



# Looking out for stable syzygy bundles<sup>☆</sup>

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## Abstract

We study (slope-)stability properties of syzygy bundles on a projective space  $\mathbb{P}^N$  given by ideal generators of a homogeneous primary ideal. In particular we give a combinatorial criterion for a monomial ideal to have a semistable syzygy bundle. Restriction theorems for semistable bundles yield the same stability results on the generic complete intersection curve. From this we deduce a numerical formula for the tight closure of an ideal generated by monomials or by generic homogeneous elements in a generic two-dimensional complete intersection ring.

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## 0. Introduction

Suppose that  $f_1, \dots, f_{d+1} \in K[U, V] = R$  are  $d + 1$  generic homogeneous polynomials of degree  $d$  in the two-dimensional polynomial ring over a field  $K$ . Since the dimension of the space of forms of degree  $d$  is  $d + 1$ , it follows that these generically chosen elements form a basis, and therefore we get the ideal inclusion  $R_{\geq d} \subseteq (f_1, \dots, f_{d+1})$  and hence also

$$R_{\geq d+1} \subseteq (f_1, \dots, f_{d+1}). \quad (*)$$

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<sup>☆</sup> With an appendix by Georg Hein: Semistability of the general syzygy bundle.

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This last statement is by no means true for other two-dimensional standard-graded domains such as  $R = K[X, Y, Z]/(G)$ . One aim of this paper is to show that  $(*)$  is true for such a two-dimensional hypersurface ring (for  $G$  generic of sufficiently high degree), if we replace the ideal  $(f_1, \dots, f_{d+1})$  on the right-hand side by its tight closure  $(f_1, \dots, f_{d+1})^*$  (Corollary 4.2). This means that  $d + 1$  generic forms of degree  $d$  are “tight generators” for  $R_{\geq d+1}$ .

The theory of tight closure has been developed by M. Hochster and C. Huneke since 1986 and plays a central role in commutative algebra [14–16]. It assigns to every ideal  $I$  in a Noetherian ring containing a field an ideal  $I^* \supseteq I$ , which is called the tight closure of  $I$ . For a domain over a field of positive characteristic  $p$  it is defined with the help of the Frobenius endomorphism, by

$$I^* := \{f \in R: \exists c \neq 0 \text{ such that } cf^q \in I^{[q]} \text{ for all } q = p^e\}.$$

The tight closure of an ideal in a regular ring is the ideal itself, and it is a typical feature of this theory that we may generalize a statement about an ideal in a regular ring to a non-regular ring if we replace the ideal by its tight closure. The tight closure version of the theorem of Briançon and Skoda is an important instance for this principle, and our stated result fits well into this picture.

There are three main ingredients for the above mentioned result and for similar results in this paper:

- (1) The geometric interpretation of tight closure and slope criteria.
- (2) Restriction theorems for stable vector bundles.
- (3) Criteria for stable syzygy bundles on a projective space.

We explain in this introduction these three items and their interplay and we give a summary on the content of this paper.

- (1) Geometric interpretation of tight closure and slope criteria.

We will use the geometric approach to the theory of tight closure in terms of vector bundles which we have developed in [5], [6] and [10]. The starting point is the cohomological characterization of tight closure due to Hochster saying that  $f \in (f_1, \dots, f_n)^*$  holds for an  $\mathfrak{m}$ -primary ideal  $(f_1, \dots, f_n)$  in an excellent normal local domain  $(R, \mathfrak{m})$  of dimension  $d$  over a field of positive characteristic if and only if  $H_{\mathfrak{m}}^d(A) \neq 0$ , where  $A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n + f)$  is the forcing algebra for these data.<sup>2</sup>

If  $R$  is a normal two-dimensional standard-graded domain over an algebraically closed field  $K$  and the data  $f_1, \dots, f_n$  and  $f$  are  $R_+$ -primary and homogeneous, then this cohomological characterization takes a simple form in terms of the locally free sheaf of syzygies  $\text{Syz}(f_1, \dots, f_n)$  on the smooth projective curve  $C = \text{Proj } R$ . This syzygy bundle is given by the short exact sequence

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(m - \deg(f_i)) \xrightarrow{f_i} \mathcal{O}_C(m) \longrightarrow 0.$$

<sup>2</sup> A remark about the characteristic: the theory arising in characteristic 0 from this cohomological characterization is called solid closure; see [13]. However solid closure has in dimension two all the good properties which we expect for a tight closure type theory and we will take it in this paper as the technical definition of tight closure and denote it henceforth with  $\star$ .

In this situation  $f \in (f_1, \dots, f_n)^*$  holds if and only if the affine-linear bundle corresponding to the cohomology class  $c = \delta(f) \in H^1(C, \text{Syz}(f_1, \dots, f_n)(m))$  (where  $m = \deg(f)$ ) is not an affine scheme.

This geometric approach allows us to apply the elaborated methods of algebraic geometry to problems coming from tight closure. In [6] we studied the ampleness and bigness properties of the dual of the syzygy bundle  $\text{Syz}(f_1, \dots, f_n)(m)$  in dependence of  $m$  and obtained both inclusion and exclusion criteria for tight closure in terms of the minimal and the maximal slope of it. These criteria together yield under the condition that the syzygy bundle is semistable (we shall recall the definitions in Section 1) the numerical characterization that

$$(f_1, \dots, f_n)^* = (f_1, \dots, f_n) + R_{\geq \frac{\deg(f_1) + \dots + \deg(f_n)}{n-1}}$$

holds in characteristic 0 (see [6, Theorem 8.1] and Remark 1.8 below for results in positive characteristic).

In order to apply this numerical formula to the computation of tight closure one has to establish the semistability property of a given syzygy bundle on the projective curve  $C = \text{Proj } R$ . This is a difficult matter in general, even if the rank of the bundle is 2 (and the number of ideal generators is 3). One result of [7] is that the syzygy bundle  $\text{Syz}(X^d, Y^d, Z^d)$  is semistable on normal domains  $R = K[X, Y, Z]/(G)$  for  $\deg(G) \geq 3d - 1$  and therefore  $(X^d, Y^d, Z^d)^* = (X^d, Y^d, Z^d) + R_{\geq 3d/2}$ .

Another and more general way to obtain semistable syzygy bundles on curves is to work on the projective plane (or a projective space or other varieties in which the curve lives) and to apply restriction theorems.

## (2) Restriction theorems for stable vector bundles.

There exist beautiful theorems due to Mehta and Ramanathan, Flenner, Bogomolov and Langer (see [23, 12, 2, 3, 18, 22]) saying that the restriction of a (semi)stable bundle on a smooth projective variety  $X$  to a general complete intersection curve of sufficiently high degree is again (semi)stable. We will present these theorems in Section 1.

We shall use mainly the easiest instance of this type of results, the restriction of stable bundles on the projective plane  $\mathbb{P}^2$  to a generic curve  $C \subset \mathbb{P}^2$ . Homogeneous elements  $f_1, \dots, f_n \in K[X, Y, Z]$  which are primary to the irrelevant ideal  $(X, Y, Z)$  define a locally free syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  on  $\mathbb{P}^2$  and its restriction to a projective curve  $C = V_+(G)$  gives the bundle which is crucial for the computation of the solid closure  $(f_1, \dots, f_n)^*$  in  $R = K[X, Y, Z]/(G)$ . So if we know that the syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  is semistable on  $\mathbb{P}^2$ , the restriction theorems yield at once that the same is true for  $\text{Syz}(f_1, \dots, f_n)|_C$  for a general curve  $C$  of sufficiently high degree. This gives us then the generic answer for  $(f_1, \dots, f_n)^*$  in a two-dimensional hypersurface ring. The result of Flenner gives a bound for the degree of the curve and the result of Bogomolov shows moreover that the restriction is in fact semistable for every smooth curve fulfilling a stronger degree condition.

So we are led to look out for stable syzygy bundles on the projective plane or more generally on a projective space. Note that the restriction theorems give us the possibility to argue on a regular polynomial ring to obtain results on tight closure, though “tight closure does nothing” (Hochster) on regular rings!

### (3) Criteria for stable syzygy bundles on a projective space.

Our main problem is now: suppose that homogeneous elements  $f_1, \dots, f_n \in K[X_0, \dots, X_N]$  are given. When is the syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  on  $\mathbb{P}^N$  semistable? Sections 2–8 are concerned with this question.

There exist surprisingly few results on stability properties of syzygy bundles. Flenner shows in [12, Corollary 2.2] (also proved by Ballico in [1, Corollary 6.5]) that the syzygy bundle for all monomials of fixed degree is semistable.

In Section 2 we shall discuss necessary numerical conditions for a syzygy bundle to be semistable. We get results by comparing the slope of  $\text{Syz}(f_1, \dots, f_n)$  with the slopes of the natural subsheaves  $\text{Syz}(f_i, i \in J)$  for subfamilies  $J \subset \{1, \dots, n\}$ . This gives at once the necessary degree condition  $d_n \leq \frac{d_1 + \dots + d_{n-1}}{n-2}$  for semistability, where  $d_n$  is the largest degree (Corollary 2.4).

The stability of the syzygy bundle implies conditions for the existence of global sections of the bundle and of its dual. These observations provide easily a characterization of semistability for bundles of rank 2 and 3, which correspond to  $n = 3$  and 4 ideal generators. We can take advantage of the fact that there exist only few line bundles on a projective space, contrary to the situation on projective curves (Section 3).

In Section 4 we study the restriction of a syzygy bundle on  $\mathbb{P}^N$  to generic lines  $\mathbb{P}^1 \subset \mathbb{P}^N$ . If these restrictions are a direct sum of line bundles of the same degree, then the bundle itself is semistable. Since this property is fulfilled for  $d+1$  generic forms of degree  $d$ , their syzygy bundle is semistable. Hence we may derive the result mentioned at the beginning of the introduction (Proposition 4.1, Corollary 4.2).

A torsion free subsheaf  $\mathcal{T} \subseteq \text{Syz}(f_i, i \in I)$  of rank  $r$  yields an invertible subsheaf  $(\bigwedge^r \mathcal{T})^{\vee\vee} \subseteq \bigwedge^r (\text{Syz}(f_i, i \in I))$ . Therefore we deal in Section 5 with exterior powers of syzygy bundles and describe them as a kernel of a suitable mapping.

In Sections 6 and 7 we settle the case of the syzygy bundle of a monomial ideal using results of A. Klyachko on toric bundles [20,21,19]. The main result is that  $\text{Syz}(X^{\sigma_i}, i \in I)$  is semistable if for every subset  $J \subseteq I$  the corresponding subsheaf  $\text{Syz}(X^{\sigma_i}, i \in J) \subseteq \text{Syz}(X^{\sigma_i}, i \in I)$  does not contradict the semistability. This provides an easy combinatorial test for semistability in the monomial case.

Finally, Section 8 addresses the case of ideals which are generated by generic forms  $f_i$  of degrees  $d_i$  fulfilling the necessary numerical conditions from Section 2. From a theorem of Bohnhorst and Spindler we deduce that the syzygy bundle of  $n$  parameters in an  $n$ -dimensional polynomial ring is semistable (Corollary 8.2) and a theorem of Hein asserts that this is also true for the syzygy bundle of  $n$  generic forms of degree  $d$  under the condition that  $n \leq d(N+1)$ . This theorem is proven by Hein in an appendix to this paper.

## 1. Stable bundles and restriction theorems

We recall the definition of semistability on a smooth projective curve  $C$  over an algebraically closed field  $K$ . Let  $\mathcal{S}$  denote a locally free sheaf on  $C$  of rank  $r$ . The degree of  $\mathcal{S}$  is defined as the degree of the corresponding invertible sheaf  $\det \mathcal{S} = \bigwedge^r \mathcal{S}$ . The number  $\mu(\mathcal{S}) = \deg(\mathcal{S})/\text{rk}(\mathcal{S})$  is called the slope of the vector bundle. A locally free sheaf  $\mathcal{S}$  is called semistable, if for every locally free subsheaf  $\mathcal{T} \subset \mathcal{S}$  the inequality  $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$  holds (and stable if  $<$  holds). This notion is due to Mumford [24] and plays a crucial role in the construction of moduli spaces for vector bundles on curves and beyond.

