to the dissipative mechanism (Bažant, 1984, 1991; Pijaudier-Cabot and Bažant, 1987; Bažant and Planas, 1998; Cazes et al., 2009; Velde et al., 2009). An alternative description of the localization process up to complete failure in nonlocal isotropic damage model can be based on averaging of the equivalent strain or displacement, see Jirásek (1998) and Jirásek and Marfia (2005).

From a computational point of view the nonlinear problem associated with elasto-damage models adopt a Newton–Raphson iterative scheme consisting in an elastic predictor phase followed by a corrector phase which closely imitates the classical scheme of computational plasticity, see e.g. Simo and Hughes (1998). A different approach for nonlocal plasticity and for nonlocal plasticity with damage is provided in Brünig et al. (2001) and Brünig and Ricci (2005) based on an inelastic (damage-plastic) predictor-elastic corrector scheme starting from the plastic predictor-elastic corrector technique presented by Nemat-Nasser (1991). The computational algorithm related to the nonlocal damage and yield conditions lead to partial differential equations solved using the finite difference method while the standard field variables are obtained by a standard FEM.

In this paper, attention is focused on a geometrically linear model of nonlocal damage of integral type and a loading function, controlling the evolution of damage, is introduced. According to the damage theory in the strain space, it is assumed that the arguments of the loading function include the strain (Jirásek and Patzák, 2002; Pamin, 2005; Jason et al., 2006; Häussler-Combe and Hartig, 2008). An important advantage of models with strain-based loading functions and explicit damage evolution laws is that the stress corresponding to a given strain can be evaluated directly, without any need for solving a nonlinear system of equations. As a consequence the numerical implementation of the nonlocal strain damage model is relatively straightforward. Of course, equilibrium iteration on the structural level cannot be avoided, same as for any other nonlinear model.

The aim of the paper is to formulate a nonlocal model of strain damage in which the stress decomposition consistently follows from the thermodynamic analysis based on a first principle written in a global form and on a second principle written in a pointwise form. The nonlocal part of the stress decomposition is a weighted average of the corresponding local variable over a certain finite neighbourhood of each material point and characterizes the state of damage of the material. Hence it is not necessary to introduce a damage tensor as an additional internal variable of the model.

A variational formulation in the complete set of local and nonlocal state variables is then provided. The methodology to recover variational formulations with different combinations of the state variables is then shown. In particular, a two-field nonlocal mixed variational formulation is derived so that the FE algorithm for the resolution of the finite-step elastic model with nonlocal damage can be properly defined and it is not based on ad hoc extensions of procedures pertaining to local damage.

The paper is organized as follows. Section 2 summarizes the basic relation of local damage. Then the proposed model is cast in a thermodynamic framework in Section 3 and the elastic constitutive model endowed with nonlocal strain damage is provided. In Section 4 the structural elastic problem with nonlocal damage is formulated and a variational formulation in the complete set of state variables is derived. A two-field nonlocal mixed variational formulation in terms of displacements and relaxation stresses is obtained and the corresponding nonlocal finite element procedure is obtained. As a special case the FE method for local damage is recovered. Section 5 shows numerical results for a 1D bar in traction and for a plane stress indented plate subjected to traction. The results are mesh independent with no boundary effects. The paper is closed by two Appendix. The formed is devoted to the proof of the nonlocal variational formulation in the complete set of state variables and the latter deals with some basic results of convex analysis adopted in the paper.

2. Local damage model

A quasi-static evolution process in a geometrically linear range is considered for an elastic body with damage subject to a given load history. The model is defined on a regular domain Ω, with boundary $\partial\Omega$, of the three-dimensional Euclidean space. A time-independent mechanical behaviour of the body is assumed so that the time is conceived as a monotonically increasing parameter which orders successive events.

Let $\mathscr{D}$ denote the linear space of strain tensors $\boldsymbol{\varepsilon}$ and $\mathscr{S}$ be the dual space of stress tensors $\boldsymbol{\sigma}$. The inner product in the dual spaces $\langle\cdot,\cdot\rangle$ has the mechanical meaning of the internal virtual work pertaining to the body Ω, that is:

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x})\, d\Omega,$$

where the scalar product between dual quantities (simple or double index saturation operation between vectors or tensors) is denoted by $*$. Bold-face letters are associated with vectors and tensors.

It is well-known that the stress for a damaged material is redistributed to the undamaged material over the effective section of resistance. Then the true stress corresponding to the undamaged material microbonds is higher than the nominal stress. In continuum damage mechanics the stress calculated over the effective area is called the *effective stress* $\boldsymbol{\sigma}^e$ (see e.g. Lemaitre, 1996).

The effective stress $\boldsymbol{\sigma}^e$ can be related to the actual stress $\boldsymbol{\sigma}$ by means of an effective-stress operator M, depending on a damage tensor $\mathbf{D}$ which characterizes the state of damage of the material, in the form:

$$\boldsymbol{\sigma}^e = M(\mathbf{D})\boldsymbol{\sigma}. \tag{1}$$

The operator M takes into account the area of the microvoids and microcracks, stress concentrations due to the microcracks and the interactions between neighbouring defects.

The mechanical behaviour of microcracks depends on their orientation so that damage is, essentially, an anisotropic phenomenon. Nevertheless damage theories based on a scalar parameter are widely used in applications due to their simplicity and their agreement with experimental behaviour of real models (see e.g. Jordon et al., 2007; Bažant, 1984).

Isotropic damage formulations (see e.g. Lemaitre and Chaboche, 1994) require that the damage operator **D** reduces to a scalar damage parameter d so that the operator M is defined in the form $1/(1-d)$. As a consequence, the effective stress is given by:

$$\boldsymbol{\sigma}^e = \frac{\boldsymbol{\sigma}}{1-d}, \tag{2}$$

where the damage parameter d belongs to the interval $[0, d_c]$. The value $d = 0$ corresponds to the undamaged material whereas a non-zero value of d in the interval $]0, d_c[$ corresponds to a damage state. The value $d = d_c < 1$ corresponds to the local rupture.

3. Nonlocal thermodynamic framework

The first principle of thermodynamics for a nonlocal behaviour is expressed in the global form (see e.g. Edelen and Laws, 1971; Polizzotto, 2007):

$$\int_\Omega \dot{u}\, d\mathbf{x} = \langle \boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}} \rangle + \int_\Omega \dot{Q}\, d\mathbf{x}, \tag{3}$$

where upper dot means time derivative and the pointwise internal energy$u$ depends on strain $\boldsymbol{\varepsilon}$, nonlocal static internal variable $\bar{\xi}$, whose mechanical meaning is clarified in the sequel, and entropy s. The heat supplied to an element of volume is $\dot{Q} = -\mathrm{div}\mathbf{q}$ being $\mathbf{q}$ the heat flux.

Note 1. The nonlocal internal variable $\bar{\xi}$ is denoted by a superposed bar and is given by:

$$\bar{\xi}(\mathbf{x}) = (R\xi)(\mathbf{x}) = \int_\Omega W(\mathbf{x}, \mathbf{y}) \xi(\mathbf{y}) d\mathbf{y} \qquad \forall \mathbf{x} \in \Omega, \tag{4}$$

where the linear regularization operator R (see e.g. Pijaudier-Cabot and Bažant, 1987; Strömber and Ristinmaa, 1996; Borino et al., 2003; Jirásek and Rolshoven, 2003) transforms the local static internal variable ξ into the related nonlocal variable $\bar{\xi}$ since its value at the point $\mathbf{x}$ of the body Ω depends on the entire field $\mathbf{y}$.

The following symmetric expression of the weight function W, similar to the one proposed in Borino et al. (2003), is considered in this paper:

$$W(\mathbf{x}, \mathbf{y}) = \left[1 - \alpha \frac{V(\mathbf{x})}{V_\infty}\right] \delta(\mathbf{x}, \mathbf{y}) + \frac{\alpha}{V_\infty} g(\mathbf{x}, \mathbf{y}), \tag{5}$$

where the symbol $\delta(\mathbf{x}, \mathbf{y})$ denotes the Dirac delta distribution, the scalar function $g(\mathbf{x}, \mathbf{y})$ is the attenuation (or influence) function depending on the material internal length scale l and $V(\mathbf{x})$ is the representative volume given by:

$$V(\mathbf{x}) = \int_\Omega g(\mathbf{x}, \mathbf{y})\, d\mathbf{y}. \tag{6}$$

Accordingly the weight function W fulfils the normalizing condition:

$$\int_\Omega W(\mathbf{x}, \mathbf{y}) d\mathbf{y} = 1, \tag{7}$$

for any $\mathbf{x}$ in Ω so that the regularization operator R coincides to the identity one for uniform fields ξ. Note that the regularization operator is self-adjoint at every point of the body including zones near to the boundary.

The value assumed by the representative volume $V(\mathbf{x})$ for an unbounded body is denoted by V_∞ and α is an adimensional scalar parameter which is concerned with the attenuation effects. The first term appearing in (5) is a local one. Setting $\alpha = 1$, it is effective for points $\mathbf{x}$ close to the boundary since, for points $\mathbf{x}$ far from the boundary, $V(\mathbf{x})$ tends to V_∞ and the local term vanishes. The second term is the classical nonlocal term associated with an unbounded body. The parameter α is added in the expression (5) of the weight function in order to control the proportion of the nonlocal addition as shown in Sections 3.1 and 3.2.

The pointwise form of (3) can be written as:

$$\dot{u} = \boldsymbol{\sigma} * \dot{\boldsymbol{\varepsilon}} + \dot{Q} + P, \tag{8}$$

where the nonlocality residual function P takes into account the energy exchanges between neighbour particles and fulfils the insulation condition:

$$\int_\Omega P\, d\mathbf{x} = 0, \tag{9}$$

since the body is a thermodynamically isolated system with reference to energy exchanges due to nonlocality. The explicit dependence on the point is dropped for simplicity.

The second principle of thermodynamics, in the present nonlocal context, is enforced in its classical pointwise form:

$$\dot{s}T + \mathrm{div}\mathbf{q} - \nabla T * \frac{\mathbf{q}}{T} \geqslant 0, \tag{10}$$

everywhere in Ω where $\dot{s}$ is the internal entropy production rate per unit volume (see Polizzotto, 2007). The symbol ∇ denotes the gradient operator and div is the divergence of $\mathbf{q}$.

Since structural models are often formulated in terms of the Helmholtz free energy ψ, the first law of thermodynamics (8) is expressed in terms of ψ which depends on the strain $\boldsymbol{\varepsilon}$, nonlocal static internal variable $\bar{\xi}$ and temperature T according to the result $\psi = u - sT$ in the form:

$$\dot{\psi} = \boldsymbol{\sigma} * \dot{\boldsymbol{\varepsilon}} - (sT)^{\cdot} + \dot{Q} + P. \tag{11}$$

The thermodynamic laws (10) and (11) then yield the non-negative total dissipation:

$$D = \boldsymbol{\sigma} * \dot{\boldsymbol{\varepsilon}} - \dot{\psi} - s\dot{T} + P - \nabla T * \frac{\mathbf{q}}{T} \geqslant 0. \tag{12}$$

Restricting the analysis to isothermal conditions, the nonlocal counterpart of the Clausius–Duhem inequality (12) is rewritten in the form:

$$D = \boldsymbol{\sigma} * \dot{\boldsymbol{\varepsilon}} - d_{\boldsymbol{\varepsilon}}\psi * \dot{\boldsymbol{\varepsilon}} - \dot{\bar{\xi}} * d_{\bar{\xi}}\psi + P = (\boldsymbol{\sigma} - d_{\boldsymbol{\varepsilon}}\psi) * \dot{\boldsymbol{\varepsilon}} - \dot{\bar{\xi}} * d_{\bar{\xi}}\psi + P \geqslant 0, \tag{13}$$

everywhere in Ω. Then, following widely used arguments (see e.g. Lemaitre and Chaboche, 1994), let an arbitrary elastic process be considered. Since no inelastic behaviours are activated, i.e. $\dot{\bar{\xi}} = \mathbf{0}$ and $P = 0$, the inequality (13) yields $\boldsymbol{\sigma} = d_{\boldsymbol{\varepsilon}}\psi(\boldsymbol{\varepsilon}, \bar{\xi})$ and the dissipation reads:

$$D = \dot{\bar{\xi}} * d_{\bar{\xi}}\psi + P \geqslant 0, \tag{14}$$

where $\bar{\xi}$ is the nonlocal static internal variable conjugate to $-d_{\bar{\xi}}\psi(\boldsymbol{\varepsilon}, \bar{\xi})$.

