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# The cubic semilocal convergence on two variants of Newton's method☆

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**Abstract**

In this paper, we discuss two variants of Newton's method without using any second derivative for solving nonlinear equations. By using the majorant function and confirming the majorant sequences, we obtain the cubic semilocal convergence and the error estimation in the Kantorovich-type theorems. The numerical examples are presented to support the usefulness and significance.


*Keywords:* Nonlinear equation; Newton's method; Cubic semilocal convergence

## 1. Introduction

It is a fundamental problem in computational mathematics for solving a nonlinear equation:

$$f(x) = 0, \tag{1.1}$$

where $f(x)$ is continuously differentiable and $f'(x) \neq 0$ in a neighborhood of a real root $x^*$. The well-known Newton's method approximates the root with quadratic convergence as the following (see [4]):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $x_0$ is some initial guess of the root.

In this paper, we consider a variant of Newton's method (see [5]):

$$x_{n+1} = x_n - \frac{f(x_n) + f(x^*_{n+1})}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

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where $x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}$ is the intermediate result from an iteration of Newton's method (1.2). And we also consider another variant of Newton's method:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{(f(x_n) - f(x_{n+1}^*))f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

which was called Newton–Secant iteration in [5]. Each of them uses one more evaluation of the function to accelerate Newton's iteration. Their cubic convergence and error equations had been obtained.

Scheme (1.3) can be recognized as a two-step method as the following:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{1.5}$$

And scheme (1.4) is rewritten as the following:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)f(y_n)}{(f(x_n) - f(y_n))f'(x_n)}. \end{cases} \tag{1.6}$$

Other methods that use the derivative $f'(x)$ to accelerate the Newton's iteration were discussed in [1,3,6]. And single-step methods that use the second derivative $f''(x)$ to reach cubic convergence were discussed in [2]. They take $N$ and $N^2$ more operations than that of (1.3) or (1.4), respectively, supposed that $f(x) = 0$ is a system of $N$ nonlinear equations.

Assume that the function $f(x)$ is defined in an open convex set $D$, $f : D \subset R \to R$. Let the majorant function be

$$h(t) = \frac{K}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}, \tag{1.7}$$

where $K$, $\beta$ and $\eta$ are positive constants, such that

$$|f'(x_0)^{-1}| \leqslant \beta, \quad |f(x_0)| \leqslant \frac{\eta}{\beta} \quad \text{for an } x_0 \in D, \tag{1.8}$$

and

$$|f'(x) - f'(y)| \leqslant K|x - y|, \quad \forall x, y \in D. \tag{1.9}$$

We have

**Lemma 1.1.** *If $\alpha = K\beta\eta \leqslant \frac{1}{2}$, then the function $h(t)$ has positive real roots $t^*$ and $t^{**}$, and*

$$\eta < t^* = \frac{1 - \sqrt{1 - 2\alpha}}{\alpha}\eta \leqslant \frac{1}{K\beta}, \quad t^* \leqslant t^{**} = \frac{1 + \sqrt{1 - 2\alpha}}{\alpha}\eta. \tag{1.10}$$

By using the majorant functions and confirming the majorant sequences for the two-step methods (1.5) and (1.6), we prove their cubic semilocal convergence and obtain the Kantorovich-type theorems too complete the convergence theories for (1.3) and (1.4) in the following sections.

## 2. Cubic semilocal convergence of (1.3)

By using scheme (1.5) to find the root of (1.7), we have

$$\begin{cases} s_n = t_n - \dfrac{h(t_n)}{h'(t_n)}, \\ t_{n+1} = s_n - \dfrac{h(s_n)}{h'(t_n)}. \end{cases} \tag{2.1}$$

**Lemma 2.1.** *If $\alpha = K\beta\eta < 6\sqrt{2} - 8$, then the sequences $\{t_k\}$ and $\{s_k\}$ from* (2.1) *satisfy*

$$0 = t_0 < s_0 < \cdots < t_k < s_k < t_{k+1} < \cdots < t^*, \tag{2.2}$$

*and*

$$t^* - t_k < \frac{\lambda^{3^k}}{\sqrt{2} - \lambda^{3^k}}(t^{**} - t^*), \quad 0 < \lambda = \sqrt{2}\frac{1 - \sqrt{1 - 2\alpha}}{1 + \sqrt{1 - 2\alpha}} < 1. \tag{2.3}$$

**Proof.** Let $u_k = t^* - t_k$, $v_k = t^{**} - t_k$, $a_k = t^* - s_k$, $b_k = t^{**} - s_k$. We prove Lemma 2.1 by induction. When $k = 0$, we have

$$s_0 - t_0 = -\frac{h(t_0)}{h'(t_0)} = \eta > 0, \quad s_0 = \eta < t^*, \quad t_1 - s_0 = -\frac{h(s_0)}{h'(t_0)} > 0.$$

