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Brauer Groups and Galois Cohomology for a Krull Scheme

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1. INTRODUCTION

In [12] the authors introduced the class group $Cl(\mathscr{C}(X))$ and the Brauer group $\mathrm{Br}(\mathscr{C}(X))$, where $\mathscr{C}(X)$ is a suitably nice category of divisorial lattices over a Krull scheme X. In this paper we show that there are exact sequences of Galois cohomology which relate the class groups and Brauer groups so defined. By choosing $\mathscr{C}(X)$ properly we can obtain as special cases several exact sequences of Galois cohomology derived by others: an exact sequence involving Brauer groups and class groups of noetherian normal domains presented by Rim [18]; one involving Brauer groups and Picard groups of commutative rings considered by Chase and Rosenberg [2], DeMeyer and Ingraham [4], and by Auslander and Brumer in an unpublished paper; an exact sequence of Yuan [23] dealing with class groups and modified Brauer groups of a noetherian normal domain; and a sequence of Lichtenbaum [13] involving divisor class groups and Brauer groups of a projective curve.

The exact sequence we shall obtain (see Theorem 4.1) contains the seven terms:

$$1 \to H^1(G, \mathcal{O}_Y(Y)^*) \to Cl(\mathscr{C}(Y)) \to H^0(G, Cl(\mathscr{C}(Y))) \to H^2(G, \mathcal{O}_Y(Y)^*)$$
$$\to Br(\mathscr{C}(Y/X)) \to H^1(G, Cl(\mathscr{C}(Y))) \to H^3(G, \mathcal{O}_Y(Y)^*),$$

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where G is a finite group and $\pi\colon Y \to X$ is a $\mathscr{C}$-Galois covering of Krull schemes with group G. This condition on π (see the definition preceding Theorem 3.2) coincides with being an ordinary Galois covering when, for a general Krull scheme W, the category $\mathscr{C}(W)$ coincides with $\mathscr{P}(W)$, the category of locally free $\mathcal{O}_W$-modules of finite type (see Proposition 4.2). But for other categories of interest the condition on π may be less restrictive than being a Galois covering: when $\mathscr{C}(\ ) = \mathscr{D}(\ )$, with $\mathscr{D}(W)$ the category of W-lattices, π is a $\mathscr{C}$-Galois covering if and only if for every affine open set $U = \mathrm{Spec}(R)$ of X, and every point x of U with R_x of Krull dimension one, $S \otimes R_x$ is a Galois extension of R_x with group G, where $\pi^{-1}(U) = \mathrm{Spec}(S)$.

When $\mathscr{C}(\ ) = \mathscr{D}(\ )$ the exact sequence can also be obtained for G a profinite group (see Proposition 4.5), and we shall use this in Section 5 to compute the ordinary Brauer group of some projective varieties.

2. A Cohomology Sequence

The setting in which we will discuss Brauer groups $Br(\mathscr{C}(X))$ and class groups $Cl(\mathscr{C}(X))$ will be that of [12], wherein the constructions of $Br(\mathscr{C}(R))$, $Cl(\mathscr{C}(R))$ given in [15] for R a Krull domain are extended to X a Krull scheme. For the convenience of the reader we will recall the definitions of the basic objects of interest—Krull domains, Krull schemes, divisorial lattices, Krull morphisms. We refer the reader to [12] and [15] for a more detailed discussion. The reader will notice that our definitions of concepts for schemes are straightforward extensions of the affine ones. We have relied on this feature in proving most of our results by giving the proof in the affine case and noting that the extension to schemes follows from the definitions. What is usually left unsaid later, but is worth noting here, is that for our quasi-compact schemes $(X, \mathcal{O}_X)$ and their modules M, the properties in our definitions hold for $\mathcal{O}_X(U)$, $M(U)$ with U an arbitrary open set if and only if they hold for all affine open sets U.

Let R be a domain with field of fractions K and let Z be the set of primes of R having height one. R is a *Krull domain* if:

(i) R_p is a D.V.R. for each p in Z.

(ii) $R = \bigcap_{p \in Z} R_p$.

(iii) Each nonzero element of K is a unit in all but finitely many R_p, p in Z.

(See [5, Chapter I] for other characterizations of a Krull domain.)

Let $(X, \mathcal{O}_X)$ be a quasi-compact, integral scheme with function field K and let Z be the set of points x of X for which $\mathcal{O}_{X,x}$ has Krull dimension one—we shall refer to these as the *height one points* of X. $(X, \mathcal{O}_X)$ is a *Krull scheme* if:

(i) $\mathcal{O}_{X,x}$ is a D.V.R. for each x in Z. We will write v_x for the corresponding valuation.

(ii) If U is an open set in X and f is a nonzero element of K such that $v_x(f) \geq 0$ for all x in $U \cap Z$, then f is in $\mathcal{O}_X(U)$.

(iii) For every nonzero element f in K, $v_x(f) = 0$ for all but finitely many x in Z.

If $X = \mathrm{Spec}(R)$ then X is a Krull scheme if and only if R is a Krull domain.

Throughout the paper $(X, \mathcal{O}_X$ and $(Y, \mathcal{O}_Y)$ will be Krull schemes. We will adopt abbreviated notation of the following sort: We will refer to the scheme X and the X-module M, rather than to the scheme $(X, \mathcal{O}_X)$ and the $\mathcal{O}_X$-module M. We will write $\mathrm{Hom}_X(M, N)$ and $M \otimes_X N$ for the X-modules usually denoted $\mathcal{H}om_{\mathcal{O}_X}(M, N)$ and $M \otimes_{\mathcal{O}_X} N$. If $f\colon Y \to X$ is a morphism of schemes and M is an X-module we will write $Y \otimes_X M$ for the Y-module usually denoted by $\mathcal{O}_Y \otimes f^{-1}(M)$, $\otimes$ being over $f^{-1}(X)$. For x a point of X we will write $\mathcal{O}_{Y,x}$ for $\mathcal{O}_{f^*(\mathcal{O}_Y),x}$, where $f^*(\mathcal{O}_Y)$ is the direct image sheaf of $\mathcal{O}_Y$ under f.

Let R be a Krull domain, K the field of fractions of R, Z the set of height one primes of R. An R-module M is *divisorial* if $M = \bigcap_{p \in Z} M_p$. For $(X, \mathcal{O}_X)$ a Krull scheme, the X-module M is *divisorial* if $M(U) = \bigcap M(U)_p$ for each open set U of X, the intersection being taken over $Z \cap U$, Z the set of height one points of X. An R-module M is an *R-lattice* if M is torsion-free, with $\dim_K(K \otimes_R M)$ finite and $M \subseteq F \subseteq K \otimes_R M$ for some R-module F of finite type (see [5, pp. 8–9] for other characterizations of a lattice). An X-module M is an *X-lattice* if for each open set U of X, $M(U)$ is an $\mathcal{O}_X(U)$-lattice.

It M and N are R-modules, the ordinary tensor product $M \otimes_R N$ need not be divisorial, but

$$M \ \tilde{\otimes}_R N = \bigcap_{p \in Z} (MN)_p$$

is a divisorial R-module, where MN is the image of $M \otimes_R N$ in $K \otimes_R M \otimes_R N$. We will usually write $M \tilde{\otimes} N$ for $M \tilde{\otimes}_R N$. This module $M \tilde{\otimes} N$ satisfies the same mapping properties with respect to divisorial R-modules as $M \otimes_R N$ does with respect to all R-modules (see [15, Proposition 1.3], or [16, Proposition 2] or [23, Lemma 4]). We will use the notation $M \tilde{\otimes}_X N$ or $M \tilde{\otimes} N$ for the corresponding construction in the case of $(X, \mathcal{O}_X)$-modules. This construction is described in [12, Proposition 1.1]. There is a canonical map $j\colon M \otimes_R N \to M \tilde{\otimes} N$, and if M,

N are divisorial R-modules with either of them R-flat, then j is an isomorphism [16, Proposition 2, (e)]. In this case we will view j as an identification, and write $M \otimes_R N = M \tilde{\otimes} N$ or $M \otimes_X N = M \tilde{\otimes} N$.

The next elementary result will be used several times in Section 3, but is naturally stated at this point.