3.1. Nonlocal constitutive relations

Let us assume that the Helmholtz free energy is expressed in the following additive form:

$$\psi(\boldsymbol{\varepsilon}, \bar{\xi}) = \phi(\boldsymbol{\varepsilon}) - \bar{\xi} * \boldsymbol{\varepsilon}. \tag{15}$$

The dissipation (13) then becomes:

$$D = (\boldsymbol{\sigma} + \bar{\xi} - d_{\boldsymbol{\varepsilon}}\phi) * \dot{\boldsymbol{\varepsilon}} + \dot{\bar{\xi}} * \boldsymbol{\varepsilon} + P \geqslant 0. \tag{16}$$

The static internal variable ξ can be identified with the *relaxation stress* tensor $\boldsymbol{\sigma}^r$ so that the relation (16) yields the constitutive relation:

$$\boldsymbol{\sigma}^e = \boldsymbol{\sigma} + \bar{\boldsymbol{\sigma}}^r = d_\varepsilon \phi(\boldsymbol{\varepsilon}, T). \tag{17}$$

The effective stress tensor $\boldsymbol{\sigma}^e$ is then given as the sum of the actual stress $\boldsymbol{\sigma}$ and of the nonlocal relaxation stress $\bar{\boldsymbol{\sigma}}^r = R\boldsymbol{\sigma}^r$ so that the long range forces arising in a damaged structure are described by the following relation:

$$\bar{\boldsymbol{\sigma}}^r(\mathbf{x}) = (R\boldsymbol{\sigma}^r)(\mathbf{x}) = \int_\Omega W(\mathbf{x},\mathbf{y})\boldsymbol{\sigma}^r(\mathbf{y})d\mathbf{y} \qquad \forall \mathbf{x} \in \Omega. \tag{18}$$

The effective stress causes the same state of deformation in a virgin material as in a damaged material and the nonlocal relaxation stress $\bar{\boldsymbol{\sigma}}^r = \boldsymbol{\sigma}^e - \boldsymbol{\sigma}$ provides the total damage accumulated during loading.

As illustrated in Fig. 1 for the case of uniaxial loading and local behaviour, i.e. $\bar{\sigma}^r = \sigma^r$, a deformation state ε caused by the stress σ in a damaged material can be achieved by the effective stress σ^e located on the virgin material curve. Comparing the one-dimensional counterpart $\sigma^e = \sigma/(1-d)$ of (2) and the equality $\sigma^e = \sigma + \sigma^r$, following from (17), the relaxation stress σ^r is related to the scalar damage parameter d by the relation $d = \sigma^r/\sigma^e$.

Accordingly the undamaged material ($d = 0$) is characterized by $\sigma^r = 0$ and the local rupture ($d = d_c$) is given by $\sigma^r = \sigma^e d_c$ so that the corresponding stress turns out to be $\sigma = \sigma^e(1 - d_c)$. It is worth noting that for $d \to 1$, the relaxation stress σ^r tends to σ^e and the corresponding stress σ tends to vanishing, i.e. $\sigma^r \to \sigma^e,\ \sigma \to 0$.

By substituting (17) into (16), the dissipation becomes:

$$D = \dot{\bar{\boldsymbol{\sigma}}}^r * \boldsymbol{\varepsilon} + P \geqslant 0. \tag{19}$$

Let us now consider that the elastic energy ϕ is a convex function of strains $\boldsymbol{\varepsilon}$ at any point $\mathbf{x}$ of the body. The constitutive relation (17) can be written in the equivalent forms:

$$\boldsymbol{\sigma}^e = d\phi(\boldsymbol{\varepsilon}), \qquad \boldsymbol{\varepsilon} = d\phi^*(\boldsymbol{\sigma}^e), \qquad \phi(\boldsymbol{\varepsilon}) + \phi^*(\boldsymbol{\sigma}^e) = \boldsymbol{\sigma}^e * \boldsymbol{\varepsilon}, \tag{20}$$

where the conjugate of the elastic energy ϕ is the complementary elastic energy ϕ^* and the relation (20) is the Fenchel's equality.

The explicit expression of the free energy for the nonlocal damage model, in isothermal conditions, in terms of the proposed weight function (5) can be obtained by evaluating the expression of the free energy $\psi[\boldsymbol{\varepsilon}(\mathbf{x}), \bar{\boldsymbol{\sigma}}^r(\mathbf{x})]$, following from (15), and after some algebra it results:

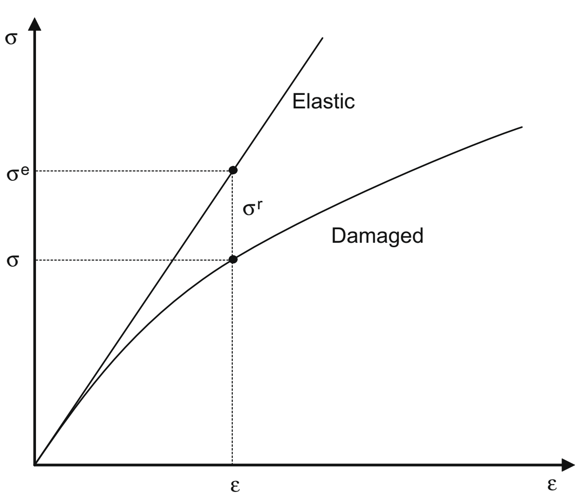

Fig. 1. Uniaxial illustration of the relaxation stress.

$$\begin{aligned}\psi[\boldsymbol{\varepsilon}(\mathbf{x}), \bar{\boldsymbol{\sigma}}^r(\mathbf{x})] &= \phi[\boldsymbol{\varepsilon}(\mathbf{x})] - \bar{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}) \\ &= \phi[\boldsymbol{\varepsilon}(\mathbf{x})] - \int_\Omega W(\mathbf{x},\xi)\boldsymbol{\sigma}^r(\mathbf{y})d\mathbf{y} * \boldsymbol{\varepsilon}(\mathbf{x}) \\ &= \psi[\boldsymbol{\varepsilon}(\mathbf{x}), \boldsymbol{\sigma}^r(\mathbf{x})] + \frac{\alpha}{V_\infty}\left[V(\mathbf{x})\boldsymbol{\sigma}^r(\mathbf{x}) - \int_\Omega g(\mathbf{x},\mathbf{y})\boldsymbol{\sigma}^r(\mathbf{y})d\mathbf{y}\right] * \boldsymbol{\varepsilon}(\mathbf{x}) \\ &= \psi[\boldsymbol{\varepsilon}(\mathbf{x}), \boldsymbol{\sigma}^r(\mathbf{x})] + \frac{\alpha}{V_\infty}\mathbf{J}(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}),\end{aligned} \tag{21}$$

where $\psi[\boldsymbol{\varepsilon}(\mathbf{x}), \boldsymbol{\sigma}^r(\mathbf{x})] = \phi[\boldsymbol{\varepsilon}(\mathbf{x})] - \boldsymbol{\sigma}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x})$ is the local Helmholtz free energy in isothermal conditions and the operator $\mathbf{J}$ is:

$$\mathbf{J}(\mathbf{x}) = \int_\Omega g(\mathbf{x},\mathbf{y})[\boldsymbol{\sigma}^r(\mathbf{x}) - \boldsymbol{\sigma}^r(\mathbf{y})]d\mathbf{y}. \tag{22}$$

From a mechanical point of view, the free energy (21) for the nonlocal damage model is given by the sum of the free energy related to the local behaviour and the free energy due to the nonlocal constitutive behaviour.

3.2. The dissipation and the residual

At every point where an irreversible mechanism develops, the dissipation (19) can be assumed in the following bilinear form:

$$D = \dot{\boldsymbol{\sigma}}^r * \boldsymbol{\gamma} \geqslant 0, \tag{23}$$

where $\boldsymbol{\sigma}^r$ is the (local) relaxation stress thermodynamically conjugated to the kinematic variable $\boldsymbol{\gamma}$ whose expression has to be identified. It is worth noting that the nonlocality residual function has disappeared from the pointwise expression (23) of the dissipation. By comparing (19) and (23), the nonlocality residual function is given by:

$$P = \dot{\boldsymbol{\sigma}}^r * \boldsymbol{\gamma} - \dot{\bar{\boldsymbol{\sigma}}}^r * \boldsymbol{\varepsilon}, \tag{24}$$

and, enforcing the insulation condition (9), it results:

$$\langle \dot{\boldsymbol{\sigma}}^r, \boldsymbol{\gamma} \rangle = \langle \dot{\bar{\boldsymbol{\sigma}}}^r, \boldsymbol{\varepsilon} \rangle. \tag{25}$$

Being $\dot{\bar{\boldsymbol{\sigma}}}^r = R\dot{\boldsymbol{\sigma}}^r$, the equality $\langle \dot{\bar{\boldsymbol{\sigma}}}^r, \boldsymbol{\varepsilon} \rangle = \langle \dot{\boldsymbol{\sigma}}^r, \bar{\boldsymbol{\varepsilon}} \rangle$, where $\bar{\boldsymbol{\varepsilon}} = R'\boldsymbol{\varepsilon} = R\boldsymbol{\varepsilon}$, can be substituted in (25) so that it results $\langle \dot{\boldsymbol{\sigma}}^r, \boldsymbol{\gamma} \rangle = \langle \dot{\boldsymbol{\sigma}}^r, \bar{\boldsymbol{\varepsilon}} \rangle$ for any relaxation stress rate $\dot{\boldsymbol{\sigma}}^r$. Hence the identification $\boldsymbol{\gamma} = \bar{\boldsymbol{\varepsilon}}$ holds true. Hence the nonlocality residual function (24) and the dissipation (23) can be given the following explicit expressions at a given point of the body Ω:

$$\begin{aligned} P &= \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}} - \dot{\bar{\boldsymbol{\sigma}}}^r * \boldsymbol{\varepsilon}, \\ D &= \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}} \geqslant 0. \end{aligned} \tag{26}$$

It is worth noting that the nonlocality in the new form $(26)_2$ of the dissipation D is due to the presence of the nonlocal strain $\bar{\boldsymbol{\varepsilon}}$.

The expression $(26)_2$ shows that the evolutive relation is described by the relaxation stress rate $\dot{\boldsymbol{\sigma}}^r$ and by its dual variable (nonlocal strain) $\bar{\boldsymbol{\varepsilon}}$.

The global mechanical dissipation for the nonlocal model, pertaining to the whole body Ω, can then be obtained by integrating $(26)_2$ over the domain Ω occupied by the body to get:

$$\mathfrak{W} = \int_\Omega D\, d\mathbf{x} = \langle \dot{\boldsymbol{\sigma}}^r, \bar{\boldsymbol{\varepsilon}} \rangle = \langle R\dot{\boldsymbol{\sigma}}^r, \boldsymbol{\varepsilon} \rangle = \langle \dot{\bar{\boldsymbol{\sigma}}}^r, \boldsymbol{\varepsilon} \rangle. \tag{27}$$

Remark 1. If the nonlocal variables are constant in space, the regularization operator R becomes the identity operator and the nonlocal variables turn out to be coincident to their local counterparts, i.e. $\bar{\boldsymbol{\sigma}}^r = \boldsymbol{\sigma}^r$ and $\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}$. As a consequence the relation $(26)_1$ shows that the nonlocality residual function P vanishes and the inequality $(26)_2$ reduces to the expression of the dissipation in terms of local variables, that is $D = \dot{\boldsymbol{\sigma}}^r * \boldsymbol{\varepsilon}$. Further the relation (17) yields the constitutive law in terms of the local relaxation stress:

$\boldsymbol{\sigma}^e = \boldsymbol{\sigma} + \boldsymbol{\sigma}^r = d_\varepsilon\phi$. Hence the constant field requirement on the regularization operator R and the insulation condition guarantee that the nonlocal damage model behaves as a local one under uniform fields.

Let us now explicitly evaluate the expressions of the dissipation and of the residual in terms of the proposed weight function (5).

- The pointwise dissipation given by $(26)_2$ can be explicitly written as:

$$\begin{aligned} D(\mathbf{x}) &= \dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \int_\Omega W(\mathbf{x},\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})d\mathbf{y} \\ &= \dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}) - \frac{\alpha}{V_\infty}V(\mathbf{x})\dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}) \\ &\quad + \frac{\alpha}{V_\infty}\dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \int_\Omega g(\mathbf{x},\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})d\mathbf{y} \\ &= \dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}) - \frac{\alpha}{V_\infty}\dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \left[V(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}) - \int_\Omega g(\mathbf{x},\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})d\mathbf{y}\right] \\ &= \dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}) - \frac{\alpha}{V_\infty}\dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \mathbf{J}_1(\mathbf{x}), \end{aligned} \tag{28}$$

where:

$$\mathbf{J}_1(\mathbf{x}) = \int_\Omega g(\mathbf{x},\mathbf{y})[\boldsymbol{\varepsilon}(\mathbf{x}) - \boldsymbol{\varepsilon}(\mathbf{y})]d\mathbf{y}. \tag{29}$$

The dissipation at a point $\mathbf{x}$ is then expressed as the sum of the dissipation related to the local damage behaviour in the strain space, i.e. $\dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x})$, and the dissipation due to the nonlocal behaviour.