On a higher dimensional smooth projective variety it is convenient to develop these notions more generally for torsion-free coherent sheaves  $\mathcal{S}$  in dependence of a fixed very ample invertible sheaf. We will work here only with the notion of  $\mu$ -stability (or Mumford–Takemoto stability), not with Gieseker stability. We take [25] as our main reference and we deal only with coherent torsion-free sheaves on a projective space  $\mathbb{P}^N$ . The determinantal bundle of such a sheaf is defined by the bidual  $\det \mathcal{S} = (\bigwedge^r \mathcal{S})^{\vee\vee}$ , which is an invertible sheaf, and the degree of  $\mathcal{S}$  is defined by  $\deg(\bigwedge^r \mathcal{S})^{\vee\vee}$ . Since  $\mathcal{S}$  is locally free outside a closed subset of codimension  $\geq 2$ , there exist projective lines  $\mathbb{P}^1 \subset \mathbb{P}^N$  such that the restriction is locally free, hence  $\mathcal{S}|_{\mathbb{P}^1} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$  and this gives another way to define the degree, as  $\sum a_i$ . The slope of  $\mathcal{S}$  is defined as before by dividing through the rank.

**Definition 1.1.** Let  $\mathcal{S}$  denote a torsion-free coherent sheaf on a projective space  $\mathbb{P}^N$ . Then  $\mathcal{S}$  is called semistable if for every coherent subsheaf  $\mathcal{T} \subseteq \mathcal{S}$  the inequality  $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$  holds.

These subsheaves are of course again torsion-free. It is enough to check this property for those subsheaves which have a torsion-free quotient (see [25, Theorem 1.2.2]).

The restriction of a semistable torsion-free sheaf to a curve is in general not semistable anymore. We will use the following restriction theorems which we cite here for the convenience of the reader. We only state the results for a bundle on a projective space and for the restriction to a complete intersection curve.

**Theorem 1.2.** (Mehta and Ramanathan, see [23, Theorem 6.1], [18, Theorem 7.2.8].) Let  $K$  denote an algebraically closed field of any characteristic and let  $\mathcal{S}$  denote a semistable torsion-free coherent sheaf on  $\mathbb{P}^N$ . Then there exists a number  $k_0$  such that for  $N - 1$  general elements  $D_1, \dots, D_{N-1} \in |\mathcal{O}(k)|$ ,  $k \geq k_0$ , the restriction  $\mathcal{S}|_C$  is again semistable on the smooth complete intersection curve  $C = D_1 \cap \cdots \cap D_{N-1}$ .

This theorem of Mehta and Ramanathan says nothing about the bound  $k_0$ . This is provided by the theorem of Flenner, but only in characteristic zero.

**Theorem 1.3.** (Flenner, see [12], [18, Theorem 7.1.1].) Let  $K$  denote an algebraically closed field of characteristic 0. Let  $\mathcal{S}$  denote a torsion-free coherent semistable sheaf of rank  $r$  on the projective space  $\mathbb{P}^N$ . Then for  $k$  fulfilling the condition that

$$\frac{\binom{k+N}{N} - (N-1)k - 1}{k} > \max\left\{\frac{r^2 - 1}{4}, 1\right\}$$

and for  $N - 1$  general elements  $D_1, \dots, D_{N-1} \in |\mathcal{O}(k)|$  the restriction  $\mathcal{S}|_C$  is again semistable on the smooth complete intersection curve  $C = D_1 \cap \cdots \cap D_{N-1}$ .

**Remark 1.4.** The degree bound in the theorem of Flenner reduces for  $N = 2$  to the condition that  $k > \frac{r^2 - 3}{2}$  and  $k \geq 2$ . So this means  $k \geq 2$  for  $r = 2$ ,  $k \geq 4$  for  $r = 3$ ,  $k \geq 7$  for  $r = 4$ .

In the surface case the theorem of Bogomolov gives even a result about the restriction to every smooth curve, not only to a general curve.

**Theorem 1.5.** (Bogomolov, see [2,3], [18, Theorem 7.3.5].) Let  $K$  denote an algebraically closed field of characteristic 0. Let  $\mathcal{S}$  denote a stable locally free sheaf of rank  $r$  on the projective plane  $\mathbb{P}^2$  with Chern classes  $c_1$  and  $c_2$ . Let  $\Delta(\mathcal{S}) = 2rc_2 - (r-1)c_1^2$  and set  $R = \binom{r}{\lfloor r/2 \rfloor} \binom{r-2}{\lfloor r/2 \rfloor - 1}$ . Then the restriction  $\mathcal{S}|_C$  is again stable for every smooth curve  $C \subset \mathbb{P}^2$  of degree  $k$  with  $2k > \frac{R}{r} \Delta(\mathcal{S}) + 1$ .

**Remark 1.6.** If  $\mathcal{S} = \text{Syz}(f_1, \dots, f_n)$  for polynomials of degree  $d_i$ , then  $c_1(\mathcal{S}) = -\sum d_i$  and  $c_2(\mathcal{S}) = \frac{(\sum d_i)^2 - \sum d_i^2}{2}$ . Therefore the discriminant is in this case

$$\Delta(\mathcal{S}) = (n-1) \left( \left( \sum d_i \right)^2 - \sum d_i^2 \right) - (n-2) \left( \sum d_i \right)^2 = \left( \sum d_i \right)^2 - (n-1) \sum d_i^2.$$

If all the degrees are constant, then  $\Delta(\mathcal{S}) = nd^2$  and Bogomolov's result yields the degree condition  $2k > 3d^2 + 1$  for  $n = 3$ ,  $2k > 4d^2 + 1$  for  $n = 4$  and  $2k > 60d^2 + 1$  for  $n = 5$ .

**Example 1.7.** Look at the syzygy bundle  $\text{Syz}(X^2, Y^2, Z^2)$  on  $\mathbb{P}^2$ . It is easy to see that this bundle is stable (see Corollary 3.2 below) and the theorem of Flenner 1.3 tells us that the restriction is semistable for a generic curve  $C$  of degree  $\deg C \geq 2$ ,  $C = \text{Proj } R$ ,  $R = K[X, Y, Z]/(G)$ . The bound in the theorem of Bogomolov tells us that the restriction to every smooth curve of degree  $\geq 7$  is semistable. Due to [7, Proposition 6.2] this is even true for degree  $\geq 5$ .

For  $\deg G = 3$  the semistability depends on the curve equation  $G$ , and so does the question whether  $XYZ \in (X^2, Y^2, Z^2)^*$  holds in  $R = K[X, Y, Z]/(G)$  or not. For the Fermat cubic  $G = X^3 + Y^3 + Z^3$ , the semistability property is easy to establish, since this curve equation gives at once the syzygy  $(X, Y, Z)$  for  $(X^2, Y^2, Z^2)$  (of total degree 3), which yields a short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \text{Syz}(X^2, Y^2, Z^2)(3) \longrightarrow \mathcal{O} \longrightarrow 0.$$

This shows that the syzygy bundle is semistable (and strongly semistable, but not stable). It follows that  $XYZ \in (X^2, Y^2, Z^2)^*$  holds in  $K[X, Y, Z]/(X^3 + Y^3 + Z^3)$  in any characteristic, which was first shown by a quite complicated computation of A. Singh; see [26].

**Remark 1.8.** We comment on the situation in positive characteristic. The best restriction theorem for semistable bundles in positive characteristic is due to A. Langer [22] and gives a Bogomolov-type restriction theorem. However, the numerical formula for tight closure mentioned in the introduction needs the assumption that the syzygy bundle is strongly semistable, meaning that every Frobenius pull-back of it is semistable. It was shown in [8] that a Bogomolov-type restriction theorem for strong semistability does not hold. It is open whether there exists a Flenner-type restriction theorem for strong semistability.

However, we may derive a slightly weaker result for prime characteristic  $p \gg 0$  from the results in characteristic zero. If we know in the relative situation, that is over  $\text{Spec } \mathbb{Z}$ , that a syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  is semistable in characteristic zero, then every twist of it of positive degree is ample, and therefore this property holds also in positive characteristic for almost all prime characteristics  $p$ . From this it follows for  $p$  large enough that for  $\deg(f) < \frac{\deg(f_1) + \dots + \deg(f_n)}{n-1}$  the element  $f$  belongs to  $(f_1, \dots, f_n)^*$  only if it belongs to the ideal itself, and for  $\deg(f) > \frac{\deg(f_1) + \dots + \deg(f_n)}{n-1}$  (not  $\geq$  as in the formula) the element belongs to the Frobenius closure of

$(f_1, \dots, f_n)$ , hence also to the tight closure. In particular, if the degree bound is not a natural number, then we get the same statement as in characteristic zero.

## 2. Numerical conditions for semistability on $\mathbb{P}^N$

Let  $f_i, i \in J$ , denote homogeneous polynomials  $\neq 0$  in  $K[X_0, \dots, X_N]$ , where  $K$  is an algebraically closed field. Their syzygy sheaf  $\text{Syz}(f_i, i \in J)$  is locally free on  $D_+(f_i, i \in J) = \bigcup_{i \in J} D_+(f_i) \subseteq \mathbb{P}^N$  and has rank  $r = |J| - 1$ . We compute first the degree of  $\text{Syz}(f_i, i \in J)(m)$ , which is by definition the degree of the invertible sheaf  $\det(\text{Syz}(f_i, i \in J)(m))$ , where  $\det(\text{Syz}(f_i, i \in J)(m)) = (\bigwedge^r (\text{Syz}(f_i, i \in J)(m)))^{\vee\vee}$ .

**Lemma 2.1.** *Let  $f_i \in K[X_0, \dots, X_N], i \in J$ , denote homogeneous polynomials  $\neq 0$  of degree  $d_i, |J| = r + 1, r \geq 1$ . Then the following hold.*

(i) *Suppose that the  $f_i$  do not have a common factor. Then*

$$\det(\text{Syz}(f_i, i \in J)(m)) \cong \mathcal{O}\left(rm - \sum_{i \in J} d_i\right)$$

*and  $\deg(\text{Syz}(f_i, i \in J)(m)) = rm - \sum_{i \in J} d_i$ .*

(ii) *Suppose that the  $f_i$  do have a highest common factor  $f$  of degree  $d$ . Then*

$$\det(\text{Syz}(f_i, i \in J)(m)) \cong \mathcal{O}\left(rm + d - \sum_{i \in J} d_i\right)$$

*and  $\deg(\text{Syz}(f_i, i \in J)(m)) = rm + d - \sum_{i \in J} d_i$ .*

**Proof.** (i). Suppose first that the  $f_i, i \in J$ , do not have a common factor, so that their zero locus  $V_+(f_i, i \in J)$  has codimension  $\geq 2$ . Thus we have the short exact (presenting) sequence

$$0 \longrightarrow \text{Syz}(f_i, i \in J)(m) \longrightarrow \bigoplus_{i \in J} \mathcal{O}(m - d_i) \longrightarrow \mathcal{O}(m) \longrightarrow 0$$

on  $D_+(f_i, i \in J)$ . We restrict this sequence to a projective line  $\mathbb{P}^1 \subset D_+(f_i, i \in J)$  and get

$$\deg(\text{Syz}(f_i, i \in J)(m)) = (r + 1)m - \sum_{i \in J} d_i - m.$$

(ii) Now suppose that the  $f_i$  do have a highest common factor, and write  $f_i = fg_i$  such that the  $g_i, i \in J$ , do not have a common factor. Then we have an isomorphism of sheaves

$$\text{Syz}(f_i, i \in J)(m) \cong \text{Syz}(g_i, i \in J)(m - d)$$

by considering a syzygy  $(s_i, i \in J)$  for  $g_i$  of total degree  $m - d$  as a syzygy for  $f_i = fg_i$  of total degree  $m$ . Therefore we get

$$\begin{aligned}
\deg(\mathrm{Syz}(f_i, i \in J)(m)) &= \deg(\mathrm{Syz}(g_i, i \in J)(m-d)) \\
&= r(m-d) - \sum_{i \in J} (d_i - d) \\
&= rm - \sum_{i \in J} d_i + d. \quad \square
\end{aligned}$$

This gives the following necessary numerical conditions for a sheaf of syzygies to be semistable.

**Proposition 2.2.** *Let  $f_i \neq 0$ ,  $i \in I$ ,  $|I| \geq 2$ , denote homogeneous elements in the polynomial ring  $K[X_0, \dots, X_N]$  of degrees  $d_i$ . For every subset  $J \subseteq I$  denote by  $d_J$  the degree of the highest common factor of the subfamily  $f_i$ ,  $i \in J$ . Suppose that the syzygy sheaf  $\mathrm{Syz}(f_i, i \in I)$  is semistable. Then for every  $J \subseteq I$ ,  $|J| \geq 2$ , we have the numerical condition*

$$\frac{d_J - \sum_{i \in J} d_i}{|J| - 1} \leq \frac{d_I - \sum_{i \in I} d_i}{|I| - 1}.$$

If  $\mathrm{Syz}(f_i, i \in I)$  is stable, then  $<$  holds for  $J \subset I$ .

