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Oscillation Caused By Impulses

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The present paper is devoted to the investigation of the oscillation of a kind of very extensively studied second order nonlinear delay differential equations with impulses, some interesting results are obtained, which illustrate that impulses play a very important role in giving rise to the oscillations of equations. © 2001 Academic Press

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Consider the impulsive delay differential equation

$$\begin{gathered}\left(a(t)|x'(t)|^{\alpha-1}x'(t)\right)' + f(t, x(t), x(t-\tau)) = 0, \qquad t \neq t_k \\ x(t_k^+) = I_k(x(t_k)), \qquad x'(t_k^+) = \hat{I}_k(x'(t_k))\end{gathered} \tag{1}$$

where $\alpha, \tau > 0$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{t\to\infty} t_k = \infty$. Suppose that

$$x'(t_k) = x'(t_k^-) = \lim_{h\to -0} \frac{x(t_k+h) - x(t_k)}{h}$$

and

$$x'(t_k^+) = \lim_{h\to +0} \frac{x(t_k+h) - x(t_k^+)}{h}.$$

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Throughout the paper, assume that the following conditions hold:

(i) $f(t,u,v)$ is continuous in $[t_0-\tau,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty)$, where $t_0\geq 0$, $uf(t,u,v)>0$ $(uv>0)$ and $\frac{f(t,u,v)}{\varphi(v)}\geq p(t)$ $(v\neq 0)$, where $p(t)$ is continuous in $[t_0-\tau,+\infty)$, $p(t)\geq 0$, and $x\varphi(x)>0$ $(x\neq 0)$, $\varphi'(x)\geq 0$;

(ii) $I_k(x)$ and $\hat{I}_k(x)$ are continuous in $(-\infty,+\infty)$; there exist positive numbers c_k, c_k^*, d_k, d_k^* such that $c_k^*\leq I_k(x)/x\leq c_k$, $d_k\leq \hat{I}_k(x)/x\leq d_k^*$;

(iii) $a(t)$ is a positive continuous function in $[t_0-\tau,+\infty)$ and $A(t)=\int_{t_0}^{t}ds/a^{1/\alpha}(s)$.

Recently, there has been increasing interest on the oscillation/non-oscillation of the first order linear delay differential equations with impulses (see paper [1–6]), and good results have been obtained. But few are on the second order nonlinear delay differential equations with impulses, e.g., [7, 8], etc.

The present paper is devoted to the study of the oscillation of a type of very extensive second order nonlinear delay differential equations with impulses. Some interesting results are gained here. In addition, some examples show that, though some delay differential equations without impulses are non-oscillatory, they may become oscillatory if some impulses are added to them. That is, in some cases, impulses play a dominating role in the oscillations of equations.

As to some related results about the oscillation of some second-order nonlinear ODE with impulses, we refer to paper [9] by Chen and Feng.

For the theory of delay differential equations and impulsive differential equations, please see the recent books by Györi and Ladas [10] and Lakshmikantham et al. [11], respectively.

We introduce the notation as follows: $PC_\beta=\{x\colon[\beta-\tau,\beta]\to R\,|\,x(t)$ is twice continuously differentiable for $t\in[\beta-\tau,\beta]\setminus\{t_k,\ k=1,2,\dots\}$, $x(t_k^+), x(t_k^-), x'(t_k^+), x'(t_k^-)$ exist and $x(t_k^-)=x(t_k), x'(t_k^-)=x'(t_k)$ for $t_k\in[\beta-\tau,\beta]\}$

$\Omega_\beta=\{x\colon[\beta-\tau,+\infty)\to R\,|\,x(t)$ is continuous first for $t\neq t_k$, $x(t_k^+)$, $x(t_k^-)$ exist and $x(t_k^-)=x(t_k)$, $x(t)$ is continuously differentiable for $t\geq\beta$, $t\neq t_k$, $t\neq t_k+\tau$ and $x'(t_k^+), x'(t_k^-)$, $x'(t_k^+ +\tau)$, $x'(t_k^- +\tau)$ exist and $x'(t_k^-)=x'(t_k)\}$.