Assuming Lemma 2.1 holds when $k \leqslant n$, we have

$$h(s_n) = \frac{K}{2}a_n b_n, \quad h'(s_n) = -\frac{K}{2}(a_n + b_n), \quad h(t_n) = \frac{K}{2}u_n v_n, \quad h'(t_n) = -\frac{K}{2}(u_n + v_n),$$

$$a_n = u_n - \frac{u_n v_n}{u_n + v_n} = \frac{u_n^2}{u_n + v_n} > 0, \quad b_n = v_n - \frac{u_n v_n}{u_n + v_n} = \frac{v_n^2}{u_n + v_n} > 0,$$

$$u_{n+1} = u_n - \frac{a_n b_n + u_n v_n}{u_n + v_n} = u_n - \frac{\dfrac{u_n^2 v_n^2}{(u_n + v_n)^2} + u_n v_n}{u_n + v_n} = \frac{u_n^4 + 2u_n^3 v_n}{(u_n + v_n)^3} > 0,$$

$$v_{n+1} = v_n - \frac{a_n b_n + u_n v_n}{u_n + v_n} = v_n - \frac{\dfrac{u_n^2 v_n^2}{(u_n + v_n)^2} + u_n v_n}{u_n + v_n} = \frac{v_n^4 + 2v_n^3 u_n}{(u_n + v_n)^3} > 0,$$

so forth $a_{n+1} > 0$, hence $s_{n+1} < t^*$, and

$$t_{n+2} - s_{n+1} = -\frac{h(s_{n+1})}{h'(t_{n+1})} > 0.$$

So, (2.2) holds. Furthermore,

$$\frac{u_k}{v_k} = \left(\frac{u_{k-1}}{v_{k-1}}\right)^3 \left(\frac{u_{k-1} + 2v_{k-1}}{2u_{k-1} + v_{k-1}}\right) < 2\left(\frac{u_{k-1}}{v_{k-1}}\right)^3 < \cdots < 2^{3^0} 2^{3^1} \cdots 2^{3^{k-1}} \left(\frac{u_0}{v_0}\right)^{3^k} = \frac{1}{\sqrt{2}}\lambda^{3^k},$$

and $0 < \lambda = \sqrt{2}\frac{1-\sqrt{1-2\alpha}}{1+\sqrt{1-2\alpha}} < 1$. Plugging $v_k = t^{**} - t^* + u_k$ in the above, we have (2.3). □

**Lemma 2.2.** *If $f(x)$ satisfies* (1.8) *and* (1.9), *$\alpha = K\beta\eta < 6\sqrt{2} - 8$, then $\{x_k\}$ and $\{y_k\}$ of* (1.5) *satisfy*

(a) $x_k \in S(x_0, t^*)$, $|f(x_k)| \leqslant h(t_k)$, $|y_k - x_k| \leqslant s_k - t_k$;
(b) $y_k \in S(x_0, t^*)$, $|f(y_k)| \leqslant h(s_k)$, $|x_{k+1} - y_k| \leqslant t_{k+1} - s_k$.

**Proof.** Since $x_k \in S(x_0, t^*)$ and $\alpha < 6\sqrt{2} - 8$, we have $|x_k - x_0| \leqslant t_k - t_0 < t^* < \frac{1}{K\beta}$. If $|f'(x_0)^{-1}| \leqslant \beta$ and $|x - x_0| < \frac{1}{K\beta}$, then

$$|f'(x) - f'(x_0)| \leqslant K|x - x_0| = h'(|x - x_0|) + \frac{1}{\beta} < \frac{1}{\beta}.$$

By Banach Lemma, $f'(x)^{-1}$ exists, and

$$|f'(x)^{-1}| \leqslant \frac{|f'(x_0)^{-1}|}{1 - |f'(x_0)^{-1}|(h'(|x - x_0|) + \frac{1}{\beta})} \leqslant -\frac{1}{h'(|x - x_0|)}. \tag{2.4}$$

By Taylor formula and using (1.5), we have

$$f(y_k) = \int_0^1 [f'(x_k + \tau(y_k - x_k)) - f'(x_k)]\,\mathrm{d}\tau(y_k - x_k), \tag{2.5}$$

$$f(x_{k+1}) = f(y_k) + f'(y_k)(x_{k+1} - y_k) + \int_0^1 [f'(y_k + \tau(x_{k+1} - y_k)) - f'(y_k)]\,\mathrm{d}\tau(x_{k+1} - y_k). \tag{2.6}$$