**Proof.** Every subset  $J \subseteq I$  defines the syzygy subsheaf  $\mathrm{Syz}(f_i, i \in J)$  sitting in

$$0 \longrightarrow \mathrm{Syz}(f_i, i \in J) \longrightarrow \mathrm{Syz}(f_i, i \in I) \longrightarrow \bigoplus_{i \notin J} \mathcal{O}(-d_i).$$

(The sequence is not exact on the right in general.) The semistability of  $\mathrm{Syz}(f_i, i \in I)$  implies that

$$\mu(\mathrm{Syz}(f_i, i \in J)) \leq \mu(\mathrm{Syz}(f_i, i \in I)),$$

and we have computed these slopes in Lemma 2.1(ii).  $\square$

**Remark 2.3.** This necessary condition for semistability is in general not sufficient, as Example 3.7 below shows. However, if the  $f_i$  are monomials, then we will see in Section 6 (Corollary 6.4) that this condition is also sufficient.

The condition in Proposition 2.2 implies the following necessary condition for the degrees of a semistable syzygy sheaf.

**Corollary 2.4.** *Let  $f_1, \dots, f_n \in K[X_0, \dots, X_N]$  denote homogeneous polynomials  $\neq 0$  without common factor of degrees  $1 \leq d_1 \leq \dots \leq d_n$ . Suppose that their syzygy sheaf is semistable. Then for every  $1 \leq r \leq n-2$  we have the numerical condition*

$$(n-r-1)(d_1 + \dots + d_{r+1}) \geq r(d_{r+2} + \dots + d_n).$$

For  $r = n-2$  this gives the necessary condition  $d_1 + \dots + d_{n-1} \geq (n-2)d_n$ . On the other hand, this last condition implies the other ones.



**Proof.** We apply Proposition 2.2 to the subset  $J = \{1, \dots, r+1\}$  and get the inequality

$$\frac{-\sum_{i=1}^{r+1} d_i}{r} \leq \frac{-\sum_{i=1}^n d_i}{n-1}$$

or equivalently that

$$(n-1) \sum_{i=1}^{r+1} d_i \geq r \sum_{i=1}^n d_i.$$

Subtracting  $r \sum_{i=1}^{r+1} d_i$  gives the result.

For the last statement, suppose that  $d_1 + \dots + d_{n-1} \geq (n-2)d_n$  holds and that we have proved already that  $(n-1) \sum_{i=1}^{r+1} d_i \geq r \sum_{i=1}^n d_i$  (descending induction on  $r$ ). We have  $(n-1)d_{r+1} \leq (n-1)d_n \leq \sum_{i=1}^n d_i$ . Therefore

$$(n-1) \sum_{i=1}^r d_i = (n-1) \sum_{i=1}^{r+1} d_i - (n-1)d_{r+1} \geq r \sum_{i=1}^n d_i - (n-1)d_{r+1} \geq (r-1) \sum_{i=1}^n d_i. \quad \square$$

### 3. Syzygy bundles of low rank

We cover now the case of a syzygy sheaf of rank 2 and 3 (corresponding to 3 or 4 ideal generators). The following criteria are known.

**Lemma 3.1.** *Suppose that  $\mathcal{S}$  is a coherent torsion-free sheaf on a projective space  $\mathbb{P}^N$  over a field. Then the following criteria for semistability hold.*

- (i) *Suppose that  $\text{rk}(\mathcal{S}) = 2$ . If  $\mathcal{S}(m)$  has no global sections  $\neq 0$  for  $m < -\mu(\mathcal{S}) = -\deg(\mathcal{S})/2$ , then  $\mathcal{S}$  is semistable.*
- (ii) *Suppose that  $\text{rk}(\mathcal{S}) = 3$ . Suppose that  $\mathcal{S}(m)$  has no global sections  $\neq 0$  for  $m < -\mu(\mathcal{S}) = -\deg(\mathcal{S})/3$  and that  $(\mathcal{S}^\vee)(k)$  has no global sections  $\neq 0$  for  $k < -\mu(\mathcal{S}^\vee) = \deg(\mathcal{S})/3$ . Then  $\mathcal{S}$  is semistable.*

**Proof.** See [25, Lemma 1.2.5, Remark 1.2.6].  $\square$

This gives the following corollaries for 3 ideal generators.

**Corollary 3.2.** *Let  $f_1, f_2, f_3 \in K[X_0, \dots, X_N]$  be a homogeneous regular sequence with degrees  $d_1 \leq d_2 \leq d_3$  such that  $d_3 \leq d_1 + d_2$ . Then the sheaf  $\text{Syz}(f_1, f_2, f_3)$  is semistable on  $\mathbb{P}^N$  (and stable for  $d_3 < d_1 + d_2$ ).*

**Proof.** The Koszul complex yields the resolution

$$0 \longrightarrow \mathcal{O}(m - d_1 - d_2 - d_3) \longrightarrow \bigoplus_{i \neq j} \mathcal{O}(m - d_i - d_j) \longrightarrow \text{Syz}(m) \longrightarrow 0.$$

Since  $N \geq 2$ , the global sections of  $\text{Syz}(m)$  come from the left, hence  $\Gamma(\mathbb{P}^N, \text{Syz}(m)) = 0$  for  $m < d_1 + d_2$ . So  $\text{Syz}(m)$  has no non-trivial sections for  $m < \frac{d_1+d_2+d_3}{2} \leq d_1 + d_2$  and the result follows from Lemma 3.1.  $\square$

**Corollary 3.3.** *Let  $K$  denote an algebraically closed field of characteristic 0. Let  $f_1, f_2, f_3 \in K[X, Y, Z]$  be homogeneous primary elements with degrees  $d_1 \leq d_2 \leq d_3$  such that  $d_3 \leq d_1 + d_2$ . Then for a general hypersurface ring  $R = K[X, Y, Z]/(G)$  for general homogeneous  $G$  of degree  $\geq 2$  we have*

$$(f_1, f_2, f_3)^* = (f_1, f_2, f_3) + R_{\geq \frac{d_1+d_2+d_3}{2}}.$$

The same is true for every  $G$  defining a smooth curve under the degree condition  $\deg(G) \geq d_1d_2 + d_1d_3 + d_2d_3 - \frac{1}{2}(d_1^2 + d_2^2 + d_3^2) + 1$ .

**Proof.** This follows from Corollary 3.2 and the restriction theorem of Flenner 1.3. The last statement follows from the restriction theorem of Bogomolov 1.5, since  $\Delta(\text{Syz}) = 2d_1d_2 + 2d_1d_3 + 2d_2d_3 - d_1^2 - d_2^2 - d_3^2$  and  $R/r = 1$ .  $\square$

**Example 3.4.** A typical example where we may apply Corollary 3.3 is for the computation of  $(X^d, Y^d, Z^d)^*$ . The generic answer is that  $(X^d, Y^d, Z^d)^* = (X^d, Y^d, Z^d) + R_{\geq 3d/2}$  and this holds for  $R = K[X, Y, Z]/(G)$  for general  $G$  of degree  $\deg G \geq 2$  and for every normal  $R$  for  $\deg G \geq \frac{3}{2}d^2 + 1$ .

**Example 3.5.** Consider the elements  $X^3, XY^2$  and  $ZY^2$  in  $K[X, Y, Z]$ . These polynomials are not  $(X, Y, Z)$ -primary and their syzygy sheaf is locally free only outside the points  $(0, 0, 1)$  and  $(0, 1, 0)$ . It fulfills the degree condition in Corollary 3.3, but it is not semistable. The syzygy  $(0, Z, -X)$  is a non-trivial global section of  $\text{Syz}(X^3, XY^2, ZY^2)(4)$ , but its degree is  $2 \cdot 4 - 9 = -1$  negative.

We consider now the case of 4 polynomials in three variables.

**Corollary 3.6.** *Let  $f_1, f_2, f_3, f_4 \in K[X, Y, Z]$  be homogeneous primary elements with ordered degrees  $d_1 \leq d_2 \leq d_3 \leq d_4$ . Suppose that  $2d_4 \leq d_1 + d_2 + d_3$  and that  $\Gamma(\mathbb{P}^2, \text{Syz}(m)) = 0$  for  $m < \frac{d_1+d_2+d_3+d_4}{3}$ . Then the syzygy bundle  $\text{Syz}(f_1, f_2, f_3, f_4)$  is semistable.*

Furthermore, for  $\text{char}(K) = 0$  and for a general hypersurface ring  $R = K[X, Y, Z]/(G)$  for  $G$  of degree  $\geq 4$  we have

$$(f_1, f_2, f_3, f_4)^* = (f_1, f_2, f_3, f_4) + R_{\geq \frac{d_1+d_2+d_3+d_4}{3}}.$$

The same is true for every  $G$  defining a smooth curve and fulfilling the degree condition  $\deg(G) \geq \sum_{i \neq j} d_i d_j - \sum_i d_i^2 + 1$ .

**Proof.** We dualize the presenting sequence and get

$$0 \longrightarrow \mathcal{O}(-m) \longrightarrow \bigoplus_{i=1}^4 \mathcal{O}(d_i - m) \longrightarrow (\text{Syz}(m))^\vee \longrightarrow 0.$$

Since  $H^1(\mathbb{P}^2, \mathcal{O}(-m)) = 0$  for all  $m \in \mathbb{Z}$ , every global section of  $(\text{Syz}(m))^\vee$  comes from a global section of  $\bigoplus_{i=1}^4 \mathcal{O}(d_i - m)$ . This means that every cosyzygy  $\text{Syz}(m) \rightarrow \mathcal{O}$  must factor through  $\bigoplus_{i=1}^4 \mathcal{O}(m - d_i)$ . Therefore for  $m > d_4$  there exists no non-trivial homomorphism  $\text{Syz}(m) \rightarrow \mathcal{O}$ . From the assumption it follows that  $\frac{d_1+d_2+d_3+d_4}{3} \geq d_4$ , hence there exists no cosyzygy for  $m > \frac{d_1+d_2+d_3+d_4}{3}$ . So both conditions in Lemma 3.1 for  $\mathcal{S} = \text{Syz}(0)$  hold true and the result follows.

The statements about solid closure follows from the theorems of Flenner 1.3 and Bogomolov 1.5.  $\square$

**Example 3.7.** Consider the four elements  $X^{10}, Y^{10}, Z^{10}$  and  $P = X^9Y + X^9Z + Y^9X + Y^9Z + Z^9X + Z^9Y$ . All the syzygy subbundles  $\text{Syz}(f_i, i \in J)$  for subsets  $J \subset \{1, 2, 3, 4\}$  do not contradict the semistability. This is clear for  $|J| = 2$  since the polynomials are pairwise coprime and for  $|J| = 3$  since the numerical degree condition in Corollary 3.6 is fulfilled. However, this syzygy bundle is not semistable. We have  $XYZP \in (X^{10}, Y^{10}, Z^{10})$  and therefore there exists a non-trivial syzygy of degree 13. But the degree of  $\text{Syz}(13)$  is  $3 \cdot 13 - 4 \cdot 10 = -1$ .

#### 4. Semistable restrictions to a generic projective line

Let  $\mathcal{S}$  denote a coherent torsion-free sheaf on  $\mathbb{P}^N$ . The slope of  $\mathcal{S}$  and of a subsheaf  $\mathcal{T} \subseteq \mathcal{S}$  can be computed on a generic line  $\mathbb{P}^1 \subset \mathbb{P}^N$ . Hence if we know that the restriction of  $\mathcal{S}$  to a generic projective line is semistable, that is of type  $\mathcal{O}(a) \oplus \cdots \oplus \mathcal{O}(a)$ , then  $\mathcal{S}$  is semistable (see [25, Remark after Lemma 2.2.1]). We derive from this observation the following semistability result for  $d + 1$  forms of degree  $d$  and we obtain the result mentioned in the introduction that  $d + 1$  general elements of degree  $d$  are “tight generators” for the next degree in a generic two-dimensional complete intersection ring.