For any $\beta\geq 0$, $\phi\in PC_\beta$, a function $x\colon[\beta-\tau,+\infty)\to R$ is called a solution of Eq. (1) satisfying the initial value condition

$$x(t)=\phi(t),\qquad t\in[\beta-\tau,\beta] \tag{2}$$

if $x\in\Omega_\beta$ and satisfies (1) and (2).

Using the method of steps, one can show that for any $\tau > 0$ and $\phi \in PC_\beta$, the initial value problem (1), (2) has a unique solution $x \in \Omega_\beta$.

A solution of (1) is said to be non-oscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

This paper is organized as follows. In Section 1 we shall offer two interesting lemmas, which will be used in Section 2 to prove our main theorems. To illustrate our results, three examples are also included in Section 3.

1. SOME LEMMAS

LEMMA 1 [9, Theorem 1.4.1]. *Assume that*

(A_0) $m \in PC^1[R_+, R]$ *and* $m(t)$ *is left-continuous at* t_k, $k = 1, 2, \ldots$

(A_1) *For* $k = 1, 2, \ldots, t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t), \qquad t \neq t_k$$
$$m(t_k^+) \leq d_k m(t_k) + b_k$$

where $q, p \in PC^1[R_+, R]$, $d_k \geq 0$ *and* b_k *are constants. Then*

$$\begin{aligned} m(t) \leq\; & m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\,ds \right) \\ & + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)\,ds \right) \right) b_k \\ & + \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\sigma)\,d\sigma \right) q(s)\,ds, \qquad t \geq t_0. \end{aligned}$$

LEMMA 2. *Let* $x(t)$ *be a solution of Eq.* (1). *Suppose that there exists some* $T \geq t_0$ *such that* $x(t) > 0$ *for* $t \geq T$. *If*

$$\int_{t_j}^{+\infty} \frac{1}{a^{1/\alpha}(s)} \prod_{t_j < t_l < s} \frac{d_i}{c_i}\,ds = +\infty \tag{3}$$

for some $t_j (\geq t_1)$, *then*

$$x'(t_k^+) \geq 0 \qquad \text{and} \qquad x'(t) \geq 0$$

for $t \in (t_k, t_{k+1}]$, *where* $t_k \geq T$.

Proof. At first, we prove that $x'(t_k) \geq 0$ for any $t_k \geq T$. If not, then there exists some j such that $t_j \geq T$, $x'(t_j) < 0$, and $x'(t_j^+) = \hat{I}_j(x'(t_j)) \leq d_j x'(t_j) < 0$. Let

$$a(t_j^+)\left|x'(t_j^+)\right|^{\alpha-1} x'(t_j^+) = -\beta^\alpha (\beta > 0) \qquad \text{and}$$

$$S(t) = a(t)|x'(t)|^{\alpha-1} x'(t).$$

By (1), for $t \in (t_{j+i-1}, t_{j+i}]$, $i = 1, 2, \ldots$, we have

$$\begin{aligned} \left(a(t)|x'(t)|^{\alpha-1} x'(t)\right)' &= -f(t, x(t), x(t-\tau)) \\ &\leq -p(t)\varphi(x(t-\tau)) \leq 0. \end{aligned} \tag{4}$$

Hence, $S(t)$ is monotonically decreasing in $(t_{j+i-1}, t_{j+i}]$. So we have

$$a(t_{j+1})\left|x'(t_{j+1})\right|^{\alpha-1} x'(t_{j+1}) \leq a(t_j^+)\left|x'(t_j^+)\right|^{\alpha-1} x'(t_j^+) = -\beta^\alpha < 0$$

and

$$\begin{aligned} a(t_{j+2})\left|x'(t_{j+2})\right|^{\alpha-1} x'(t_{j+2}) &\leq a(t_{j+1}^+)\left|x'(t_{j+1}^+)\right|^{\alpha-1} x'(t_{j+1}^+) \\ &= a(t_{j+1})\left|\hat{I}_{j+1}(x'(t_{j+1}))\right|^{\alpha-1} \hat{I}_{j+1}(x'(t_{j+1})) \\ &\leq (d_{j+1})^\alpha a(t_{j+1})\left|x'(t_{j+1})\right|^{\alpha-1} x'(t_{j+1}) \\ &\leq -(d_{j+1})^\alpha \beta^\alpha < 0. \end{aligned}$$