Now, we prove Lemma 2.2 by induction. When $k = 0$, (a) and (b) hold. In fact,

$$x_0 \in S(x_0, t^*), \quad |f'(x_0)^{-1}| \leqslant \beta = -\frac{1}{h'(t_0)}, \quad |f(x_0)| \leqslant \frac{\eta}{\beta} = h(t_0),$$

$$|y_0 - x_0| = \left|-\frac{f(x_0)}{f'(x_0)}\right| \leqslant -\frac{h(t_0)}{h'(t_0)} = s_0 - t_0 < t^*, \quad y_0 \in S(x_0, t^*), \quad |f'(y_0)^{-1}| \leqslant -\frac{1}{h'(s_0)}.$$

By (2.5), we have

$$|f(y_0)| \leqslant \frac{K}{2}|y_0 - x_0|^2 \leqslant \frac{K}{2}(s_0 - t_0)^2 = h(t_0) + h'(t_0)(s_0 - t_0) + \frac{K}{2}(s_0 - t_0)^2,$$

and

$$|x_1 - y_0| = \left|-\frac{f(y_0)}{f'(x_0)}\right| \leqslant -\frac{h(s_0)}{h'(t_0)} = t_1 - s_0.$$

When $k \leqslant n$, assuming (a) and (b) hold, we have

$$|x_{n+1} - x_0| \leqslant \sum_{k=0}^{n}(t_{k+1} - t_k) = t_{n+1} - t_0 < t^*, \quad x_{n+1} \in S(x_0, t^*).$$

By (2.4), we have

$$|f'(x_{n+1})^{-1}| \leqslant -\frac{1}{h'(t_{n+1})}.$$

By (2.6), we have

$$\begin{aligned}
|f(x_{n+1})| &= \left| f(y_n) - f'(y_n)\frac{f(y_n)}{f'(x_n)} + \int_0^1 [f'(y_n + \tau(x_{n+1} - y_n)) - f'(y_n)]\,\mathrm{d}\tau(x_{n+1} - y_n) \right| \\
&\leqslant h(s_n)\left|\frac{f'(y_n) - f'(x_n)}{f'(x_n)}\right| + \frac{K}{2}(t_{n+1} - s_n)^2 \\
&\leqslant -h(s_n)\frac{K(s_n - t_n)}{h'(t_n)} + \frac{K}{2}(t_{n+1} - s_n)^2 \\
&= h(s_n)\frac{h'(t_n) - h'(s_n)}{h'(t_n)} + \frac{K}{2}(t_{n+1} - s_n)^2 \\
&= h(s_n) + h'(s_n)(t_{n+1} - s_n) + \frac{K}{2}(t_{n+1} - s_n)^2 = h(t_{n+1}).
\end{aligned}$$

By (1.5), we have

$$|y_{n+1} - x_{n+1}| = \left| -\frac{f(x_{n+1})}{f'(x_{n+1})} \right| \leqslant -\frac{h(t_{n+1})}{h'(t_{n+1})} = s_{n+1} - t_{n+1};$$

$$|y_{n+1} - x_0| \leqslant s_{n+1} - t_{n+1} + \sum_{k=0}^{n} (t_{k+1} - t_k) = s_{n+1} - t_0 < t^*, \quad y_{n+1} \in S(x_0, t^*);$$

$$|f'(y_{n+1})^{-1}| \leqslant -\frac{1}{h'(s_{n+1})}.$$

By (2.5), we have

$$|f(y_{n+1})| \leqslant \frac{K}{2} |y_{n+1} - x_{n+1}|^2 \leqslant \frac{K}{2} (s_{n+1} - t_{n+1})^2$$

$$= h(t_{n+1}) + h'(t_{n+1})(s_{n+1} - t_{n+1}) + \frac{K}{2}(s_{n+1} - t_{n+1})^2 = h(s_{n+1});$$

and

$$|x_{n+2} - y_{n+1}| = \left| -\frac{f(y_{n+1})}{f'(x_{n+1})} \right| \leqslant -\frac{h(s_{n+1})}{h'(t_{n+1})} = t_{n+2} - s_{n+1}. \quad \square$$

**Theorem 2.1.** *If a nonlinear function $f(x)$ satisfies* (1.8) *and* (1.9), *and $\alpha = K\beta\eta < 6\sqrt{2} - 8$, then the sequence $\{x_k\}$ from* (1.3) *remains in $S(x_0, t^*)$ and converges cubically to the unique root $x^* \in \overline{S(x_0, t^*)}$, and satisfies*

$$|x^* - x_n| \leqslant t^* - t_n < \frac{\lambda^{3^n}}{\sqrt{2} - \lambda^{3^n}} (t^{**} - t^*) \quad \textit{where } 0 < \lambda = \sqrt{2} \frac{1 - \sqrt{1 - 2\alpha}}{1 + \sqrt{1 - 2\alpha}} < 1. \tag{2.7}$$