**Proposition 4.1.** *Let  $f_1, \dots, f_{d+1} \in K[X_0, \dots, X_N]$  be  $d + 1$  forms of degree  $d$  over an algebraically closed field  $K$ . Suppose that there exists a linear morphism  $K[X_0, \dots, X_N] \rightarrow K[U, V]$  such that the images of these forms are linearly independent in  $K[U, V]_d$ . Then the syzygy bundle  $\text{Syz}(f_1, \dots, f_{d+1})$  is semistable on  $\mathbb{P}^N$ . This is in particular true for  $d + 1$  generic forms of degree  $d$ .*

**Proof.** Let the linear mapping be given by  $X_j \mapsto a_j U + b_j V$ . We may write the images of  $f_i$  as  $\tilde{f}_i = \sum_{k=0}^d p_{i,k}(a_j, b_j) U^k V^{d-k}$ , where the coefficients  $p_{i,k}$  are polynomials in  $a_j, b_j$ . Since the determinant of the  $(d + 1) \times (d + 1)$  matrix of this polynomial entries is  $\neq 0$  for a special value of  $(a_j, b_j)$ , the determinant is not the zero polynomial. This means that the images of these forms are a bases of  $K[U, V]_d$  for generic choice of  $(a_j, b_j)$ . Therefore we have on a generic line

$$\text{Syz}(f_1, \dots, f_{d+1})(d + 1)|_{\mathbb{P}^1} \cong \text{Syz}(U^d, U^{d-1}V, \dots, V^d)(d + 1) \cong \mathcal{O}_{\mathbb{P}^1}^d.$$

So the restriction is semistable and hence the bundle itself on the projective space is semistable.

A generic set of  $d + 1$  forms of degree  $d$  is generic on a generic line.  $\square$

**Corollary 4.2.** *Let  $K$  denote an algebraically closed field of characteristic 0. Let  $f_1, \dots, f_{d+1} \in K[X_0, \dots, X_N]$  be  $d + 1$  forms of degree  $d$ . Suppose that there exists a linear morphism  $K[X_0, \dots, X_N] \rightarrow K[U, V]$  such that the images of these forms are linearly independent in*

$K[U, V]_d$ . Then on a generic complete intersection ring  $R = K[X_0, \dots, X_N]/(G_1, \dots, G_{N-1})$ , where  $G_j$  are generic forms of sufficiently high degree, we have

$$(f_1, \dots, f_{d+1})^* = (f_1, \dots, f_{d+1}) + R_{\geq d+1}.$$

This holds in particular for  $d + 1$  generic forms of degree  $d$ .

**Proof.** From Proposition 4.1 and the Restriction Theorems 1.2 or 1.3 it follows that the syzygy bundle is semistable on the smooth projective complete intersection curve defined by  $(G_1, \dots, G_{N-1})$  for generic  $G_j$  of sufficiently high degree. Hence the numerical formula from the introduction holds with the degree bound  $\sum_{i=1}^{d+1} \deg(f_i)/d = d(d+1)/d = d+1$ .  $\square$

**Example 4.3.** The preceding proposition and corollary are applicable for  $d + 1$  forms of type

$$X^d + ZP_0(X, Y, Z), \quad X^{d-1}Y + ZP_1(X, Y, Z), \quad \dots, \quad Y^d + ZP_d(X, Y, Z),$$

where  $P_i$  are polynomials of degree  $d - 1$ . By setting  $Z = 0$ , these forms yield all monomials of  $K[X, Y]_d$ .

The easiest instance is given by setting  $P_i = 0$ , which gives just all monomials in  $X$  and  $Y$ . Here the equality  $\text{Syz}(X^d, \dots, Y^d)(d+1) \cong \mathcal{O}^d$  holds already on  $D_+(X, Y) \subset \mathbb{P}^2$  and the stated result is true for every curve  $V_+(G) \subset D_+(X, Y)$  (this condition means that  $X$  and  $Y$  are parameters in  $R = K[X, Y, Z]/(G)$ ). An element  $f$  of degree  $m$  yields a cohomology class in  $H^1(V_+(G), \mathcal{O}(m - d - 1)^d)$  and one may argue on the components. In this special case the formula in Corollary 4.2 is also clear from [17, Theorem 5.11].

**Example 4.4.** Consider  $X^3, Y^3, Z^3, X^2Y$ . Setting  $Z = X + Y$ , the restriction yields four independent polynomials. Hence the bundle is semistable and it follows that  $R_{\geq 4} \subseteq (X^3, Y^3, Z^3, X^2Y)^*$  in a generic hypersurface ring  $K[X, Y, Z]/(G)$ .

**Question 4.5.** Let  $n$  monomials in  $K[X_0, \dots, X_N]$  of the same degree  $d$  ( $n \leq d + 1$ ) be given. When does there exist a linear mapping

$$K[X_0, \dots, X_N] \longrightarrow K[U, V]$$

such that the images of the monomials are linearly independent?

**Example 4.6.** Consider the five monomials

$$X^4, Y^4, Z^4, X^3Y, X^3Z \in K[X, Y, Z]$$

of degree four and let  $\mathcal{S} = \text{Syz}(X^4, Y^4, Z^4, X^3Y, X^3Z)$  denote their syzygy bundle. The images of the monomials  $X^4, X^3Y$  and  $X^3Z$  are linearly dependent for every linear homomorphism  $K[X, Y, Z] \rightarrow K[U, V]$ . It follows that for every line  $\mathbb{P}^1 \subset \mathbb{P}^2$  the restriction  $\mathcal{S}|_{\mathbb{P}^1}$  is not semistable, since the dependence yields non-trivial sections in  $(\mathcal{S}|_{\mathbb{P}^1})(4)$  (but the degree of  $\mathcal{S}(4)$  is  $-4$ ).

The syzygy bundle  $\mathcal{S}$  on  $\mathbb{P}^2$  has global sections of total degree 5, and  $\Gamma(\mathbb{P}^2, \mathcal{S}(5))$  is spanned by

$$(Z, 0, 0, 0, -X), \quad (Y, 0, 0, -X, 0) \quad \text{and} \quad (0, 0, 0, Z, -Y).$$

These global syzygies span the syzygy subsheaf  $\mathrm{Syz}(X^4, X^3Y, X^3Z)(5) \cong \mathrm{Syz}(X, Y, Z)(2)$ . Its slope is  $1/2$ , whereas  $\mathcal{S}(5)$  has slope 0, hence  $\mathcal{S}$  is not semistable.

We want to compute its Harder–Narasimhan filtration. We have the exact sequence (see the proof of Proposition 2.2)

$$0 \longrightarrow \mathrm{Syz}(X, Y, Z)(m-3) \longrightarrow \mathcal{S}(m) \xrightarrow{(p_2, p_3)} \mathcal{O}(m-4) \oplus \mathcal{O}(m-4).$$

The syzygy subbundle on the left is semistable. The image of the last mapping is a torsion-free subsheaf of rank 2. This quotient is given locally by

$$\mathcal{Q}(m) = \{(s, t) \in \mathcal{O}(m-4) \oplus \mathcal{O}(m-4) : sY^4 + tZ^4 \in (X^3)\}.$$

For  $m$  large enough,  $\mathcal{Q}(m)$  is generated by its global sections, hence we look at this condition on  $K[X, Y, Z]$ . Then either both  $s$  and  $t$  are multiples of  $X^3$  or  $sY^4 + tZ^4 = 0$ . This gives the resolution

$$0 \longrightarrow \mathcal{O}(m-11) \longrightarrow \mathcal{O}(m-7) \oplus \mathcal{O}(m-7) \oplus \mathcal{O}(m-8) \longrightarrow \mathcal{Q}(m) \longrightarrow 0,$$

where the surjection is given by  $1 \mapsto (X^3, 0)$ ,  $1 \mapsto (0, X^3)$  and  $1 \mapsto (Z^4, -Y^4)$  and the injection is given by  $1 \mapsto (-Z^4, Y^4, X^3)$ . The quotient sheaf has degree  $\deg \mathcal{Q}(m) = 2m - 11$  and it is semistable, since its first non-trivial section is for  $m = 7$ . So we have found the Harder–Narasimhan filtration of  $\mathcal{S} = \mathrm{Syz}(X^4, Y^4, Z^4, X^3Y, X^3Z)$ .

## 5. Wedge criteria for stability

Let  $\mathcal{S}$  denote a coherent torsion-free sheaf on  $\mathbb{P}^N$ . A coherent subsheaf  $\mathcal{T} \subseteq \mathcal{S}$  of rank  $r$  yields  $\bigwedge^r \mathcal{T} \rightarrow \bigwedge^r \mathcal{S}$ . The bidual of  $\bigwedge^r \mathcal{T}$  is an invertible sheaf and its degree is by definition the degree of  $\mathcal{T}$ . Therefore the maximum over the degrees of all subsheaves  $\mathcal{T} \subseteq \mathcal{S}$  of rank  $r$  is directly related to the existence of sections  $\neq 0$  of  $(\bigwedge^r \mathcal{S})(k)$ . In particular we have the following criterion for semistability, see [4, Proposition 1.1 and the following remark there].

**Proposition 5.1.** *Let  $\mathcal{S}$  denote a locally free sheaf on  $\mathbb{P}^N$  over an algebraically closed field  $K$  of characteristic 0. Then  $\mathcal{S}$  is semistable if and only if for every  $r < \mathrm{rk}(\mathcal{S})$  and every  $k < -r\mu(\mathcal{S})$  there does not exist a global section  $\neq 0$  of  $(\bigwedge^r \mathcal{S})(k)$ .*

**Proof.** If  $\mathcal{S}$  is semistable, then all its exterior powers  $\bigwedge^r \mathcal{S}$  are also semistable (in characteristic 0) due to [18, Corollary 3.2.10]. Hence  $(\bigwedge^r \mathcal{S}) \otimes \mathcal{O}(k)$  does not have global sections  $\neq 0$  for  $\mu((\bigwedge^r \mathcal{S}) \otimes \mathcal{O}(k)) < 0$ , which means that  $k < -\mu(\bigwedge^r \mathcal{S}) = -r\mu(\mathcal{S})$ .

Now suppose that the condition on the global sections is fulfilled (this direction holds in any characteristic), and let  $\mathcal{T} \subset \mathcal{S}$  denote a coherent subsheaf of rank  $r$ . Then  $\bigwedge^r \mathcal{T} \subset \bigwedge^r \mathcal{S}$  and  $(\bigwedge^r \mathcal{T})^{\vee\vee} \cong \mathcal{O}(m)$  is an invertible sheaf on  $\mathbb{P}^N$ , where  $m = \deg(\mathcal{T})$ . But then also  $\bigwedge^r \mathcal{T} \cong \mathcal{O}(m)$  outside a closed subset of codimension  $\geq 2$ . Since  $\mathcal{S}$  is locally free, this gives a non-trivial morphism  $\mathcal{O}(m) \rightarrow \bigwedge^r \mathcal{S}$ . Therefore  $(\bigwedge^r \mathcal{S})(-m)$  has a global section  $\neq 0$ , so  $-m \geq -r\mu(\mathcal{S})$  by assumption and hence  $\mu(\mathcal{T}) = \frac{m}{r} \leq \mu(\mathcal{S})$ .  $\square$

So if we want to apply this stability criterion we need to get control on the exterior powers of a syzygy bundle  $\text{Syz}(f_i, i \in I)$  and their global sections. From the embedding

$$\text{Syz}(f_i, i \in I) \hookrightarrow \bigoplus_{i \in I} \mathcal{O}(-d_i)$$

we get the canonical embedding

$$\bigwedge^r (\text{Syz}(f_i, i \in I)) \longrightarrow \bigwedge^r \left( \bigoplus_{i \in I} \mathcal{O}(-d_i) \right) \cong \bigoplus_{J \subseteq I, |J|=r} \mathcal{O} \left( -\sum_{i \in J} d_i \right).$$

Here the identification on the right is given by sending

$$s_1 \wedge \cdots \wedge s_r \longmapsto \det((s_{ji})_{j=1, \dots, r, i \in J}).$$

**Lemma 5.2.** *Let  $K$  denote a field. Let  $f_i \in R = K[X_0, \dots, X_N]$ ,  $i \in I = \{1, \dots, n\}$ , denote homogeneous,  $R_+$ -primary polynomials. Then the sequence*

$$0 \longrightarrow \bigwedge^r (\text{Syz}(f_i, i \in I)) \longrightarrow \bigoplus_{|J|=r} \mathcal{O} \left( -\sum_{i \in J} d_i \right) \xrightarrow{\varphi} \bigoplus_{|K|=r-1} \mathcal{O} \left( -\sum_{i \in K} d_i \right)$$

is exact on  $\mathbb{P}^N$ , where  $\varphi$  sends  $e_J \mapsto \sum_{k \in J} \text{sign}(k, J) f_k e_{J-\{k\}}$  (we use the induced order on  $J \subseteq I$  to define  $\text{sign}(k, J)$  to be 1 if  $k$  is an even element in  $J$  and  $-1$  otherwise).