- The explicit expression of the residual P at a point $\mathbf{x}$ follows from $(26)_1$:

$$P(\mathbf{x}) = D(\mathbf{x}) - \dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \boldsymbol{\varepsilon}(\mathbf{x}), \tag{30}$$

and, after some calculations, is given by:

$$P(\mathbf{x}) = \frac{\alpha}{V_\infty}\dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \int_\Omega g(\mathbf{x},\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})d\mathbf{y} - \frac{\alpha}{V_\infty}\int_\Omega g(\mathbf{x},\mathbf{y})\dot{\boldsymbol{\sigma}}^r(\mathbf{y})d\mathbf{y} * \boldsymbol{\varepsilon}(\mathbf{x}). \tag{31}$$

The nonlocality residual P turns out to be a homogeneous function of the relaxation stress rate. Accordingly, for a given strain field $\boldsymbol{\varepsilon}$ corresponding to a prescribed configuration of the body, the residual$P$ can be rewritten in the following form:

$$P(\mathbf{x}) = \int_\Omega \dot{\boldsymbol{\sigma}}^r(\mathbf{y}) * \mathbf{f}_1(\mathbf{x},\mathbf{y})d\mathbf{y} + \dot{\boldsymbol{\sigma}}^r(\mathbf{x}) * \mathbf{F}_1(\mathbf{x}), \tag{32}$$

where:

$$\begin{aligned} \mathbf{f}_1(\mathbf{x},\mathbf{y}) &= -\frac{\alpha}{V_\infty}g(\mathbf{x},\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{x}), \\ \mathbf{F}_1(\mathbf{x}) &= \frac{\alpha}{V_\infty}\int_\Omega g(\mathbf{x},\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} = -\int_\Omega \mathbf{f}_1(\mathbf{y},\mathbf{x})d\mathbf{y}. \end{aligned} \tag{33}$$

The nonlocality residual P can then be evaluated by means of the functions $\mathbf{f}_1$ and $\mathbf{F}_1$ once the relaxation stress rate $\dot{\boldsymbol{\sigma}}^r$ is assigned. Plots regarding the functions $\mathbf{f}_1$ and $\mathbf{F}_1$, given by (33), are reported in the example developed in Section 5.1 with reference to a one-dimensional bar.

Remark 2. According to a mechanical requirement, it is useful to check whether the considered nonlocal model tends to the elastic model with local damage if the material length scale l tends to zero, i.e. $l \to 0$. Since the attenuation function $g(\mathbf{x},\mathbf{y})$ tends to the Dirac distribution $\delta(\mathbf{x},\mathbf{y})$ for a vanishing internal length l, it results $V(\mathbf{x}) = V_\infty \to 1$. Hence from (22) and (29) the following relations hold: $\mathbf{J}(\mathbf{x}) \to 0$ and $\mathbf{J}_1(\mathbf{x}) \to 0$ for any $\mathbf{x} \in \Omega$. Accordingly, the nonlocal free energy (21) and the nonlocal dissipation (28) reduce to the corresponding local terms since the nonlocal quantities vanish. Moreover, being $g(\mathbf{x},\mathbf{y}) \to \delta(\mathbf{x},\mathbf{y})$, the relation (31) shows that the residual $P(\mathbf{x})$ identically vanishes. Hence the nonlocal model of damage tends to the local damage for the internal length l tending to zero.

3.3. The flow rule

Adopting the approach described above, the constitutive behaviour is entirely defined by the specification of two potentials. The first is the Helmholtz free energy ψ and the second is either the dissipation function D or the damage function as hereafter shown.

In fact the flow rule for the nonlocal damage model can be deduced from the properties of the dissipation according to the procedures shown, for local plasticity, in Eve et al. (1990), Romano et al. (1993) and Houlsby and Puzrin (2006). Hereafter such a procedure is briefly summarized in the proposed context of nonlocal damage.

Assuming that the sublinear function D is lower-semicontinuous (Rockafellar, 1970), the dissipation D turns out to be the support function of a closed convex domain K, that is:

$$D(\dot{\boldsymbol{\sigma}}^r) = \sup_{\bar{\boldsymbol{\varepsilon}} \in K} \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}}, \tag{34}$$

where the closed convex domain K is given by:

$$K = \{\bar{\boldsymbol{\varepsilon}} \in \mathscr{D} : D(\dot{\boldsymbol{\sigma}}^r) \geqslant \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}} \quad \forall \dot{\boldsymbol{\sigma}}^r \in \mathscr{S}\}. \tag{35}$$

In mechanical terms, the domain K is the set of admissible nonlocal strains $\bar{\boldsymbol{\varepsilon}}$ and its boundary represents the damage surface in the strain space.

Since the conjugate of the support function of K is the indicator of the domain K (Rockafellar, 1970):

$$\sqcup_K(\bar{\boldsymbol{\varepsilon}}) = \begin{cases} 0 & \text{if } \bar{\boldsymbol{\varepsilon}} \in K, \\ +\infty & \text{otherwise}, \end{cases} \tag{36}$$

the evolutive law of damage in the nonlocal context can be expressed in the following equivalent forms by virtue of the relations between the conjugate functions $D(\dot{\boldsymbol{\sigma}}^r)$ and $\sqcup_K(\bar{\boldsymbol{\varepsilon}})$:

$$\begin{cases} \dot{\boldsymbol{\sigma}}^r \in N_K(\bar{\boldsymbol{\varepsilon}}) = \partial\sqcup_K(\bar{\boldsymbol{\varepsilon}}), \\ \bar{\boldsymbol{\varepsilon}} \in \partial D(\dot{\boldsymbol{\sigma}}^r), \\ D(\dot{\boldsymbol{\sigma}}^r) + \sqcup_K(\bar{\boldsymbol{\varepsilon}}) = \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}}, \end{cases} \tag{37}$$

where the symbol N_K represents the normal cone to the damage domain and ∂ denotes the subdifferential operator (Rockafellar, 1970).

The relation $(37)_1$ represents the normality rule to the domain K for the nonlocal damage model. In fact if the nonlocal strain $\bar{\boldsymbol{\varepsilon}}$ belongs to the interior of the damage domain K, the normal cone to K at the point $\bar{\boldsymbol{\varepsilon}}$ is the null set so that no relaxation stress flow $\dot{\boldsymbol{\sigma}}^r$ is allowed. On the contrary if the nonlocal strain $\bar{\boldsymbol{\varepsilon}}$ belongs to the damage surface (the boundary of the damage domain K), the normal cone to K at the point $\bar{\boldsymbol{\varepsilon}}$ is non-vanishing and the relaxation stress flow $\dot{\boldsymbol{\sigma}}^r$ belongs to the normal cone of$K$ at the point $\bar{\boldsymbol{\varepsilon}}$. Hence a nonlocal damage standard material in the strain space is recovered. The relation $(37)_2$ represents the damage evolutive relation in terms of the dissipation and the expression $(37)_3$ is the Fenchel's equality.

The following proposition proves that the non-negativeness of the dissipation D is equivalent to require that the null nonlocal strain is admissible.

Proposition 3 (Non-negative maximum damage dissipation for the nonlocal model). *Let the nonlocal strain $\bar{\boldsymbol{\varepsilon}}$ and the relaxation stress rate $\dot{\boldsymbol{\sigma}}^r$ fulfil one of the expressions (37) of the damage flow rule. The dissipation attains its maximum at the point $\bar{\boldsymbol{\varepsilon}}$ and can be written in the form $D(\dot{\boldsymbol{\sigma}}^r) = \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}}$. Moreover, the dissipation is non-negative if and only if the null nonlocal strain belongs to the damage domain.*

Proof.. If the state variables $(\dot{\boldsymbol{\sigma}}^r, \bar{\boldsymbol{\varepsilon}})$ fulfil the damage flow rule, Fenchel's equality $(37)_3$ is met. Noting that the indicator function $\sqcup_K(\bar{\boldsymbol{\varepsilon}})$ is zero since the nonlocal strain $\bar{\boldsymbol{\varepsilon}}$ belongs to the damage domain, the expression $D(\dot{\boldsymbol{\sigma}}^r) = \dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}}$ of the dissipation, see $(26)_2$, is thus recovered.

If the null nonlocal strain belongs to the damage domain K, the expression (34) implies that the dissipation D is non-negative. The converse implication is proved per absurdum. Actually if the null nonlocal strain $\bar{\boldsymbol{\varepsilon}}$ does not belong to the damage domain K, the Hahn–Banach theorem ensures that there exists a relaxation stress rate $\dot{\boldsymbol{\sigma}}^r$ such that $\dot{\boldsymbol{\sigma}}^r * \bar{\boldsymbol{\varepsilon}} < 0$ for any $\bar{\boldsymbol{\varepsilon}}$ belonging to K in contrast with the assumption. □

In applications, a loading function is usually introduced at any point $\mathbf{x}$ of the body Ω so that the damage domain K is the convex set of nonlocal strains $\bar{\boldsymbol{\varepsilon}}$ defined as the level set of the damage mode g at the value $g_o(\mu)$ in the following form:

$$K = \{\bar{\boldsymbol{\varepsilon}} \in \mathscr{D} : g(\bar{\boldsymbol{\varepsilon}}) \leqslant g_o(\mu)\}. \tag{38}$$

The damage mode g is concave and the current damage threshold g_o is a non-negative monotone convex function. Their choice depends on the particular damage criterion adopted for the material at hand.

The parameter μ can be interpreted as a scalar measure of the amount of damage accumulated during the loading process. A more precise identification of μ is given hereafter.

Setting the damage function $h(\bar{\boldsymbol{\varepsilon}}) = g(\bar{\boldsymbol{\varepsilon}}) - g_o(\mu)$, it is well-known that the evolutive relation $(37)_1$ can then be rewritten in the equivalent form:

$$\dot{\boldsymbol{\sigma}}^r = \lambda dh(\bar{\boldsymbol{\varepsilon}}) \quad \text{subject to } \lambda \in N_{\Re^-}[h(\bar{\boldsymbol{\varepsilon}})]. \tag{39}$$

The time derivative of the parameter μ is assumed to be coincident to the value of the damage multiplier λ, i.e. $\dot{\mu} = \lambda$, so that the evolutive relation (39) can be rewritten as:

$$\dot{\boldsymbol{\sigma}}^r = \dot{\mu} dh(\bar{\boldsymbol{\varepsilon}}) \qquad \dot{\mu} \geqslant 0, \quad h(\bar{\boldsymbol{\varepsilon}}) \leqslant 0, \quad \dot{\mu} h(\bar{\boldsymbol{\varepsilon}}) = 0, \tag{40}$$

which is the explicit expression of the damage flow rule in terms of the complementarity conditions. The relation (40) shows that $\dot{\mu}$ is zero if $h(\bar{\boldsymbol{\varepsilon}}) < 0$ and is non-negative if $h(\bar{\boldsymbol{\varepsilon}}) = 0$. The sign constraint on $\dot{\mu}$ implies that the parameter μ is a non-decreasing function of the time. Hence the current damage threshold $g_o(\mu)$ is non-decreasing during the evolution of damage phenomena.

If the damage function h fulfils the condition $\|dh(\bar{\boldsymbol{\varepsilon}})\| = 1$, it results from (40) that $\|\dot{\boldsymbol{\sigma}}^r\| = \dot{\mu}\|dh(\bar{\boldsymbol{\varepsilon}})\| = \dot{\mu}$. Hence the actual value of the damage parameter μ, at the time t, can be related to the accumulated damage in the following form $\mu = \int_0^t \|\dot{\boldsymbol{\sigma}}^r(\tau)\| d\tau$. In this case the damage parameter μ has the physical meaning of the effective relaxation stress (in analogy with the effective plastic strain in plasticity) so that the size of the damage domain is driven by the relaxation stress.

Remark 4. In the nonlocal damage theory reported in Pijaudier-Cabot and Bažant (1987), the evolution of damage is driven by the damage energy release rate $\phi(\boldsymbol{\varepsilon}) = 1/2\mathbf{E}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}$. In strain damage, the damage energy release rate is transformed into a variable that has the dimension of strain in the form (Jirásek and Patzák, 2002): $g(\boldsymbol{\varepsilon}) = \sqrt{2\phi(\boldsymbol{\varepsilon})/E} = \sqrt{(\mathbf{E}\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon})/E}$. The expression of $g(\varepsilon)$ is not sensitive to the sign of the principal strains and the model exhibits a symmetric behaviour in tension and in compression. To take into account the fact that damage in quasi-brittle materials is driven mainly by tension, the positive part ε^+ of strain ε is introduced in $g(\varepsilon)$ by applying the positive part operator to the principal values in the spectral decomposition of ε (see Mazars, 1986). Hence the damage mode can be defined as the norm of the positive part of strain, i.e. $g(\boldsymbol{\varepsilon}) = \|\boldsymbol{\varepsilon}^+\| = \sqrt{\boldsymbol{\varepsilon}^+ * \boldsymbol{\varepsilon}^+}$.

An alternative definition of the damage mode g, corresponding to the Rankine criterion of maximum principal stress, is provided in Jirásek and Patzák (2002) and is based on the positive part of the effective stress in the form: $g(\boldsymbol{\varepsilon}) = \sqrt{\boldsymbol{\sigma}^{e+} * \boldsymbol{\sigma}^{e+}}/E = \sqrt{(\mathbf{E}\boldsymbol{\varepsilon})^+ * (\mathbf{E}\boldsymbol{\varepsilon})^+}/E$ where the positive part $(\mathbf{E}\boldsymbol{\varepsilon})^+$ of $\mathbf{E}\varepsilon$ is obtained by applying the positive part operator to the principal values in the spectral decomposition of $\mathbf{E}\varepsilon$.

In order to develop the structural model, it is convenient to formulate the relevant relations in a global form, i.e. in terms of quantities pertaining to the whole structure. In the sequel such quantities will be referred to as fields. For a continuous model such fields are functional defined in the domain Ω occupied by the body and belong to suitable functional spaces.

