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The Brézis–Nirenberg Problem on $\mathbb{S}^3$

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In this paper we study existence and nonexistence of solutions to the Brézis–Nirenberg problem for different values of λ in geodesic spheres on $\mathbb{S}^3$. The picture differs considerably from the one in the Euclidean space. It is shown that large spheres containing the hemisphere have two different type of radial solutions for negative values of λ. Numerical results indicate that for λ very small the solutions have a maximum near the boundary, whereas for larger values of λ the maximum is at the origin. The techniques used are: Pohozaev type identities, concentration-compactness lemma and numerical methods. © 2002 Elsevier Science

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1. INTRODUCTION

Let D be a geodesic ball on $\mathbb{S}^3$. Consider the problem

$$-\Delta_{\mathbb{S}^3} u = u^5 + \lambda u \qquad \text{on } D, \tag{1}$$

with $u \geqslant 0$ and Dirichlet boundary conditions, i.e.,

$$u = 0 \qquad \text{on } \partial D. \tag{2}$$

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Here $\Delta_{\mathbb{S}^3}$ is the Laplace–Beltrami operator on $\mathbb{S}^3$, and 5 is the corresponding critical Sobolev exponent. We are interested in determining the range of values of λ for which there exists a positive solution of (1) and (2). The equivalent problem for a ball in Euclidean space was solved a long time ago by Brézis and Nirenberg [6]. In the Euclidean space there is a vast literature on many different extensions of the problem considered by Brézis and Nirenberg (see, e.g., [15], Chapter 3; see also [4] and the references therein). It follows immediately from the maximum principle that no positive solutions exist if $\lambda \geqslant \lambda_1$ where λ_1 is the first eigenvalue of the Laplace–Beltrami operator with Dirichlet boundary conditions.

Our main result is the following theorem.

THEOREM 1. *Let $D \subset \mathbb{S}^3$ be a ball whose geodesic radius will be denoted by θ_1. Then the following statements are true* (*cf. Fig.* 2).

(i) *If*

$$\mu_1 = \frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \lambda_1 = \frac{\pi^2 - \theta_1^2}{\theta_1^2}, \tag{3}$$

there is a unique positive solution to (1) *and* (2).

(ii) *If D is contained in the hemisphere, i.e. if $\theta_1 \leqslant \pi/2$, and if $\lambda \leqslant \frac{\pi^2-4\theta_1^2}{4\theta_1^2}$, there is no nontrivial solution to* (1) *and* (2).

(iii) *If D contains the hemisphere, there exists a function $v\colon (\frac{\pi}{2}, \pi) \to (-\frac{3}{4}, -\infty)$, $v(t) \to -\frac{3}{4}$ as $t \to \pi$ such that for $v(\theta_1) < \lambda \leqslant \frac{\pi^2-4\theta_1^2}{4\theta_1^2}$ there is no nontrivial solution to* (1) *and* (2). *Numerical computations indicate that there are solutions if $\lambda < v(\theta_1)$.*

Remarks 1. (i) In particular for a hemisphere of S^3 there is a solution of (1) and (2), if and only if $0 < \lambda < 3$.

(ii) On the other hand, in the limit as the geodesic radius of D, θ_1 goes to zero, one recovers the Brézis–Nirenberg result, i.e., there is a positive solution if and only if $\lambda_1/4 < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of the ball.

(iii) For $\lambda = \frac{\pi^2-4\theta_1^2}{4\theta_1^2}$ no solution exists. It is not known if there is a solution for $\lambda = v(\theta_1)$.

(iv) The picture for the geodesic balls contained in a hemisphere is not surprising. It was expected after the recent results of Bandle and Peletier [2] on best critical constants for the Sobolev embeddings in $\mathbb{S}^3$ in the case $\lambda = 0$.

(v) An interesting open problem is to prove the existence of solutions in the range $\lambda < v(\theta_1)$. Numerical calculations show that in contrast to the solutions in the range $\frac{\pi^2-4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2-\theta_1^2}{\theta_1^2}$ their maximum is not at the origin (cf. Fig. 3). They become singular at the boundary as λ tends to $v(\theta_1)$. Arguments based on symmetrization [1] show that those solutions cannot be minimizers of the associated energy functional $S_{p,\lambda}$ defined in the next section.

In Section 2, we will discuss the existence of nontrivial positive solutions, while in Section 3 we will use an appropriately modified Pohozaev identity to show nonexistence of positive solutions. In Section 4 we investigate numerically the region $\lambda < -3/4$ for balls beyond the hemisphere. A boundary value problem relevant to the proof of the existence of positive solutions is briefly discussed in the Appendix.