By induction, we obtain

$$a(t)|x'(t)|^{\alpha-1} x'(t) \leq -\left(d_{j+1} d_{j+2} \cdots d_{j+n}\right)^\alpha \beta^\alpha = -\beta^\alpha \prod_{t_j < t_k < t} d_k^\alpha < 0, \tag{5}$$

for $t \in (t_{j+n}, t_{j+n+1}]$. Therefore

$$x'(t) \leq -\frac{\beta \prod_{t_j < t_k < t} d_k}{a^{1/\alpha}(t)}.$$

In view of condition (ii), we have $x(t_k^+) \le c_k x(t_k)$. Applying Lemma 1, we obtain

$$x(t) \le x(t_j^+) \prod_{t_j<t_k<t} c_k + \beta \int_{t_j}^{t} \prod_{s<t_k<t} c_k \prod_{t_j<t_1<s} d_i \frac{1}{a^{1/\alpha}(s)}\, ds, \qquad t > t_j. \tag{6}$$

In view of the fact that $\prod_{t_j<t_k<t} c_k = \prod_{t_j<t_l<s} c_i \prod_{s<t_l<t} c_l$, we have

$$x(t) \le \prod_{t_j<t_k<t} c_k \left\{ x(t_k^+) - \beta \int_{t_j}^{t} \frac{1}{a^{1/\alpha}(s)} \prod_{t_j<t_l<s} \frac{d_i}{c_i}\, ds \right\}, \qquad t > t_j. \tag{7}$$

Since $x(t_k) > 0$ $(t_k \ge T)$, one can find that (7) contradicts (3) as $t \to \infty$. Therefore

$$x'(t_k) \ge 0 \qquad (t_k \ge T).$$

By condition (ii), we have, for any $t_k \ge T$, $x'(t_k^+) \ge d_k x'(t_k) \ge 0$. Because $S(t)$ is decreasing in $(t_{j+i-1}, t_{j+i}]$, we get, for $t \in (t_{j+i-1}, t_{j+i}]$, $S(t) \ge 0$, which implies $x'(t) \ge 0$. The proof of this lemma is complete.

Remark 1. In the case that $x(t)$ is eventually negative, if (3) holds true, then $x'(t_k^+) \le 0$ and $x'(t) \le 0$, for $t \in (t_{j+i-1}, t_{j+i}]$ where $t_k \ge T$.

2. MAIN RESULTS

THEOREM 1. *Assume that* (3) *holds and there exists a positive integer* k_0 *such that* $c_k^* \ge 1$ *for* $k \ge k_0$. *If*

$$\int_{t_0}^{+\infty} \prod_{t_0<t_{0,n}<s} \frac{1}{\theta_{0,n}} p(s)\, ds = +\infty, \tag{8}$$

where

$$\theta_{0,n} = \begin{cases} 1 & t_{0,n} = t_k + \tau \ne t_m \quad (m > k) \\ (d_k^*)^{\alpha} & t_{0,n} = t_k \\ (d_m^*)^{\alpha} & t_{0,n} = t_k + \tau = t_m \end{cases} \tag{9}$$

and $t_{0,n} = t_k$ *or* $t_k + \tau$ $(t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,n} < t_{0,n+1} < \cdots)$, *then every solution of* (1) *is oscillatory.*