The global elastic energy Φ is defined to be the functional of the strain field $\boldsymbol{\varepsilon}$ obtained by integrating the pointwise elastic energy ϕ over the whole body Ω:

$$\Phi(\boldsymbol{\varepsilon}) = \int_\Omega \phi[\boldsymbol{\varepsilon}(\mathbf{x})] d\mathbf{x}. \tag{41}$$

An analogous definition holds for any other functional to be defined over the body Ω. Note that convexity (concavity) of functions is preserved by the global ones in the corresponding fields.

4. The nonlocal structural problem

Let us assume that displacements $\mathbf{u}$ belong to the Hilbert space $\mathscr{U}$. The kinematic operator $\mathbf{B} \in \text{Lin}\{\mathscr{U}, \mathscr{D}\}$ is a bounded linear operator from $\mathscr{U}$ to the Hilbert space of square integrable strain fields $\boldsymbol{\varepsilon} \in \mathscr{D}$ (Showalter, 1997; Romano, 2002).

Denoting by $\mathscr{F}$ the subspace of external forces, which is dual of $\mathscr{U}$, the continuous operator $\mathbf{B}' \in \text{Lin}\{\mathscr{S}, \mathscr{F}\}$, dual of $\mathbf{B}$, is the equilibrium operator and provides the force system in equilibrium with a given stress field. Stress and strain spaces can be identified with a pivot Hilbert space (square integrable fields). The duality pairing between $\mathscr{U}$ and its dual $\mathscr{F}$ is denoted by $\langle \cdot, \cdot \rangle$ having the physical meaning of external virtual work. For avoiding proliferation of symbols, the internal and external virtual works are denoted by the same symbol.

The external relation between reactions and displacements is assumed to be given by the equivalent relations:

$$\mathbf{r} \in \partial\Upsilon(\mathbf{u}) \Longleftrightarrow \mathbf{u} \in \partial\Upsilon^*(\mathbf{r}) \Longleftrightarrow \Upsilon(\mathbf{u}) + \Upsilon^*(\mathbf{r}) = \langle \mathbf{r}, \mathbf{u} \rangle, \tag{42}$$

being $\Upsilon : \mathscr{U} \to \Re \cup \{-\infty\}$ a concave functional. The concave functional $\Upsilon^* : \mathscr{F} \to \Re \cup \{-\infty\}$ represents the conjugate of Υ and the relation $(42)_3$ represents the Fenchel's equality.

Different expressions can be given to the functional Υ depending on the type of external constraints such as bilateral, unilateral, elastic or convex. For future reference the expressions of Υ and Υ^* are reported in the case of external frictionless bilateral constraints with non-homogeneous boundary conditions.

The admissible set of displacements is the subspace $\mathscr{L} = \mathbf{w} + \mathscr{L}_o$ where $\mathscr{L}_o$ collects conforming displacements which satisfy the homogeneous boundary conditions and $\mathbf{w} \in \mathscr{U}$ represents a displacement field which fulfils the non-homogeneous boundary conditions. The subspace of the external constraint reactions $\mathscr{R}$ is the orthogonal complement of $\mathscr{L}_o$, that is $\mathscr{R} = \mathscr{L}_o^\perp$. Then the functional Υ turns out to be the indicator of $\mathscr{L}_o$ defined in the form:

$$\Upsilon(\mathbf{u}) = \sqcap_{\mathscr{L}_o}(\mathbf{u}) = \begin{cases} 0 & \text{if } \mathbf{u} \in \mathscr{L}_o, \\ -\infty & \text{otherwise,} \end{cases} \tag{43}$$

and its conjugate Υ^* is given by:

$$\Upsilon^*(\mathbf{r}) = \sqcap_{\mathscr{L}_o^\perp}(\mathbf{r}) = \begin{cases} 0 & \text{if } \mathbf{r} \in \mathscr{L}_o^\perp = \mathscr{R}, \\ -\infty & \text{otherwise.} \end{cases} \tag{44}$$

Let $\ell = \{\mathbf{t}, \mathbf{b}\} \in \mathscr{F}$ be the load functional where $\mathbf{t}$ and $\mathbf{b}$ denote the tractions and the body forces. The relations governing the elastic structural problem with nonlocal damage for a given load history $\ell(t)$ are:

$$\begin{cases} \mathbf{B}'\boldsymbol{\sigma} = \ell + \mathbf{r} & \text{equilibrium,} \\ \mathbf{B}(\mathbf{u}+\mathbf{w}) = \boldsymbol{\varepsilon} & \text{compatibility,} \\ \boldsymbol{\sigma}^e = \boldsymbol{\sigma} + \bar{\boldsymbol{\sigma}}^r & \text{additivity of stresses,} \\ \boldsymbol{\varepsilon} = d\Phi^*(\boldsymbol{\sigma}^e) & \text{constitutive relation,} \\ \dot{\boldsymbol{\sigma}}^r \in N_K(\bar{\boldsymbol{\varepsilon}}) = \partial \sqcup_K(\bar{\boldsymbol{\varepsilon}}) & \text{damage flow rule,} \\ \mathbf{u} \in \partial \Upsilon^*(\mathbf{r}) & \text{external relation.} \end{cases} \tag{45}$$

A solution of the elastic structural problem with nonlocal damage can be achieved by a FE approach together with a time discretization so that a sequence of damage problems are solved in which the load increment is applied and the state variables are updated at the end of each increment.

4.1. Variational formulations

Attention is focused on a single step of the procedure for which the load increment is given. Accordingly one needs to evaluate the finite increments of the unknown variables corresponding to the increment of strain when their values are assigned at the beginning of the step. Let $\boldsymbol{\sigma}_o^r$ denote the known relaxation stress $\boldsymbol{\sigma}^r$ at the beginning of each step. In order to formulate the finite-step counterpart of the flow rule $(45)_5$, the time derivative is replaced by the finite increment ratio $\Delta\boldsymbol{\sigma}^r = \boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r$. Adopting a fully implicit time integration scheme, the flow rule for the nonlocal model is enforced at the end of the step according to the relation $\Delta\boldsymbol{\sigma}^r/\Delta t \in N_K(\bar{\boldsymbol{\varepsilon}})$ which, being $N_K(\bar{\boldsymbol{\varepsilon}})$ a convex cone, can be rewritten in the equivalent forms:

$$\begin{cases} \boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r \in N_K(\bar{\boldsymbol{\varepsilon}}) = \partial \sqcup_K(\bar{\boldsymbol{\varepsilon}}), \\ \bar{\boldsymbol{\varepsilon}} \in \partial D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r), \\ \sqcup_K(\bar{\boldsymbol{\varepsilon}}) + D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r) = \langle \boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r, \bar{\boldsymbol{\varepsilon}} \rangle. \end{cases} \tag{46}$$

The finite-step elastic structural model with nonlocal damage is then given by:

$$\begin{cases} \mathbf{B}'\boldsymbol{\sigma} = \ell + \mathbf{r} & \text{equilibrium,} \\ \mathbf{B}(\mathbf{u}+\mathbf{w}) = \boldsymbol{\varepsilon} & \text{compatibility,} \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^e - \bar{\boldsymbol{\sigma}}^r & \text{additivity of stresses,} \\ \boldsymbol{\varepsilon} = d\Phi^*(\boldsymbol{\sigma}^e) & \text{constitutive relation,} \\ \bar{\boldsymbol{\varepsilon}} \in \partial D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r) & \text{finite-step damage flow rule,} \\ \mathbf{u} \in \partial \Upsilon^*(\mathbf{r}) & \text{external relation.} \end{cases} \tag{47}$$

Introducing the product space $\mathscr{V} = \mathscr{U} \times \mathscr{S} \times \mathscr{S} \times \mathscr{S} \times \mathscr{D} \times \mathscr{F}$ and its dual space $\mathscr{V}'$, the finite-step nonlocal structural problem (47) can be arranged to build up a global multi-valued structural operator $\mathbf{S} : \mathscr{V} \to \mathscr{V}'$ governing the whole problem:

$$\mathbf{0} \in \mathbf{S}(\mathbf{v}) - \mathbf{v}_o, \qquad \mathbf{v} \in \mathscr{V}, \quad \mathbf{v}_o \in \mathscr{V}'.$$

The explicit expressions of the structural operator $\mathbf{S}$ and of the vectors $\mathbf{v}$ and $\mathbf{v}_o$ are given by:

$$\mathbf{S} = \begin{bmatrix} 0 & \mathbf{B}' & 0 & 0 & 0 & -I_{\mathscr{F}} \\ \mathbf{B} & 0 & 0 & 0 & -I_{\mathscr{D}} & 0 \\ 0 & 0 & -d\Phi^* & 0 & I_{\mathscr{D}} & 0 \\ 0 & 0 & 0 & \partial D & -R & 0 \\ 0 & -I_{\mathscr{S}} & I_{\mathscr{S}} & -R & 0 & 0 \\ -I_{\mathscr{U}} & 0 & 0 & 0 & 0 & \partial\Upsilon^* \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^e \\ \boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r \\ \boldsymbol{\varepsilon} \\ \mathbf{r} \end{bmatrix}, \quad \mathbf{v}_o = \begin{bmatrix} \ell \\ -\mathbf{Bw} \\ \mathbf{0} \\ \mathbf{0} \\ R\boldsymbol{\sigma}_o^r \\ \mathbf{0} \end{bmatrix}, \tag{48}$$

where $I_\bullet$ is the identity operator in the spaces $\mathscr{U}$, $\mathscr{F}$, $\mathscr{D}$ or $\mathscr{S}$. The conservativity of the structural operator follows from the duality existing between the pairs $(\mathbf{B}, \mathbf{B}')$, $(I_{\mathscr{D}}, I_{\mathscr{S}})$, $(I_{\mathscr{U}}, I_{\mathscr{F}})$, the conservativity of $d\Phi$ and the conservativity of the subdifferentials ∂D and $\partial\Upsilon^*$.

The related potential can be evaluated by summing up the potentials of each component operator so that it turns out to be:

$$P(\mathbf{v}) = \int_0^1 \langle \mathbf{S}(\zeta\mathbf{v}), \mathbf{v} \rangle d\zeta - \langle \ell, \mathbf{u} \rangle - \langle R\boldsymbol{\sigma}_o^r, \boldsymbol{\varepsilon} \rangle + \langle \boldsymbol{\sigma}, \mathbf{Bw} \rangle.$$

Hence it results:

$$P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}) = -\Phi^*(\boldsymbol{\sigma}^e) + D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r) + \langle \boldsymbol{\sigma}, \mathbf{B}(\mathbf{u}+\mathbf{w}) - \boldsymbol{\varepsilon} \rangle + \langle \boldsymbol{\sigma}^e - R\boldsymbol{\sigma}^r, \boldsymbol{\varepsilon} \rangle + \Upsilon^*(\mathbf{r}) - \langle \ell + \mathbf{r}, \mathbf{u} \rangle. \tag{49}$$

The potential P turns out to be linear in $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})$, convex with respect to the state variable $\boldsymbol{\sigma}^r$ and jointly concave with respect to $(\boldsymbol{\sigma}^e, \mathbf{r})$. The nonlocality lies in the presence of the regularization operator R. Accordingly it results:

Proposition 5. *The set* $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r})$ *is a solution of the saddle problem*

$$\min_{\boldsymbol{\sigma}^r} \max_{(\boldsymbol{\sigma}^e, \mathbf{r})} \operatorname*{stat}_{(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}),$$

if and only if it is a solution of the finite-step elastic problem with nonlocal damage (47).

A family of potentials can be recovered from the potential P by enforcing the field equations, the constitutive relations and the external relation. All these functionals assume the same value when they are evaluated at a solution point of the elastic structural problem with nonlocal damage. For sake of conciseness, such formulations are not derived in this paper.

In the following a nonlocal variational formulation which allows a derivation of a nonlocal finite element procedure for the nonlocal problem of damage in the strain space is provided.

Imposing in the expression of the potential P the external relation $(47)_6$ in terms of Fenchel's equality $(42)_3$, the constitutive relation $(47)_4$ in terms of Fenchel's equality, i.e. $\Phi(\boldsymbol{\varepsilon}) + \Phi^*(\boldsymbol{\sigma}^e) = \langle \boldsymbol{\sigma}^e, \boldsymbol{\varepsilon} \rangle$, and the compatibility condition $(47)_2$, the following two-field nonlocal mixed potential is obtained:

$$P_1(\mathbf{u}, \boldsymbol{\sigma}^r) = \Phi[\mathbf{B}(\mathbf{u}+\mathbf{w})] + D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r) - \langle R\boldsymbol{\sigma}^r, \mathbf{B}(\mathbf{u}+\mathbf{w}) \rangle - \Upsilon(\mathbf{u}) - \langle \ell, \mathbf{u} \rangle, \tag{50}$$

which is jointly convex with respect to the state variables $(\mathbf{u}, \boldsymbol{\sigma}^r)$. The stationarity conditions of P_1 yield the finite-step nonlocal structural problem (47) in the two fields $(\mathbf{u}, \boldsymbol{\sigma}^r)$:

$$\begin{cases} \mathbf{B}' d\Phi[\mathbf{B}(\mathbf{u}+\mathbf{w})] - \mathbf{B}' R\boldsymbol{\sigma}^r - \ell \in \partial\Upsilon(\mathbf{u}), \\ R\mathbf{B}(\mathbf{u}+\mathbf{w}) \in \partial D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r) \iff \boldsymbol{\sigma}^r - \boldsymbol{\sigma}_o^r = \Delta\mu \, dH[R\mathbf{B}(\mathbf{u}+\mathbf{w})], \end{cases} \tag{51}$$

under the complementarity conditions $\Delta\mu \geqslant 0$, $H[\mathbf{B}(\mathbf{u}+\mathbf{w})] \leqslant 0$ and $\Delta\mu H[\mathbf{B}(\mathbf{u}+\mathbf{w})] = 0$.