2. EXISTENCE OF POSITIVE SOLUTIONS

It is well-known that for any domain $D \subset \mathbb{R}^3$, $W_0^{1,2}(D)$ is continuously embedded into $L^6(D)$. This means that there exists a positive number $S_1(D)$ given by

$$S_1(D) = \inf_v \int_D |\nabla v|^2\, dx, \quad \text{where } v \in W_0^{1,2}(D) \qquad \text{and} \qquad \int_D v^6\, dx = 1. \tag{4}$$

$S_1(D)$ is called the best Sobolev constant. For domains $D \neq \mathbb{R}^3$ the minimum is never attained. Every minimizing sequence contains a subsequence which concentrates. The constant $S_1(D)$ is independent of the domain and has the value

$$S^* = 3\left(\frac{\pi}{2}\right)^{4/3}. \tag{5}$$

The situation is different on $\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid |x| = 1\}$. If we map $\mathbb{S}^3$ stereographically onto $\mathbb{R}^3$, a domain $D' \subset \mathbb{S}^3$ is mapped onto a domain $D \subset \mathbb{R}^3$. Since the transformation is conformal, the line element of $\mathbb{S}^3$ is proportional to the line element of the Euclidean space, i.e.,

$$ds = p(x)\, dx \qquad p(x) \equiv \frac{2}{1+|x|^2}. \tag{6}$$

The best Sobolev constant for D' is then given by

$$S_p(D) = \inf_v \int_D |\nabla v|^2 \, p(x) \, dx,$$

$$\text{where} \quad v \in W_0^{1,2}(D) \quad \text{and} \quad \int_D v^6 p^3 \, dx = 1. \tag{7}$$

A geodesic ball in $\mathbb{S}^3$ and centered at the north pole is mapped onto a ball $B_R = \{x \in \mathbb{R}^3 \mid |x| < R\}$ (cf. Fig. 1). In particular B_1 is the image of the hemisphere under the stereographic projection. Recently Bandle and Peletier [2] proved that $S_p(B_R)$ is never attained for $R \leqslant 1$ (in fact, $S_p(B_R) = S^*$, if $R \leqslant 1$ and every minimizing sequence has a subsequence which concentrates at a single point). On the other hand, if $R > 1$ there exists a unique minimizer and $S_p(B_R) < S^*$.

As in the Euclidean case [6], set

$$S_{p,\lambda}(D) = \inf \left\{ \int_D |\nabla v|^2 \, p(x) \, dx - \lambda \int_D v^2 p^3 \, dx \mid v \in X_p \right\} \tag{8}$$

with

$$X_p = \left\{ v \in W_0^{1,2}(D) \,\middle|\, \int_D v^6 p^3 \, dx = 1 \right\}. \tag{9}$$

Clearly $S_{p,0}(D) = S^*$.

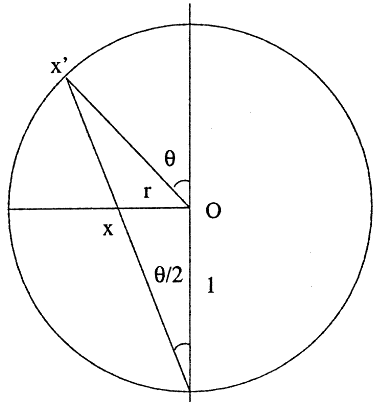

FIG. 1. Relation between stereographic projection and geodesic coordinates.

LEMMA 1. *Let D' be a geodesic ball centered at the north pole of $\mathbb{S}^3$, and let D be its stereographic projection (from the south pole) onto $\mathbb{R}^3$. Let θ_1 be the geodesic radius of D'. Then, for all $\lambda > (\pi^2 - 4\theta_1^2)/4\theta_1^2$, we have*

$$S_{p,\lambda}(D) < S^*. \tag{10}$$

Proof. We will estimate the quotient

$$Q_{\lambda,p}(v) \equiv \frac{\int_D |\nabla v|^2 \, p(x)\, dx - \lambda \int_D v^2 p^3 \, dx}{(\int_D v^6 p^3 \, dx)^{1/3}}, \tag{11}$$

for the family of functions

$$v_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^2)^{1/2}}, \tag{12}$$

$r = |x|$ and $\varepsilon > 0$. Here, $\varphi(r)$ is a fixed smooth function satisfying $\varphi(0) = 1$, $\varphi'(0) = 0$, $\varphi(R) = 0$, which will be chosen appropriately later. We will compute each of the three terms appearing in the quotient (11), to leading order in ε, as ε goes to zero. We start with

$$\int_D v_\varepsilon^2 p^3 \, dx = 4\pi \int_0^R \frac{\varphi(r)^2}{(\varepsilon + r^2)} p(r)^3 \, r^2 \, dr = 4\pi \int_0^R \varphi(r)^2 \, p(r)^3 \, dr - I_1, \tag{13}$$

where the remainder term is given by

$$I_1 = 4\pi \int_0^R \varphi(r)^2 \, p(r)^3 \frac{\varepsilon}{\varepsilon + r^2} \, dr. \tag{14}$$