Proof. Without loss of generality, we can assume $k_0 = 1$. If (1) has a non-oscillatory solution $x(t)$, we might as well assume that $x(t) > 0$ $(t \geq t_0)$. It follows from Lemma 2 that $x'(t) \geq 0$ for $t \in (t_k, t_{k+1}]$, where $k = 1, 2, \ldots$. Let

$$w(t) = \frac{a(t)|x'(t)|^{\alpha-1}x'(t)}{\varphi(x(t-\tau))}. \tag{10}$$

Then $w(t_k^+) \geq 0$ $(k = 1, 2, \ldots)$, $w(t) \geq 0$ for $t \geq t_0$. Using condition (i) and Eq. (1), we get

$$\begin{aligned} w'(t) &= -\frac{f(t, x(t), x(t-\tau))}{\varphi(x(t-\tau))} \\ &\quad - \frac{a(t)|x'(t)|^{\alpha-1}x'(t)\varphi'(x(t-\tau))x'(t-\tau)}{\varphi^2(x(t-\tau))} \leq -p(t). \end{aligned} \tag{11}$$

It follows from the continuity of $a(t)$, condition (ii), $c_k^* \geq 1$, and $\varphi'(x) \geq 0$ that

$$\begin{aligned} w(t_k^+) &= \frac{a(t_k^+)|x'(t_k^+)|^{\alpha-1}x'(t_k^+)}{\varphi(x(t_k^+ - \tau))} \\ &\leq \begin{cases} \dfrac{(d_k^*)^\alpha a(t_k)|x'(t_k)|^{\alpha-1}x'(t_k)}{\varphi(x(t_k-\tau))} = (d_k^*)^\alpha w(t_k) \\ \qquad t_k - \tau \neq t_m \quad (0 < m < k) \\ \dfrac{(d_k^*)^\alpha a(t_k)|x'(t_k)|^{\alpha-1}x'(t_k)}{\varphi(c_m^* x(t_m))} \\ \qquad \leq \dfrac{(d_k^*)^\alpha a(t_k)|x'(t_k)|^{\alpha-1}x'(t_k)}{\varphi(x(t_k-\tau))} = (d_k^*)^\alpha w(t_k) \\ \qquad t_k - \tau = t_m \quad (0 < m < k) \end{cases} \end{aligned} \tag{12}$$

and

$$w(t_k^+ + \tau) = \frac{a(t_k^+ + \tau)|x'(t_k^+ + \tau)|^{\alpha-1}x'(t_k^+ + \tau)}{\varphi(x(t_k^+))}$$

$$\leq \begin{cases} \dfrac{a(t_k + \tau)|x'(t_k^+ + \tau)|^{\alpha-1}x'(t_k^+ + \tau)}{\varphi(c_k^* x(t_k))} \\ \quad \leq \dfrac{a(t_k + \tau)|x'(t_k + \tau)|^{\alpha-1}x'(t_k + \tau)}{\varphi(x(t_k))} \\ \quad = w(t_k + \tau) \\ \qquad t_k + \tau \neq t_m \quad (m > k) \\ \dfrac{a(t_m)|\hat{I}_m(x'(t_m))|^{\alpha-1}\hat{I}_m(x'(t_m))}{\varphi(c_k^* x(t_k))} \\ \quad \leq \dfrac{(d_m^*)^\alpha a(t_m)|x'(t_m)|^{\alpha-1}x'(t_m)}{\varphi(c_k^* x(t_k))} \\ \quad \leq \dfrac{(d_m^*)^\alpha a(t_k + \tau)|x'(t_k + \tau)|^{\alpha-1}x'(t_k + \tau)}{\varphi(x(t_k))} \\ \quad = (d_m^*)^\alpha w(t_k + \tau) \\ \qquad t_k + \tau = t_m \quad (m > k) \end{cases} \tag{13}$$