In fact the relation $(51)_1$ shows that there exist a strain $\boldsymbol{\varepsilon} = \mathbf{B}(\mathbf{u}+\mathbf{w})$ and an elastic stress $\boldsymbol{\sigma}^e = d\Phi(\boldsymbol{\varepsilon})$ such that the associated stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^e - R\boldsymbol{\sigma}^r$ fulfils the equilibrium equation and the external constraint $\mathbf{B}'\boldsymbol{\sigma} - \ell = \mathbf{r} \in \partial\Upsilon(\mathbf{u})$. The relation $(51)_2$ provides the finite-step damage flow rule for the nonlocal model in terms of the dissipation or, equivalently, in terms of the damage functional H and of the complementarity conditions.

Accordingly the following two-field nonlocal mixed variational principle can be stated.

Proposition 6. *The pair* $(\mathbf{u}, \boldsymbol{\sigma}^r)$ *is a solution of the minimum problem*

$$\min_{\mathbf{u}, \boldsymbol{\sigma}^r} P_1(\mathbf{u}, \boldsymbol{\sigma}^r),$$

if and only if it is a solution of the finite-step elastic problem with nonlocal damage (47).

Note that the global damage condition is expressed in terms of the functional H in the form:

$$H(\boldsymbol{\varepsilon}) = \sup_{\mathbf{x}\in\Omega} h[\boldsymbol{\varepsilon}(\mathbf{x})], \tag{52}$$

and is equivalent to the damage condition at every point of the body Ω (in the case of local plasticity see Romano and Alfano, 1995).

4.2. Finite element incremental problem

The resolution of a continuum dissipative structural problem can be achieved by two different discretization techniques. The former requires the integration along the loading path and has been provided by the Euler backward difference scheme. The latter is a space discretization and is usually performed by using a finite element procedure.

Usually a finite element approach to solve the nonlinear problem associated with the elasto-damage model is based on the elastic predictor-damage corrector procedure. The predictor phase is a linear elastic analysis that provides displacements and, then, strains at the end of the step. The resulting strains are the starting point to impose the constitutive consistency in the damage corrector phase. Once the complementarity damage problem is solved, the updated stress is computed and the resulting out-of-balance forces are used to start a new elastic prediction phase of the iterative procedure for the assigned load step.

A different approach (Brünig et al., 2001; Brünig and Ricci, 2005) is based on an inelastic (plastic-damage) predictor-elastic corrector which amounts to solve partial differential equations for the nonlocal plastic-damage condition and the standard variables are obtained by a standard FEM.

On the contrary, an important advantage of models cast in the framework of damage with strain-based loading functions is that the stress corresponding to a given strain can be evaluated directly. Accordingly it is not necessary to solve a nonlinear system of equations in the damage corrector phase as shown hereafter.

Let Ω_e, with $e = 1,\ldots,\mathscr{N}$, be the domain decomposition induced by the meshing of the domain Ω fulfilling the conditions $\cup_{e=1}^{\mathscr{N}}\overline{\Omega}_e = \overline{\Omega}$ and $\Omega_e \cap \Omega_j = \emptyset$ for any $e \neq j$. A simplified damage condition (52) can conveniently be enforced from a computational point of view by checking it only at the Gauss points $\mathbf{x}_g$:

$$H(\boldsymbol{\varepsilon}) = \max_{g=1,\ldots,n} H_g(\boldsymbol{\varepsilon}) = \max_{g=1,\ldots,n} \int_{\Pi_g} h[\boldsymbol{\varepsilon}(\mathbf{x})]d\mathbf{x}, \tag{53}$$

where Π_g, with $g = 1,\ldots,n$, is the gth subdomain of the finite element mesh. The subdomain $\Pi_g \subseteq \Omega_e$ is chosen in such a way that it contains in its interior only one of the n_g^e Gauss points of the element Ω_e, $\cup_{g=1}^{n_g^e}\overline{\Pi}_g = \overline{\Omega}_e$ and $\Pi_i \cap \Pi_j = \emptyset$ for any $i \neq j$.

Using a conforming FE discretization, the unknown displacement field $\mathbf{u}(\mathbf{x})$ is given, for each element, in the interpolated form $\mathbf{u}_h^e(\mathbf{x}) = \mathbf{N}_e(\mathbf{x})\mathbf{p}_u^e$ with $\mathbf{x} \in \Omega_e$ where $\mathbf{p}_u^e$ is the vector collecting the nodal displacements of the eth FE and $\mathbf{N}_e(\mathbf{x})$ is the chosen shape-function matrix.

The unknown relaxation stress field $\boldsymbol{\sigma}^r(\mathbf{x})$ is given, for each element, in the interpolated form $\boldsymbol{\sigma}_h^{re}(\mathbf{x}) = \mathbf{M}_e(\mathbf{x})\mathbf{p}_{\sigma^r}^e$ with $\mathbf{x} \in \Omega_e$ where $\mathbf{p}_{\sigma^r}^e$ is the vector collecting the relaxation stress at the n_g^e Gauss points of the eth FE. Considering the previously introduced partition of Ω in subdomains $\Pi_g \subseteq \Omega_e$, the shape-function matrix $\mathbf{M}_e(\mathbf{x})$ has the form $\mathbf{M}_e(\mathbf{x}) = \left[f_1(\mathbf{x})\mathbf{I}\, f_2(\mathbf{x})\mathbf{I}\, \ldots\, f_{n_g^e}(\mathbf{x})\mathbf{I}\right]$ where $\mathbf{I}$ is the identity matrix and f_g is the step function defined as $f_g(\mathbf{x}) = 1$ if $\mathbf{x} \in \Pi_g$ and $f_g(\mathbf{x}) = 0$ otherwise.

The conforming displacement field $\mathbf{u}_h = \{\mathbf{u}_h^1, \mathbf{u}_h^2, \ldots, \mathbf{u}_h^{\mathscr{N}}\}$ satisfies the homogeneous boundary conditions and the interelement continuity conditions so that rigid-body displacements are ruled out. The displacement parameters $\mathbf{p}_u^e$ can be expressed in terms of nodal parameters $\mathbf{q}_u$ by means of the standard assembly operator $\mathscr{A}_e$ according to the parametric expression $\mathbf{p}_u^e = \mathscr{A}_e\mathbf{q}_u$.

On the contrary, the relaxation stress parameters $\mathbf{p}_{\sigma^r}^e$ are simply collected in the global lists $\mathbf{q}_{\sigma^r}$ according to the expression $\mathbf{p}_{\sigma^r}^e = \mathscr{J}_e\mathbf{q}_{\sigma^r}$ where the operator $\mathscr{J}_e$ is the canonical extractor which picks up, from the global list $\mathbf{q}_{\sigma^r}$, the local parameter $\mathbf{p}_{\sigma^r}^e$.

The interpolated counterpart of the functional P_1, in the case of external frictionless bilateral constraints, is obtained by adding-up the contributions of each non-assembled element and imposing that the interpolating displacement $\mathbf{u}_h$ satisfies the conformity requirement to get:

$$P_{1h}(\mathbf{u}_h, \boldsymbol{\sigma}_h^r) = \frac{1}{2}\langle \mathbf{EB}(\mathbf{u}_h + \mathbf{w}_h), \mathbf{B}(\mathbf{u}_h + \mathbf{w}_h)\rangle + D(\boldsymbol{\sigma}_h^r - \boldsymbol{\sigma}_{ho}^r) - \langle R\boldsymbol{\sigma}_h^r, \mathbf{B}(\mathbf{u}_h + \mathbf{w}_h)\rangle - \langle \ell, \mathbf{u}_h\rangle. \tag{54}$$

The stationarity conditions of P_{1h} yield the mixed variational problem in the two interpolating fields $(\mathbf{u}_h, \boldsymbol{\sigma}_h^r)$ whose matrix form is:

$$\begin{cases} \sum_{e=1}^{\mathscr{N}} \mathscr{A}_e^T \mathbf{K}_{ee}^l \mathscr{A}_e \mathbf{q}_u - \sum_{e=1}^{\mathscr{N}} \mathscr{A}_e^T \left(\mathbf{L}_{ee}^l - \mathbf{L}_{ee}^{nl}\right)\mathscr{J}_e \mathbf{q}_{\sigma^r} \\ \quad - \sum_{e=1}^{\mathscr{N}} \sum_{m=1}^{\mathscr{N}} \mathscr{A}_e^T \mathbf{L}_{em}^{nl} \mathscr{J}_m \mathbf{q}_{\sigma^r} = \sum_{e=1}^{\mathscr{N}} \mathscr{A}_e^T \mathbf{f}_e, \\ \mathbf{p}_{\sigma^r}^e - \mathbf{p}_{\sigma_o^r}^e = \mathrm{diag}[dh_g]\Delta\boldsymbol{\mu}_e, \\ \Delta\mu_g \geqslant 0, \quad h_g \leqslant 0, \Delta\mu_g h_g = 0 \quad \text{with } g = 1,\ldots,n_g^e, \\ \qquad\qquad\qquad\qquad e = 1,\ldots,\mathscr{N}, \end{cases} \tag{55}$$

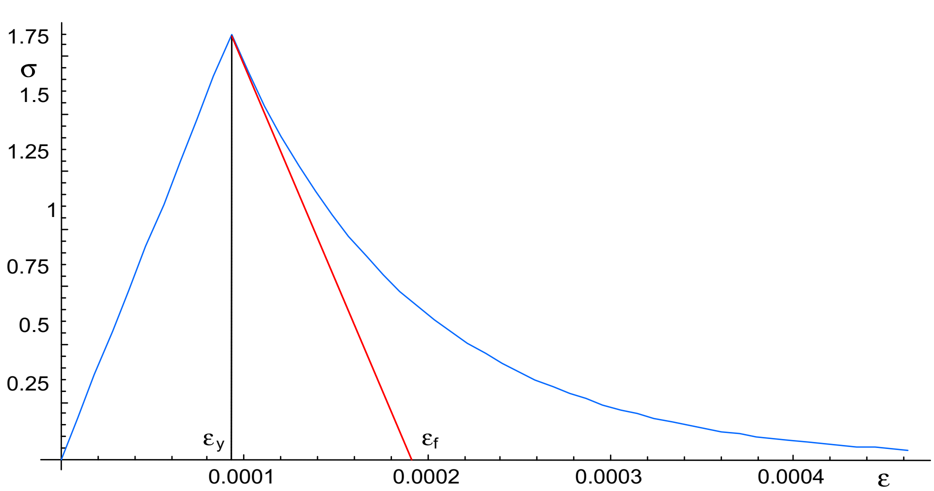

Fig. 2. The stress–strain diagram with exponential softening.

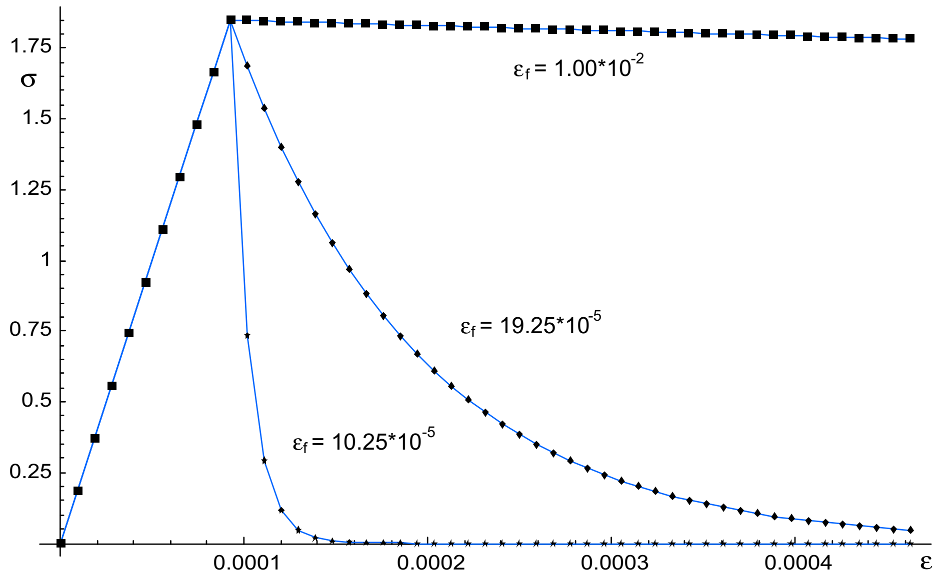

Fig. 3. Different stress–strain curves with exponential softening for increasing values of the strain parameter ε_f.

where the vector $\Delta\boldsymbol{\mu}_e = \left[\Delta\mu_1 \ldots \Delta\mu_{n_g^e}\right]^T$ collects the damage parameters at the n_g^e Gauss points of the eth element, diag$[dh_g]$ denotes the diagonal matrix given by the derivative of the damage function h evaluated at the gth Gauss point of the eth element and h_g is the value of the damage function h at the gth Gauss point of the eth element.