LEMMA 2.1. *Let R be a Krull domain. Let M, N be torsion-free R-modules and let L be a divisorial R-lattice. Let $f\colon M \otimes_R N \to L$ be an R-module homomorphism such that for each p in $Z(R)$ the induced map $f_p\colon M_p \otimes_{R_p} N_p \to L_p$ is an isomorphism. Then f induces a map $\tilde{f}$: $M \tilde{\otimes}_R N \to L$ which is an isomorphism.*

Proof. Because L is divisorial there is an R-module homomorphism $\tilde{f}$ making this diagram commutative

$$\begin{tikzcd} M \otimes_R N \arrow[rr, "j"] \arrow[dr, "f"'] & & M \tilde{\otimes}_R N \arrow[dl, "\tilde{f}"] \\ & L & \end{tikzcd}$$

where j is the canonical map described above. Our hypotheses imply that M_p and N_p are torsion-free R_p-modules and since R_p is a D.V.R. they are flat R_p-modules. The flatness implies that the canonical map $M_p \otimes_{R_p} N_p \to M_p \tilde{\otimes}_{R_p} N_p$ is an isomorphism. Since f_p is an isomorphism, and since $(M \tilde{\otimes}_R N)_p = M_p \tilde{\otimes}_{R_p} N_p$ [16, Proposition 2, (k)], it follows that $\tilde{f}_p\colon (M \tilde{\otimes}_R N)_p \to L_p$ is an isomorphism for all p in $Z(R)$. But $\tilde{f}$ is a map between divisorial R-modules and it follows easily that $\tilde{f}_p$ being an isomorphism for all p implies $\tilde{f}$ is an isomorphism.

For W a Krull scheme let $Z(W)$ denote the set of height one points of W, $\mathscr{D}(W)$ the category of divisorial W-lattices an W-module morphisms. As discussed in [12, Section 3] there does not seem to be a natural way to arrange for an arbitrary morphism of Krull schemes, $\pi\colon Y \to X$, to induce a functor $\mathscr{D}(X) \to \mathscr{D}(Y)$ such that the composite of morphisms gives the composite of functors. But by putting conditions on π one can overcome this lack of functoriality. Let $\pi\colon Y \to X$ be a morphism of Krull schemes (π being our abbreviation for $(\pi, \pi^{\#})\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$). We say π is a *Krull morphism* if the generic point of Y maps to the generic point η of X and $\pi(Z(Y)) \subseteq Z(X) \cup \{\eta\}$. The following result summarizes information which we will need (see [12], especially Section 3, for a more detailed discussion, and refer to our foregoing notational conventions).

PROPOSITION 2.2. *Let $\pi\colon Y \to X$ be a morphism of Krull schemes.*

(a) *π is a Krull morphism if and only if $\pi_*(\mathcal{O}_Y)$ is divisorial as an X-module.*

(b) *If* π *is a Krull morphism then for any divisorial* $\mathcal{O}_X$*-lattice* M *the* $\mathcal{O}_Y$*-module* $Y \tilde{\otimes} M$ *is a divisorial* $\mathcal{O}_Y$*-lattice.* π *induces a functor* $\boldsymbol{\pi}: \mathcal{D}(X) \to \mathcal{D}(Y)$.

For W *a Krull scheme let* $\mathcal{P}(W)$ *denote the category of locally free* W*-modules of finite type. If* M *is an object in* $\mathcal{P}(X)$ *then* $Y \tilde{\otimes} M = Y \otimes_X M$ *and* π *induces a functor* $\boldsymbol{\pi}: \mathcal{P}(X) \to \mathcal{P}(Y)$.

(c) *If* $\theta: W \to Y$ *is another morphism of Krull schemes and* $\phi = \pi\theta$ *then* $\boldsymbol{\phi} = \boldsymbol{\pi} \circ \boldsymbol{\theta}$ *(up to natural equivalence of functors from* $\mathcal{D}(W)$ *to* $\mathcal{D}(X)$*).*

This result, together with properties of divisorial lattices discussed in [12, Section 2 and 3] and [15, Sections 1 and 3] can be seen to imply that if we write $\mathscr{C}(W)$ for $\mathcal{D}(W)$, this gives one way of assigning to each Krull scheme W a subcategory $\mathscr{C}(W)$ of $\mathcal{D}(W)$ such that the following axioms hold:

(A1) For X a Krull scheme, X is an object in $\mathscr{C}(X)$, and $\mathscr{C}(X)$ is a subcategory of $\mathcal{D}(X)$.

(A2) If M and N are objects in $\mathscr{C}(X)$ then so are $M \oplus N$, $M \tilde{\otimes} N$ and $\mathrm{Hom}_X(M, N)$.

(A3) If M and $M \tilde{\otimes} N$ are in $\mathscr{C}(X)$ and N is in $\mathcal{D}(X)$ then N is in $\mathscr{C}(X)$.

(A4) If $\pi: Y \to X$ is a Krull morphism of Krull schemes then for any object M in $\mathscr{C}(X)$, $Y \tilde{\otimes} M$ is in $\mathscr{C}(Y)$.

In [12] axioms (A1) and (A2) are shown to be sufficient to permit the construction of a Brauer group $Br(\mathscr{C}(X))$ and a class group $Cl(\mathscr{C}(X))$. Axiom (A4) is shown to imply that the assignments $X \to Br(\mathscr{C}(X))$, $X \to Cl(\mathscr{C}(X))$ define functors from the category of Krull schemes and Krull morphisms to the category of abelian groups. Axiom (A3) occurs in a more general form in [12], involving a pair of categories $\mathscr{C}_1(X) \subseteq \mathscr{C}_2(X)$, rather than $\mathscr{C}(X) \subseteq \mathcal{D}(X)$, and is used there to get an exact sequence linking the groups $Br(\mathscr{C}_i(X))$, $Cl(\mathscr{C}_i(X))$, $i = 1, 2$. In the current paper (A3) will be crucial for establishing our cohomological exact sequence.

Let $\mathscr{C}\ell(\mathscr{C}(X))$ be the subcategory of $\mathscr{C}(X)$ consisting of objects which have rank one as X-modules. Since rank is multiplicative with respect to $\tilde{\otimes}$, $\mathscr{C}\ell(\mathscr{C}(X))$ is closed under the operation sending (M, N) to $M \tilde{\otimes} N$. The operation $\tilde{\otimes}$ is commutative and associative, meaning that for L, M and N any X-modules, there exist canonical isomorphisms

$$M \tilde{\otimes} N \to N \tilde{\otimes} M, \qquad L \tilde{\otimes} (M \tilde{\otimes} N) \to (L \tilde{\otimes} M) \tilde{\otimes} N$$

which arise from the maps (in the affine case and for ordinary tensor products) sending $x \otimes y$ to $y \otimes x$ and $x \otimes (y \otimes z)$ to $(x \otimes y) \otimes z$. For M in

$\mathscr{C}\ell(\mathscr{C}(X))$, the dual module $\mathrm{Hom}_X(M, X)$ is again in $\mathscr{C}\ell(\mathscr{C}(X))$. This module is usually denoted M^* for an arbitrary module M, but for M in $\mathscr{C}\ell(\mathscr{C}(X))$ we shall also write M^{-1} for M^*. There is a canonical isomorphism

$$M \tilde{\otimes} M^{-1} \to \mathcal{O}_X$$

which arises from the map (in the affine case and for ordinary tensor products) sending $x \otimes h$ in $M \otimes \mathrm{Hom}_R(M, R)$ to $h(X)$. We shall write

$$\mathbf{X}: \mathscr{C}\ell(\mathscr{C}(X)) \to \mathscr{C}\ell(\mathscr{C}(X)),$$

$$(\ )^{-1}: \mathscr{C}\ell(\mathscr{C}(X)) \to \mathscr{C}\ell(\mathscr{C}(X))$$

for the functors which send an object M in $\mathscr{C}\ell(\mathscr{C}(X))$ to $\mathcal{O}_X$ and M^{-1}, respectively, and an isomorphism $j: M \to N$ to Id_X and $(\ ) \circ j^{-1}$, respectively.

The automorphism groups of objects of $\mathscr{C}\ell(\mathscr{C}(X))$ play a role in the cohomology referred to below, so some remarks about them are in order. For any divisorial lattice M there is a canonical isomorphism from $M \tilde{\otimes} M^*$ to $\mathrm{End}_X(M)$, related to the map which in the affine case and for ordinary tensor products sends $x \otimes h$ to $h(\ )x$. When M has rank one it follows that $\mathrm{End}_X(M)$ is canonically isomorphic of $\mathcal{O}_X$, and $\mathrm{Aut}_X(M)$ is canonically isomorphic to the sheaf $\mathcal{O}_X(\ )^*$ of units of $\mathcal{O}_X$.

