Local and global uniform convergence for elliptic problems on varying domains

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Abstract

The aim of the paper is to prove optimal results on local and global uniform convergence of solutions to elliptic equations with Dirichlet boundary conditions on varying domains. We assume that the limit domain be stable in the sense of Keldyš [Amer. Math. Soc. Transl. 51 (1966) 1–73]. We further assume that the approaching domains satisfy a necessary condition in the inside of the limit domain, and only require $L^2$-convergence outside. As a consequence, uniform and $L^2$-convergence are the same in the trivial case of homogenisation of a perforated domain. We are also able to deal with certain cracking domains.

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1. Introduction

Given a sequence of open sets \( \Omega_n \subset \mathbb{R}^N \), \( \lambda > 0 \) and \( f_n \in L^\infty(\mathbb{R}^N) \) we let \( u_n \) be the unique (weak) solution of

\[
-\Delta u + \lambda u = f \quad \text{in } \Omega_n, \\
\quad u = 0 \quad \text{on } \partial \Omega_n.
\]

We extend \( u_n \) by zero outside \( \Omega_n \) to get a sequence of functions defined on \( \mathbb{R}^N \).

The aim of this paper is to study necessary and sufficient conditions on \( \Omega_n \) implying uniform convergence, that is, convergence in \( L^\infty(\mathbb{R}^N) \) of \( u_n \) to the solution of

\[
-\Delta u + \lambda u = f \quad \text{in } \Omega, \\
\quad u = 0 \quad \text{on } \partial \Omega
\]
on a limit domain \( \Omega \). Convergence in \( L^2 \) has obtained a lot of attention (see, for instance, [9,10,12,18,22–26]), but there are not many results on uniform convergence if \( \Omega \) is perturbed singularly (for smooth perturbations, see [19]). We make extensive use of sophisticated comparison arguments, so the techniques cannot be applied to Neumann boundary conditions, and in fact many results are not true in that case (see [3]). The results in this paper generalise and complement earlier results in [1,4]. Related results proved by completely different techniques appear in [8]. Our results can be applied to semi-linear elliptic equations and also to linear and non-linear parabolic equations in \( L^\infty \) as shown in [1,13].

Throughout, we allow \( \Omega_n, \Omega \) to be disconnected or unbounded. We will deal with two cases, namely local and global uniform convergence, that is, convergence in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \) or \( L^\infty(\mathbb{R}^N) \). Denote by \( u := R_{\Omega}(\lambda)f \) the solution of (1.2) extended by zero outside \( \Omega \). We will only consider \( f \in L^\infty(\mathbb{R}^N) \), but emphasise that we could use \( f \in L^p(\mathbb{R}^N) \) for \( p > N/2 \) as shown in Corollary 3.5.

We will show in Section 3 that \( R_{\Omega_n}(\lambda)f \to R_{\Omega}(\lambda)f \) in \( L^\infty(\mathbb{R}^N) \) or \( L^\infty_{\text{loc}}(\mathbb{R}^N) \) for all \( f \in L^\infty(\mathbb{R}^N) \) and all \( \lambda > 0 \) if and only if this is the case for \( f \equiv 1 \) and some \( \lambda > 0 \). This motivates the following definitions:

**Definition 1.1.** Let \( \Omega_n, \Omega \) be open subsets of \( \mathbb{R}^N \). We write

1. \( \Omega_n \xrightarrow{\text{gu}} \Omega \) if \( R_{\Omega_n}(\lambda)1 \to R_{\Omega}(\lambda)1 \) in \( L^\infty(R^N) \) for some \( \lambda > 0 \) and say \( \Omega_n \) converges to \( \Omega \) globally uniformly;
2. \( \Omega_n \xrightarrow{\text{lu}} \Omega \) if \( R_{\Omega_n}(\lambda)1 \to R_{\Omega}(\lambda)1 \) in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \) for some \( \lambda > 0 \) and say \( \Omega_n \) converges to \( \Omega \) locally uniformly.

Some sufficient conditions (regular convergence), such that \( \Omega_n \xrightarrow{\text{lu}} \Omega \) are given in [1] (not to be confused with the “regular perturbations” discussed in [19]). We will extend them significantly. Uniform convergence from the interior can be characterised by requiring that there are no holes of non-zero capacity cut into \( \Omega \) (see Theorem 8.3).
Quite surprisingly, and contrary to our initial intuition, uniform convergence from the outside only requires a mild regularity assumption on the limit domain $\Omega$, but not on the domains $\Omega_n$! The condition is that $\overline{\Omega}$ be stable in the sense of Keldyš [21, Section V]. We will say $\Omega$ is uniformly stable. Note that this is not the same as the stability of $\Omega$ in [16] as used for $L^2$-convergence in most papers on $L^2$-convergence mentioned above (see the appendix). Our proof works by localisation, separating the part of $\Omega_n$ at a positive distance from $\Omega$ and the part of $\Omega_n$ close to $\Omega$. The study of the part away from $\Omega$ leads to the case where the limit problem is trivial, and we only require $L^2$-convergence. The part close to $\Omega$ is dealt with by using the stability of $\overline{\Omega}$. We refer to Section 8 for precise statements of these results.