Making the change of variables $r = \varepsilon^{1/2} s$ in (14), inserting the expression for $p(r)$ given by (6) and using the smoothness of φ, we get

$$I_1 = 4\pi \int_0^{R/\varepsilon^{1/2}} \frac{\varphi(\varepsilon^{1/2} s)^2}{(1 + s^2)} \frac{8}{(1 + \varepsilon s^2)^3} \varepsilon^{1/2} \, ds = O(\varepsilon^{1/2}), \tag{15}$$

as ε goes to zero. Hence, from (13) and (15), we have

$$\int_D v_\varepsilon^2 p^3 \, dx = 4\pi \int_0^R \varphi(r)^2 \, p(r)^3 \, dr + O(\varepsilon^{1/2}). \tag{16}$$

Next, consider

$$\int_D v_\varepsilon^6 p^3 \, dx = 4\pi \int_0^R \frac{\varphi(r)^6}{(\varepsilon + r^2)^3} p(r)^3 \, r^2 \, dr = I_2 + I_3 + I_4, \tag{17}$$

with

$$I_2 \equiv 4\pi \int_0^R \frac{\varphi(r)^6 - 1}{(\varepsilon + r^2)^3}\, p(r)^3\, r^2\, dr, \tag{18}$$

$$I_3 \equiv 4\pi \int_0^R \frac{p^3 - 8}{(\varepsilon + r^2)^3}\, r^2\, dr, \tag{19}$$

and,

$$I_4 \equiv 4\pi \int_0^R \frac{8}{(\varepsilon + r^2)^3}\, r^2\, dr. \tag{20}$$

Since $\varphi(0) = 1$ and $\varphi'(0) = 0$ and $p^3 \leqslant 8$, we have

$$|I_2| \leqslant C \int_0^R \frac{r^4}{(\varepsilon + r^2)^3}\, dr = O(\varepsilon^{-1/2}). \tag{21}$$

Also, $8 - p^3 = 8(r^6 + 3r^4 + 3r^2)/(1 + r^2)^3$, and $0 \leqslant r \leqslant R$. Thus, $8 - p(r)^3 \leqslant C_1 r^2$, where C_1 is a constant depending on R. Using this in (19) we get

$$|I_3| \leqslant C_1 \int_0^R \frac{r^4}{(\varepsilon + r^2)^3}\, dr = O(\varepsilon^{-1/2}). \tag{22}$$

Whereas, making the same change of variables as before, $r = \varepsilon^{1/2} s$, we get from (20)

$$I_4 = 4\pi \int_0^{R/\varepsilon^{1/2}} \frac{8}{\varepsilon^{3/2}} \frac{s^2}{(1 + s^2)^3}\, ds = \frac{32\pi}{\varepsilon^{3/2}} \int_0^\infty \frac{s^2 ds}{(1 + s^2)^3} + O(1) = \frac{2\pi^2}{\varepsilon^{3/2}} + O(1). \tag{23}$$

Hence, from (17), (21), (22), and (23), we conclude that

$$\int_D v_\varepsilon^6 p^3\, dx = \frac{2\pi^2}{\varepsilon^{3/2}} + O(\varepsilon^{-1/2}). \tag{24}$$

Finally, we must consider

$$\int_D |\nabla v_\varepsilon|^2\, p\, dx = 4\pi \int_0^R \left(\frac{\partial v_\varepsilon}{\partial r}\right)^2 p(r)\, r^2\, dr \tag{25}$$

When we use the expression (12) for $v_\varepsilon(r)$ in the previous equation, and after integration by parts, we can decompose

$$\int_D |\nabla v_\varepsilon|^2 \, p \, dx = I_5 + I_6 + I_7, \tag{26}$$

where

$$I_5 \equiv 12\pi \int_0^R \varphi(r)^2 \frac{\varepsilon r^2}{(\varepsilon + r^2)^3} \, p(r) \, dr, \tag{27}$$

$$I_6 \equiv 4\pi \int_0^R \frac{\varphi'(r)^2}{\varepsilon + r^2} \, p(r) \, r^2 \, dr, \tag{28}$$

and,

$$I_7 \equiv 4\pi \int_0^R \varphi(r)^2 \frac{r^3 p'(r)}{(\varepsilon + r^2)^3} \, dr. \tag{29}$$