It follows from inequalities (11)–(13) that

$$w'(t) \leq -p(t), \qquad t \neq t_{0,n}$$
$$w(t_{0,n}^+) \leq \theta_{0,n} w(t_{0,n}),$$

where $t_{0,n} = t_k$ or $t_k + \tau$ $(t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,n} < t_{0,n+1} < \cdots)$ and $\theta_{0,n}$ is defined by (9). Then, applying Lemma 1, we obtain

$$w(t) \leq \prod_{t_0 < t_{0,n} < t} \theta_{0,n} \left\{ w(t_0^+) - \int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{\theta_{0,n}} p(s)\, ds \right\}, \qquad t \geq t_0 \tag{14}$$

In view of (8), (14), and $w(t) \geq 0$, we get a contradiction as $t \to \infty$. Hence, every solution of (1) is oscillatory. The proof of Theorem 1 is complete.

THEOREM 2. *Assume that* (3) *holds and* $\varphi(ab) \geq \varphi(a)\varphi(b)$ *for any* $ab > 0$. *If*

$$\int_{t_0}^{+\infty} \prod_{t_0 < t_{0,n} < s} \frac{1}{\mu_{0,n}} p(s)\, ds = +\infty, \tag{15}$$

where

$$\mu_{0,n} = \begin{cases} \dfrac{(d_m^*)^\alpha}{\varphi(c_k^*)} & t_{0,n} = t_k + \tau = t_m \quad (m > k) \\ (d_k^*)^\alpha & t_{0,n} = t_k \quad \text{and} \quad t_k - \tau \neq t_m \\ \dfrac{1}{\varphi(c_k^*)} & t_{0,n} = t_k + \tau \neq t_m \quad (m > k) \\ \dfrac{(d_k^*)^\alpha}{\varphi(c_m^*)} & t_{0,n} = t_k \quad \text{and} \quad t_k - \tau = t_m \quad (0 < m < k), \end{cases} \tag{16}$$

then every solution of (1) *is oscillatory.*

Proof. If (1) has a non-oscillatory solution $x(t)$, without loss of generality, we can assume $x(t) > 0$ $(t \geq t_0)$. Let $w(t)$ be defined by (10). Then

$$w(t_k^+) \geq 0 \qquad (k = 1, 2, \dots), \quad w(t) \geq 0 \quad (t \geq t_0).$$

It is easy to see that

$$\begin{aligned} w(t_k^+) &= \frac{a(t_k^+)\left|x'(t_k^+)\right|^{\alpha-1} x'(t_k^+)}{\varphi(x(t_k^+ - \tau))} \\ &\leq \begin{cases} \dfrac{(d_k^*)^\alpha a(t_k)\left|x'(t_k)\right|^{\alpha-1} x'(t_k)}{\varphi(x(t_k - \tau))} = (d_k^*)^\alpha w(t_k) \\ \qquad t_k - \tau \neq t_m \quad (0 < m < k) \\ \dfrac{(d_k^*)^\alpha a(t_k)\left|x'(t_k)\right|^{\alpha-1} x'(t_k)}{\varphi(c_m^*)\varphi(x(t_m))} \\ \qquad \leq \dfrac{(d_k^*)^\alpha a(t_k)\left|x'(t_k)\right|^{\alpha-1} x'(t_k)}{\varphi(c_m^*)\varphi(x(t_k - \tau))} = \dfrac{(d_k^*)^\alpha}{\varphi(c_m^*)} w(t_k) \\ \qquad t_k - \tau = t_m \quad (0 < m < k) \end{cases} \end{aligned} \tag{17}$$

and

$$w(t_k^+ + \tau) = \frac{a(t_k^+ + \tau)|x'(t_k^+ + \tau)|^{\alpha-1}x'(t_k^+ + \tau)}{\varphi(x(t_k^+))}$$