**Proof.** This is a global version of the local fact that  $\bigwedge^r (V \oplus R) \cong \bigwedge^r V \oplus \bigwedge^{r-1} V$  for a free  $R$ -module  $V$ .  $\square$

**Remark 5.3.** With the results of this section it is in principle possible to decide whether a given syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  is semistable or not. The exterior bundles  $\bigwedge^r \text{Syz}$  are given as kernels of some mappings between splitting bundles, hence the minimal degree of a global section  $\neq 0$  is computable with Groebner basis techniques. An algorithm for this was developed and implemented by A. Kaid in the computer algebra system CoCoA [11].

**Remark 5.4.** Let  $\mathcal{S} \hookrightarrow \bigoplus_{i \in I} \mathcal{O}(-d_i)$  be a subsheaf of rank  $r$ . We describe a method to compute  $\deg(\mathcal{S})$ . Let  $s_1, \dots, s_r \in \Gamma(\mathbb{P}^N, \mathcal{S}(m))$  be  $r$  global sections which are linearly independent in at least one point (hence on an open subset). This gives a mapping  $\mathcal{O}^r \rightarrow \mathcal{S}(m) \hookrightarrow \bigoplus_{i \in I} \mathcal{O}(m - d_i)$ . Let these sections be given as  $s_j = (g_{ji})$ ,  $g_{ji} \in \Gamma(\mathbb{P}^N, \mathcal{O}(m - d_i))$ ,  $j = 1, \dots, r$ . For every  $J \subseteq I$ ,  $|J| = r$ , we look at the projection  $\bigoplus_{i \in I} \mathcal{O}(-d_i) \rightarrow \bigoplus_{i \in J} \mathcal{O}(-d_i)$  and the induced mapping  $\mathcal{O}^r \rightarrow \mathcal{S}(m) \rightarrow \bigoplus_{i \in J} \mathcal{O}(m - d_i)$ , and at

$$\mathcal{O} \cong \bigwedge^r \mathcal{O}^r \longrightarrow \bigwedge^r (\mathcal{S}(m)) \longrightarrow \bigwedge^r \left( \bigoplus_{i \in J} \mathcal{O}(m - d_i) \right) \cong \mathcal{O} \left( rm - \sum_{i \in J} d_i \right),$$

which is given by  $1 \mapsto \det((g_{ji})_{1 \leq j \leq r, i \in J}) =: h_J$ . These  $h_J$  give together a non-zero map  $\mathcal{O} \cong \bigwedge^r \mathcal{O}^r \rightarrow \bigwedge^r (\mathcal{S}(m)) \rightarrow \bigoplus_{J \subseteq I, |J|=r} \mathcal{O}(rm - \sum_{i \in J} d_i)$ . The degree of  $\bigwedge^r (\mathcal{S}(m))$  can be computed by counting the zeroes of codimension one of this section in  $\bigwedge^r (\mathcal{S}(m))$ . This

can be estimated by counting the zeroes of codimension one of this section in the direct sum, which is the degree of the highest common factor of all  $h_J$ ,  $|J| = r$ . So we get the estimate  $\deg(\mathcal{S}(m)) \leq \deg(\text{hcf}(h_J, |J| = r))$ .

Suppose now that we have a syzygy bundle. Global sections of  $\text{Syz}(f_i, i \in I)(m)$  (where  $m := \sum_{i \in I} d_i$ ) are sometimes (see the proof of Theorem 6.3) given as

$$s = \left( a_i \prod_{k \neq i} f_k \right)_{i \in I}$$

with  $\sum_{i \in I} a_i = 0$ ,  $a_i \in K$ . Suppose that  $r$  such sections  $s_1, \dots, s_r$  are given and that they are global sections of a subsheaf  $\mathcal{S}(m)$  of  $\text{Syz}(f_i)(m)$  of rank  $r$  which are linearly independent in a point. Write  $s_j = (a_{ji} \prod_{k \neq i} f_k)_{i \in I}$ . Hence for a subset  $J \subseteq I$  with  $r$  elements we get  $h_J = \det((a_{ji} \prod_{k \neq i} f_k)_{1 \leq j \leq r, i \in J})$ , and this is (as the expressions  $\prod_{k \neq i} f_k$  are constant in the column corresponding to  $i \in J$ )

$$\left( \prod_{k \in I} f_k \right)^{r-1} \left( \prod_{k \in I-J} f_k \right) \cdot \det((a_{ji})_{1 \leq j \leq r, i \in J}).$$

So here the highest common factor of the expressions  $\prod_{k \in I-J} f_k$  for  $|J| = r$  and  $\det((a_{ji})_{1 \leq j \leq r, i \in J}) \neq 0$  is crucial. We get then

$$\begin{aligned} \deg \mathcal{S} &= \deg(\mathcal{S}(m)) - rm \\ &\leq (r-1)m + \deg \left( \text{hcf} \left( \prod_{k \in I-J} f_k, |J| = r, \det((a_{ji})_{1 \leq j \leq r, i \in J}) \neq 0 \right) \right) - rm \\ &= - \sum_{k \in I} d_k + \deg \left( \text{hcf} \left( \prod_{k \in I-J} f_k, |J| = r, \det((a_{ji})_{1 \leq j \leq r, i \in J}) \neq 0 \right) \right). \end{aligned}$$

## 6. Stability of syzygies bundles of monomial ideals

We consider now the case where  $f_i \in R = K[X_0, \dots, X_N]$ ,  $i \in I$ , are monomials and we will write  $f_i = X^{\sigma_i} = X_0^{\sigma_{i0}} \cdots X_N^{\sigma_{iN}}$ , where  $\sigma_i \geq 0$  are integral lattice points in  $\mathbb{N}^{N+1}$ . Their degrees are  $d_i = |\sigma_i| = \sum_{j=0}^N \sigma_{ij}$ . We will apply the theory of toric bundles which has been developed by Klyachko (see [20,21,19]). We consider the projective space  $\mathbb{P}^N$  as a toric variety with the torus  $T = G_m^N = (\mathbb{A}^\times)^N$  acting as

$$(t_1, \dots, t_N)(x_0, \dots, x_N) = (x_0, t_1 x_1, \dots, t_N x_N).$$

A toric bundle is a vector bundle  $V \rightarrow \mathbb{P}^N$  with an action of  $T$  on  $V$  compatible with the basic torus action.

For every tuple  $v = (v_0, \dots, v_N)$  we can make the line bundle  $\mathcal{O}(\sum_j v_j) \rightarrow \mathbb{P}^N$  into a toric line bundle by the action

$$(t_1, \dots, t_N)(x_0, \dots, x_N; z) = (x_0, t_1 x_1, \dots, t_N x_N; t_1^{v_1} \cdots t_N^{v_N} z),$$

which we denote by  $\mathcal{O}(\nu)$ . Recall that  $\mathcal{O}(k)$  ( $k \in \mathbb{Z}$ ) itself (disregarding any toric structure) is given by dividing through the  $\mathbb{A}^\times$ -action  $u(x_0, \dots, x_N; z) = (ux_0, \dots, ux_N; u^k z)$ . For a given family of monomials  $X^{\sigma_i}$  also  $\bigoplus_{i \in I} \mathcal{O}(-\sigma_i)$  is a toric bundle and the monomials define a toric morphism of this sum to  $\mathcal{O}$ , hence  $\text{Syz}(X^{\sigma_i}, i \in I)$  is a toric subbundle. Explicitly,  $(t_1, \dots, t_N)$  sends a point  $(x_0, \dots, x_N; z_i, i \in I)$  (with  $\sum_{i \in I} z_i x^{\sigma_i} = 0$ ) over  $(x_0, \dots, x_N)$  to  $(x_0, t_1 x_1, \dots, t_N x_N; t_1^{-\sigma_{i1}} \dots t_N^{-\sigma_{iN}} z_i, i \in I)$ . Note that

$$\sum_{i \in I} t_1^{-\sigma_{i1}} \dots t_N^{-\sigma_{iN}} z_i \cdot x_0^{\sigma_{i0}} (t_1 x_1)^{\sigma_{i1}} \dots (t_N x_N)^{\sigma_{iN}} = \sum_{i \in I} z_i x^{\sigma_i} = 0.$$

Klyachko studies toric bundles  $\mathcal{E}$  with the help of families of filtrations in the special fiber  $E_P = \mathcal{E} \otimes \kappa(P)$ , where  $P$  is a closed point outside of any toric hypersurface. Every toric hypersurface  $H_\alpha$  determines a decreasing filtration  $E^\alpha(m)$ ,  $m \in \mathbb{Z}$ , of vector subspaces in  $E_P$ . For  $\mathbb{P}^N$ , these toric hypersurfaces are just  $H_\alpha = V_+(X_\alpha)$ ,  $\alpha = 0, \dots, N$ . The toric line bundle  $\mathcal{O}(\nu)$  on  $\mathbb{P}^N$  corresponds to the family of filtrations on  $K$  given by  $K^\alpha(m) = K$  for  $m \leq \nu_\alpha$  and  $K^\alpha(m) = 0$  for  $m > \nu_\alpha$  (compare [19, Example 2.3]). The category of toric vector bundles on a toric variety is equivalent to the category of vector spaces with such families of filtrations fulfilling certain compatibility conditions (see [19, Theorem 2.2.1]).

We collect some of the main properties which we need in the sequel of this section. Due to the setting of Klyachko's work we restrict to characteristic zero.

**Lemma 6.1.** *Let  $\mathcal{E}$  be a toric bundle on  $\mathbb{P}_K^N$ ,  $K$  an algebraically closed field, and set  $E = E_P$  with the corresponding filtrations  $E^\alpha(m)$ ,  $\alpha = 0, \dots, N$ , where  $P$  is a closed point in the torus. Then the following hold.*

- (i) *Let a vector  $w \in E$  be given. Let  $n_\alpha$  be the maximal integer (take  $\infty$  for  $w = 0$ ) such that  $w \in E^\alpha(n_\alpha)$ . Then the  $K$ -linear mapping  $K \rightarrow E$ ,  $1 \mapsto w$ , extends to a toric bundle morphism  $\mathcal{O}(\nu) \rightarrow \mathcal{E}$  under the condition that  $\nu_\alpha \leq n_\alpha$  for  $\alpha = 0, \dots, N$ .*
- (ii) *Let a linear form  $\psi: E \rightarrow K$  be given and let  $m_\alpha$  be the smallest number such that  $E^\alpha(m_\alpha) \subseteq \ker \psi$  (take  $-\infty$  for  $\psi = 0$ ). Then  $\psi$  extends to a toric bundle morphism  $\mathcal{E} \rightarrow \mathcal{O}(\nu)$  for  $\nu = (\nu_\alpha)$ ,  $\nu_\alpha \geq m_\alpha$ .*
- (iii) *Let  $F \subseteq E$  be a vector subspace. If  $\psi_k: E \rightarrow K$  is a family of linear forms such that  $F = \bigcap_k \ker \psi_k$ , then the kernel sheaf (which is not locally free in general) of the toric mapping  $(\psi_k): \mathcal{E} \rightarrow \bigoplus_k \mathcal{O}(m_k)$  (as constructed in (ii)) is a toric subsheaf  $\mathcal{F}$  (in the sense that it is a toric subbundle over an open toric subvariety which contains all points of codimension one) such that  $\mathcal{F} \otimes \kappa(P) = F$ . In particular, subspaces of the special fiber correspond to toric subsheaves which are locally free in codimension one.*
- (iv) *For  $w \in F \subseteq E$  the global morphism (constructed in (i)) factors through  $\mathcal{F}$ .*
- (v) *The maximal destabilizing subsheaf of  $\mathcal{E}$  is given by a toric subbundle which is defined on an open toric subvariety containing all points of codimension one.*

**Proof.** (i) Let  $\varphi: K \rightarrow E$  be the map given by  $1 \mapsto w$ . Then  $\varphi(K^\alpha(m)) \subseteq E^\alpha(m)$  for all  $m$  if and only if  $\varphi(K^\alpha(\nu_\alpha)) \subseteq E^\alpha(\nu_\alpha)$  if and only if  $w \in E^\alpha(\nu_\alpha)$  if and only if  $\nu_\alpha \leq n_\alpha$  (this holds for every  $\alpha$ ). Such a filtered linear mapping corresponds to a toric bundle morphism  $\mathcal{O}(\nu) \rightarrow \mathcal{E}$  by Klyachko's theorem [19, Theorem 2.2.1].