We now change notation slightly so that we are working with a Krull scheme Y, rather than X. Let G be a group acting on Y as automorphisms of schemes, the action being faithful. Each σ in G gives rise to a functor $\boldsymbol{\sigma}$ from $\mathscr{C}\ell(\mathscr{C}(Y))$ to $\mathscr{C}\ell(\mathscr{C}(Y))$, which is a category equivalence. The basic description of this functor is in the affine case, where $Y = \mathrm{Spec}(S)$, so that G acts as ring automorphisms of S. An element of $\mathscr{C}\ell(C(S))$ is represented by a rank one S-module M in $\mathscr{C}(S)$. For such an M let $\boldsymbol{\sigma}(M)$ be the rank one S-module whose underlying set is M but with S-action given by

$$s \cdot x = \sigma(s)x, \qquad s \in S, x \in M. \tag{1}$$

For h a morphism in $\mathscr{C}(S)$, $\boldsymbol{\sigma}(h)$ is the set map h itself. For Y not necessarily affine we can set $\boldsymbol{\sigma}(M)$ to be the direct image sheaf of M with respect to σ^{-1}. This description, $\boldsymbol{\sigma}(M) = \sigma_*^{-1}(M)$, is consistent with the affine one.

Now let $Cl(\mathscr{C}(Y))$ be the set of isomorphism classes $\{M\}$ of objects M in $\mathscr{C}\ell(\mathscr{C}(Y))$. The operation $\{M\}\{N\} = \{M \tilde{\otimes} N\}$ makes $Cl(\mathscr{C}(Y))$ into an abelian group. The inverse of $\{M\}$ is $\{M^{-1}\}$. For σ in G, the functor $\boldsymbol{\sigma}$ on $\mathscr{C}\ell(\mathscr{C}(Y))$ induces a G-action on $Cl(\mathscr{C}(Y))$.

Having noted these basic facts we can now turn to their cohomological consequences as developed by Ulbrich [21]. We can apply Section 3 of that paper in our setting, substituting $\mathscr{C}\ell(\mathscr{C}(Y))$ for $\mathscr{P}ic(S)$ and making the necessary allowances for differences in notation. Ulbrich's analysis (or the analysis of Hattori [8] on which Ulbrich's is based) applies mutatis mutandis, and yields that there is an exact sequence

$$\begin{aligned} 1 &\to H^1(G, \mathcal{O}_Y(Y)^*) \to \mathbf{H}^0 \to H^0(G, Cl(\mathscr{C}(Y))) \to H^2(G, \mathcal{O}_Y(Y)^*) \\ &\to \mathbf{H}^1 \to H^1(G, Cl(\mathscr{C}(Y))) \to H^3(G, \mathcal{O}_Y(Y)^*) \to \mathbf{H}^2 \to \cdots. \end{aligned} \tag{2}$$

The groups of the form $H^i(G, A)$ refer to group cohomology of G acting on A. The groups $\mathbf{H}^i$ are defined in [21] following a model presented in [8]. In the next section we shall examine $\mathbf{H}^0$ and $\mathbf{H}^1$ in more detail.

3. Analysis of $\mathbf{H}^0$ and $\mathbf{H}^1$

Let Y continue to denote a Krull scheme, $\mathscr{C}(Y)$ a category which depends on Y in the manner circumscribed by axioms (A1) to (A4) of Section 2. Let $Z(Y)$ denote the set of height one points of Y. Let G be a finite groups of automorphisms of the scheme Y over the Krull scheme X. We will restrict ourselves first to the affine case in order to make the definitions which will be the basis for the non-affine version of the concepts introduced. To this end we will assume $Y = \operatorname{Spec}(S)$, $X = \operatorname{Spec}(R)$, so that G is a finite group of R-algebra automorphisms of S and both R and S are Krull domains. We will write $\mathscr{C}(R)$ for $\mathscr{C}(X)$, etc.

Let $D(S, G)$ denote the trivial crossed product of S with G. This means $D(S, G)$ is $\bigoplus_\sigma Su_\sigma$ (σ varies over G) as an S-module and is a ring in which $(su_\sigma)(tu_\tau) = s\sigma(t)\, u_{\sigma\tau}$. Let $j\colon D(S, G) \to \operatorname{End}_R(S)$ be determined by $j(su_\sigma)(t) = s\sigma(t)$. We will say S is a *$\mathscr{C}$-Galois extension* of R with group G if three conditions hold:

(1) j is an isomorphism.

(2) If M is in $\mathscr{C}(S)$ then M is in $\mathscr{C}(R)$; in particular S is in $\mathscr{C}(R)$.

LEMMA 3.1. *Let $R \subseteq S$ be Krull domains and assume S is a $\mathscr{C}$-Galois extension of R with group G. Then*:

(a) *$R \to S$ is a Krull morphism.*

(b) $S^G = R$.

(c) *Let M be a $D(S, G)$-module in $\mathscr{D}(S)$. Let M^G be the set of elements of M fixed under the action of each σ in G. Then there is a natural isomorphism $S \tilde{\otimes} M^G \to M$ ($\tilde{\otimes}$ is the product referred to in Section 2). If M is in $\mathscr{C}(S)$ then M^G is in $\mathscr{C}(R)$.*

Proof. (a) Condition (2) implies that S is a divisorial R-module, since $\mathscr{C}(R)\subseteq\mathscr{D}(R)$ (axiom (A1) of Section 2). Hence the inclusion of R in S satisfies condition (PDE) (i.e., the contraction to R of a height one prime of S has height at most one) [15, Theorem 1]. Thus is the definition of $R\to S$ being a Krull morphism.

(b) Let p be any height one prime of R. Write A for R_p, B for S_p (by which we mean the ring of fractions $S\otimes_R R_p$ with respect to the multiplicative set $R-p$). Conditions (1) and (2) imply that B is a free A-module (A is a discrete valuation ring) and that $j\colon D(B,G)\to \mathrm{End}_A(B)$ is an isomorphism. Because B is A-free, there is an A-homomorphism $t\colon B\to A$ such that $t(b)=b$ for b in A. Since j is an isomorphism, $t=\sum b_\sigma$ for some b_σ in B. Then $\sum b_\sigma=1$ and for b in B^G, $b=\sum b_\sigma\sigma(b)=t(b)$, so b is in A. Thus $B^G=A$. Thus, for Z the set of height one primes of R,

$$R\subseteq S^G\subseteq \bigcap_{p\in Z} S_p^G=\bigcap_{p\in Z} R_p=R,$$

the last equality being valid because R is a Krull domain.

(c) There is a natural map $f\colon S\otimes_R M^G\to M$ satisfying $f(s\otimes x)=sx$. It is easy to see that for p any height one prime of R we have $(M^G)_p=(M_p)^G$. It follows that f_p is an isomorphism. This is true because R_p is a discrete valuation ring; hence S_p is a Galois extension of R_p with Galois group G, in the sense of [2]. We would like to apply Lemma 2.1 to deduce the existence of the desired isomorphism $S\mathbin{\tilde{\otimes}} M^G\to M$. To apply this lemma we need to know M is a divisorial R-lattice. But M is a divisorial S-lattice (it is in $\mathscr{D}(S)$) and S is a divisorial R-lattice (condition (2) plus the assumed inclusion $\mathscr{C}(R)\subseteq\mathscr{D}(R)$) and this implies M is a divisorial R-lattice [15, Corollary 1 to Theorem 1].

For a covering of Krull schemes, $\pi\colon Y\to X$, in which X and Y are not necessarily affine and on which G acts as a group of automorphisms, we say π is a *$\mathscr{C}$-Galois covering* with group G if π is an affine map (i.e., $\pi^{-1}(U)$ is affine for U an affine open set of X) and for each open set U in X, $\mathscr{O}_Y(\pi^{-1}(U))$ is a $\mathscr{C}$-Galois extension of $\mathscr{O}_X(U)$ with group G. The assertions of Lemma 3.1 have corresponding versions in this situation, which we shall not state explicitly.

THEOREM 3.2. *Let X and Y be Krull schemes and $\pi\colon Y\to X$ a $\mathscr{C}$-Galois covering with group G. Then $\mathbf{H}^0\simeq Cl(\mathscr{C}(X))$.*

Proof. We will work with Hattori's formulation [8] for the cohomology group involved in the case where $\mathscr{P}ic(S)$ replaces $\mathscr{C}\ell(\mathscr{C}(Y))$. We note that in Hattori's paper $\mathbf{H}^0$ is called $\mathbf{H}^1(S,G)$. We shall adjust Hattori's indexing so it conforms to ours, which is that used by Ulbrich [21].

Hence $\mathbf{Z}^2(S/R)$ will become $\mathbf{Z}^1$, etc. In Section 5 of his paper Hattori sketches the construction of an isomorphism from $\mathbf{H}^0$ to $\mathrm{Pic}(R)$. A careful reading of the proof shows that his construction can be adapted to our setting. We shall outline below the additional facts that are needed. For the sake of simplicity, and to keep our notation parallel to that in [8] we shall reduce to the affine case, and work with R and S rather than X and Y.