The component submatrices and subvectors appearing in (55) are defined as:

$$\begin{aligned}
\mathbf{K}_{ee}^{l} &= \int_{\Omega_e} (\mathbf{BN}_e)^T(\mathbf{x})\mathbf{E}(\mathbf{x})(\mathbf{BN}_e)(\mathbf{x})\,d\mathbf{x},\\
\mathbf{L}_{ee}^{l} &= \int_{\Omega_e} (\mathbf{BN}_e)^T(\mathbf{x})\mathbf{M}_e(\mathbf{x})d\mathbf{x},\\
\mathbf{L}_{ee}^{nl} &= \frac{\alpha}{V_\infty}\int_{\Omega_e} V(\mathbf{x})(\mathbf{BN}_e)^T(\mathbf{x})\mathbf{M}_e(\mathbf{x})\,d\mathbf{x},\\
\mathbf{L}_{em}^{nl} &= \frac{\alpha}{V_\infty}\int_{\Omega_e}\int_{\Omega_m} (\mathbf{BN}_e)^T(\mathbf{x})g(\mathbf{x},\xi)\mathbf{M}_m(\xi)d\xi\,d\mathbf{x},\\
\mathbf{f}_e &= \int_{\Omega_e} \mathbf{N}_e^T(\mathbf{x})\mathbf{b}(\mathbf{x})d\mathbf{x} + \int_{S_e} \mathbf{N}_e^T(\mathbf{x})\mathbf{t}(\mathbf{x})\,d\mathbf{x},\\
&\quad - \int_{\Omega_e} (\mathbf{BN}_e)^T(\mathbf{x})\mathbf{E}(\mathbf{x})(\mathbf{BN}_e)(\mathbf{x})\,d\mathbf{x}\;\mathbf{w}_e.
\end{aligned} \tag{56}$$

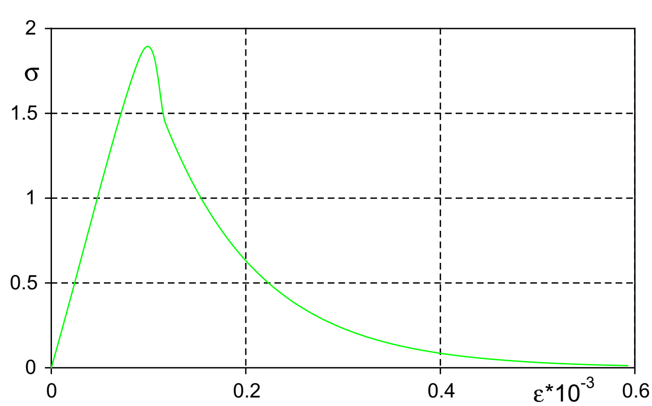

Fig. 4. The stress–strain diagram with exponential softening and local damage referred to the element of the FE discretization in the middle of the bar.

The integrations appearing in $(56)_{1-2}$ are performed elementwise so that the matrices $\mathbf{K}_{ee}^{l}$ and $\mathbf{L}_{ee}^{l}$ turn out to be the standard stiffness matrix and the standard matrix coupling stresses and displace-

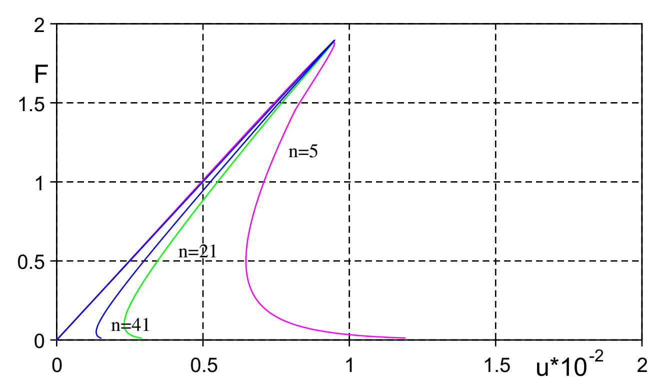

Fig. 5. The force–displacement diagram for the model with local damage with reference to the end section of the bar.

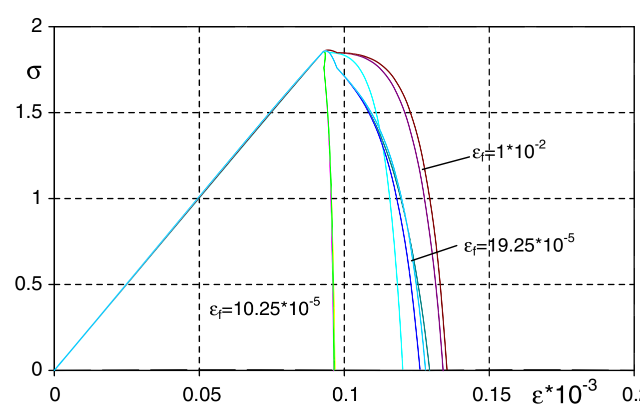

Fig. 6. The stress–strain diagram with exponential softening and nonlocal damage is plotted with reference to the element of the FE discretization in the middle of the bar. For each value of ε_f, three meshes with $n = 41$, $n = 101$ and $n = 201$ elements are considered.

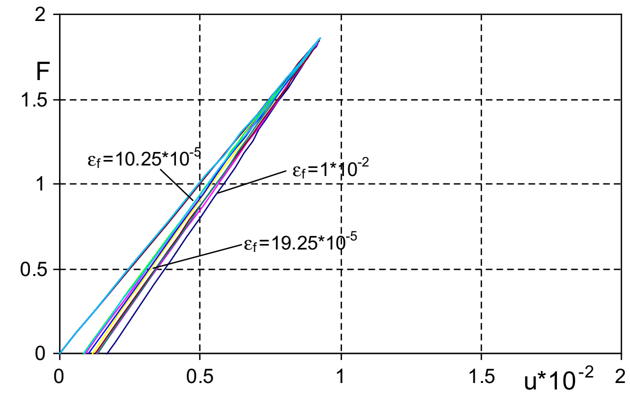

Fig. 7. The force–displacement response curve for nonlocal damage is plotted with reference to the end section of the bar. For each value of ε_f, three meshes with $n = 41$, $n = 101$ and $n = 201$ elements are considered.

ments, respectively. The matrices $\mathbf{L}^{nl}_{ee}$ and $\mathbf{L}^{nl}_{em}$ turn out to be the nonlocal matrices reflecting the nonlocality of the model. The elements of the matrix $\mathbf{L}^{nl}_{em}$ vanishes if the related elements Ω_e and Ω_m are too far with respect to the influence distance$r$. Accordingly the assembled matrix $\mathbf{L} = \mathbf{L}^{l} - \mathbf{L}^{nl}$ is banded with a band width larger than in the corresponding local matrix.

Hence the solving linear equation system follows from (55) and is given by:

$$\begin{cases} \mathbf{K}\mathbf{q}_u - \mathbf{L}\mathbf{q}_{\sigma^r} = \mathbf{f}, \\ \mathbf{p}^e_{\sigma^r} - \mathbf{p}^e_{\sigma^r_o} = \operatorname{diag}[dh_g]\Delta\boldsymbol{\mu}_e, \\ \Delta\mu_g \geqslant 0, \quad h_g \leqslant 0, \quad \Delta\mu_g h_g = 0 \quad \text{with } g = 1,\ldots,n^e_g, \\ \qquad\qquad\qquad\qquad\qquad\qquad e = 1,\ldots,\mathcal{N}, \end{cases} \tag{57}$$

where the global stiffness matrix $\mathbf{K}$ is symmetric and positive definite.

In the case of a local elastic behaviour, the nonlocal terms disappear so that $\mathbf{L}^{nl}_{ee} = \mathbf{L}^{nl}_{em} = \mathbf{0}$ and the solving equation system reduces to the standard mixed FEM with local strain damage given by $\mathbf{K}\mathbf{q}_u - \mathbf{L}^l\mathbf{q}_{\sigma^r} = \mathbf{f}$.

5. Numerical results

5.1. One-dimensional bar in traction

The first numerical application regard a one-dimensional bar in simple uniform traction. The bar has a unit cross-section and a length $L = 100$ mm. It is clamped at the end $x = 0$ and is subjected to a given displacement w at the other end $x = L$. The material parameters are: elastic modulus $E = 2 \times 10^4$ MPa, internal length $l = 2$ mm, influence distance $r = 12$ mm and $\alpha = 1$. The tensile strength is $\sigma_y = Eg_o(0) = 1.887$ MPa and, in order to trigger the damage localization, the tensile strength of the central element of the FE discretization of the bar is reduced of 2% to get $\sigma_y = Eg_o(0) = 1.85$ MPa.

The space weight function W is defined in terms of the bi-exponential attenuation function $g(x,y) = \exp(-|x-y|/l)/2l$ which has an unbounded support but it decays very fast for increasing $|x-y|/l$. Hence from a computational point of view, it is possible to assume that the attenuation function is vanishing if $|x-y| > r$.

The numerical simulations are performed by one-dimensional finite elements with linear displacement interpolation functions and constant relaxation stress interpolation functions. The 1-Gauss point numerical quadrature of 1D problems is considered.

The damage domain (38) is given by the relation $\bar{\varepsilon} \leqslant g_o(\mu)$ and the damage function is $h(\bar{\varepsilon}) = \bar{\varepsilon} - g_o(\mu) \leqslant 0$. The constitutive finite-step elastic model with nonlocal damage can then be re-cast in the form $\sigma = \sigma^e - \bar{\sigma}^r$, $\varepsilon = E^{-1}\sigma^e$, $\Delta\sigma^r = \Delta\mu dh(\bar{\varepsilon}) = \Delta\mu$.

The function g_o^{-1} can be identified from the uniaxial exponential softening stress–strain curve (Jirásek and Patzák, 2002) and it can be written in the form:

$$\mu = g_o^{-1}(\bar{\varepsilon}) = \begin{cases} 0 & \text{if } \bar{\varepsilon} < \varepsilon_y, \\ E\bar{\varepsilon} - E\varepsilon_y \exp\left(-\frac{\bar{\varepsilon}-\varepsilon_y}{\varepsilon_f-\varepsilon_y}\right) & \text{if } \bar{\varepsilon} \geqslant \varepsilon_y. \end{cases} \tag{58}$$

Considering a local behaviour, the stress $\sigma = \sigma^e - \mu$, related to the exponential softening curve (58), is reported in Fig. 2 assuming that the limit elastic strain is $\varepsilon_y = g_o(0) = \sigma_y/E = 9.25 \times 10^{-5}$. The elastic behaviour is followed by a softening post-peak branch. The strain $\varepsilon_f = 19.25 \times 10^{-5}$ represents a parameter affecting the ductility of the response and is related to the fracture energy. In

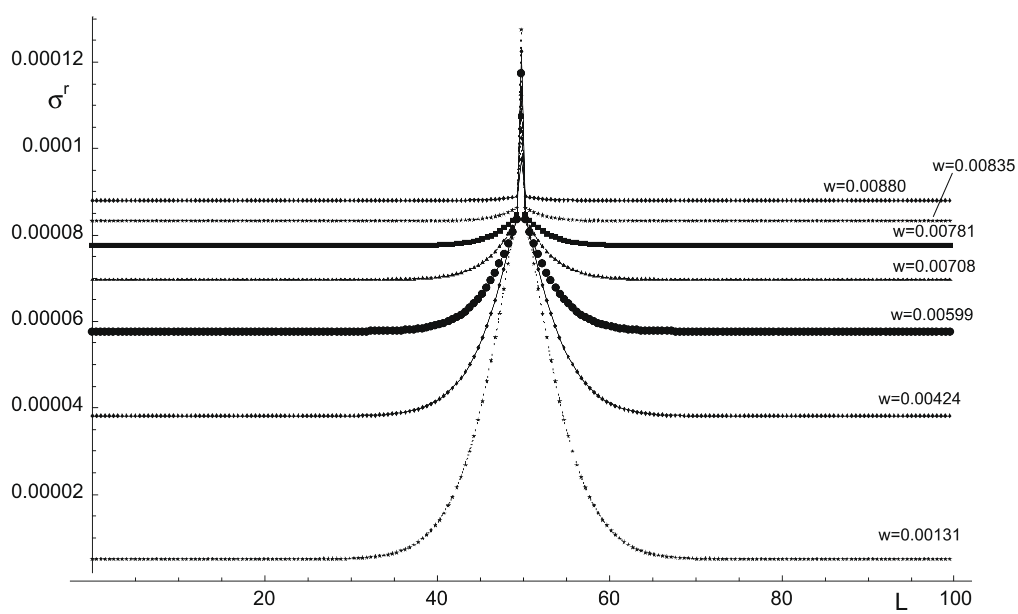

Fig. 8. Evolution of the damage localization in the bar due to subsequent imposed displacements w at the end section. The bar is discretized with $n = 201$ elements and $\varepsilon_f = 19.25 \times 10^{-5}$.