(ii) The linear mapping  $\psi: E \rightarrow K$  is for  $\nu = (\nu_\alpha)$ ,  $\nu_\alpha \geq m_\alpha$ , compatible with the filtrations, since for  $m < m_\alpha$  we have  $m < \nu_\alpha$  and so  $K^\alpha(m) = K$ , and for  $m \geq m_\alpha$  we have  $\psi(E^\alpha(m)) = 0$ .



Hence again by [19, Theorem 2.2.1]  $L$  extends to a toric morphism  $\mathcal{E} \rightarrow \mathcal{O}(v)$  for  $v = (v_\alpha)$ ,  $v_\alpha \geq m_\alpha$ .

(iii) The linear forms  $\psi_k: E \rightarrow K$  induce by (ii) a toric morphism  $\mathcal{E} \rightarrow \bigoplus_k \mathcal{O}(m_k)$ , where the  $m_k = m_{k,\alpha}$  are defined as in (ii). The kernel sheaf is locally free in codimension one and its special fiber is  $F$ . Since the toric automorphisms respect the kernel it is a toric subsheaf. In particular, every subspace  $F \subset E$  is the special fiber of a toric subsheaf. On the other hand, let two toric subsheaves  $\mathcal{F}_1$  and  $\mathcal{F}_2$  of  $\mathcal{E}$  with the same special fiber be given. Then the induced filtrations are the same and so they are isomorphic as toric bundles on a certain open toric subvariety. By [19, Theorem 2.2.1] they must be the same subbundle, since the embedding is determined by the filtered linear mapping.

(iv) The composed mapping  $\mathcal{O}(v) \rightarrow \mathcal{E} \rightarrow \bigoplus_k \mathcal{O}(m_k)$  is the zero map on the special fiber, since  $w \in F = \bigcap_k \ker \psi_k$ . Hence, again by [19, Theorem 2.2.1], it is the zero map and so it factors through the kernel, which is  $\mathcal{F}$  by part (iii).

(v) For  $\mathbb{P}^2$  this is proven in [20, Theorem 3.2.2], but in general we have to be a bit more careful. Let  $\mathcal{F}$  be the maximal destabilizing subsheaf of  $\mathcal{E}$ . As this is uniquely determined, we must have  $t^*(\mathcal{F}) = \mathcal{F}$  as a subsheaf of  $\mathcal{E}$  for (the automorphism given by)  $t \in T$ . In particular,  $\mathcal{F}$  is locally free on an open toric subvariety which contains all points of codimension one. The action  $t: \mathcal{E} \rightarrow \mathcal{E}$  must send  $\mathcal{F} \otimes \kappa(P)$  to  $\mathcal{F} \otimes \kappa(t(P))$ . Hence the action restricts to  $\mathcal{F}$  and so  $\mathcal{F}$  is toric (but not necessarily a bundle on the whole).  $\square$

**Lemma 6.2.** *Let  $v_i, i \in I$ , be a set of spanning vectors  $\neq 0$  in a vector space  $U$  of dimension  $r$  and with  $\sum_{i \in I} v_i = 0$ . Then there exists a partition  $I = I_1 \uplus \dots \uplus I_\ell \uplus \tilde{I}$  such that  $\sum_{\lambda=1}^\ell (|I_\lambda| - 1) = r$  and such that, setting  $V_\lambda = \langle v_i, i \in I_\lambda \rangle$ , the following holds: the set of vectors  $\{v_i: i \in I_\lambda\}$  is linearly dependent modulo the subspace  $V_1 + \dots + V_{\lambda-1}$ , but all strict subsets are independent.*

**Proof.** Note first that for every hyperplane  $H \subset U$  there exist at least two vectors outside  $H$ , because of the sum property. We do induction on  $r$ . For  $r = 1$  any  $I_1 = \{i, j\}$  and  $\tilde{I} = I - I_1$  will do. So let  $r \geq 2$ . Reorder so that  $v_1, \dots, v_r$  are a basis of  $V$ . Take  $v_{r+1}$  (which must exist because of the sum property) and consider  $\{v_1, \dots, v_r, v_{r+1}\}$ . If this set has the property that every strict subset is independent, then we take  $I_1 = \{1, \dots, r+1\}$  ( $\ell = 1$ ) and we are done. In the other case there exists a dependent (strict) subset, which must contain  $v_{r+1}$ , since the first  $r$  vectors  $v_i$  are independent. Then either this set has the property or we decrease the set further until we arrive at a set with the required properties (the smallest possibility is that of  $\{v_i, v_{r+1}\}$  being dependent).

Let  $I_0 \subseteq \{1, \dots, r, r+1\}$  be such an index set and let  $V_0 = \langle v_i, i \in I_0 \rangle (\neq 0)$  be the subspace. Let  $I' \subset I$  be the subset consisting of all  $i$  such that  $v_i$  do not belong to  $V_0$ . Then the quotient space  $V/V_0$  and the set of residue classes  $\{\bar{v}_i: i \in I'\}$  fulfill also all the assumptions and is of smaller dimension. Hence we apply the induction hypothesis to get a partition  $I' = I'_1 \uplus \dots \uplus I'_\ell \uplus \tilde{I}'$  with the desired properties. Then the sets  $I_0, I_\lambda := I'_\lambda (\lambda = 1, \dots, \ell)$  and  $\tilde{I} = \tilde{I}' \cup (I - (I_0 \cup I'))$  form a partition of  $I$  with the desired properties.  $\square$

Recall that the maximal slope of a vector bundle is the maximum over the slopes of all subsheaves.

**Theorem 6.3.** *Let  $f_i = X^{\sigma_i}$ ,  $i \in I = \{1, \dots, n\}$ , denote a set of primary monomials in  $K[X_0, \dots, X_N]$  of degree  $d_i = |\sigma_i|$ . For  $J \subseteq I$  denote by  $d_J$  the degree of the highest common*

factor of  $f_i$ ,  $i \in J$ . Then the maximal slope of  $\text{Syz}(f_i, i \in I)$  is

$$\mu_{\max}(\text{Syz}(f_i, i \in I)) = \max_{J \subseteq I, |J| \geq 2} \left\{ \frac{d_J - \sum_{i \in J} d_i}{|J| - 1} \right\}.$$

**Proof.** It is clear that  $\geq$  holds. By Lemma 6.1(v) we only have to consider toric subsheaves  $\mathcal{F} \subseteq \text{Syz}(f_i, i \in I)$  (i.e., toric subbundles defined on an open toric subvariety containing all points of codimension one). These are in one-to-one correspondence (Lemma 6.1(iii)) with subspaces  $F \subseteq E$  inside the special fiber  $E$  of the syzygy bundle over the point  $P = (1, \dots, 1)$  ( $E$  itself is the hypersurface in  $K^n$  given by  $\sum_{i=1}^n a_i = 0$ ). So let  $F \subseteq E$  be a subspace of dimension  $r$ , given by  $r$  linearly independent vectors  $w_1, \dots, w_r$ , where  $w_j = \sum_{i=1}^n a_{ji} e_i$ ,  $\sum_{i=1}^n a_{ji} = 0$ . We look at the global sections

$$s_j = \left( a_{ji} \prod_{k \in I, k \neq i} f_k \right)_{i \in I} \in \Gamma(\mathbb{P}^N, \text{Syz}(f_i, i \in I)(m))$$

(where  $m = \sum_{i=1}^n d_i$ ), which have  $w_j$  as their values at  $(1, \dots, 1)$ . These sections are toric sections (where  $\text{Syz}(m)$  has the natural toric structure induced by  $\bigoplus_{i \in I} \mathcal{O}(\sum_{k \neq i} \sigma_k)$ ), hence they coincide (up to the twist) with the sections constructed in Lemma 6.1(i). These sections factor through the toric subsheaf  $\mathcal{F}(m)$  (Lemma 6.1(iv)).

By Remark 5.4 we get the estimate

$$\deg(\mathcal{F}) \leq - \sum_{k \in I} d_k + \deg \left( \text{hcf} \left( \prod_{k \in I-J} f_k, |J| = r, \det((a_{ji})_{1 \leq j \leq r, i \in J}) \neq 0 \right) \right).$$

Set  $v_i = (a_{ji})$ ,  $i \in I$ , considered in the vector space  $K^r$ . Note that  $\sum_{i \in I} v_i = 0$ . By Lemma 6.2 there exists a partition  $I = I_1 \uplus \dots \uplus I_\ell \uplus \tilde{I}$  such that  $\sum_{\lambda=1}^\ell (|I_\lambda| - 1) = r$  and such that, setting  $V_\lambda = \langle v_i, i \in I_\lambda \rangle$ , the following holds: the set  $\{v_i: i \in I_\lambda\}$  is linearly dependent modulo the subspace  $V_1 + \dots + V_{\lambda-1}$ , but all strict subsets are independent. Then for all subsets

$$J = J_1 \uplus \dots \uplus J_\ell, \quad J_\lambda \subset I_\lambda, \quad |J_\lambda| = |I_\lambda| - 1$$

the vectors  $v_i$ ,  $i \in J$ , are linearly independent and so the determinantal coefficients for these  $J$  are  $\neq 0$ . Hence

$$\deg \left( \text{hcf} \left( \prod_{k \in I-J} f_k, |J| = r, \det((a_{ji})_{1 \leq j \leq r, i \in J}) \neq 0 \right) \right) \leq \deg \left( \text{hcf} \left( \prod_{k \in I-J} f_k, J \text{ as above} \right) \right).$$

The products on the right can be written as  $(\prod_{i \in \tilde{I}} f_i) f_{j_1} \dots f_{j_\ell}$  for any choice  $j_1 \in I_1, \dots, j_\ell \in I_\ell$ . So their highest common factor is  $(\prod_{i \in \tilde{I}} f_i) \cdot \text{hcf}(f_i, i \in I_1) \dots \text{hcf}(f_i, i \in I_\ell)$ . Therefore

$$\begin{aligned} \deg(\mathcal{F}) &\leq - \sum_{i \in I_1 \cup \dots \cup I_\ell} d_i + \deg(\text{hcf}(f_i, i \in I_1)) + \dots + \deg(\text{hcf}(f_i, i \in I_\ell)) \\ &= \sum_{\lambda=1}^\ell \left( \sum_{i \in I_\lambda} -d_i + \deg(\text{hcf}(f_i, i \in I_\lambda)) \right). \end{aligned}$$

On the right we have the degree of the subsheaf  $\text{Syz}(f_i, i \in I_1) \oplus \cdots \oplus \text{Syz}(f_i, i \in I_\ell)$  of rank  $r$ . Its slope can be estimated by the maximum of the slopes of its direct summands, which are  $\frac{d_{I_\lambda} - \sum_{i \in I_\lambda} d_i}{|I_\lambda| - 1}$ .  $\square$

We can now state our combinatorial criterion for a monomial family to have a semistable syzygy bundle (the necessity of the condition was already established in Proposition 2.2).

**Corollary 6.4.** *Let  $f_i = X^{\sigma_i}$ ,  $i \in I$ , denote a set of primary monomials in  $K[X_0, \dots, X_N]$  of degree  $d_i = |\sigma_i|$ . Suppose that for every subset  $J \subseteq I$ ,  $|J| \geq 2$ , the inequality*

$$\frac{d_J - \sum_{i \in J} d_i}{|J| - 1} \leq \frac{-\sum_{i \in I} d_i}{|I| - 1}$$

*holds, where  $d_J$  is the degree of the highest common factor of  $f_i$ ,  $i \in J$ . Then the syzygy bundle  $\text{Syz}(f_i, i \in I)$  is semistable (and stable if  $<$  holds).*

**Proof.** This follows at once from Theorem 6.3.  $\square$

**Corollary 6.5.** *Let  $f_i = X^{\sigma_i}$ ,  $i = 1, \dots, n$  denote a set of primary monomials of the same degree  $d$  in  $K[X_0, \dots, X_N]$ . For every monomial  $X^\nu$  of degree  $e = |\nu| \leq d$  let  $s_\nu$  denote the number of monomials in the family which are multiples of  $X^\nu$ . Then the syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  is semistable if and only if for every  $\nu$  the following inequality holds:*

$$\frac{s_\nu - 1}{d - e} \leq \frac{n - 1}{d}.$$

**Proof.** Let  $J \subseteq I = \{1, \dots, n\}$  denote the subset of monomials which are multiples of  $X^\nu$ . We may assume that  $X^\nu$  is the highest common factor of this family. The numerical semistability condition is that (setting  $e = |\nu|$ ,  $s = |J|$ )

$$\mu(\text{Syz}(f_i, i \in J)) = \frac{e - sd}{s - 1} \leq \frac{-nd}{n - 1} = \mu(\text{Syz}(f_i, i \in I)).$$

This is equivalent with  $e(n - 1) - sd(n - 1) \leq -(s - 1)nd$  and hence with  $sd \leq nd - e(n - 1)$  and with  $(s - 1)d \leq (n - 1)d - e(n - 1) = (n - 1)(d - e)$ , so the result follows.  $\square$

## 7. Examples of monomial ideals

We first deduce the following result of Flenner (see [12, Corollary 2.2] and [1, Corollary 6.5]) from our numerical criterion.