$$\leq \begin{cases} \dfrac{a(t_k + \tau)|x'(t_k^+ + \tau)|^{\alpha-1}x'(t_k^+ + \tau)}{\varphi(c_k^* x(t_k))} \\ \quad \leq \dfrac{a(t_k + \tau)|x'(t_k + \tau)|^{\alpha-1}x'(t_k + \tau)}{\varphi(c_k^*)\varphi(x(t_k))} \\ \quad = -\dfrac{1}{\varphi(c_k^*)} w(t_k + \tau) \\ \qquad t_k + \tau \neq t_m \quad (m > k) \\ \dfrac{a(t_m)|\hat{I}_m(x'(t_m))|^{\alpha-1}\hat{I}_m(x'(t_m))}{\varphi(c_k^* x(t_k))} \\ \quad \leq \dfrac{(d_m^*)^{\alpha} a(t_m)|x'(t_m)|^{\alpha-1}x'(t_m)}{\varphi(c_k^*)\varphi(x(t_k))} \\ \quad \leq \dfrac{(d_m^*)^{\alpha} a(t_k + \tau)|x'(t_k + \tau)|^{\alpha-1}x'(t_k + \tau)}{\varphi(c_k^*)\varphi(x(t_k))} \\ \quad = \dfrac{(d_m^*)^{\alpha}}{\varphi(c_k^*)} w(t_k + \tau) \\ \qquad t_k + \tau = t_m \quad (m > k). \end{cases} \tag{18}$$

It follows from inequalities (10), (17), and (18) that

$$w'(t) \leq -p(t), \qquad t \neq t_{0,n}$$
$$w(t_{0,n}^+) \leq \mu_{0,n} w(t_{0,n}),$$

where $t_{0,n} = t_k$ or $t_k + \tau$ ($t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,n} < t_{0,n+1} < \cdots$) and $\mu_{0,n}$ is defined by (16). Then, applying Lemma 1, we obtain

$$w(t) \leq \prod_{t_0 < t_{0,n} < t} \mu_{0,n} \left\{ w(t_0^+) - \int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{\mu_{0,n}} p(s)\, ds \right\}, \qquad t \geq t_0. \tag{19}$$

In view of (15), (19), and $w(t) \geq 0$, we get a contradiction as $t \to \infty$. Hence, every solution of (1) is oscillatory. The proof of Theorem 2 is complete.

Using Theorems 1 and 2, we can obtain some corollaries as follows:

COROLLARY 1. *Assume that* (3) *holds and there exists a positive integer* k_0 *such that* $c_k^* \geq 1$, $d_k^* \leq 1$ *for* $k \geq k_0$. *If*

$$\int^{+\infty} p(s)\, ds = +\infty, \tag{20}$$

then every solution of (1) *is oscillatory.*

Proof. Without loss of generality, let $k_0 = 1$. By $c_k^* \geq 1$, $d_k^* \leq 1$, we know that $1/\theta_{0,n} \geq 1$. Therefore

$$\int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{\theta_{0,n}} p(s)\, ds \geq \int_{t_0}^{t} p(s)\, ds. \tag{21}$$

Let $t \to \infty$; it follows from (20) and (21) that (8) holds. By Theorem 1, we get that all solutions of (1) are oscillatory.

COROLLARY 2. *Assume that* (3) *holds and there exist a positive integer* k_0 *and a constant* $\gamma > 0$ *such that*

$$c_k^* \geq 1, \qquad \frac{1}{(d_k^*)^{\alpha}} \geq \left(\frac{t_{k+1}}{t_k} \right)^{\gamma} \qquad \text{for } k \geq k_0 \tag{22}$$

