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A Note on Rational Approximation to $(1-x)^\sigma$

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THEOREM 1. *For* $0<\sigma\leqslant 1$, $0\leqslant x\leqslant 1$ *and* $n\geqslant 0$, *we have*

$$0\leqslant S_n^{-1}(x)-(1-x)^\sigma\leqslant\frac{x^{n+1}}{S_n(x)}\leqslant\binom{\sigma+n}{n}^{-1}, \tag{1}$$

where $S_n(x)=\sum_{j=0}^n\binom{\sigma+j-1}{j}x^j$. (Note that $(1-x)^{-\sigma}=\sum_{j=0}^\infty\binom{\sigma+j-1}{j}x^j$ *for* $0<\sigma\leqslant 1$, $0\leqslant x<1$.)

Proof. For $0<\sigma\leqslant 1$, $0\leqslant x<1$, $j\geqslant 0$, and $n\geqslant 0$, we obtain, by oberserving the fact that

$$\binom{n+\sigma+j}{n+1+j}\leqslant\binom{\sigma+j-1}{j},$$

$$0\leqslant S_n^{-1}(x)-(1-x)^\sigma=\frac{\sum_{j=n+1}^\infty\binom{\sigma+j-1}{j}x^j}{S_n(x)(1-x)^{-\sigma}}$$

$$=\frac{x^{n+1}\sum_{j=0}^\infty\binom{n+\sigma+j}{n+1+j}x^j}{S_n(x)(1-x)^{-\sigma}}$$

$$\leqslant\frac{x^{n+1}\sum_{j=0}^\infty\binom{\sigma+j-1}{j}x^j}{S_n(x)(1-x)^{-\sigma}}=\frac{x^{n+1}}{S_n(x)}.$$

Let $0<\sigma\leqslant 1$, $n\geqslant 0$. As

$$(n+1)\,S_n(x)>xS_n'(x)\qquad\text{for}\quad x>0,$$

$$\left(\frac{x^{n+1}}{S_n(x)}\right)'>0\qquad\text{for}\quad x>0,$$

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and hence $x^{n+1}/S_n(x)$ is increasing on $[0, \infty)$. Thus, for $0 \leqslant x < 1$,

$$0 \leqslant [S_n(x)]^{-1} - (1-x)^\sigma \leqslant \frac{x^{n+1}}{S_n(x)} < \frac{1}{S_n(1)}$$
$$= \left(\sum_{j=0}^{n} \binom{\sigma+j-1}{j} \right)^{-1} = \binom{\sigma+n}{n}^{-1}.$$

The last two equalities prove (1) also for $x = 1$. It follows that

$$\| (1-x)^\sigma - S_n^{-1}(x) \|_{L^\infty[0,1]} \leqslant \binom{\sigma+n}{n}^{-1}. \tag{2}$$

Remarks on Theorem 1. (i) For $\sigma = \frac{1}{2}$, (2) was obtained in [1].

(ii) For $0 < \sigma < 1$, (2) is sharper than the estimate given in [2], as for $n = 1, 2, \ldots$,

$$\binom{\sigma+n}{n}^{-1} = \frac{\Gamma(\sigma+1)\,\Gamma(n+1)}{\Gamma(\sigma+n+1)} < \frac{n}{n+1} \frac{\Gamma(\sigma)\,\Gamma(n)}{\Gamma(\sigma+n)}.$$

THEOREM 2. *Let $P(x)$ and $Q(x)$ be polynomials of degree at most n $(\geqslant 1)$ having only real, non-negative coefficients $(Q(0) > 0)$ and let $0 < \sigma < 1$. Then*

$$\left\| (1-x)^\sigma - \frac{P(x)}{Q(x)} \right\|_{L^\infty[0,1]}$$
$$\geqslant (n+1)^{-\sigma} \left(1 + \left(\frac{n+1}{n} \right)^n \right)^{-1} > \frac{(n+1)^{-\sigma}}{1+e}. \tag{3}$$

Proof. Set

$$\left\| (1-x)^\sigma - \frac{P(x)}{Q(x)} \right\|_{L^\infty[0,1]} = \varepsilon.$$