Let $((P, p))$ be a cocycle in $\mathbf{Z}^0$. Then P is an object in $\mathscr{C}\ell(\mathscr{C}(S))$ and p consists of a family of isomorphisms $p_\sigma\colon {}^\sigma P \tilde{\otimes} P^{-1} \to S$, with ${}^\sigma P$ defined by (1) of Section 2. Thus, for each σ in G we get an isomorphism $q_\sigma\colon {}^\sigma P \to P$. Let each σ in G act on P by $\sigma x = q_\sigma(x)$. The cocycle condition on $((P, p))$ implies that $\sigma(\tau x) = (\sigma\tau)x$, hence that P is a $D(S, G)$-module. By (c) of Lemma 3.1, there is an isomorphism $S \tilde{\otimes} P^G \to P$, and P^G is in $\mathscr{C}\ell(\mathscr{C}(R))$. The correspondence sending the cohomology class of $((P, p))$ to the isomorphism class of P^G in $Cl(\mathscr{C}(R))$ yields an isomorphism of $\mathbf{H}^0$ with $Cl(\mathscr{C}(R))$. We refer the reader to [8] for verification of the details. This completes the proof of Theorem 3.2.

Our analysis of $\mathbf{H}^1$ also follows the model provided in [8, Section 5], which is also related to work of Kanzaki [10]. But the interpretation of $\mathbf{H}^1$ is more cumbersome than that of $\mathbf{H}^0$ and we shall have to indicate why the work of Hattori and Kanzaki can be carried over to our setting.

Hattori demonstrates that a group he calls $\mathbf{H}^2(S, G)$ or $\mathbf{H}^2(S/R)$ is isomorphic to the relative Brauer group $Br(S/R)$ for S/R a Galois extension with group G. The proof proceeds by establishing a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & Pr(S/R) & \to & \hat{B}r(S/R) & \to & Br(S/R) & \to & 1, \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 1 & \to & \mathbf{B}^2(S/R) & \to & \mathbf{Z}^2(S/R) & \to & \mathbf{H}^2(S/R) & \to & 1 \end{array} \tag{3}$$

with exact rows. We shall re-examine these correspondences in our setting.

First we indicate the changes we need from Hattori's notation. $Br(R)$ will be replaced by $Br(\mathscr{C}(R))$, $Br(S/R)$ by $Br(\mathscr{C}(S/R))$, $B^2(S/R)$ by B^1, $Z^2(S/R)$ by Z^1, $H^2(S/R)$ by H^1, $Pr(S/R)$ by $Pr(\mathscr{C}(S/R))$. We will explain what these objects are as we go along.

$Br(\mathscr{C}(R))$ is the Brauer group of equivalence classes $[A]$ of $\mathscr{C}[R]$-Azymaya algebras A [12, 15]. Such algebras A are characterized by being in $\mathscr{C}(R)$ as R-modules, having R as center and having the natural map

$$\eta_A\colon A \tilde{\otimes} A^0 \to \mathrm{End}_R(A)$$

be an isomorphism. The equivalence relation is defined by setting $A \sim B$ if

$$A \tilde{\otimes} \mathrm{End}_R(P) \simeq B \tilde{\otimes} \mathrm{End}_R(Q)$$

with P and Q in $\mathscr{C}(R)$. For $R \subseteq S$ a Krull morphism (it is understood of Krull domains) there is a map from $Br(\mathscr{C}(R))$ to $Br(\mathscr{C}(S))$ induced by sending $[A]$ to $[S \tilde{\otimes} A]$. The kernel of this map is denoted $Br(\mathscr{C}(S/R))$.

Since the usual Brauer group arises when $\mathscr{C}(R)$ is the category of projective R-modules of finite type we shall refer to this situation as the projective case. Since any projective module (or even any flat one) is divisorial [16, Proposition 2] this classical setting is one in which our hypotheses apply. It will be useful to remember that in this case the modified tensor product $\tilde{\otimes}$ agrees with the ordinary tensor product $\otimes_R$ for objects in $\mathscr{C}(R)$ [16, Proposition 2, (e)]. In the projective case $\mathscr{C}$-Galois extensions coincide with Galois extensions in the sense of [2]. The projective case of the next result plays a crucial role in Hattori's proof that $H^2(S/R) \cong Br(S/R)$.

PROPOSITION 3.3. *Let $R \subseteq S$ be Krull domains, with S a $\mathscr{C}$-Galois extension of R. A $\mathscr{C}(R)$-Azumaya algebra A represents an element in $Br(\mathscr{C}(S/R))$ if and only if $A \sim B$, with B a $\mathscr{C}(R)$-Azumaya algebra containing S as a maximal commutative subalgebra.*

Proof. This argument parallels the one in the projective case (see [4, Theorem 5.5, p. 64]). Let A be in $Br(\mathscr{C}(S/R))$ and identify $S \tilde{\otimes} A^0$ with $\mathrm{End}_S(E)$, E in $\mathscr{C}(S)$; the trivial element of $Br(\mathscr{C}(S))$ is of this form by the concluding remark in the proof of Theorem 3.1 of [15]. By condition (2) for $\mathscr{C}$-Galois extensions, E is in $\mathscr{C}(R)$, hence $C = \mathrm{End}_R(E)$ is a $\mathscr{C}(R)$-Azumaya algebra, and we have

$$S \tilde{\otimes} A^0 = \mathrm{End}_S(E) \subseteq C.$$

Let $B = C^{A^0}$. The multiplication map $B \otimes_R A^0 \to C$ induces a map from $B \tilde{\otimes} A^0$ to C which is an isomorphism in the projective case, hence an isomorphism in our setting too, by Lemma 2.1. The same sort of argument, plus the use of axiom (A3), shows that B is a $\mathscr{C}(R)$-Azumaya algebra, that $[B] = [A]$ in $Br(\mathscr{C}(R))$ and that S is a maximal commutative subalgebra of B.

Conversely, if B is a $\mathscr{C}(R)$-Azumaya algebra containing S as a maximal commutative subalgebra, then we define a map from $S \otimes_R A_0$ to $\mathrm{End}_S(A)$ by sending $s \otimes a^0$ to $s(\ )a$. In the projective case this map is an isomorphism, hence in our setting we have an isomorphism of $S \otimes A^0$ with $\mathrm{End}_S(A)$ by the now usual localization argument. This completes our proof.

Following [8] we will call a pair (A, α) a $\mathscr{C}(S/R)$-Azumaya algebra when A is a $\mathscr{C}(R)$-Azumaya algebra and α: $S \to A$ is an embedding of S as a maximal commutative subalgebra of A. The S-isomorphism classes of such pairs form a group $\hat{Br}(\mathscr{C}(S/R))$. The product is given by $(A, \alpha) * (B, \beta) = (C, \gamma)$, where $C = e(A \tilde{\otimes} B)e$. In the projective case e is the separability idempotent of $S \otimes_R S$—it can be described as $h^{-1}(e_1)$, where h is the isomorphism from $S \otimes_R S$ to $\mathrm{Maps}(G, S)$ given by $h(s \otimes t)(\sigma) = s\sigma(t)$, and $e_1(\sigma) = \delta_{\sigma,1}$ (the Kronecker delta). In our setting we have a corresponding e in $S \tilde{\otimes} S$, again by a localization argument using Lemma 2.1—e is now $\tilde{h}^{-1}(e_1)$. The natural map from $S \otimes_R S$ to $A \otimes_R B$ induces one from $S \tilde{\otimes} S$ to $A \tilde{\otimes} B$, so the expression $e(A \tilde{\otimes} B)e$ makes sense.

There is a surjective map from $\hat{Br}(\mathscr{C}(S/R))$ to $Br(\mathscr{C}(S/R))$ defined by sending the isomorphism class of (A, α) to $[A]$ (see Proposition 3.3). The kernel of this map is called $Pr(\mathscr{C}(S/R))$, and consists of classes of pairs $(\mathrm{End}_R(E), \alpha)$ with E in $\mathscr{C}(R)$. The embedding $\alpha: S \to \mathrm{End}_R(E)$ makes E into an S-module, and because S is a $\mathscr{C}$-Galois extension, E is in $\mathscr{C}(S)$. A rank argument shows E is in $\mathscr{C}\ell(\mathscr{C}(S))$. So we have an exact sequence

$$1 \to Pr(\mathscr{C}(S/R)) \to \hat{Br}(\mathscr{C}(S/R)) \to Br(\mathscr{C}(S/R)) \to 1. \tag{4}$$

The isomorphism $\mathbf{H}^1 \simeq Br(\mathscr{C}(S/R))$ which we are after is obtained by showing that $\mathbf{Z}^1 \simeq \hat{Br}(\mathscr{C}(S/R))$ under an isomorphism which carries $\mathbf{B}^1$ to $Pr(\mathscr{C}(S/R))$.