**Proof.** When $K > 0, \beta > 0, \eta > 0$ and $\alpha < 6\sqrt{2} - 8$, we have $0 < \lambda < 1$. By Lemmas 2.1 and 2.2, $\{x_n\}$ remains in the ball $S(x_0, t^*)$; $\{t_n\}$ is a Cauchy sequence; hence $\{x_n\}$ is a Cauchy sequence and converges to the unique solution $x^* \in \overline{S(x_0, t^*)}$, since $|f(x_n)| \leqslant h(t_n) \to 0$ as $n \to \infty$. Moreover, since

$$|x_m - x_n| \leqslant \sum_{k=n}^{m-1} |x_{k+1} - x_k| \leqslant \sum_{k=n}^{m-1} (t_{k+1} - t_k) = t_m - t_n,$$

by taking limit, we obtain (2.7). $\square$

## 3. Cubic semilocal convergence of (1.4)

By using scheme (1.6) to find the root of (1.7), we have

$$\begin{cases} s_n = t_n - \dfrac{h(t_n)}{h'(t_n)}, \\ t_{n+1} = s_n - \dfrac{h(t_n)h(s_n)}{(h(t_n) - h(s_n))h'(t_n)}. \end{cases} \tag{3.1}$$

**Lemma 3.1.** *If $\alpha = K\beta\eta < \frac{1}{2}$, then $\{t_k\}$ and $\{s_k\}$ of* (3.1) *satisfy*

$$0 = t_0 < s_0 < \cdots < t_k < s_k < t_{k+1} < \cdots < t^*; \tag{3.2}$$

*and*

$$t^* - t_k = \frac{\lambda^{3^k}}{1-\lambda^{3^k}}(t^{**} - t^*), \quad 0 < \lambda = \frac{1-\sqrt{1-2\alpha}}{1+\sqrt{1-2\alpha}} < 1. \tag{3.3}$$

**Proof.** Let $u_k = t^* - t_k$, $v_k = t^{**} - t_k$, $a_k = t^* - s_k$, $b_k = t^{**} - s_k$. We prove Lemma 3.1 by induction. When $k = 0$, we have

$$s_0 - t_0 = -\frac{h(t_0)}{h'(t_0)} = \eta > 0, \quad s_0 = \eta < t^*, \quad t_1 - s_0 = -\frac{h(t_0)h(s_0)}{(h(t_0) - h(s_0))h'(t_0)} > 0.$$

Assuming Lemma 3.1 holds when $k \leqslant n$, we have

$$h(s_n) = \frac{K}{2}a_n b_n, \quad h'(s_n) = -\frac{K}{2}(a_n + b_n), \quad h(t_n) = \frac{K}{2}u_n v_n, \quad h'(t_n) = -\frac{K}{2}(u_n + v_n),$$

$$a_n = u_n - \frac{u_n v_n}{u_n + v_n} = \frac{u_n^2}{u_n + v_n}, \quad b_n = v_n - \frac{u_n v_n}{u_n + v_n} = \frac{v_n^2}{u_n + v_n},$$

$$u_{n+1} = u_n - \frac{u_n^2 v_n^2}{(u_n v_n - a_n b_n)(u_n + v_n)} = u_n - \frac{u_n^2 v_n^2}{\left(u_n v_n - \dfrac{u_n^2 v_n^2}{(u_n + v_n)^2}\right)(u_n + v_n)} = \frac{u_n^3}{u_n^2 v_n + u_n v_n + v_n^2},$$

$$v_{n+1} = v_n - \frac{u_n^2 v_n^2}{(u_n v_n - a_n b_n)(u_n + v_n)} = v_n - \frac{u_n^2 v_n^2}{\left(u_n v_n - \dfrac{u_n^2 v_n^2}{(u_n + v_n)^2}\right)(u_n + v_n)} = \frac{v_n^3}{u_n^2 v_n + u_n v_n + v_n^2},$$

so forth $a_{n+1} > 0$, hence $s_{n+1} < t^*$, and

$$t_{n+2} - s_{n+1} = -\frac{h(t_{n+1})h(s_{n+1})}{(h(t_{n+1}) - h(s_{n+1}))h'(t_{n+1})} > 0.$$

So (3.2) holds. Furthermore,

$$\frac{u_k}{v_k} = \left(\frac{u_{k-1}}{v_{k-1}}\right)^3 = \cdots = \left(\frac{u_0}{v_0}\right)^{3^k} = (\lambda)^{3^k},$$

and $0 < \lambda = \frac{1-\sqrt{1-2\alpha}}{1+\sqrt{1-2\alpha}} < 1$. Plugging $v_k = t^{**} - t^* + u_k$ in the above, we have (3.3). □