**Corollary 7.1.** *Let  $K$  denote a field. Then the syzygy bundle of the family of all monomials in  $K[X_0, \dots, X_N]$  of fixed degree  $d$  is semistable.*

**Proof.** We want to apply Corollary 6.5, so let  $X^\nu$  be a monomial of degree  $|\nu| = e \leq d$ . Every monomial of degree  $d - e$  gives a multiple of  $X^\nu$  of degree  $d$ , so we have to show that

$$\frac{\binom{N+d-e}{N} - 1}{d - e} \leq \frac{\binom{N+d}{N} - 1}{d}.$$

We may assume successively that  $e = 1$ , so we have to show that  $((\binom{N+d}{N} - 1)(d - 1) \geq d((\binom{N+d-1}{N} - 1))$ , which is equivalent with  $(\binom{N+d}{N})(d - 1) \geq d(\binom{N+d-1}{N} - 1)$ . This is true for  $N = 1$  or  $d = 1$  and follows for  $N, d \geq 2$  from  $(N + d)(d - 1) \geq d^2$ .  $\square$

For a family consisting only of some powers of the variables we have the following result, which is also a special case of Corollary 8.2 and follows also from Theorem 8.1 of Bohnhorst and Spindler.

**Corollary 7.2.** *Consider the family  $X_i^{d_i}$ ,  $i = 0, \dots, N$ , in  $K[X_0, \dots, X_N]$ . Suppose that  $1 \leq d_0 \leq \dots \leq d_N$ . Then the syzygy bundle  $\text{Syz}(X_0^{d_0}, \dots, X_N^{d_N})$  is semistable on  $\mathbb{P}^N$  if and only if  $(N - 1)d_N \leq \sum_{j=0}^{N-1} d_j$  holds.*

**Proof.** The numerical condition is necessary due to Corollary 2.4. On the other hand, again due to Corollary 2.4 the necessary numerical conditions for smaller ranks are also fulfilled, so the result follows from Corollary 6.4.  $\square$

We give some examples of small families of monomials in three variables and check whether their syzygy bundles are stable or not.

**Corollary 7.3.** *Let  $X^{d_1}, Y^{d_2}, Z^{d_3}$  and  $X^{a_1}Y^{a_2}Z^{a_3}$  be four monomials in  $K[X, Y, Z]$ ,  $a_j < d_j$ . Set  $d_4 = a_1 + a_2 + a_3$ . Then the syzygy bundle is semistable if and only if the following two numerical conditions hold:*

- (i)  $3 \max(d_1, d_2, d_3, d_4) \leq d_1 + d_2 + d_3 + d_4$ ;
- (ii)  $\min(a_1 + a_2 + d_3, a_1 + d_2 + a_3, d_1 + a_2 + a_3, d_1 + d_2, d_1 + d_3, d_2 + d_3) \geq \frac{\sum_{i=1}^4 d_i}{3}$ .

**Proof.** We apply the semistability criterion Corollary 6.4 for subsets  $J$  with  $|J| = 2$  or  $3$ . For  $|J| = 3$  we have  $d_J = 0$ , so the condition is that  $\frac{-\sum_{i \in J} d_i}{2} \leq \frac{-\sum_{i=1}^4 d_i}{3}$ , which is equivalent with  $-\sum_{i \neq k} d_i \leq -2d_k$  for every  $k$ , so this is condition (i). For  $|J| = 2$  the condition is  $d_J - \sum_{i \in J} d_i \leq \frac{-\sum_{i=1}^4 d_i}{3}$  or  $\sum_{i \in J} d_i - d_J \geq \frac{\sum_{i=1}^4 d_i}{3}$  for all subsets  $J$ ,  $|J| = 2$ , which is condition (ii).  $\square$

**Example 7.4.** Consider  $X^4, Y^4, Z^4, XYZ^2$ . The first condition is clearly satisfied. The minimum in the second condition is 6 which is  $\geq 16/3$ , so the syzygy bundle is semistable. If we replace however  $XYZ^2$  by  $XZ^3$ , then the first condition is again satisfied, but the minimum in the second condition is now 5 and so this syzygy bundle is not semistable. For  $X^3, Y^3, Z^3, XY^2Z^2$  the second condition is fulfilled, but the first is not fulfilled.

We consider now examples of more than four monomials.

**Example 7.5.** Consider now the monomials  $X^6, Y^6, Z^6, X^2Y^2Z^2, XY^2Z^3$ . Their syzygy bundle is not semistable, since its slope is  $-\frac{30}{4} = -7.5$ , but the subbundle  $\text{Syz}(X^2Y^2Z^2, XY^2Z^3)$  has slope  $5 - 12 = -7$ . This is also the maximal slope of this bundle.

For the monomials  $X^6, Y^6, Z^6, X^2Y^2Z^2, X^3Z^3$  the slope is again  $-\frac{30}{4} = -7.5$ . For  $|J| = 2$  the highest common factor has maximal degree 4, which gives slope  $4 - 12 = -8$ . For  $|J| = 3$

the common factor has degree at most 2, which gives the slope  $\frac{2-18}{2} = -8$ , so this syzygy bundle is stable.

**Example 7.6.** Consider the monomial family given by  $X^5, X^4Z, Y^5, Y^4Z, Z^5$ , so that the slope is  $-6.25$ . The subsheaf

$$\mathrm{Syz}(X^5, X^4Z) \oplus \mathrm{Syz}(Y^5, Y^4Z) \subset \mathrm{Syz}(X^5, X^4Z, Y^5, Y^4Z, Z^5)$$

has slope  $-6 > -6.25$  and it is the maximal destabilizing subsheaf. The subsheaves given by a subfamily of three elements do not contradict semistability.

**Example 7.7.** For a fixed  $r$  the minimal degree shift of a global section of  $\bigwedge^r \mathrm{Syz}(f_i, i \in I)$  does in general not arise from a subsheaf of rank  $r$  of the form  $\bigoplus_{\lambda=1}^{\ell} \mathrm{Syz}(f_i, i \in I_{\lambda})$  (though the maximal slope can be computed using only these subsheaves). Look at the example given by the six monomials

$$X^4Y^2, \quad X^4Z^2, \quad Y^3Z^3, \quad Y^5, \quad Z^5, \quad X^7.$$

Their syzygy bundle is semistable according to the monomial criterion (but not stable). For  $r = 2$ , the subfamilies of three elements yield global sections of  $(\bigwedge^2 \mathrm{Syz}(f_i, i \in I))(15)$ , but not of smaller degree, and the subsheaves of form  $\mathrm{Syz}(f_i, f_j) \oplus \mathrm{Syz}(f_s, f_t)$  yield only global sections of degree  $\geq 16$ .

There exists however also a section of degree 14 of  $\bigwedge^2 \mathrm{Syz}(f_i, i \in I)$ . The subfamily  $(X^4Y^2, X^4Z^2, Y^3Z^3)$  yields the section of degree 18

$$s_1 = -Y^3Z^3e_{\{1,2\}} + X^4Z^2e_{\{1,3\}} - X^4Y^2e_{\{2,3\}}$$

(given in terms of Lemma 5.2) and  $\mathrm{Syz}(X^4Y^2, Y^5) \oplus \mathrm{Syz}(X^4Z^2, Z^5)$  yields the section

$$s_2 = Y^3Z^3e_{\{1,2\}} - X^4Y^3e_{\{1,5\}} + X^4Z^3e_{\{2,4\}} + X^8e_{\{4,5\}}.$$

Then  $s_1 + s_2$  is a multiple of  $X^4$  and yields the section of degree 14

$$Z^2e_{\{1,3\}} - Y^3e_{\{1,5\}} - Y^2e_{\{2,3\}} + Z^3e_{\{2,4\}} + X^4e_{\{4,5\}}.$$

**Question 7.8.** Does there exist for every  $d$  and every  $n \leq \binom{N+d}{N}$  a family of  $n$  monomials in  $K[X_0, \dots, X_N]$  of degree  $d$  such that their syzygy bundle is semistable? This is due to Corollary 6.4 a purely combinatorial problem. A positive answer to this question would imply that also the syzygy bundle for generic polynomials of constant degree is semistable. For  $N = 1$  (two variables) this is clearly true for the family  $X_0^d, X_0^{d-1}X_1, \dots, X_0^{d-n+1}X_1^{n-1}$ . In three variables this is proved in [9].

## 8. Syzygy bundles of generic forms

What can we say about stability properties of  $\mathrm{Syz}(f_1, \dots, f_n)$  for generic homogeneous forms  $f_1, \dots, f_n \in K[X_0, \dots, X_N]$  of given degrees  $d_i$ ? There is no hope for semistable syzygy bundles unless the degrees satisfy the necessary numerical condition described in Proposition 2.4.

On the other hand, if these numerical conditions are fulfilled, e.g. if the degrees are constant, then it is not clear at all whether there exist semistable syzygy bundle of this degree type. The degrees determine the Chern classes of the syzygy bundle and therefore the question is equivalent to the following. Does the moduli space  $\mathcal{M}(n-1, c_j)$  of rank  $n-1$  stable vector bundles on  $\mathbb{P}^N$  with Chern classes  $c_j$  contain syzygy bundles?

We will give here some partial results for semistability using results of Bohnhorst and Spindler [4] and of Hein (see Appendix A).

**Theorem 8.1** (Bohnhorst–Spindler). *Let  $\mathcal{E}$  denote a vector bundle of rank  $N$  on the projective space  $\mathbb{P}^N$  over an algebraically closed field of characteristic 0. Suppose that  $\mathcal{E}$  has a resolution*

$$0 \longrightarrow \bigoplus_{i=1}^k \mathcal{O}(a_i) \longrightarrow \bigoplus_{j=1}^{N+k} \mathcal{O}(b_j) \longrightarrow \mathcal{E} \longrightarrow 0$$

and suppose that the pair  $(a, b)$  is admissible, that means that  $a_1 \geq \dots \geq a_k$ ,  $b_1 \geq \dots \geq b_{N+k}$  and  $a_i < b_{N+i}$  for  $i = 1, \dots, k$ . Then the following are equivalent.

- (i)  $\mathcal{E}$  is semistable.
- (ii)  $b_1 \leq \mu(\mathcal{E}) = \frac{1}{N}(\sum_{j=1}^{N+k} b_j - \sum_{i=1}^k a_i)$ .

**Proof.** See [4, Theorem 2.7].  $\square$

**Corollary 8.2.** *Let  $K$  denote an algebraically closed field of characteristic 0. Suppose that  $f_1, \dots, f_{N+1} \in K[X_0, \dots, X_N]$  are homogeneous polynomials of degree  $d_1 \geq \dots \geq d_{N+1} \geq 1$ . Suppose that  $d_1 \leq \frac{\sum_{i=2}^{N+1} d_i}{N-1}$  and that the  $f_i$  are parameters. Then the syzygy bundle  $\text{Syz}(f_1, \dots, f_{N+1})$  is semistable.*

**Proof.** Since the  $f_i$  are parameters their syzygy bundle is locally free and the presenting sequence

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_{N+1}) \longrightarrow \bigoplus_{j=1}^{N+1} \mathcal{O}(-d_j) \longrightarrow \mathcal{O} \longrightarrow 0$$

is exact on the right. Its dual is then also exact and we are in the situation of the theorem of Bohnhorst and Spindler with  $k = 1$ ,  $a_1 = 0$  and  $b_j = d_j$ . This pair is clearly admissible. The numerical condition in the assumption is equivalent to the numerical condition in Theorem 8.1(ii). Hence  $\text{Syz}^\vee$  is semistable and then by [25, Lemma II.1.2.4] also  $\text{Syz}(f_1, \dots, f_{N+1})$  is semistable.  $\square$

**Corollary 8.3.** *Let  $K$  denote an algebraically closed field of characteristic 0. Suppose that  $f_1, \dots, f_{N+1} \in K[X_0, \dots, X_N]$  are homogeneous parameters with degrees  $d_1 \geq \dots \geq d_{N+1} \geq 1$  such that  $d_1 \leq \frac{\sum_{i=2}^{N+1} d_i}{N-1}$ . Then for generic forms  $G_1, \dots, G_{N-1} \in K[X_0, \dots, X_N]$  of sufficiently high degree the equation*

$$(f_1, \dots, f_{N+1})^* = (f_1, \dots, f_{N+1}) + R \quad \geq \frac{\sum_{i=1}^{N+1} d_i}{N}$$

holds in  $R = K[X_0, \dots, X_N]/(G_1, \dots, G_{N-1})$ .