To associate an element of $\hat{Br}(\mathscr{C}(S/R))$ to a cocycle $((P, p))$ in $\mathbf{Z}^1$ we can use exactly the same construction Hattori uses, namely the crossed product algebra. If $((P, p))$ is in $\mathbf{Z}^1$, then for each σ in G we have that $P(\sigma)$ is an object in $\mathscr{C}\ell(\mathscr{C}(S))$. Let $S_\sigma = \{s_\sigma \mid s \text{ in } S\}$, with left S-action $ts_\sigma = (ts)_\sigma$ and right S-action $s_\sigma t = (s\sigma(t))_\sigma$. Define $J_\sigma = P(\sigma) \tilde{\otimes} S_\sigma$, where $\tilde{\otimes}$ denotes the modified tensor product associated to $\otimes_S$, i.e., the operation in $\mathscr{C}(S)$. The cocycle condition on $((P, p))$ implies that there are (S, S)-bimodule isomorphisms $j_{\sigma,\tau}: J_\sigma \tilde{\otimes}_S J_\tau \to J_{\sigma\tau}$ such that the definition $\Delta(J, j) = \bigoplus_\sigma J_\sigma$ yields an associative algebra. As in the projective case this crossed product algebra is a $\mathscr{C}(R)$-Azumaya algebra containing S as a maximal commutative subring—the fact that S is $\mathscr{C}$-Galois is used to verify this, just as in the projective case S being an ordinary Galois extension is used. Thus the S-isomorphism class of $\Delta(J, j)$ is an element of $\hat{Br}(\mathscr{C}(S/R))$. The correspondence $((P, p)) \to \Delta(J, j)$ gives an isomorphism of $\mathbf{Z}^1$ with $\hat{Br}(\mathscr{C}(S/R))$. Verifying the details used in this assertion can be done by the methods used in the projective case plus the sort of arguments we have employed in the foregoing discussion. Similarly, $\mathbf{B}^1$ corresponds bijectively to $Pr(\mathscr{C}(S/R))$ under the isomorphism sending $((P, p))$ to $\Delta(J, j)$. This establishes that there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & Pr(\mathscr{C}(S/R)) & \to & \hat{Br}(\mathscr{C}(S/R)) & \to & Br(\mathscr{C}(S/R)) & \to & 1 \\
 & & \wr\wr & & \wr\wr & & \wr\wr & & \\
1 & \to & \boldsymbol{B}_1 & \to & \mathbf{Z}^1 & \to & \mathbf{H}^1 & \to & 1
\end{array}$$

with exact rows. This affine argument can be extended to Krull schemes by using the assumption that a $\mathscr{C}$-Galois covering $\pi: Y \to X$ is an affine map. Hence we have the next result.

THEOREM 3.4. *Let X and Y be Krull schemes and $\pi: Y \to X$ a $\mathscr{C}$-Galois covering with group G. Then $\mathbf{H}^1 \simeq Br(\mathscr{C}(Y/X))$.*

We would like to thank the referee for some observations relating to other generalizations of the Brauer group and to their use in extending our analysis. Fröhlich and Wall's work on equivariant Brauer groups and class groups [6, 7] is related to that of Hattori [8, 9], some of which we exploited to obtain our exact sequence (2) of Section 2. The connection of Hattori's work with that of Fröhlich and Wall is noticeable by a reading of Section 8 of [7] and Sections 2 and 3 of [9]. One might hope that in a setting more general than our Galois coverings an interpretation of the groups $\mathbf{H}^i$, $i = 0, 1$, might be given by building on and extending the notions considered in the affine case by Fröhlich and Wall. Such an interpretation does not seem to be presently available. For an illuminating overview of various general approaches to the exact sequence (2) of Section 2 the reader is referred to the papers of Takeuchi [19], Takeuchi and Ulbrich [20] and Ulbrich [21]. As a more concrete background motivating these approaches one may consult the papers of Chase and Rosenberg [2], Childs [3] and Villamayor and Zelinsky [22].

4. The Exact Sequence and Special Cases

The hypotheses on $\mathscr{C}$, and our notational conventions, are those of the previous section. In particular, G is at the outset a finite group.

THEOREM 4.1. *Let X and Y be Krull schemes. Let $\pi: Y \to X$ be a $\mathscr{C}$-Galois covering with finite group G. Then there is an exact sequence*

$$\begin{aligned} 1 &\to H^1(G, \mathcal{O}_Y(Y)^*) \to Cl(\mathscr{C}(X)) \to Cl(\mathscr{C}(Y))^G \to H^2(G, \mathcal{O}_Y(Y)^*) \\ &\to Br(\mathscr{C}(Y/X)) \to H^1(G, Cl(\mathscr{C}(Y))) \to H^3(G, \mathcal{O}_Y(Y)^*). \end{aligned} \tag{5}$$

Proof. This is an immediate consequence of the exact sequence (2) of Section 2, used together with Theorems 3.2 and 3.4 and with the relation $H^0(G, A) = A^G$.

We will be concerned with applying Theorem 4.1 in the cases where $\mathscr{C}(X) = \mathscr{D}(X)$, the category of divisorial X-lattices, and $\mathscr{C}(X) = \mathscr{P}(X)$, the category of projective X-lattices (which are the locally free $\mathcal{O}_X$-modules of finite type, since in the affine case a projective lattice is automatically finitely generated [16, Corollary to Proposition 2]). It will be useful to have a characterization of $\mathscr{C}$-Galois extensions for these categories.

PROPOSITION 4.2. *Let $\pi\colon Y \to X$ be an affine Krull morphism of Krull schemes, G a finite group of automorphisms of Y.*

(a) *Each of the following conditions is equivalent to π being a $\mathscr{P}$-Galois covering with group G:*

(i) *For each affine open set $U = \mathrm{Spec}(R)$ of X and $\pi^{-1}(U) = \mathrm{Spec}(S)$, S is a Galois extension of R with group G (in the usual sense, as used in* [1, 2, 4, 8, 10, 21], *for example).*

(ii) *For each x in X, $\mathcal{O}_{Y,x}$ is a Galois extension of $\mathcal{O}_{X,x}$ with group G (we are writing $\mathcal{O}_{Y,x}$ for $\mathcal{O}_{\pi_*(\mathcal{O}_Y),x}$).*

(b) *π is a $\mathscr{D}$-Galois covering with group G if and only if for each height one point x in X, $\mathcal{O}_{Y,x}$ is a Galois extension of $\mathcal{O}_{X,x}$ with group G.*

Proof. (a) Follows easily from the definitions and from the well-known local criteria for separability and projectivity (see [4, Theorem 7.1], for example).

(b) Suppose π is a $\mathscr{D}$-Galois covering with group G. For U an affine open set in X let $U = \mathrm{Spec}(R)$ and let $\pi^{-1}(U) = \mathrm{Spec}(S)$. Then S is a $\mathscr{D}(R)$-Galois extension of R with group G. Thus $j\colon D(S, G) \to \mathrm{End}_R(S)$ is an isomorphism and S is a divisorial R-lattice. Hence for x in U we have that j_x is an isomorphism and that S_x is a divisorial R_x-lattice (this last assertion holds by [16, Proposition 3 and (e) of Proposition 2]). But if x is a height one point, R_x is a discrete valuation ring, hence S_x is then a projective R_x-module of finite type. Hence $\mathcal{O}_{Y,x}$ (which is S_x, or equivalently, $S \otimes R_x$ or $S \otimes \mathcal{O}_{X,x}$) is a Galois extension of $\mathcal{O}_{X,x}$ with group G.

Conversely, suppose that $\mathcal{O}_{Y,x}$ is a Galois extension of $\mathcal{O}_{X,x}$ with group G for every x of height one in X. We need to show that for R and S as above, S is a $\mathscr{D}(R)$-Galois extension of R with group G, i.e., conditions (1) and (2) given prior to Lemma 3.1 hold. But S is divisorial as an R-module since π being a Krull morphism implies $\pi_*(\mathcal{O}_Y)$ is divisorial as an $\mathcal{O}_X$-module [12, Proposition 3.1]. Also $j\colon D(S, G) \to \mathrm{End}_R(S)$ is an isomorphism since each j_x is an isomorphism for x of height one [15, Lemma 1.1]. Thus (1) holds. Condition (2) holds by the result that for S a divisorial R-module, any divisorial R-module is divisorial over R [16, Corollary 1 to Theorem 1]. This completes the proof.