**Lemma 3.2.** *If $f(x)$ satisfies* (1.8) *and* (1.9), $\alpha = K\beta\eta < \frac{1}{2}$, *then* $\{x_k\}$ *and* $\{y_k\}$ *of* (1.6) *satisfy*

(a) $x_k \in S(x_0, t^*)$, $|f(x_k)| \leqslant h(t_k)$, $|y_k - x_k| \leqslant s_k - t_k$, $y_k \in S(x_0, t^*)$, $|f(y_k)| \leqslant h(s_k)$;
(b) $f'(x_k + \tau(y_k - x_k)) \geqslant -h'(t_k + \tau(s_k - t_k)) > 0$, *or* $f'(x_k + \tau(y_k - x_k)) \leqslant h'(t_k + \tau(s_k - t_k)) < 0$, *for any* $0 \leqslant \tau \leqslant 1$, *and* $|x_{k+1} - y_k| \leqslant t_{k+1} - s_k$.

**Proof.** Since $0 < \alpha < \frac{1}{2}$ and $x_k \in S(x_0, t^*)$, we have $|x_k - x_0| < t^* < \frac{1}{K\beta}$. If $|f'(x_0)^{-1}| \leqslant \beta$, $|x - x_0| < \frac{1}{K\beta}$, then

$$|f'(x) - f'(x_0)| \leqslant K|x - x_0| = h'(|x - x_0|) + \frac{1}{\beta} < \frac{1}{\beta}.$$

By Banach Lemma, $f'(x)^{-1}$ exists, and

$$|f'(x)^{-1}| \leqslant \frac{|f'(x_0)^{-1}|}{1-|f'(x_0)^{-1}|\left(h'(|x-x_0|)+\dfrac{1}{\beta}\right)} \leqslant -\frac{1}{h'(|x-x_0|)}, \tag{3.4}$$

i.e., $f'(x) \geqslant -h'(|x-x_0|) > 0$, or $f'(x) \leqslant h'(|x-x_0|) < 0$. By Taylor formula and (1.6), we have

$$f(y_k) = \int_0^1 [f'(x_k+\tau(y_k-x_k)) - f'(y_k)]\,d\tau(y_k-x_k), \tag{3.5}$$

$$f(x_{k+1}) = f(y_k) + f'(y_k)(x_{k+1}-y_k) + \int_0^1 [f'(y_k+\tau(x_{k+1}-y_k)) - f'(y_k)]\,d\tau(x_{k+1}-y_k). \tag{3.6}$$

Now, we prove Lemma 3.2 by induction. When $k=0$, (a) and (b) hold. In fact,

$$x_0 \in S(x_0,t^*), \quad |f'(x_0)^{-1}| \leqslant \beta = -\frac{1}{h'(t_0)}, \quad |f(x_0)| \leqslant \frac{\eta}{\beta} = h(t_0),$$

$$|y_0-x_0| = \left|-\frac{f(x_0)}{f'(x_0)}\right| \leqslant -\frac{h(t_0)}{h'(t_0)} = s_0 - t_0 < t^*, \quad y_0 \in S(x_0,t^*), \quad |f'(y_0)^{-1}| \leqslant -\frac{1}{h'(s_0)},$$

and for any $0 \leqslant \tau \leqslant 1$,

$$f'(x_0+\tau(y_0-x_0)) \geqslant -h'(t_0+\tau(s_0-t_0)) > 0 \quad \text{or} \quad f'(x_0+\tau(y_0-x_0)) \leqslant h'(t_0+\tau(s_0-t_0)) < 0.$$

By (3.5), we have

$$|f(y_0)| \leqslant \frac{K}{2}|y_0-x_0|^2 \leqslant \frac{K}{2}|s_0-t_0|^2 = h(t_0) + h'(t_0)(s_0-t_0) + \frac{K}{2}|s_0-t_0|^2.$$

By the mean value theorem of Cauchy, we have

$$\begin{aligned}
|x_1-y_0| &= \left|-\frac{f(x_0)f(y_0)}{(f(x_0)-f(y_0))f'(x_0)}\right| = \left|\frac{f(y_0)(y_0-x_0)}{h(s_0)-h(t_0)}\,\frac{h(s_0)-h(t_0)}{f(x_0)-f(y_0)}\right| \\
&= \left|\frac{f(y_0)(y_0-x_0)}{h(s_0)-h(t_0)}\,\frac{h(t_0+(s_0-t_0))-h(t_0)}{f(x_0+(y_0-x_0))-f(x_0)}\right| = \left|\frac{f(y_0)}{h(s_0)-h(t_0)}\,\frac{h'(\theta s_0)(s_0-t_0)}{f'(x_0+\theta(y_0-x_0))}\right| \\
&\leqslant \frac{h(s_0)(s_0-t_0)}{h(s_0)-h(t_0)}\,\frac{h'(\theta s_0)}{h'(\theta s_0)} = t_1 - s_0, \quad 0<\theta<1.
\end{aligned}$$