and

$$\int^{+\infty} t^r p(t)\, dt = +\infty, \tag{23}$$

then every solution of (1) *is oscillatory.*

Proof. Without loss of generality, let $k_0 = 1$. Then we have

$$\begin{aligned}
&\int_{t_0}^{t} \prod_{t_0 < t_{0,n} < s} \frac{1}{\theta_{0,n}} p(s)\, ds \\
&\quad = \int_{t_0}^{t_1} p(s)\, ds + \frac{1}{(d_a^*)^{\alpha}} \int_{t_1}^{t_2} p(s)\, ds + \frac{1}{(d_1^* d_2^*)^{\alpha}} \int_{t_2}^{t_3} p(s)\, ds + \cdots \\
&\qquad + \frac{1}{(d_1^* d_2^* \cdots d_n^*)^{\alpha}} \int_{t_n}^{t} p(s)\, ds \\
&\quad \geq \frac{1}{(d_1^*)^{\alpha}} \int_{t_1}^{t_2} p(s)\, ds + \frac{1}{(d_1^* d_2^*)^{\alpha}} \int_{t_2}^{t_3} p(s)\, ds \\
&\qquad + \frac{1}{(d_1^* d_2^* \cdots d_n^*)^{\beta} a} \int_{t_n}^{t} p(s)\, ds
\end{aligned}$$

$$\geq \frac{1}{t_1^\gamma}\left[\int_{t_1}^{t_2} t_2^\gamma p(s)\,ds + \int_{t_2}^{t_3} t_3^\gamma p(s)\,ds + \cdots + \int_{t_n}^{t} t_{n+1}^\gamma p(s)\,ds\right]$$

$$\geq \frac{1}{t_1^\gamma}\left[\int_{t_0}^{t_1} s^\gamma p(s)\,ds + \int_{t_1}^{t_2} s^\gamma p(s)\,ds + \cdots + \int_{t_n}^{t} s^\gamma p(s)\,ds\right]$$

$$= \frac{1}{t_1^\gamma}\int_{t_0}^{t} s^\gamma p(s)\,ds \tag{24}$$

for $t \in (t_n, t_{n+1}]$. Let $t \to \infty$; it follows from (23) and (24) that (8) holds. According to Theorem 1, we get a conclusion that Eq. (1) is oscillatory.

COROLLARY 3. *Assume that* (3) *and* (23) *hold, and* $\varphi(ab) \geq \varphi(a)\varphi(b)$ *for any* $ab > 0$. *Furthermore, suppose that there exist a positive integer* k_0 *and a constant* $\gamma > 0$ *such that*

$$t_{k+1} - t_k > \tau, \quad \frac{\varphi(c_k^*)}{(d_k^*)^\alpha} \geq \left(\frac{t_{k+1}}{t_k}\right)^\gamma \qquad \text{for } k \geq k_0. \tag{25}$$

Then every solution of Eq. (1) *is oscillatory.*

Corollary 3 can be deduced from Theorem 2. Its proof is similar to that of Corollary 2 and it is omitted.

Remark 2. Using the same technique and the same argument as above, one also can obtain new criteria about the oscillation of the advanced differential equation with impulses

$$\begin{gathered}\left(a(t)|x'(t)|^{\alpha-1}x'(t)\right)' + f(t, x(t), x(t+\tau)) = 0, \qquad t \neq t_k \\ x(t_k^+) = I_k(x(t_k)), \quad x'(t_k^+) = \hat{I}_k(x'(t_k)).\end{gathered} \tag{26}$$

3. EXAMPLES

EXAMPLE 1. Consider the impulsive delay differential equation

$$\begin{gathered}x'' + \frac{1}{4t^2}x\left(t - \frac{1}{5}\right) = 0, \qquad t \neq k,\ k = 1, 2, 3, \dots \\ x(k^+) = x(k), \quad x'(k^+) = (k/(k+1))x'(k), \qquad k = 1, 2, \dots,\end{gathered} \tag{27}$$

where $d_k = d_k^* = k/(k+1)$, $c_k = c_k^* = 1$, $p(t) = 1/4t^2$, $t_k = k$, and $\varphi(x) = x$.