After making the standard stretching of variables, i.e., $r = \varepsilon^{1/2} s$, using the smoothness of $\varphi(r)$ together with the fact that $\varphi'(0) = 0$, we conclude that

$$I_6 = 4\pi \int_0^R p\varphi'^2 \, dr + O(\varepsilon). \tag{30}$$

Using the same argument, and the fact that $p' = -rp^2$, we also have

$$I_7 = -16\pi \int_0^R \frac{\varphi^2(r)}{(1 + r^2)^2} \, dr + O(\varepsilon) = -4\pi \int_0^R p(r)^2 \, \varphi(r)^2 \, dr + O(\varepsilon). \tag{31}$$

Finally, using the stretching of variables, the smoothness of both $p(r)$ and $\varphi(r)$, and the fact that $\varphi(0) = 1$ and $p(0) = 2$, we obtain

$$I_5 = \frac{24\pi}{\varepsilon^{1/2}} \int_0^\infty \frac{s^2 ds}{(1 + s^2)^3} + O(\varepsilon^{1/2}) = \frac{3\pi^2}{2\varepsilon^{1/2}} + O(\varepsilon^{1/2}). \tag{32}$$

Therefore, from (26), (30), (31), and (32) we obtain

$$\int_D \nabla v_\varepsilon^2 \, p \, dx = \frac{3\pi^2}{2\varepsilon^{1/2}} + 4\pi \int_0^R p\varphi'^2 \, dr - 4\pi \int_0^R p^2 \varphi^2 \, dr + O(\varepsilon^{1/2}). \tag{33}$$

Using (16), (24), and (33) in the quotient (11), with $v = v_\varepsilon$ we get

$$Q_{\lambda,p}(v_\varepsilon) = S^* + \frac{\varepsilon^{1/2}}{(2\pi^2)^{1/3}} F(\varphi, \lambda) + O(\varepsilon), \tag{34}$$

as ε goes to zero. Here,

$$F(\varphi, \lambda) \equiv 4\pi \int_0^R p\varphi'^2 \, dr - 4\pi \int_0^R p^2\varphi^2 \, dr - 4\pi\lambda \int_0^R p^3\varphi^2 \, dr, \tag{35}$$

and S^* is given by (5). Now choose $\varphi(r) = \varphi_1(r) \equiv (1+r^2)\cos(\pi \arctan r/\theta_1)$. This function satisfies $\varphi_1(0) = 1$, $\varphi_1'(0) = 0$ and $\varphi_1(R) = 0$, since $\tan \theta_1/2 = R$. Because of Lemma 3 and the Remark in the Appendix, we have

$$F(\varphi_1, \lambda) = 4\pi(\mu_1 - \lambda) \int_0^R p^3\varphi_1^2 \, dr < 0 \tag{36}$$

if $\lambda > \mu_1(\theta_1)$, where

$$\mu_1(\theta_1) \equiv \frac{\pi^2 - 4\theta_1^2}{4\theta_1^2}.$$

Thus, if $\lambda > \mu_1(\theta_1)$, $Q_{\lambda,p}(v_\varepsilon) < S^*$. ▮

This lemma together with the concentration-compactness alternative (cf. for instance [1, 5]) implies that $S_{p,\lambda}$ is attained and that the minimizer solves (1), (2). This proves the existence of a solution in Theorem 1(i). In order to prove the uniqueness we use a result of Kwong and Li [11] where the case $\Delta u + u^5 + q(r)\, u = 0$ in B_1, $u = 0$ on ∂B_1 has been studied. It turns out [Theorem 2] that, if there exists a number $r_0 \in (0, 1]$ such that $r^2 q(r)$ is nondecreasing in $(0, r_0)$ and nonincreasing in $(r_0, 1]$ then there is at most one solution. As observed in [1] the Euler–Lagrange equation associated to $S_{p,\lambda}(D)$ can always be brought into such a form. Indeed if u is a minimizer then $\chi = \sqrt{p}\, u$ satisfies $\Delta\chi + \frac{3+4\lambda}{(1+r^2)^2}\chi + S_{p,\lambda}(D)\, \chi^5 = 0$ in D, $\chi = 0$ on ∂D. If $3 + 4\lambda > 0$ then the theorem of Kwong and Li applies and establishes the second part of Theorem 1(i).