Rim [18] proved result (b) below under the more restrictive hypotheses that $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ with R and S noetherian normal domains and S a Galois extension of R.

THEOREM 4.3. *Let $\pi\colon Y \to X$ be an affine Krull morphism of Krull schemes, G a finite group of automorphisms of Y.*

(a) *Suppose π is a Galois covering with group G (by which we mean that π is a $\mathscr{P}$-Galois covering with group G). Then there is an exact sequence*

$$1 \to H^1(G, \mathcal{O}_Y(Y)^*) \to Pic(X) \to Pic(Y)^G \to H^2(G, \mathcal{O}_Y(Y)^*)$$
$$\to Br(Y/X) \to H^1(G, Pic(Y)) \to H^3(G, \mathcal{O}_Y(Y)^*).$$

(b) *Suppose that for each height one point x of X, $\mathcal{O}_{Y,x}$ is a Galois extension of $\mathcal{O}_{X,x}$ with group G. Then there is an exact sequence*

$$1 \to H^1(G, \mathcal{O}_Y(Y)^*) \to Cl(X) \to Cl(Y)^G \to H^2(G, \mathcal{O}_Y(Y)^*)$$
$$\to \bigcap_{x \in Z} Br(\mathcal{O}_{Y,x}/\mathcal{O}_{X,x}) \to H^1(G, Cl(Y)) \to H^3(G, \mathcal{O}_Y(Y)^*), \tag{6}$$

where Z is the set of height one points of X.

Proof. We apply Theorem 4.1 with $\mathscr{C} = \mathscr{P}$ for (a) and $\mathscr{C} = \mathscr{D}$ for (b), keeping Proposition 4.2 in mind. Then (a) follows at once by noting that $Cl(\mathscr{P}(X)) = Pic(X)$, the group of invertible $\mathcal{O}_X$-modules. In (b) we have that $Cl(\mathscr{D}(X)) = Cl(X)$, where $Cl(X)$ is defined as $\mathrm{Div}(X)/\mathrm{Prin}(X)$. $\mathrm{Div}(X)$ is the group of divisors on X, i.e., the free abelian group on the height one points of X and $\mathrm{Prin}(X)$ is the subgroup of principal divisors $\mathrm{div}(x)$ arising from the elements x in the generic stalk K of X. For details see [5]. With these remarks almost all of sequence (6) is a direct transcription of sequence (5), provided we can write

$$Br(\mathscr{D}(Y/X)) = \bigcap_{x \in Z} Br(\mathcal{O}_{Y,x}/\mathcal{O}_{X,x}), \tag{7}$$

where the Brauer groups on the right are the ordinary ones. The group $Br(\mathscr{D}(X))$ was denoted as $\beta(X)$ in [12], where it was shown that

$$Br(\mathscr{D}(X)) = \bigcap_{x \in Z} Br(\mathcal{O}_{X,x}); \tag{8}$$

$Br(\mathcal{O}_{X,x})$ embeds into $Br(K)$, where K is the generic stalk of X, since $\mathcal{O}_{X,x}$ is a regular domain with field of fractions K, and the intersection in (8) is understood to occur inside $Br(K)$. From (8) it is not difficult to derive (7): If $[A]$ is in $Br(\mathscr{D}(Y/X))$ then $[A_x]$ is in $Br(\mathcal{O}_{X,x})$ for each x in Z, by (8). But then for L the generic stalk of Y, $[K \otimes A_y] = 1$ in $Br(L)$. Since $\mathcal{O}_{Y,x}$ is a Galois extension of $\mathcal{O}_{X,x}$, and the latter is regular, $\mathcal{O}_{Y,x}$ is regular too. Hence $[A_y] = 1$, so that $[A]$ is in $Br(\mathcal{O}_{Y,x}/\mathcal{O}_{X,x})$ for all x in Z. The reverse inclusion is established similarly.

We shall now extend our considerations to profinite groups G. Let $R \subseteq S$ be a divisorial morphism of Krull domains, and G a profinite group of R-algebra automorphisms of S. Let $G = \varprojlim G_\alpha$ be the representation of G as the inverse limit of its finite quotient groups, and let I denote the indexing set of finite quotients G_α of G. Let $\theta_\alpha : G \to G_\alpha$, $\theta_{\alpha,\beta} : G_\beta \to G_\alpha$ be the projections associated with this representation, for α, β in I and $\alpha \leqslant \beta$. Write H_α for $\mathrm{Ker}(\theta_\alpha)$ and $H_{\alpha,\beta}$ for $\mathrm{Ker}(\theta_{\alpha,\beta})$. Let S_α be the set of elements of S fixed by all elements of H_α. We shall say that S is a *$\mathscr{C}$-Galois extension* of R with group G if the following conditions hold:

1. S_α is a Krull domain for each α in I.
2. $S = \varinjlim S_\alpha$, i.e., $S = \bigcup_\alpha S_\alpha$.
3. S_α is a $\mathscr{C}$-Galois extension of R with group G_α.
4. For $\alpha \leqslant \beta$, S_β is a $\mathscr{C}$-Galois extension of S_α with group $H_{\alpha,\beta}$.

For $\pi : Y \to X$ a morphism of Krull schemes, suppose G is a group of automorphisms of Y over X. We shall say π is a *$\mathscr{C}$-Galois covering* with group G if π is an affine morphism and for each open set U in X, $\mathscr{O}_Y(\pi^{-1}(U))$ is a $\mathscr{C}$-Galois extension of $\mathscr{O}_X(U)$ with group G. In this case we can represent Y as $\varprojlim Y_\alpha$ with Y_α a Krull scheme, and we will write $\pi_\alpha : Y \to Y_\alpha$, $\pi_{\alpha,\beta} : Y_\beta \to Y_\alpha$ for the morphisms which arise.

In the situation just described the morphisms π_α and $\pi_{\alpha,\beta}$ are Krull morphisms, by (c) of Lemma 3.1. They therefore induce maps $Cl(\mathscr{C}(Y_\alpha)) \to Cl(\mathscr{C}(Y))$, $Cl(\mathscr{C}(Y_\alpha)) \to Cl(\mathscr{C}(Y_\beta))$, and corresponding ones for the Brauer groups. We shall now consider the question of whether $Cl(\mathscr{C}(Y)) = \varinjlim Cl(\mathscr{C}(Y_\alpha))$ and $Br(\mathscr{C}(Y)) = \varinjlim Br(\mathscr{C}(Y_\alpha))$ with respect to the maps so induced.

LEMMA 4.4. *Let I be a directed set, R_i a Krull domain for each i in I, $f_{i,j} : R_i \to R_j$ a Krull morphism for each $i \leqslant j$. Let $R = \varinjlim R_i$ and assume that for each i in I, $f_i : R_i \subseteq R$ is a Krull morphism of Krull domains. Then*

$$Cl(R) = \varinjlim Cl(R_i), \qquad Pic(R) = \varinjlim Pic(R_i),$$
$$\beta(R) = \varinjlim \beta(R_i), \qquad Br(R) = \varinjlim Br(R_i).$$

Proof. The assertions for $Pic(R)$ and $Br(R)$ are well-known, and do not require the hypotheses that the f_i and the $f_{i,j}$ are Krull morphisms–the crucial observation is that being a projective R-module of finite type (respectively an Azumaya R-algebra A) depends on the existence of a finite set of data such as a projective basis (respectively a projective basis plus data for $A \otimes_R A^0 \to \mathrm{End}_R(A)$ to be an isomorphism).

Recall that $\beta(R)$ denotes $Br(\mathscr{D}(R))$, where $\mathscr{D}(R)$ is the category of divisorial R-lattices. Let i be in I and write R' for R_i. Let M' be a divisorial

R'-lattice. The passage from $Cl(R')$ to $Cl(R)$ involves assigning to the isomorphism class of M' not the class of $R \otimes_{R'} M'$, but rather the class of $R \tilde{\otimes}_{R'} M'$. Because the inclusion of R' in R is a Krull morphism, $R \tilde{\otimes}_{R'} M'$ can be described in the alternative forms

$$R \tilde{\otimes}_{R'} M' = \bigcap_{p \in Z} (R \otimes_{R'} M')_p = \bigcap_{q \in Z'} (R \otimes_{R'} M')_q$$

where Z is the set of height one points of R and Z' the set of height one points of R' [16, Proposition 4]. We remind the reader that the modified tensor product $\tilde{\otimes}$ has the property that $M \tilde{\otimes}_R N = M \otimes_R N$ if M is divisorial over R and N is R-flat—this fact will be used implicitly later in the proof.