When $k \leqslant n$, assuming (a) and (b) hold, we have

$$|x_{n+1}-x_0| \leqslant \sum_{k=0}^{n}(t_{k+1}-t_k) = t_{n+1}-t_0 < t^*, \quad x_{n+1} \in S(x_0,t^*).$$

By (3.6), we have

$$
\begin{aligned}
|f(x_{n+1})| &= \left| f(y_n) - f'(y_n)\frac{f(x_n)f(y_n)}{(f(x_n)-f(y_n))f'(x_n)} \right. \\
&\quad \left. + \int_0^1 [f'(y_n+\tau(x_{n+1}-y_n)) - f'(y_n)]\,\mathrm{d}\tau (x_{n+1}-y_n) \right| \\
&\leqslant h(s_n)\left|1 - f'(y_n)\frac{y_n-x_n}{f(y_n)-f(x_n)}\right| + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&= h(s_n)\left|\frac{\int_0^1 [f'(x_n+\tau(y_n-x_n)) - f'(y_n)]\,\mathrm{d}\tau}{\int_0^1 f'(x_n+\tau(y_n-x_n))\,\mathrm{d}\tau}\right| + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&\leqslant -h(s_n)\frac{K\int_0^1 |x_n+\tau(y_n-x_n)-y_n|\,\mathrm{d}\tau}{\int_0^1 h'(t_n+\tau(s_n-t_n))\,\mathrm{d}\tau} + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&\leqslant -h(s_n)\frac{K\int_0^1 [s_n-t_n-\tau(s_n-t_n)]\,\mathrm{d}\tau}{\int_0^1 h'(t_n+\tau(s_n-t_n))\,\mathrm{d}\tau} + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&= h(s_n)\frac{K\int_0^1 [t_n+\tau(s_n-t_n)-s_n]\,\mathrm{d}\tau}{\int_0^1 h'(t_n+\tau(s_n-t_n))\,\mathrm{d}\tau} + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&= h(s_n)\frac{\int_0^1 h'(t_n+\tau(s_n-t_n))\,\mathrm{d}\tau - \int_0^1 h'(s_n)\,\mathrm{d}\tau}{\int_0^1 h'(t_n+\tau(s_n-t_n))\,\mathrm{d}\tau} + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&= h(s_n)\left[1 - \frac{h'(s_n)(s_n-t_n)}{h(s_n)-h(t_n)}\right] + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&= h(s_n)\left[1 - h'(s_n)\frac{h(t_n)}{(h(t_n)-h(s_n))h'(t_n)}\right] + \frac{K}{2}(t_{n+1}-s_n)^2 \\
&= h(s_n) + h'(s_n)(t_{n+1}-s_n) + \frac{K}{2}(t_{n+1}-s_n)^2 = h(t_{n+1}).
\end{aligned}
$$

By (1.6), we have

$$
|y_{n+1}-x_{n+1}| = \left|-\frac{f(x_{n+1})}{f'(x_{n+1})}\right| \leqslant -\frac{h(t_{n+1})}{h'(t_{n+1})} = s_{n+1}-t_{n+1}, \quad y_{n+1}\in S(x_0,t^*).
$$

By (3.5), we have

$$
\begin{aligned}
|f(y_{n+1})| &\leqslant \frac{K}{2}(y_{n+1}-x_{n+1})^2 \leqslant \frac{K}{2}(s_{n+1}-t_{n+1})^2 \\
&= h(t_{n+1}) + h'(t_{n+1})(s_{n+1}-t_{n+1}) + \frac{K}{2}(s_{n+1}-t_{n+1})^2 = h(s_{n+1}).
\end{aligned}
$$

By (3.4), we have

$$
f'(x_{n+1}+\tau(y_{n+1}-x_{n+1})) \geqslant -h'(t_{n+1}+\tau(s_{n+1}-t_{n+1})) > 0 \quad \text{for any } 0\leqslant\tau\leqslant 1,
$$

or

$$
f'(x_{n+1}+\tau(y_{n+1}-x_{n+1})) \leqslant h'(t_{n+1}+\tau(s_{n+1}-t_{n+1})) < 0 \quad \text{for any } 0\leqslant\tau\leqslant 1.
$$