Remarks 2. (i) The lower bound for λ is related to the Green's function G of the operator $\Delta_{\mathbb{S}^3} + \lambda$ with Dirichlet boundary conditions. If the singularity is in the center of the geodesic ball, it depends only on θ and has the form $G(\theta, 0) = \text{const.}(\sin(\sqrt{1+\lambda}\,(\theta_1 - \theta))/\sin\theta)$. It can be split into a singular and a regular part $G(\theta, 0) = \text{const.}((\sin(\theta_1\sqrt{1+\lambda})/\sin\theta) + h(\theta))$

where $h(\theta) = -\frac{1}{\sin\theta}(\cos(\sqrt{1+\lambda}\,(2\theta_1 - \theta)/2)\sin(\theta\sqrt{1+\lambda}/2))$. If $\lambda > \frac{\pi^2 - 4\theta_1^2}{4\theta_1^2}$ then $h(0) > 0$. This is in accordance with a result of Schoen [14].

(ii) The existence of solutions for arbitrary domains $\Omega \subset \mathbb{S}^3$ was studied in [1]. It turns out that there is an interval $I = (\mu_1(\Omega), \lambda_1(\Omega))$ such that for $\lambda \in I$ Problem (1), (2) in Ω has a solution. If B_0 is the outer and B_i the inner ball of Ω a monotonicity argument yields $\mu_1(B_0) < \mu_1(\Omega) < \mu_1(B_i)$ and $\lambda_1(B_0) < \lambda_1(\Omega) < \lambda_1(B_i)$. By means of symmetrization the lower bounds can be improved. Indeed if Ω^* denotes the geodesic ball of the same Riemannian volume as Ω then $\mu_1(\Omega^*) < \mu_1(\Omega)$ and $\lambda_1(\Omega^*) < \lambda_1(\Omega)$.

3. NONEXISTENCE OF SOLUTIONS

Consider the equation

$$-\Delta_{\mathbb{S}^3} u = u^5 + \lambda u \tag{37}$$

on a geodesic ball of radius θ_1, with $u \geqslant 0$, and Dirichlet boundary conditions. Then we have the following nonexistence result.

LEMMA 2. *Assume* $\lambda \leqslant \frac{\pi^2 - 4\theta_1^2}{4\theta_1^2}$.

(i) *If* $\theta_1 \leqslant \pi/2$ *then, there is no solution of* (37).

(ii) *If* $\theta_1 \geqslant \pi/2$ *and if in addition* $\lambda \geqslant -\frac{3}{4}$ *then, there is no solution of* (37).

Proof. This time the proof is simpler if we use geodesic coordinates rather than the stereographic projection used in the proof of Lemma 1. For that purpose, let us choose the north pole of $\mathbb{S}^3$ as the center of the geodesic ball. Given a point on $\mathbb{S}^3$, let θ be the azimuthal angle of that point (i.e., if we consider rays coming from the center of the ball to the north pole and to the given point, respectively, θ is the angle between those two rays, cf. Fig. 1). By a result of Padilla [12] (extending the classical result of Gidas, Ni, and Nirenberg [9] to domains on manifolds of constant curvature), a solution u to (37) is symmetric, i.e., it only depends on the azimuthal angle θ. Writing $u(x) = u(\theta)$, where θ is the azimuthal angle of x, (37) can be written as

$$-u'' - 2\cot\theta\, u' = u^5 + \lambda u, \tag{38}$$

with $u(0)$ finite, $u'(0) = 0$ and $u(\theta_1) = 0$. The proof of this lemma is a Pohozaev type argument. We first multiply (38) by $\sin^2\theta\, g(\theta)\, u'(\theta)$ and integrate the result in θ from 0 to θ_1. Here, $g(\theta)$ is a smooth function satisfying $g(0) = 0$, $g(\theta) > 0$ for $\theta \in (0, \theta_1)$, and otherwise arbitrary.

Integrating by parts, using the boundary conditions on u and $g(0)=0$, we obtain

$$\int_0^{\theta_1} (u')^2\, h(\theta)\, d\theta - \tfrac{1}{2}u'^2(\theta_1)\, g(\theta_1)\sin^2(\theta_1)$$

$$= -\int_0^{\theta_1} (2\sin\theta\cos\theta\, g(\theta) + \sin^2\theta\, g'(\theta))(\tfrac{1}{6}u^6 + \tfrac{1}{2}\lambda u^2)\, d\theta, \qquad (39)$$

where $h(\theta) = (1/2)\, g'(\theta)\sin^2\theta - \sin\theta\cos\theta\, g(\theta)$. Then we multiply (38) by $h(\theta)$ just defined. Thus we obtain,

$$\int_0^{\theta_1} u'^2 h(\theta)\, d\theta - \int_0^{\theta_1} u^2(\tfrac{1}{4}g''' + g')\sin^2\theta\, d\theta = \int_0^{\theta_1} (u^6 + \lambda u^2)\, h(\theta)\, d\theta. \quad (40)$$