Let M be a divisorial R-lattice. To show $Cl(R) = \varinjlim Cl(R_i)$ we wish to show that for some index i in I there is a divisorial R_i-lattice M' such that $M \simeq R \tilde{\otimes}_{R_i} M'$. Let K be the field of fractions of R, and let $V = K \otimes_R M = KM$. Because M is an R-lattice there is a free R-module F in V with $M \subseteq F$ and $KF = V$. Fix an R-basis $\mathscr{B} = \{v_1, ..., v_n\}$ for F, where $n = \dim_K V$. For any subring S of K we will write $S\mathscr{B}$ to denote the free S-module with basis $\mathscr{B}$. Thus $F = R\mathscr{B}$ and for $S \subseteq T \subseteq K$ we have $T(S\mathscr{B}) = T\mathscr{B}$.

For p in Z and i in I write $R_{i,p'}$ for $\mathscr{S}^{-1}R_i$, where $\mathscr{S}$ is the multiplicative set R_i-$(p \cap R_i)$. It is easy to see that $R_p = \varinjlim R_{i,p'}$. Note that $R_{i,p'}$ is a discrete valuation ring, since the hypothesis that f_i is a Krull morphism implies $p \cap R_i$ is a height one prime of R_i.

For p in Z, M_p is a free R_p-module on a basis $\mathscr{B}_p$, and for p not in J we may take $\mathscr{B}_p = \mathscr{B}$. It is easy to see that since $\mathscr{B}_p$ is a finite set there is for each p in J an index $i(p)$ such that $\mathscr{B}_p \subseteq R_{i(p),p'}\mathscr{B}$. Since J is a finite set there is therefore an index $i(0)$ such that, writing R' for $R_{i(0)}$, we have

$$R'_{p'}\mathscr{B}_p \subseteq R'_{p'}\mathscr{B}$$

for p in Z, with equality if p is not in J.

Let J' denote the set of elements of Z' of the form $p' = p \cap R'$ with p in J. For q in J' let $p_1, ..., p_t$ be the elements p of J for which $p' = q$. Let $\mathscr{S} = R - (p_1 \cup \cdots \cup p_t)$, a multiplicative subset of R. Then $\mathscr{S}^{-1}R$ is a Krull domain of Krull dimension one, hence a Dedekind domain [5, Theorem 13.1], and is semi-local as well, hence is a principal ideal domain. The $\mathscr{S}^{-1}R$-module

$$N(q) = \bigcap_{i=1}^{t} M_{p_i}$$

has the property that $R_{p_i}N(q) = M_{p_i}$ for $i = 1, ..., t$ [5, Proposition 5.2]. It is torsion-free, and is of finite type as an $\mathscr{S}^{-1}R$-module, being contained in the intersection $\bigcap R_{p_i}\mathscr{B}$, which equals $\mathscr{S}^{-1}R\mathscr{B}$. Thus $N(q)$ is free, on a basis we will call $\mathscr{B}_q$. Let $M(q) = R'_q\mathscr{B}_q$ for q in J', and $M(q) = R'_q\mathscr{B}$ for q in Z' but not in J'. Let K' denote the field of fractions of R', and let $V' = K'\mathscr{B}$. We then have:

> For each q in Z', $M(q)$ is a free R'_q-module satisfying $K'M(q) = V'$ and $M(q) = R'_q\mathscr{B}$, with equality if $q \in Z' - J'$. Furthermore, if $p' = q$, then $R_p M(q) = M_p$.

Now define an R'-module M' by

$$M' = \bigcap_{q \in Z'} M(q).$$

We claim that M' is a divisorial R'-lattice for which $R \tilde{\otimes}_{R'} M' \simeq M$. The divisoriality of M' follows from two features of the family $\{M(q)\}$ [16, Lemma 1]: (i) each $M(q)$ is a divisorial R'_q-module; (ii) the family $\{M(q)\}$ is of finite character. The last statement means that $K'M(q) = V'$ is independent of q, and each element of V' is in all but finitely many $M(q)$ (a condition which holds here because $M(q) = R'_q\mathscr{B}$ for all but finitely many q in Z').

To see that $R \tilde{\otimes}_{R'} M' \simeq M$ first observe that because $R' \to R$ is a Krull morphism, $R \tilde{\otimes}_{R'} M'$ is a divisorial R-lattice [16, Proposition 3]. We note that $M' \subseteq M$ because

$$M' \subseteq \bigcap_{q \in J'} M(q) \subseteq \bigcap_{p \in Z} M_p = M,$$

since M is divisorial over R. Thus we have an R-homomorphism $g\colon R \otimes_{R'} M' \to M$ satisfying $g(r \otimes m') = rm'$ for r in R, m' in M'. Since M is a divisorial R-module, g induces an R-homomorphism $h\colon R \tilde{\otimes}_{R'} M' \to M$ [16, Proposition 2]. To show h is an isomorphism it suffices by Lemma 2.1 to show h_p is an isomorphism for each p in Z. But

$$\begin{aligned}
(R \tilde{\otimes}_{R'} M')_p &= (R \otimes_{R'} M')_p \qquad [5, \text{Proposition } 5.2]\\
&= R_p \otimes_{R'_{p'}} M'_{p'}\\
&= R_p M(p')\\
&= M_p.
\end{aligned}$$

This completes the proof of the fact that $Cl(R) = \varinjlim Cl(R_i)$. To show the corresponding fact for the Brauer groups of the categories $\mathscr{D}(R)$ and $\mathscr{D}(R_i)$ one can use a similar argument, and we shall not provide the details here.

PROPOSITION 4.5. *Let X and Y be Krull schemes and $\pi: Y \to X$ a morphism of schemes. Let G be a profinite group of automorphisms of Y over X.*

(a) *Suppose π is a Galois covering with group G. Then there is an exact sequence*

$$1 \to H^1(G, \mathcal{O}_Y(Y)^*) \to Pic(X) \to Pic(Y)^G \to H^2(G, \mathcal{O}_Y(Y)^*)$$
$$\to Br(Y/X) \to H^1(G, Pic(Y)) \to H^3(G, \mathcal{O}_Y(Y)^*).$$

(b) *Suppose that for each height one point x of X, $\mathcal{O}_{Y,x}$ is a Galois extension of $\mathcal{O}_{X,x}$ with group G. Then there is an exact sequence*

$$1 \to H^1(G, \mathcal{O}_Y(Y)^*) \to Cl(X) \to Cl(Y)^G \to H^2(G, \mathcal{O}_Y(Y)^*)$$
$$\to \bigcap_{x \in Z} Br(\mathcal{O}_{Y,x}/\mathcal{O}_{X,x}) \to H^1(G, Cl(Y)) \to H^3(G, \mathcal{O}_Y(Y)^*),$$

where Z is the set of height one point of X.

Proof. These results follow easily from Corollary 4.2, Lemma 4.3 and the fact that $H^n(G, \varinjlim A_i) = \varinjlim H^n(G, A_i)$.

COROLLARY 4.6. *Let X, Y and π be as in* (a) *of Proposition* 4.5 *If we further assume that X and Y are locally factorial then*

$$Br(Y/X) \simeq \bigcap_{x \in Z} Br(\mathcal{O}_{Y,x}/\mathcal{O}_{X,x}),$$

where Z is the set of height one points of X.

Proof. The assumption that X and Y are locally factorial implies that $Pic(X) = Cl(X)$ and $Pic(Y) = Cl(Y)$ [5, Corollary 18.5]. Comparing the exact sequences obtained from (a) and (b) of Proposition 4.5, we see that they give the same groups at all places save possibly one. The same groups must occur at this spot as well, which gives the desired result.

5. BRAUER GROUPS OF SOME COMPLETE VARIETIES

In this section we will apply the last result of Section 4 to obtain information about $Br(X)$ for certain complete varieties X.

PROPOSITION 5.1. *Let X be a complete normal geometrically connected variety over a field K. Let L be a Galois extension of K with group G and let*

$Y = X \times_{\operatorname{Spec} K} \operatorname{Spec} L$. *Let Z denote the set of height one points of X. Then there exist exact sequences*

$$1 \to Cl(X) \to Cl(Y)^G \to Br(L/K) \to \bigcap_{x \in Z} Br(\mathcal{O}_{Y,x}/\mathcal{O}_{X,x})$$
$$\to H^1(G, Cl(Y)) \to H^3(G, L^*),$$
$$1 \to Pic(X) \to Pic(Y)^G \to Br(L/K) \to Br(Y/X)$$
$$\to H^1(G, Pic(Y)) \to H^3(G, L^*).$$

Proof. Since X is a complete variety over K, Y is a complete variety over L, hence $\mathcal{O}_Y(Y)^* = L^*$. Since $H^1(G, L^*) = 1$ and $H^2(G, L^*) = Br(L/K)$ the exact sequences can be obtained from Proposition 4.5.