Obviously, $\alpha = 1$, the conditions (i) and (ii) are satisfied and

$$
\begin{aligned}
&\int_{t_j}^{+\infty} \frac{1}{a^{1/\alpha}(s)} \prod_{t_j < t_l < s} \frac{d_i}{c_i}\, ds \\
&\quad = A(t_{j+1}) - A(t_j) - \frac{d_{j+1}}{c_{j+1}}\big(a(t_{j+2}) - A(t_{j+1})\big) \\
&\qquad + \cdots + \frac{d_{j+1}d_{j+2}\cdots d_{j+n}}{c_{j+1}c_{j+2}\cdots c_{j+n}}\big(A(t_{j+n+1}) - A(t_{j+n})\big) + \cdots \\
&\quad = 1 + \frac{j+1}{j+2} + \frac{j+1}{j+3} + \cdots + \frac{j+1}{j+n} + \cdots \\
&\quad = +\infty.
\end{aligned}
$$

Let $k_0 = 1$, $\gamma = 1$. Then

$$
\frac{1}{(d_k^*)^\alpha} = \frac{k+1}{k} = \frac{t_{k+1}}{t_k}
$$

and

$$
\int^{+\infty} t^\gamma p(t)\, dt = \int^{+\infty} tp(t)\, dt = \int^{+\infty} t \times \frac{1}{2t^2}\, dt = +\infty.
$$

By Corollary 2, we know that every solution of Eq. (27) is oscillatory.

EXAMPLE 2. Consider the super-linear impulsive equation

$$
\begin{aligned}
&x'' + \frac{1}{t^3} x^{2n-1}\left(t - \frac{1}{3}\right) = 0, \qquad t \neq k, \quad k = 1, 2, \ldots \\
&x(k^+) = (k + 1/k)x(k), \quad x'(k^+) = x'(k), \qquad k = 1, 2, \ldots,
\end{aligned}
\tag{28}
$$

where $n > 1$ is a natural number, and $c_k = c_k^* = (k+1)/k$, $d_k = d_k^* = 1$, $p(t) = 1/t^3$, $t_k = k$, and $\varphi(x) = x^{2n-1}$. Let $k_0 = 1$, $\gamma = 3$. Obviously, the conditions (i), (ii), and (3) are satisfied and

$$
\frac{\varphi(c_k^*)}{(d_k^*)^\alpha} = \left(\frac{k+1}{k}\right)^{2n-1} \geq \left(\frac{t_{k+1}}{t_k}\right)^3
$$

and

$$\int^{+\infty} t^{\gamma} p(t)\, dt = \int^{+\infty} t^{3} p(t)\, dt = \int^{+\infty} t^{3} \times \frac{1}{t^{3}}\, dt = +\infty.$$

By Corollary 3, we know that every solution of Eq. (28) is oscillatory. But the delay differential equation $x'' + (1/t^3)x^{2n-1}(t - \frac{1}{3}) = 0$, by paper [12], is non-oscillatory.

EXAMPLE 3. Consider the sub-linear impulsive equation

$$\begin{aligned} & x'' + \frac{1}{t^2} x^{1/3}\left(t - \frac{1}{12}\right) = 0, \qquad t \neq k, \quad k = 1, 2, \ldots \\ & x(k^+) = x(k), \quad x'(k^+) = (k/(k+1))x'(k), \qquad k = 1, 2, \ldots, \end{aligned} \tag{29}$$

where $c_k = c_k^* = 1$, $d_k = d_k^* = k/(k+1)$, $p(t) = 1/t^2$, $t_k = k$, and $\varphi(x) = x^{1/3}$. Let $k_0 = 1$, $\gamma = 1$. Obviously, the conditions (i), (ii), and (3) are satisfied and $\varphi(ab) = \varphi(a)\varphi(b)$ for any $ab > 0$,

$$\frac{\varphi(c_k^*)}{(d_k^*)^{\alpha}} = \frac{k+1}{k} \geq \frac{t_{k+1}}{t_k},$$

and

$$\int^{+\infty} t^{\gamma} p(t)\, dt = \int^{+\infty} t p(t)\, dt = \int^{+\infty} t \times \frac{1}{t^{2}}\, dt = +\infty.$$

By Corollary 3, we know that every solution of Eq. (29) is oscillatory. But the delay differential equation $x'' + (1/t^2)x^{1/3}(t - \frac{1}{12}) = 0$, by paper [12], is non-oscillatory.

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