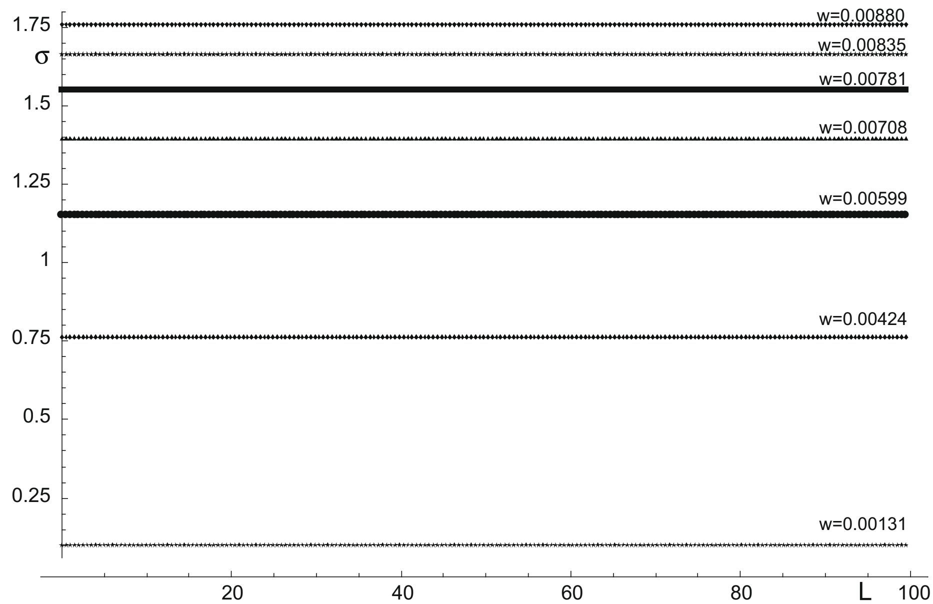

Fig. 9. Evolution of the stress profile σ along the bar due to subsequent imposed displacements w at the end section. The bar is discretized with $n = 201$ elements and $\varepsilon_f = 19.25 \times 10^{-5}$.

fact the area under the exponential curve in Fig. 2 is given by $E\varepsilon_y(\varepsilon_f - \varepsilon_y/2)$ and has the mechanical meaning of the energy dissipated per unit volume of totally damaged material under uniaxial tension.

Different stress–strain curves can be obtained from the exponential law of damage provided in (58) by assuming different values of the parameter ε_f as reported in Fig. 3. If $\varepsilon_f = 10.25 \times 10^{-5}$, the stress–strain curve leads to a very abrupt softening similar to the one reported in Pijaudier-Cabot (1996). Setting $\varepsilon_f = 19.25 \times 10^{-5}$, the stress–strain curve leads to a more smooth softening behaviour. Setting $\varepsilon_f = 1.00 \times 10^{-2}$, the stress–strain curve tends to a bi-linear branch as in Peerlings et al. (1998).

The stress–strain relation in traction for the element of the FE discretization in the middle of the bar is plotted in Fig. 4 by considering the local elasto-damage model. The analysis is performed by using a (local) FE formulation following from the one reported in Section 4.2 by setting $\bar{\sigma}^r = \sigma^r$. Different meshes are considered by discretizing the bar with a different number of elements, all of equal size, namely $n = 5$, $n = 21$ and $n = 41$. The force–displacement diagram referred to the bar end section is plotted in Fig. 5 and the dependence of the response on the number of elements in FE discretization is visible.

The traction test on the bar having the proposed nonlocal behaviour is reported in Figs. 6 and 7.

The stress–strain diagram for the element of the FE discretization in the middle of the bar is plotted in Fig. 6 for different values of the strain ε_f and, for each value of ε_f, three different meshes with a different number of elements, all of equal size, $n = 41$, $n = 101$ and $n = 201$ are considered.

The stress–strain curves in Fig. 6 show a smooth softening for $\varepsilon_f = 1.00 \times 10^{-2}$ and a more abrupt softening branch for decreasing values of ε_f. The stress–strain curves do not depend on the

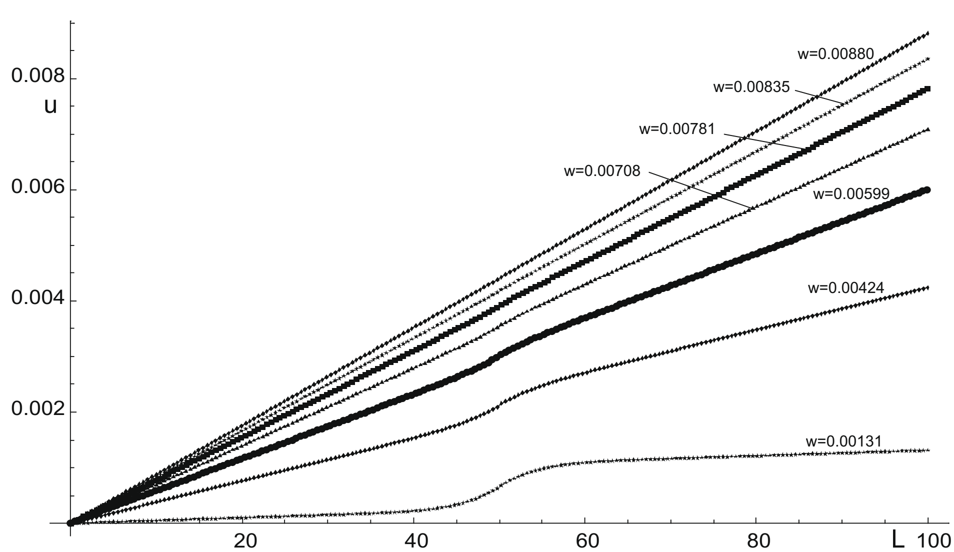

Fig. 10. Evolution of the displacements in the bar due to subsequent imposed displacements w at the end section. The bar is discretized with $n = 201$ elements and $\varepsilon_f = 19.25 \times 10^{-5}$.

chosen mesh. The stress–strain diagram obtained by the mesh with $n = 41$ elements and $\varepsilon_f = 1.00 \times 10^{-2}$ provides, for a given stress, a slight lower strain in the softening behaviour than the corresponding strain obtained by the meshes with $n = 101$ or $n = 201$ elements and the same value of ε_f.

The force–displacement response curves for the bar end section are reported in Fig. 7 and show an increasing of the slope of the descendant branch for decreasing ε_f. Three different meshes with 41, 101, 201 elements are considered for each value of ε_f.

The stress–strain and force–displacement curves show that the obtained solutions do not depend on the number of elements used in the analysis.

The results in Fig. 8 are relative to the discretization with $n = 201$ elements and $\varepsilon_f = 19.25 \times 10^{-5}$. It can be observed that the plots depict the evolution of the damage distribution provided by the nonlocal relaxation stress $\bar{\sigma}^r$ in the bar due to the sequence of imposed displacements at the end cross-section of the bar. The results show that no mesh dependence or boundary effects are pointed out by the considered nonlocal model.

No spurious oscillations or boundary effects are apparent in Fig. 9 where the stress $\sigma = \sigma^e - \sigma^r$ along the bar is plotted in terms of the imposed displacement of the end section. The stress is constant in the bar as required by equilibrium considerations.

The evolution of the displacements in the bar in terms of the imposed displacement of the end section is reported in Fig. 10. The displacement plot is linear in the two parts of the bar which have an elastic behaviour and has a nonlinear shape in the central damaged zone. The nonlinear behaviour becomes more apparent as the damage grows up.

The plots of the two-dimensional function f_1 and of the one-dimensional function F_1, see (33), pertaining to the considered nonlocal model of damage are reported in Fig. 11(a) and (b) with reference to the bar discretized by $n = 101$ elements and $\varepsilon_f = 19.25 \times 10^{-5}$. The strains along the bar, used to evaluate the

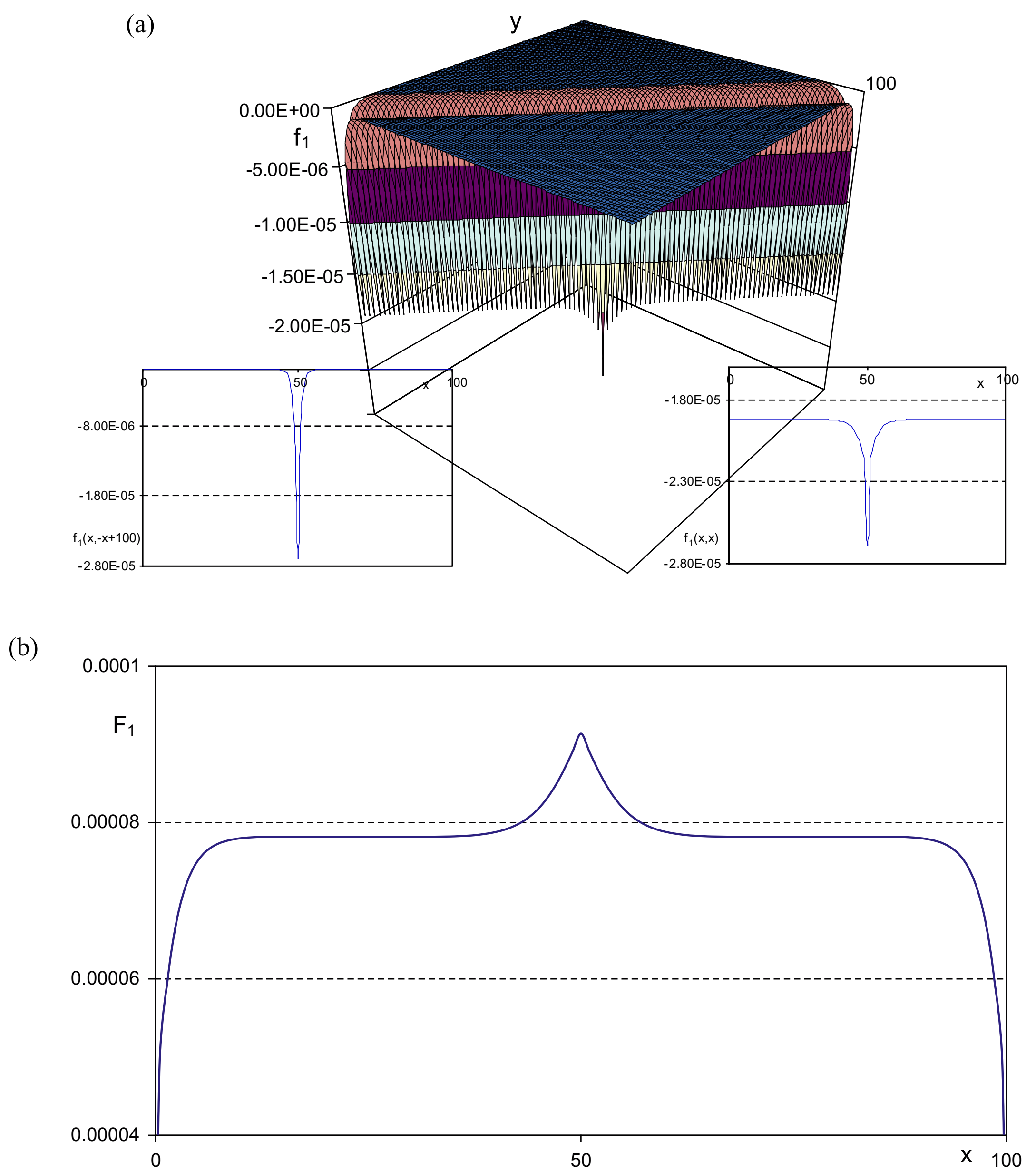

Fig. 11. Plots of the functions f_1 and F_1 providing the nonlocality residual P for the nonlocal model of damage ($n = 201$ elements and $\varepsilon_f = 19.25 \times 10^{-5}$): (a) the two-dimensional function f_1; (b) the one-dimensional function F_1.

functions f_1 and F_1 according to (33), corresponds to the imposed displacement $w = 0.00775$ cm and the bar is in the softening regime.

5.2. A 2D plate in tension

The second example regards a traction test of a four notched concrete specimen of unitary thickness. Fig. 12(a) and (b) shows the geometry and the loading conditions of the sample.

The numerical analysis is carried out under the hypothesis of plane stress condition setting the elastic modulus $E = 1.83 \times 10^4$ MPa, the internal length $l = 2$ mm, the influence distance $r = 6$ mm and $\alpha = 1$. The tensile strength is $\sigma_y = Eg_o(0) = 4.97$ MPa.

Due to the symmetry of the geometry and of the load, only one half of the sample is studied and the analysis is performed with reference to the three different meshes shown in Fig. 12(b).

In Fig. 13 the stress–strain diagram is provided with reference to a one-dimensional bar subject to a homogeneous state of strain. The corresponding curve is similar to the constitutive law proposed by Pijaudier-Cabot and Benallal (1993) for concrete-like materials. It is worth noting that the area under the softening branch of the two curves differs from 0.01%.