COROLLARY 5.2 (Lichtenbaum [13]). *Let X be a proper, smooth and geometrically connected curve over a field K. Let $\bar{K}$ be the separable closure of K, $G = \operatorname{Gal}(\bar{K}/K)$ and $\bar{X} = X \times_{\operatorname{Spec} K} \operatorname{Spec} L$. Then there is an exact sequence*

$$1 \to Pic(X) \to Pic(\bar{X})^G \to Br(K) \to Br(X)$$
$$\to H^1(G, Pic(\bar{X})) \to H^3(G, L^*).$$

Now let X be any variety over a field K. If X has a K-rational point then $Br(K)$ embeds as a direct summand of $Br(X)$ via the composition of morphisms $\operatorname{Spec} K \to X \to \operatorname{Spec} K$. For K a local field arising from an algebraic number field and X a curve, $Br(K) \to Br(X)$ is one-one if and only if X has a K-rational point (see [13]). The next proposition shows that the conclusion is not always true: take X the projective variety in $\mathbb{P}^4(Q)$ defined by the quadratic form $x_1^2 + \cdots + x_5^2$. Then X does not have a K-rational point, but $Br(X) \simeq Br(Q)$.

PROPOSITION 5.3. *Let X be a projective variety in $\mathbb{P}^{n-1}(K)$ defined by a non-degenerate n-ary quadratic form over a field K of characteristic zero, where $n \geqslant 5$. Then $Br(X) \simeq Br(K)$.*

Proof. By a linear change of variables we may assume the form is diagonal, and we may write $X = \operatorname{Proj}(R)$, where

$$R = K[x_1, \ldots, x_n]/(a_1 x_1^2 + \cdots + a_n x_n^2).$$

Let L be the algebraic closure of K and let $Y = \operatorname{Proj}(S)$, where

$$S = L[x_1, \ldots, x_n]/(a_1 x_1^2 + \cdots + a_n x_n^2).$$

We observe that:

(a) X and Y are regular varieties [14, p. 214].

(b) R and S are unique factorization domains.

Let $G = \mathrm{Gal}(L/K)$. We will show that the following hold:

(c) $Cl(X) = H^0(G, Cl(Y)) = Cl(Y) \simeq \mathbb{Z}$.

(d) $Br(Y) = (0)$.

To prove (c) let D be a divisor on Y. Then $D = \sum_i n_i p_i$, where the p_i are height one homogeneous prime ideals of S and only finitely many n_i are not zero. Since S is a unique factorization domain, $p_i = (g_i)$ for some homogeneous element g_i in S and $D = \mathrm{div}(\prod_i g_i^{n_i})$. Define the integer $\deg(D)$ to be $\sum_i n_i(\deg(g_i))$. Let p be the homogeneous prime ideal generated by the residue class in S of x_1. We claim that if $\deg(D) = d$, then $D \sim dp$ in $\mathrm{Div}(Y)$, where $\sim$ denotes linear equivalence of divisors. To see this write $D = D_1 - D_2$, where D_1 and D_2 are effective divisors. If $D_1 = \mathrm{div}(h_1)$ and $D_2 = \mathrm{div}(h_2)$ then $d = \deg(h_1) - \deg(h_2)$. Since $D - dp = \mathrm{div}(h_1/x_1^d h_2)$, and since $\deg(h_1) = d + \deg(h_2) = \deg(x_1^d h_2)$, it follows that $h_1/x_1^d h_2$ is in the function field $L(Y)$ of Y. This shows $D \sim dp$. Also, for every σ in G, $\sigma(p) = p$ and G acts trivially on $Cl(Y)$. This proves (c).

To prove (d), let $S_n = L[z_1,..., z_{n-1}]/(z_1 z_2 - G(z_3,..., z_{n-1}))$, where $z_i = y_i/y_n$ and the y_i are chosen so that

$$S = L[y_1,..., y_n]/(y_1^2 + \cdots + y_n^2).$$

Then the assignment from $S_n[1/z_n]$to $L[t_1, 1/t_1, t_3,..., t_{n-1}]$ given by sending z_i to t_i for $i = 1, 3,..., n-1$ and z_2 to $G(t_3,..., t_{n-1})/t_1$ is an isomorphism (the inverse map is defined by sending t_i to z_i, $i = 1, 3,..., n-1$). Hence

$$\begin{aligned} Br(S_n[1/z_1]) &\simeq Br(L[t_1, 1/t_1, t_3,..., t_{n-1}]) \\ &\simeq B_r(L[t_1, 1/t_1]), \end{aligned}$$

the last isomorphism being valid because $L[t_1, 1/t_1]$ is a regular ring of characteristic zero [17, Corollary 8.8., p. 100]. By regularity of $L[t_1)$ the group $Br(L[t_1, 1/t_1])$ embeds in $Br(L(t_1)$, but the latter is zero by Tsen's theorem. Again, by regularity of Y we have

$$Br(Y) \subseteq Br(S_n) \subseteq Br(S_n[1/z_1]),$$

and (d) follows.

Since X is regular, every finite Galois extension of $\mathcal{O}_{X,x}$ is regular for each x in X. It follows that X and Y are locally factorial, hence $Cl(X) = Pic(X)$

and $Cl(Y)=Pic(Y)$. Since projective varieties are complete we may use (c) and (d), together with Proposition 5.1, to conclude that $Br(X)\simeq Br(K)$.

In the next example we consider the projective variety defined by a non-degenerate quadratic form with a unique singular point.

LEMMA 5.4. *Let X be a projective variety in $\mathbb{P}^n(K)$ defined by a non-degenerate n-ary quadratic form, with $n\geqslant 5$ and K a field of characteristic zero. Let L denote the algebraic closure of K and let $Y=X\times_{\mathrm{Spec}\,K}\mathrm{Spec}\,L$. Then for each x in X and each y in Y the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are geometrically factorial (i.e., their strict henselizations are factorial).*

Proof. We may assume $X=\mathrm{Proj}(R)$ and $Y=\mathrm{Proj}(S)$, where

$$R=K[x_1,..., x_{n+1}]/(a_1x_1^2+\cdots+a_nx_n^2)$$

and $S=L\otimes_K R$. Let η denote the homogeneous prime ideal generated by the residue classes of $x_1,..., x_n$ (by abuse of notation we will use η whether we are working with X or Y). By the Jacobian criterion η is the unique singular point in X or Y. Since the completion $\hat{\mathcal{O}}_{Y,\eta}$ is a faithfully flat $\mathcal{O}^{\mathrm{sh}}_{Y,\eta}$-module and $Cl(\mathcal{O}^{\mathrm{sh}}_{Y,\eta})\to Cl(\mathcal{O}_{Y,\eta})$ is an embedding (A^{sh} denotes the strict henselization of A)

$$\mathcal{O}_{Y,\eta}\simeq L[[x_1,..., x_n]]/(x_1^2+\cdots+x_n^2)$$

has a unique singular maximal ideal by the Nagata Jacobian criterion and since $n\geqslant 5$, $\mathcal{O}_{Y,\eta}$ is a unique factorization domain [5, Proposition 18.5]. Now the lemma follows since $\mathcal{O}^{\mathrm{sh}}_{Y,\eta}=\mathcal{O}^{\mathrm{sh}}_{X,\eta}$.

PROPOSITION 5.5. *Let X be as in Lemma 5.4. Then $Br(X)\simeq Br(K)$.*

Proof. By Lemma 5.4, every Galois extension of each $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ is a UFD. $Br(Y)$ embeds in $Br(F)$, where F is the function field of Y [12, Theorem 5.2]. Write

$$S=L[y_1,..., y_{n+1}]/(y_1y_2-G(y_3,..., y_n)),$$

where $G(y_1,..., y_n)$ is a quadratic form in $n-2$ variables, and let

$$S_{n+1}=L[y_1,..., y_n]/(y_1y_2-G(y_3,..., y_n)).$$

Then $\mathrm{Spec}(S_{n+1})$ is isomorphic to an affine open set of Y and

$$S_{n+1}[1/y_t]\simeq L[t_1, 1/t_1, t_3,..., t_n].$$

Now we conclude $Br(Y)=1$, from the commutative diagram

$$\begin{tikzcd} Br(S_{n+1}) \arrow[r] \arrow[d] \arrow[dr, "\beta"] & Br(S_{n+1}[1/y_1])=1 \arrow[d] \\ Br(Y) \arrow[r, "\alpha"'] & Br(F) \end{tikzcd}$$

where α and β are one-one. From the exact sequence in Proposition 5.1 we can conclude $Br(X) \simeq Br(K)$.

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