Substracting (40) from (39) we get

$$\int_0^{\theta_1} u^2 \sin^2\theta(\tfrac{1}{4}g''' + g'(1+\lambda))\, d\theta$$

$$= \tfrac{1}{2}u'^2(\theta_1)\, g(\theta_1)\sin^2\theta_1 + \tfrac{2}{3}\int_0^{\theta_1} u^6 \sin\theta(\cos\theta\, g(\theta) - \sin\theta\, g'(\theta))\, d\theta. \qquad (41)$$

From this point on we have to distinguish two cases: (i) $\lambda < -3/4$, $\theta_1 \leqslant \pi/2$ and (ii) $\lambda \geqslant -3/4$.

Case (i). $\lambda < -3/4$, $\theta_1 \leqslant \pi/2$. For these values of λ choose $g(\theta) \equiv \sin\theta$. Then, $\cos\theta g(\theta) - \sin\theta g'(\theta) \equiv 0$. Moreover,

$$\tfrac{1}{4}g'''(\theta) + (1+\lambda)\, g'(\theta) = \tfrac{1}{4}\cos\theta(3+4\lambda) < 0, \qquad (42)$$

for $0 \leqslant \theta < \theta_1 \leqslant \pi/2$. Since $g(\theta_1) \geqslant 0$, from (41) and (42) we get a contradiction. Therefore, there are no solutions of (38) in this case.

Case (ii). $\lambda \geqslant -3/4$. For these values of λ, $\omega \equiv \sqrt{4(1+\lambda)} > 0$. Now choose $g(\theta) = \sin(\omega\theta)$, so that

$$\tfrac{1}{4}g'''(\theta) + (1+\lambda)\, g'(\theta) \equiv 0. \qquad (43)$$

Clearly, $g(\theta) > 0$ if $0 < \theta \leqslant \theta_1$ and ω is such that the product $\omega\theta_1$ is less than π, i.e., if $\lambda < (\pi^2 - 4\theta_1^2)/4\theta_1^2$. Also,

$$F(\theta) \equiv \cos\theta\, g(\theta) - \sin\theta\, g'(\theta) = \cos\theta\sin(\omega\theta) - \omega\sin\theta\cos(\omega\theta) > 0, \qquad (44)$$

for $0 < \theta < \theta_1$, since $F(0) = 0$ and $F'(\theta) = (3+4\lambda) \sin\theta \sin(\omega\theta) > 0$, whenever $0 < \theta < \theta_1$, $\omega\theta_1 < \pi$ and $3+4\lambda > 0$. Using (43) and (44) in (41), noticing that $g(\theta_1) > 0$, we get a contradiction again, and this lemma is proved. ∎

On the other hand, we also have the rather standard nonexistence result.

LEMMA 3. *If*

$$\lambda \geqslant \lambda_1(\theta_1) \equiv \frac{\pi^2 - \theta_1^2}{\theta_1^2}, \tag{45}$$

with $0 < \theta_1 \leqslant \pi$, *then, there are no positive solutions of* (37).

Remarks 3. (i) Here $\lambda_1(\theta_1)$ is the lowest Dirichlet eigenvalue of a geodesic cap with geodesic radius θ_1. The corresponding eigenfunction is given by $u_1(\theta) = \sin(\pi\theta/\theta_1)/\sin(\theta)$, which is positive inside the cap and symmetric.

(ii) The spectrum of the Laplace–Beltrami operator of geodesic caps on the n dimensional sphere has been considered by several people (see, e.g., [3, 7, 8, 13]).

Proof. Since the positive solutions of (37) are symmetric, we can just consider (38). Multiplying (38) by $u_1 \sin^2(\theta)$, integrating by parts, and using the boundary conditions on u and u_1 we get

$$(\lambda - \lambda_1) \int_0^{\theta_1} u u_1 \sin^2\theta \, d\theta + \int_0^{\theta_1} u^5 u_1 \sin^2\theta \, d\theta = 0, \tag{46}$$

which proves the lemma. ∎

4. BEYOND THE HEMISPHERE WITH $\lambda < -3/4$

In the previous sections, we have determined the existence and nonexistence of solutions for (1) and (2) on a geodesic cap in two cases: (i) for $\theta_1 \leqslant \pi/2$ and all real values of λ, and (ii) for $\theta_1 > \pi/2$, and $\lambda \geqslant -3/4$.