In Fig. 14 the load resultant Q is plotted versus the vertical displacement of the point A of the sample for the three considered meshes. In order to compare the numerical results with existing ones published in the literature, the load–displacement curves are plotted in Fig. 14 together with the curve (broken line) pro-

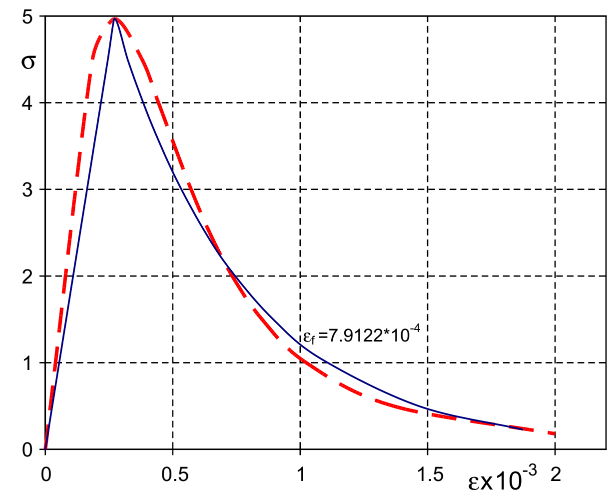

Fig. 13. The stress–strain diagram with exponential softening and local damage referred to a one-dimensional bar subject to a homogeneous deformation state.

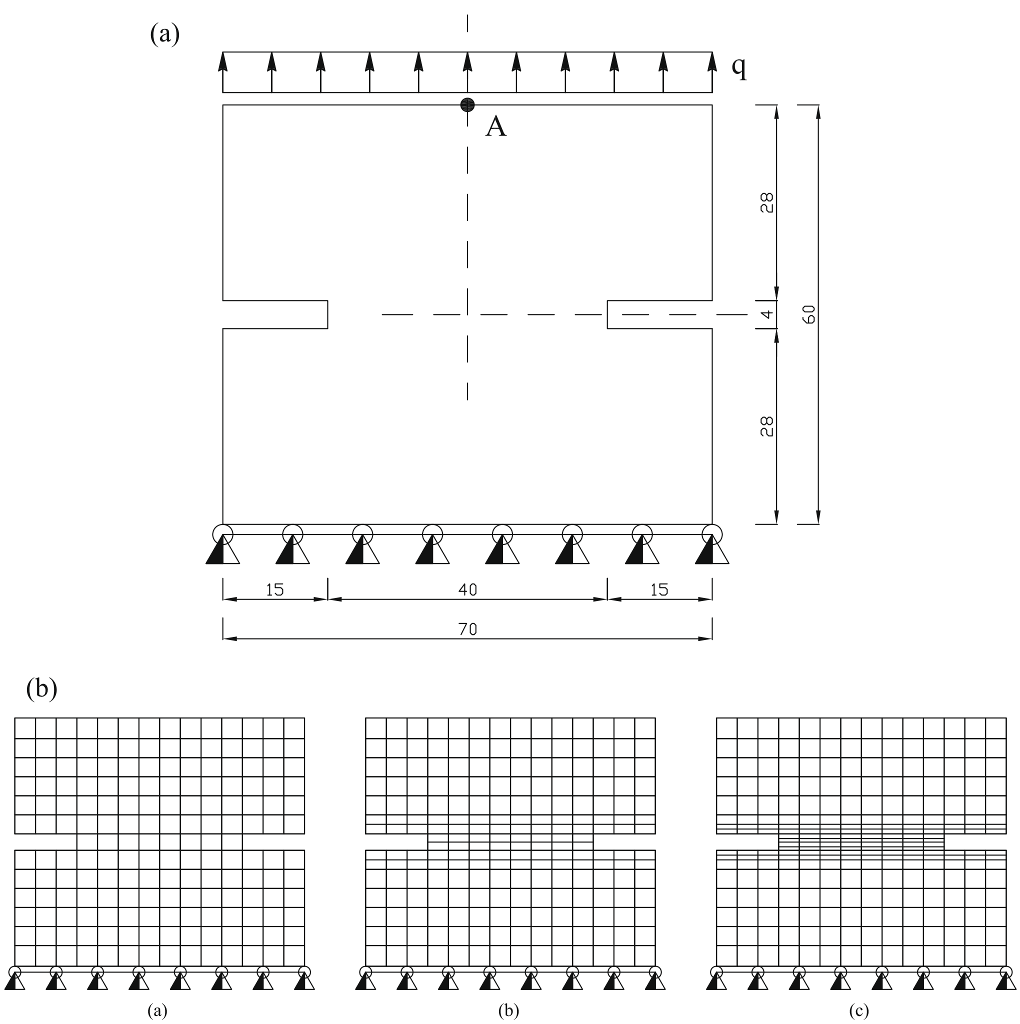

Fig. 12. (a) Two-dimensional specimen subjected to distributed tensile load. Geometric features in mm. (b) Three different meshes adopted in the finite element simulations: mesh (a) 176 elements, mesh (b) 212 elements, mesh (c) 256 elements.

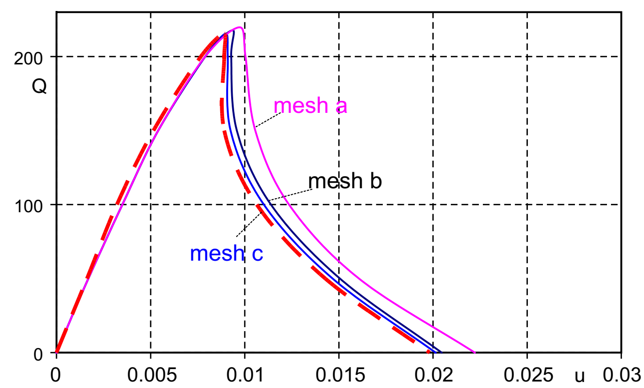

Fig. 14. The load–displacement response curve of point *A* for nonlocal damage obtained for different meshes: mesh (a) $n = 176$ elements, mesh (b) $n = 212$ elements and mesh (c) $n = 256$ elements. A related load–displacement plot (broken line) contributed in the literature is reported for comparison.

vided by Benvenuti et al. (2002) for the fine mesh (c) of Fig. 12(b). In that paper the nonlocal problem is solved by means of a non-symmetric LCP approach.

It can be observed a good agreement between the two procedures and a satisfactory mesh independent result of the proposed method.

6. Closure

The response of a structural elastic problem with nonlocal strain damage under assigned loads is provided.

The nonlocal damage model is cast in the framework of the internal variable theories and is formulated in a thermodynamic framework so that the constitutive relations for the nonlocal model are derived without ad hoc assumptions.

The stress decomposition in the effective stress and in the nonlocal relaxation stress, characterizing the state of damage of the material, consistently follows from the thermodynamic analysis.

A variational basis is then provided to the nonlocal model and two mixed variational principles with different combinations of state variables are obtained. It is worth noting that many other variational formulations can be obtained following the procedure outlined in this paper.

The FE algorithm for the resolution of the finite-step elastic model with nonlocal damage is properly defined starting from the mixed two-field variational principle and it is not based on extensions of procedures pertaining to local damage.

The proposed treatment of nonlocal damage can provide a basis for further developments as numerical analyses and additional computational comparisons with existing models which are the subjects of ongoing studies.

Acknowledgments

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Appendix A

It is reported the proof of the variational principle stated in Proposition 5.

The stationary conditions of P enforced at the point $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r})$ is given in the form $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \in \partial P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r})$ and yield the following relations:

$$\begin{cases} \mathbf{0} \in \partial_{\mathbf{u}} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}), \\ \mathbf{0} \in \partial_{\boldsymbol{\sigma}} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}), \\ \mathbf{0} \in \partial_{\boldsymbol{\sigma}^e} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}), \\ \mathbf{0} \in \partial_{\boldsymbol{\sigma}^r} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}), \\ \mathbf{0} \in \partial_{\boldsymbol{\varepsilon}} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}), \\ \mathbf{0} \in \partial_{\mathbf{r}} P(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^e, \boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}, \mathbf{r}) \end{cases} \Longleftrightarrow \begin{cases} \mathbf{B}'\boldsymbol{\sigma} = \ell + \mathbf{r}, \\ \mathbf{B}(\mathbf{u} + \mathbf{w}) = \boldsymbol{\varepsilon}, \\ \boldsymbol{\varepsilon} = d\Phi^*(\boldsymbol{\sigma}^e), \\ R\boldsymbol{\varepsilon} \in \partial D(\boldsymbol{\sigma}^r - \boldsymbol{\sigma}^r_o), \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^e - R\boldsymbol{\sigma}^r, \\ \mathbf{u} \in \partial \Upsilon^*(\mathbf{r}). \end{cases} \tag{59}$$

The relation $(59)_1$ provides the equilibrium equation. The relation $(59)_2$ represents the compatibility condition. The relation $(59)_3$ yields the constitutive relation in terms of strains $\boldsymbol{\varepsilon}$ and elastic stresses $\boldsymbol{\sigma}^e$. The relation $(59)_4$ shows that the nonlocal strain $\bar{\boldsymbol{\varepsilon}} = R\boldsymbol{\varepsilon}$ fulfils the finite-step damage flow rule. The relation $(59)_5$ provides the additive decomposition of the stress in which the relaxation stress $\bar{\boldsymbol{\sigma}}^r = R\boldsymbol{\sigma}^r$ is nonlocal and the relation $(59)_5$ provides the external constraint. The damage structural model is thus obtained. By reverting the steps above, a solution of the finite-step elastic problem with nonlocal damage makes the functional P stationarity.

Appendix B

Some basic definitions and properties of convex analysis which are referred to in the paper are briefly recalled here. A comprehensive treatment of the subject can be found in Rockafellar (1970).

Let (X, X') be a pair of locally convex topological vector spaces placed in separating duality by a bilinear form $\langle \cdot, \cdot \rangle$. The subdifferential of the convex functional $f : X \to \Re \cup \{+\infty\}$ is the set $\partial f \subseteq X'$ given by

$$x^* \in \partial f(x_o) \Longleftrightarrow f(y) - f(x_o) \geqslant \langle x^*, y - x_o \rangle \qquad \forall y \in X.$$

In particular, if the functional f is differentiable at $x_o \in X$, the subdifferential is a singleton and reduces to the usual differential.

The conjugate of a convex functional f is the convex functional $f^* : X' \to \Re \cup \{+\infty\}$ defined by:

$$f^*(x^*) = \sup \{\langle x^*, y \rangle - f(y) \quad \text{with } y \in X\},$$

so that Fenchel’s inequality holds:

$$f(y) + f^*(x^*) \geqslant \langle x^*, y \rangle \qquad \forall y \in X, \forall x^* \in X'.$$

The elements x, x^* for which Fenchel’s inequality holds as an equality are said to be conjugate and the following relations are equivalent if f is closed:

$$f(x) + f^*(x^*) = \langle x^*, x \rangle, \qquad x^* \in \partial f(x), \quad x \in \partial f(x^*).$$

Analogous results holds for concave functional by interchanging the role of $+\infty$, $\geqslant$ and sup with those of $-\infty$, $\leqslant$ and inf. The prefix *sub* used in the convex case has to be replaced by *super*. The same symbol ∂ is used to denote subdifferential (superdifferential) of a convex (concave) functional when no ambiguity can arise.

A relevant case of conjugate functionals associated with a convex set C is provided by the indicator functional:

$$\sqcup_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

and the support functional:

$$D(x^*) = \sup \{\langle x^*, x \rangle \quad \text{with } x \in C\}.$$

It is worth noting that the subdifferential of the indicator of a convex set C at a point $x \in C$ coincides with the normal cone to C at x:

$$\partial \sqcup_C(x) = N_C(x) = \begin{cases} \{x^* \in X' : \langle x^*, y - x \rangle \leqslant 0 \quad \forall y \in X\} & \text{if } x \in C, \\ \emptyset & \text{othertwise.} \end{cases}$$

Finally, the following rules holds for subdifferentiability (Romano, 1995).

Chain rule – Given a differentiable operator $A{:}X \rightarrow Y$ and a convex functional $f : Y \rightarrow \Re \cup \{+\infty\}$ which turns out to be subdifferentiable at $y = A(x)$, it results:

$$\partial(f \circ A)(x) = [dA(x)]'\partial f(A(x)),$$

where $dA(x)$ is the derivative of the operator A at the point x and $[dA(x)]'$ is the dual operator;

Additivity – Given two convex functional $f_1 : X \rightarrow \Re \cup \{+\infty\}$ and $f_2 : X \rightarrow \Re \cup \{+\infty\}$ which are subdifferentiable at $x \in X$, it results:

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x).$$

A functional $k : X \times Y \rightarrow \bar{\Re}$ is said to be saddle (convex–concave) if $k(x,y)$ is a convex functional of $x \in X$ for each $y \in Y$ and a concave functional of y for each x. The subdifferential of the convex functional $k(\cdot, y)$ at x is defined as $\partial_1 k(x,y)$ or $\partial_x k(x,y)$ and the superdifferential of the concave functional $k(x,\cdot)$ at y is defined as $\partial_2 k(x,y)$ or $\partial_y k(x,y)$. The subdifferential of the saddle functional k at the point (x,y) is defined as follows:

$$\partial k(x,y) = \partial_x k(x,y) \times \partial_y k(x,y).$$

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