Then

$$\varepsilon \geqslant \left(1 - \frac{n}{n+1} \right)^\sigma - \frac{P\left(\frac{n}{n+1}\right)}{Q\left(\frac{n}{n+1}\right)}$$
$$\geqslant \frac{1}{(n+1)^\sigma} + \left(0 - \frac{P(1)}{Q(1)} \right) \left(\frac{n+1}{n} \right)^n$$
$$\geqslant \frac{1}{(n+1)^\sigma} - \varepsilon \left(\frac{n+1}{n} \right)^n$$

Hence

$$\varepsilon \geqslant (n+1)^{-\sigma}\left(1+\left(\frac{n+1}{n}\right)^{n}\right)^{-1}.$$

Remarks on Theorem 2. (i) The estimate (3) is sharper than these obtained in [2, 3]. The technique used here is different from the one used in [3].

(ii) Actually the inequality $P(1) \geqslant \max(0, P(n/(n+1)))$ suffices for the proof of Theorem 2 rather than assuming $P(x)$ has only real, non-negative coefficients.

THEOREM 3. *Let $P(x)$ and $Q(x)$ be polynomials of degree at most n $(\geqslant 2)$ and let $P(x)$ have only real, non-negative coefficients. Let $Q(x) \neq 0$ throughout $[0, 1]$ and let $0 < \sigma \leqslant 1$. Then*

$$\left\| (1-x)^{\sigma} - \frac{P(x)}{Q(x)} \right\|_{L^{\infty}[0,1]} \geqslant (16n^{2\sigma})^{-1}.$$

We need the following well known

LEMMA [5, p. 68, Eq. (9)]. *If a real polynomial $Q_n(x)$ of degree $\leqslant n$ $(n \geqslant 0)$ satisfies the inequality $|Q_n(x)| \leqslant L$ on $[a, b]$ $(-\infty < a < b < \infty)$, then at each real x outside $[a, b]$ we have*

$$|Q_n(x)| \leqslant L \left| T_n\left(\frac{2x-a-b}{b-a}\right)\right|,$$

where T_n is the nth *degree Chebychev polynomial of the first kind.*

Proof of Theorem 3. Suppose

$$\left\| (1-x)^{\sigma} - \frac{P(x)}{Q(x)} \right\|_{L^{\infty}[0,1]} < (16n^{2\sigma})^{-1}. \tag{4}$$

Then on $[0, 1-n^{-2}]$,

$$\left|\frac{P(x)}{Q(x)}\right| > (1-x)^{\sigma} - \frac{1}{16n^{2\sigma}} \geqslant \frac{1}{n^{2\sigma}} - \frac{1}{16n^{2\sigma}} = \frac{15}{16n^{2\sigma}},$$

$$|Q(x)| < \frac{16n^{2\sigma}}{15} P(1). \tag{5}$$

In (5) we have used the assumption that $P(x)$ has only real, non-negative coefficients. Now, by applying the lemma to (5) with $a = 0$, $b = 1 - n^{-2}$, we get on [0, 1], by familiar elementary properties of T_n, and by [4, p. 38, Problem 172],

$$\begin{aligned}\operatorname*{Max}_{[0,1]} |Q(x)| &\leqslant \frac{16n^{2\sigma}}{15} P(1)\, T_n\left(\frac{n^2+1}{n^2-1}\right)\\ &< \frac{16n^{2\sigma}}{15} P(1)\left(\frac{n^2+1}{n^2-1}+\sqrt{\left(\frac{n^2+1}{n^2-1}\right)^2-1}\right)^n\\ &= \frac{16n^{2\sigma}}{15} P(1)\left(\frac{n+1}{n-1}\right)^n\\ &\leqslant \frac{16n^{2\sigma}}{15} P(1)\cdot 9 = \frac{48}{5} n^{2\sigma} P(1). \qquad (6)\end{aligned}$$

From (6) and (4) we obtain $\frac{5}{48} < n^{2\sigma}\,|P(1)/Q(1)| < \frac{1}{16}$, which is false.

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