An interesting phenomena occurs when the geodesic cap is larger than the hemisphere. In fact, we have found numerical evidence that for those caps, not only we have the solutions embodied in our Theorem 1(i), but also for a given value of $\theta_1 > \pi/2$, there are positive solutions for all sufficiently negative values of the parameter λ. This is somewhat reminiscent of the analogous problem on an annulus in Euclidean space (see [10]). However, the situation is not completely similar, because here, for a fixed value of geodesic radius $\theta_1 > \pi/2$, there will be a gap between the values of λ for which we have the positive solutions described by Theorem 1(i) and the (sufficiently negative) values of λ for which these new positive solutions exist (see Fig. 2). There is no such a gap in the classical example of the annulus in Euclidean space [10].

Before we go into the discussion of the general case, notice that for the full sphere (i.e., for $\theta_1 = \pi$) there is a trivial positive solution for every negative value of λ, namely the constant $u(\theta) \equiv (-\lambda)^{1/4}$.

For the general case (i.e., for $\pi/2 < \theta_1 < \pi$ and $\lambda < -3/4$), we should also like to determine the region of these parameters for which there exist positive solutions. We have partially solved this problem as we will now describe.

LEMMA 4 (Nonexistence of Positive Solutions). *There is a curve in the* (θ_1, λ)*-plane, denoted by* $\lambda = \nu(\theta_1)$ *(see Fig. 2), such that if* $-3/4 \geqslant \lambda > \nu(\theta_1)$*, then there are no positive solutions of* (1), (2).

Proof. We use the identity (41) that we obtained in Section 3. We distinguish three different (although related) cases.

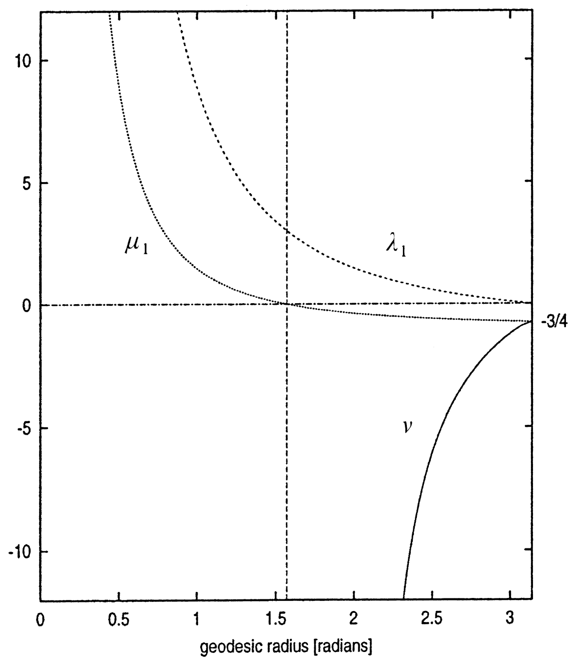

FIG. 2. Range of values of λ for the existence of positive solutions.

(i) $\lambda = -1$. We take $g(\theta) = a\theta - \theta^2$, and choose a in such a way that both $g(\theta) \geqslant 0$ and $\cos\theta\, g(\theta) - \sin\theta\, g'(\theta) \geqslant 0$ for $\theta < a$. Numerically we find $a = 3.04238$. Now, since $\lambda = -1$, $g(0) = 0$, and $g''' \equiv 0$, we get a contradiction from (41) if $\theta_1 < 3.04238$. Hence there are no positive solutions for $\lambda = -1$ and $\theta_1 < 3.04238$. Thus we set $v(3.04238) = -1$.

(ii) If $-3/4 > \lambda > -1$. Let $w = \sqrt{4(1+\lambda)}$ and take $g(\theta) = \sin w\theta - a(\cos w\theta - 1)$, and as in the previous case we choose a in such a way that it gives the largest range of values of θ for which both, $g(\theta)$ and $\cos\theta\, g(\theta) - \sin\theta\, g'(\theta) \geqslant 0$. Denote by $v^{-1}(\lambda)$ this maximal range. Then, for all $\theta_1 < v^{-1}(\lambda)$, since $g'''/4 + (1+\lambda)\, g \equiv 0$, and $g(0) = 0$, we get a contradiction when using (41), and therefore, there are no positive solutions when $-3/4 > \lambda > -1$ and $\theta_1 < v^{-1}(\lambda)$.