By the mean value theorem of Cauchy, we have

$$|x_{n+2}-y_{n+1}| = \left|-\frac{f(x_{n+1})f(y_{n+1})}{(f(x_{n+1})-f(y_{n+1}))f'(x_{n+1})}\right| = \left|-\frac{f(y_{n+1})(y_{n+1}-x_{n+1})}{h(s_{n+1})-h(t_{n+1})}\,\frac{h(s_{n+1})-h(t_{n+1})}{f(y_{n+1})-f(x_{n+1})}\right|$$
$$= \left|-\frac{f(y_{n+1})}{h(s_{n+1})-h(t_{n+1})}\,\frac{h'(t_{n+1}+\theta(s_{n+1}-t_{n+1}))(s_{n+1}-t_{n+1})}{f'(x_{n+1}+\theta(y_{n+1}-x_{n+1}))}\right|$$
$$\leqslant \frac{h(s_{n+1})(s_{n+1}-t_{n+1})}{h(s_{n+1})-h(t_{n+1})}\,\frac{h'(t_{n+1}+\theta(s_{n+1}-t_{n+1}))}{h'(t_{n+1}+\theta(s_{n+1}-t_{n+1}))} = t_{n+2}-s_{n+1}, \quad 0<\theta<1. \qquad \square$$

**Theorem 3.1.** *If a nonlinear function* $f(x)$ *satisfies* (1.8) *and* (1.9) *and* $\alpha = K\beta\eta < \frac{1}{2}$, *then the sequence* $\{x_n\}$ *from* (1.4) *remains in* $S(x_0, t^*)$ *and converges cubically to the unique root* $x^* \in \overline{S(x_0, t^*)}$, *and satisfies*

$$|x^*-x_k| \leqslant t^*-t_k = \frac{\lambda^{3^k}}{1-\lambda^{3^k}}(t^{**}-t^*) \quad \textit{where } 0<\lambda=\frac{1-\sqrt{1-2\alpha}}{1+\sqrt{1-2\alpha}}<1. \tag{3.7}$$

**Remark** (*see Zhu and Han [7]*). Under the assumptions in Theorem 3.1, the sequence $\{x_n\}$ from Newton's method (1.2) remains in $S(x_0, t^*)$ and converges quadratically to the unique root $x^* \in \overline{S(x_0, t^*)}$, and satisfies

$$|x^*-x_k| \leqslant t^*-t_k = \frac{\lambda^{2^k}}{1-\lambda^{2^k}}(t^{**}-t^*) \quad \text{where } 0<\lambda=\frac{1-\sqrt{1-2\alpha}}{1+\sqrt{1-2\alpha}}<1. \tag{3.8}$$

## 4. Numerical examples

According to Theorems 2.1 and 3.1, by (1.3) and (1.4), $\{x_n\}$ remains in $S(x_0, t^*)$ and cubically converges to the unique root $x^* \in \overline{S(x_0, t^*)}$ supposed that the assumptions are satisfied respectively. We do not need to call for $f''(x)$ to reach the cubic convergence. So, $f''(x)$ does not need to be continuous or even to exist near $x^*$.

**Example 4.1.** Let $f(x) = x^3 \sin 1/x + 2\sin x$, $[a, b] = [-1, 1]$, $x_0 = 0.5$. We have $x^* = 0$, and

$$f'(x) = \begin{cases} 3x^2\sin(1/x) - x\cos(1/x) + 2\cos x, & x \neq 0, \\ 2, & x = 0. \end{cases}$$

Therefore, $f'(x^*) = 2 \neq 0$ and $f''(x^*)$ does not exist. However, Schemes (1.2)–(1.4) all can be used and can converge cubically (see Table 1).

**Example 4.2.** Let

$$f(x) = \begin{cases} (x-0.2)\left(x^3\sin\dfrac{1}{x} + 2\cos x - 1\right), & x \neq 0, \\ -0.2, & x = 0, \end{cases}$$

$[a, b] = [-0.5, 0.5]$ and $x_0 = 0$. We have $x^* = 0.2$. And from

$$\left(x^3\sin\frac{1}{x} + 2\cos x - 1\right)' = \begin{cases} 3x^2\sin(1/x) - x\cos(1/x) - 2\sin x, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

we have $f'(x_0) = 1 \neq 0$ and $f''(x_0)$ does not exist. But Schemes (1.2)–(1.4) still can be used and converge to $x^*$ by one step of Newton's iteration $x_1^* = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-0.2}{1} = 0.2$.

We can compute the convergence ball of (1.2)–(1.4) according to Theorems 2.1, 3.1 and the Remark.