**Proof.** This follows from Corollary 8.2, the restriction theorems and the numerical formula for tight closure.  $\square$

**Corollary 8.4.** *Let  $K$  denote an algebraically closed field of characteristic 0. Let  $f_1, \dots, f_{N+1} \in K[X_0, \dots, X_N]$  denote  $N + 1$  generic homogeneous polynomials of the same degree  $d \geq 1$ . Let  $G_1, \dots, G_{N-1} \in K[X_0, \dots, X_N]$  denote generic forms of sufficiently high degree. Then*

$$(f_1, \dots, f_{N+1})^* = (f_1, \dots, f_{N+1}) + R_{\geq \frac{(N+1)d}{N}}$$

holds in  $R = K[X_0, \dots, X_N]/(G_1, \dots, G_{N-1})$ .

**Proof.**  $N + 1$  generic homogeneous elements are parameters in  $K[X_0, \dots, X_N]$ , so this follows from Corollary 8.3.  $\square$

**Remark 8.5.** Corollaries 8.2 and 8.3 generalize the case  $N = 2$  treated in Corollaries 3.2 and 3.3. Corollary 7.2 is also a special case of Corollary 8.2. On the other hand, we may deduce Corollary 8.4 from Corollary 7.2 without using the result of Bohnhorst and Spindler: since semistability is an open property in a flat family it is enough to establish the semistability property for a single choice of homogeneous forms with given degree.

**Theorem 8.6 (Hein).** *Let  $K$  denote an algebraically closed field and let  $f_1, \dots, f_n \in K[X_0, \dots, X_N]$ ,  $N \geq 2$ , denote generic homogeneous polynomials of the same degree  $d$ . Suppose that  $n \leq d(N + 1)$ . Then their syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  is semistable on  $\mathbb{P}^N$ .*

**Proof.** See Theorem A.1 of Appendix A by G. Hein.  $\square$

**Corollary 8.7.** *Let  $K$  denote an algebraically closed field of characteristic 0. Let  $f_1, \dots, f_n \in K[X_0, \dots, X_N]$  denote generic homogeneous forms of degree  $d$ ,  $n \leq d(N + 1)$ . Then for a generic complete intersection ring  $R = K[X_0, \dots, X_N]/(G_1, \dots, G_{N-1})$  of sufficiently high degree we have*

$$(f_1, \dots, f_n)^* = (f_1, \dots, f_n) + R_{\geq \frac{dn}{n-1}}.$$

**Proof.** This follows from Theorem 8.6 with the help of the restriction theorems and the numerical formula for tight closure from the introduction.  $\square$

**Example 8.8.** We consider the case of  $n$  generic polynomials of degree  $d = 30$ ,  $3 \leq n \leq 31$ . The following table shows how the degree bound behaves as  $n$  varies (we only list  $n$  if the degree bound drops).

$n$ (number of gen. generators)	3	4	5	6	7	9	11	16	31
$\frac{n}{n-1}d$ (degree bound)	45	40	37.5	36	35	33.75	33	32	31

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## Appendix A. Semistability of the general syzygy bundle

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In this appendix we prove three results about the (semi)stability of a syzygy bundle. Theorem A.1 implies Theorem 8.6 of this article. Here we show stability (resp. semistability) by showing that the restriction of a sheaf to a given curve is stable (resp. semistable). In Theorem A.1 we use an elliptic curve. This gives us the least restrictive conditions on the integer parameters  $n$  and  $d$ . However, we cannot show stability, because there exist no stable vector bundles of given rank  $r$  and degree  $d$  on an elliptic curve unless  $r$  and  $d$  are coprime.

Thus, to obtain slope stable coherent sheaves we have to consider curves of genus greater than 1. This is done in Theorems A.2 and A.3. It should be remarked that the kernel of a morphism  $\varphi: \mathcal{O}_{\mathbb{P}^N}^{\oplus n} \rightarrow \mathcal{O}_{\mathbb{P}^N}(d)$  is no vector bundle for  $n \leq N$ . However, even in these cases we can deduce (semi)stability.

The strategy of all proofs is as follows:

- (1) We take a suitable (semi)stable sheaf  $G$  on a curve  $C \subset \mathbb{P}^N$ .
- (2) We show that there exists a short exact sequence

$$0 \rightarrow G \rightarrow \mathcal{O}_C^{\oplus n} \xrightarrow{\bar{\varphi}} \mathcal{O}_{\mathbb{P}^N}(d)|_C \rightarrow 0.$$

- (3) We show that  $\bar{\varphi}$  is the restriction of a morphism  $\mathcal{O}_{\mathbb{P}^N}^{\oplus n} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^N}(d)$  to the curve  $C$ .
- (4) Now the kernel  $F = \ker(\varphi)$  is a coherent sheaf on  $\mathbb{P}^N$  which is a vector bundle in an open set containing the curve  $C$ .
- (5) This implies (see [4]) that the restriction of  $F$  to the generic curve in the Hilbert scheme of curves is (semi)stable.
- (6) From that we eventually conclude that  $F$  is (semi)stable, because the restriction of an unstable sheaf to the generic curve in  $\mathbb{P}^N$  is unstable too.

To show (3) it is sufficient to take projectively normal curves  $C \subset \mathbb{P}^N$ . We use the theorem of Castelnuovo, Mattuck and Mumford which states that on a curve  $C$  of genus  $g_C$  every line bundle  $L$  of degree  $\deg(L) > 2 \cdot g_C$  is normally generated (see [2]). This implies that the embedding  $C \rightarrow \mathbb{P}(H^0(L))$  is projectively normal.

**Theorem A.1.** *Let  $E \subset \mathbb{P}^N$  be a smooth projective elliptic curve embedded by a complete linear system of degree  $N + 1$ . If the integers  $n$  and  $d$  satisfy  $2 \leq n \leq d(N + 1)$ , then the kernel of a general morphism  $\varphi \in \text{Hom}(\mathcal{O}_{\mathbb{P}^N}^{\oplus n}, \mathcal{O}_{\mathbb{P}^N}(d))$  is a semistable vector bundle when restricted to  $E$ . This implies that  $\ker(\varphi)$  is a slope semistable coherent sheaf for  $\varphi$  generic.*

**Proof.** Let  $F$  be a semistable vector bundle on the elliptic curve  $E$  with  $\text{rk}(F) = n - 1$  and  $\det(F) \cong \mathcal{O}_{\mathbb{P}^N}(d)|_E$ . This implies  $\deg(F) = d(N + 1)$ . The existence of such a vector bundle follows from Atiyah's work [1]. The inequality  $n \leq d(N + 1)$  implies that  $\mu(F) = \frac{\deg(F)}{\text{rk}(F)} > 1$ .

Let  $P \in E$  be an arbitrary geometric point of  $E$ . We consider the vector bundle  $F(-P) = F \otimes \mathcal{O}_E(-P)$ . We compute that the slope  $\mu(F(-P)) = \mu(F) - 1 > 0$ . This implies that  $H^1(E, F(-P)) = 0$ . Thus, we conclude from the long exact cohomology sequence associated to  $0 \rightarrow F(-P) \rightarrow F \rightarrow F \otimes k(P) \rightarrow 0$  that  $F$  is globally generated in the point  $P$ . We eventually obtain the surjectivity of the evaluation map  $H^0(E, F) \otimes \mathcal{O}_E \rightarrow F$ .

By the Riemann–Roch theorem we have  $h^0(F) = d(N + 1) \geq n$ . Suppose now that  $h^0(F) > n$  holds. We claim that for a general  $n$ -dimensional subspace  $V \subset H^0(E, F)$  the evaluation morphism  $\text{ev}_V: V \otimes \mathcal{O}_E \rightarrow F$  is surjective. This is done by a dimension count. The dimension of

the Grassmannian variety of all  $n$  dimensional subspaces of  $H^0(E, F)$  is  $n(h^0(F) - n)$ . Next we consider a surjection  $F \xrightarrow{\alpha} k(P)$  and denote its kernel by  $F'$ . Since  $F$  is globally generated  $h^0(F') = h^0(F) - 1$ . We deduce that the Grassmannian of all  $n$ -dimensional subspaces  $V$  of  $H^0(E, F)$ , such that the image of the evaluation map  $\text{ev}_V$  is contained in  $F'$ , is of dimension  $n(h^0(F) - n - 1)$ . Since the surjections  $F \rightarrow k(P)$  are parametrized by  $\mathbb{P}(F)$ , and  $\dim(\mathbb{P}(F)) = \text{rk}(F) = n - 1$ , we conclude the claim.

Now we take a surjection  $\beta: \mathcal{O}_E^{\oplus n} \rightarrow F$ . The kernel of this surjection is the line bundle  $\mathcal{O}_{\mathbb{P}^N}(-d)|_E$ . Thus, considering the dual of  $\beta$  we obtain the following short exact sequence of semistable vector bundles on  $E$ :

$$0 \longrightarrow F^\vee \xrightarrow{\beta^\vee} \mathcal{O}_E^{\oplus n} \xrightarrow{\bar{\varphi}} \mathcal{O}_{\mathbb{P}^N}(d)|_E \longrightarrow 0$$

If we can show that the surjection  $\bar{\varphi}$  is the restriction of a homomorphism  $\varphi \in \text{Hom}(\mathcal{O}_{\mathbb{P}^N}^{\oplus n}, \mathcal{O}_{\mathbb{P}^N}(d))$ , then our theorem is proven. In order to conclude our proof, we have to show the surjectivity of the restriction map  $\text{Hom}(\mathcal{O}_{\mathbb{P}^N}^{\oplus n}, \mathcal{O}_{\mathbb{P}^N}(d)) \rightarrow \text{Hom}(\mathcal{O}_E^{\oplus n}, \mathcal{O}_{\mathbb{P}^N}(d)|_E)$  which is equivalent to the surjectivity of  $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^N}(d)|_E)$ . However, this is fulfilled since  $E$  is projectively normal.  $\square$

**Theorem A.2.** *Let  $C$  be a smooth quartic in  $\mathbb{P}_k^2$ . If the integers  $n$  and  $d$  fulfill the inequality  $2 \leq n \leq \frac{4}{3}d + 1$ , then the kernel of a general morphism  $\varphi \in \text{Hom}(\mathcal{O}_{\mathbb{P}^2}^{\oplus n}, \mathcal{O}_{\mathbb{P}^2}(d))$  is a stable vector bundle when restricted to  $C$ . This implies that  $\ker(\varphi)$  is a slope stable coherent sheaf for a general morphism  $\varphi$ .*

**Proof.** The only new ingredient in our proof is the existence of stable vector bundles with given determinant on the curve  $C$  of genus 3. This may be deduced from [3]. Indeed, we need a rank  $n - 1$  stable vector bundle  $F$  of determinant  $\omega_C^{\otimes d}$ . The slope of  $F$  is

$$\mu(F) = \frac{\deg(\omega^{\otimes d})}{n - 1} = \frac{4d}{n - 1} \geq 5.$$

This implies the global generatedness of  $F$  and we can proceed as in the above proof, because  $C$  is projectively normal.  $\square$

**Theorem A.3.** *Let  $C \subset \mathbb{P}^N$  be a smooth curve of genus two embedded by a complete linear system of degree  $N + 2$  for  $N \geq 3$ . If the integers  $n$  and  $d$  suffice  $2 \leq n \leq \frac{N+2}{3}d + 1$ , then the restriction of the kernel of a general morphism  $\varphi \in \text{Hom}(\mathcal{O}_{\mathbb{P}^N}^{\oplus n}, \mathcal{O}_{\mathbb{P}^N}(d))$  to  $C$  is a stable vector bundle. Thus,  $\ker(\varphi)$  is a slope stable coherent sheaf on  $\mathbb{P}^N$ .*

**Proof.** As in the proof of Theorem A.2 we have stable vector bundles  $F$  with given determinant on  $C$ . Since  $C$  is of genus two, every stable vector bundle  $F$  with  $\mu(F) \geq 3$  is globally generated. The projective normality of  $C$  is deduced again by the theorem of Castelnuovo, Mattuck and Mumford.  $\square$

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