(iii) If $-1 > \lambda$. We proceed exactly as in the previous case, but this time we set $w = \sqrt{-4(1+\lambda)}$ and choose $g(\theta) = \sinh w\theta - a(\cosh w\theta - 1)$. The rest of the argument is the same as before, which allows us to complete the curve $\lambda = v(\theta_1)$. ▌

Unfortunately we do not have at the moment and existence theorem for values of λ below the curve $v(\theta_1)$ However, we have performed extensive numerical computations that indicate the existence of positive solutions for

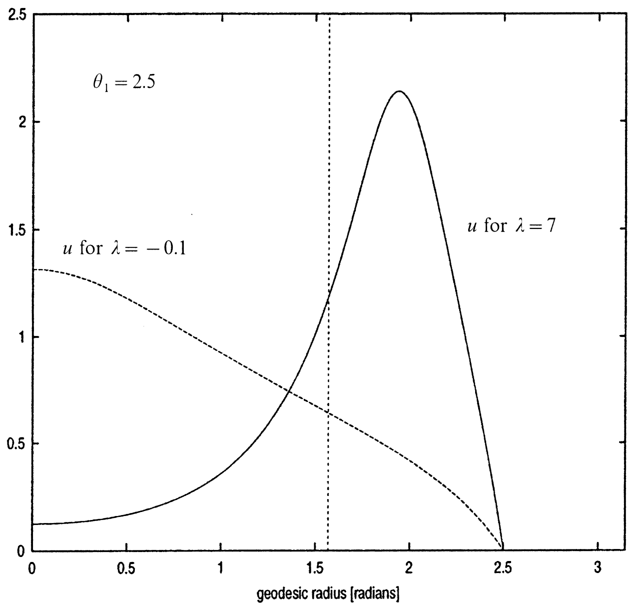

FIG.3. Numerical solution for $\theta_1 = 2.5$, and two different values of λ: $\lambda = -0.1$ (with $u(0) = 1.31474$) and $\lambda = -7$ (with $u(0) = 0.12425$).

values of λ below this curve, and in fact we can get very close to the curve we determined. Just as an example of our numerical solutions, in Fig. 3, we exhibit a solution for $\theta_1 = 2.5$, and $\lambda = -7$. For this solution $u(0) = 0.12425$. Notice that the solution is not decreasing in θ. Quite on the contrary it is highly peaked beyond the hemisphere. For comparison we also shown in Fig. 3 a positive solution with $\lambda = -0.1$ (with $u(0) = 1.31474$), which is decreasing.

Before we conclude this section, we would like to point out that at the special point $\lambda = -3/4$, where we know that there are no positive solutions for any value of θ_1 except at $\theta_1 = \pi$, there is an interesting singular solutions satisfying (1), namely

$$u(\theta) = \frac{1}{\sqrt{2}} \frac{1}{(\sin \theta)^{1/2}}.$$

This may just be a coincidence, however, it might play a role later in the solution to the existence problem for values of $\lambda < -3/4$.

5. APPENDIX

Consider the boundary value problem

$$-(p\varphi')' - p^2\varphi = \mu p^3 \varphi \qquad \text{in } (0, R), \tag{47}$$

with boundary conditions

$$\varphi'(0) = 0 \qquad \text{and} \qquad \varphi(R) = 0. \tag{48}$$

Here, as before,

$$p(r) = \frac{2}{1 + r^2}. \tag{49}$$

Then we have,

LEMMA 5. *The groundstate of the boundary value problem* (47), (48) *is given by*

$$\varphi_1(r) = (1 + r^2) \cos\left(\frac{\pi}{\theta_1} \arctan r\right) \tag{50}$$

where $\theta_1 = 2 \arctan R$. *The corresponding eigenvalue is given by*

$$\mu_1 = \frac{\pi^2}{4\theta_1^2} - 1. \tag{51}$$

Proof. The boundary value problem (47), (48) is defined by a regular Sturm–Liouville operator. It is straightforward to check that $\varphi_1(r)$ satisfies (47) and the boundary conditions (48). Moreover, $\varphi_1(r) > 0$ in $(0, R)$. By the Perron Frobenius theorem, φ_1 is the ground state. ∎

Remark 1. Since φ_1 is the groundstate of the Sturm–Liouville operator defining the boundary value problem (47), (48), we have

$$\int_0^R p(r)\, \varphi'(r)^2\, dr - \int_0^R p(r)^2\, \varphi(r)^2\, dr \geqslant \mu_1 \int_0^R p(r)^3\, \varphi(r)^2\, dr \tag{52}$$

for any smooth function $\varphi(r)$ satisfying the boundary conditions (48). Equality is attained in (52) if and only if $\varphi = \varphi_1$.

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