Table 1
Solving $f(x) = x^3 \sin\frac{1}{x} + 2\sin x$ with $x_0 = 0.5$

| Method | $n$ | $x_n$ | $f(x_n)$ | $\delta x_n$ | $\mathrm{COC}_n$ | $e_n$ | $\mathrm{OC}_n$ | $\lvert x_n - x_0\rvert$ |
|---|---|---|---|---|---|---|---|---|
| (1.2) | 1 | 9.454538509290417e − 02 | 1.8804e − 01 | 4.0545e − 01 | | 9.4545e − 02 | 3.4028 | 4.0545e − 01 |
| | 2 | 7.614108474485876e − 04 | 1.5228e − 03 | 9.3784e − 02 | 2.6217 | 7.6141e − 04 | 3.0442 | 4.9924e − 01 |
| | 3 | −2.860862477321044e − 07 | −5.7217e − 07 | 7.6170e − 04 | 3.0337 | 2.8609e − 07 | 2.0984 | 5.0000e − 01 |
| | 4 | −3.902345998949588e − 14 | −7.8047e − 14 | 2.8609e − 07 | 2.0985 | 3.9023e − 14 | 2.0492 | 5.0000e − 01 |
| | 5 | −7.509955817704032e − 28 | −1.5020e − 27 | 3.9023e − 14 | 2.0492 | 7.5100e − 28 | 2.0229 | 5.0000e − 01 |
| (1.3) | 1 | 2.345956130917615e − 02 | 4.6902e − 02 | 4.7654e − 01 | | 2.7309e − 07 | 5.4137 | 4.7654e − 01 |
| | 2 | 2.730865325713261e − 07 | 5.4617e − 07 | 2.3459e − 02 | 5.0627 | 2.4321e − 04 | 4.0276 | 5.0000e − 01 |
| | 3 | 3.599890025830776e − 21 | 7.1998e − 21 | 2.7309e − 07 | 4.0276 | 3.5999e − 21 | 3.1147 | 5.0000e − 01 |
| (1.4) | 1 | 8.346923774930792e − 03 | 1.6694e − 02 | 4.9165e − 01 | | 8.3469e − 03 | 6.9045 | 4.9165e − 01 |
| | 2 | −8.632805754971162e − 11 | −1.7266e − 10 | 8.3469e − 03 | 6.7408 | 8.6328e − 11 | 4.8419 | 5.0000e − 01 |
| | 3 | 0 | 0 | 8.6328e − 11 | 4.8419 | 0 | Inf | 5.0000e − 01 |

$e_n = |x_n - x^*|$, $\mathrm{OC}_n = \log(e_n)/\log(e_{n-1})$, $\delta x_n = |x_n - x_{n-1}|$, $\mathrm{COC}_n = \log(\delta x_n)\,/\,\log(\delta x_{n-1})$.

Table 2
Semilocal convergence of (1.2), (1.3) and (1.4) for $f(x) = \mathrm{e}^x - 1$

| Method | $\lvert x_1 - x_0\rvert$ | $\lvert x_2 - x_0\rvert$ | $\lvert x_3 - x_0\rvert$ | $\lvert x_4 - x_0\rvert$ | $x_3$ or $x_4$ |
|---|---|---|---|---|---|
| (1.2) | 0.2591817793182822 | 0.2991781564264502 | 0.2999996623790672 | 0.2999999999999430 | 5.6961e − 14 |
| (1.3) | 0.2900462929684838 | 0.2999995121884677 | 0.3000000000000000 | | −2.3095e − 17 |
| (1.4) | 0.2942186370970631 | 0.2999999518302611 | 0.2999999999999999 | | 9.2071e − 17 |

**Example 4.3.** Let $f(x) = \mathrm{e}^x - 1$, $[a, b] = [-0.2, 0.6]$ and $x_0 = 0.3$, where $x^* = 0$. By taking $K = \max_{[a,b]} |f''(x)|$, we have $K = 1.8221$, $\beta = 0.74082$, $\eta = 0.34985$ and $\alpha = K\beta\eta = 0.47224$. After verifying the conditions

$$\alpha < 1/2, \quad \alpha < 6\sqrt{2} - 8 \simeq 0.48528 \quad \text{and} \quad \alpha < 1/2,$$

we obtain the convergence ball with the radius $\rho = t^* = \frac{1-\sqrt{1-2\alpha}}{\alpha}\eta = 0.56631$ for (1.2), (1.3) and (1.4), respectively. We can see in the above that the conditions for (1.2) and (1.4) are the same and are weaker than that for (1.3). And we can see that $x_n$ remains in the convergence ball in Table 2. Actually, the property of convergence ball is obviously showed by the sequence $\{|x_n - x_0|\}$ both in Tables 1 and 2.

## 5. Conclusions

For solving nonlinear equations, Halley's method uses second derivatives to arrive at the cubic convergence. The methods such as (1.3) and (1.4) have their advantage in case the second derivatives are hard to be used. These kind of methods find an equilibrium between the high velocity and the operational cost and can find their application. The results of their cubic semilocal convergence have fundamental importance in theory and in practice.

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