In Section 5, we extensively discuss the case where the limit problem is trivial, that is, $R_{\Omega_n}(\lambda)1 \to 0$. We call this the vanishing case. The interesting fact is, that then, $L^2$- and $L^\infty$-convergence are equivalent. In particular, our results show that in the trivial case in homogenisation theory (see, for instance, [5, Theorem 1.3] or [23, Section 4]) convergence to zero is not just in $L^2$, but uniform! Also, our results show that in the “vanishing case” discussed in [7, Proposition 3.5] convergence is not only in $\mathcal{L}(L^2(\mathbb{R}^N))$ but even in $\mathcal{L}(L^\infty(\mathbb{R}^N))$. A special case also appears in [4, Example 2.17]. Note however, that we do not require the measure of $\Omega_n$ to converge to zero! A standard example is a sequence of periodically perforated domains as shown in Fig. 1. More precisely, $\Omega_n$ is an open rectangle $U$ with $n$ closed balls of radius $r_n$ removed. If they are such that $nr_n^{N-2} \to \infty$ if $N \geq 3$ and $n/|\log r_n| \to \infty$ if $N = 2$, then it is well known (see [23, Section 4]) that the solutions of (1.1) converge to zero in $L^2(\mathbb{R}^N)$. If we choose $r_n > 0$ as above and such that $nr_n^N \to 0$, then the total measure (but not capacity) of the balls converges to zero. Hence, $\text{meas}(\Omega_n) \to \text{meas}(U) \neq 0$ as $n \to \infty$. Our theory shows that convergence is automatically in $L^\infty(\mathbb{R}^N)$. Note that the geometric criteria from [1] do not apply to the above example.

As mentioned above, if the limit domain $\Omega$ is not trivial, then we need vanishing for $\Omega_n \cap \overline{\Omega}^c$, the uniform stability of $\Omega$ and the necessary condition from the inside. Hence we can deal with situations like that shown in Fig. 2.

In fact, we only need a stability condition on that part of the boundary where we come from the outside. This means we can also deal with fairly general cracking domains as shown in Fig. 3.

An outline of the paper is as follows: in Section 2, we introduce the framework and prove some basic inequalities. In Section 3, we prove some general characterisations of local and global uniform convergence. One of the highlights is, that $\Omega_n \overset{lu}{\to} \Omega$ if and only if $\Omega_n \cap B \overset{lu}{\to} \Omega \cap B$ for all bounded open sets $B \subset \mathbb{R}^N$. Hence, to prove local uniform
convergence it is sufficient to look at global uniform convergence of $\Omega_n \cap B$. Another result is that weak*\-convergence of $f_n$ implies local uniform convergence of $R_{\Omega_n}(\lambda)f_n$. This is essential to deal with semi-linear problems as done in [1, Section 8]. Section 4 shows the expected connections to $L^p$\-convergence. As discussed above, Section 5 is concerned with the vanishing case. Sections 6 and 7 provide localisation tools to prove the main results. We make extensive use of the semigroup generated by the Dirichlet Laplacian and the Laplace transform representation of the resolvent $R_{\Omega}(\lambda)$. In Section 8 we state our main convergence criteria. Finally, there is an appendix showing that our notion of uniform stability of $\Omega$ coincides with the stability of $\overline{\Omega}$ introduced in Keldyš [21].

2. Preliminary results

In this section, we briefly discuss properties of the elliptic equation on $L^\infty$ and then prove some key inequalities used throughout the paper. We start by giving a proper formulation of the elliptic problem on arbitrary open sets $\Omega \subset \mathbb{R}^N$. More details can be found in [1]. It is well known that for every $\lambda > 0$ and $f \in L^2(\mathbb{R}^N)$ the problem

$$
\lambda u - \Delta u = f \quad \text{in} \quad \mathcal{D}'(\Omega),
$$

$$
u \in H^1_0(\Omega)
$$

has a unique solution. We write $u = R_{2,\Omega}(\lambda)f$. If $f \in L^2(\Omega) \cap L^\infty(\Omega)$, such that $0 \leq f \leq 1$, then $0 \leq \lambda R_{2,\Omega}(\lambda)f \leq 1$. It follows that the operator has a unique extension $R_{\Omega}(\lambda) \in \mathcal{L}(L^\infty(\Omega))$ which is weak*-continuous. This means that $R_{\Omega}(\lambda)f = R_{\Omega,2}(\lambda)f$ for all $f \in L^2(\Omega) \cap L^\infty(\Omega)$ and that $R_{\Omega}(\lambda)f_n \rightharpoonup^* R_{\Omega}(\lambda)f$ weak* whenever $f_n \rightharpoonup^* f$ weak* in $L^\infty(\Omega)$.
By duality we define an operator on $L^1(\Omega)$ by setting $R_1 \Omega(\lambda) := (R_\Omega(\lambda))'$. By interpolation we get operators $R_p, \Omega(\lambda)$ on $L^p(\Omega)$ for all $p \in (1, \infty)$. As $R_2, \Omega(\lambda)$ is self-adjoint we have $R_{p, \Omega}(\lambda) f = R_q, \Omega(\lambda) f$ for all $f \in L^p(\Omega) \cap L^q(\Omega)$. If no confusion seems likely we simply write $R_\Omega(\lambda)$ for $R_{p, \Omega}(\lambda)$. Consequently,

$$\| R_\Omega(\lambda) \|_{L(L^p(\Omega))} \leq \frac{1}{\lambda} \quad (2.1)$$

for all $1 \leq p \leq \infty$ and $\lambda > 0$. Also, for every $f \in L^\infty(\Omega)$ we have $u := R_\Omega(\lambda) f \in C(\Omega)$, but $u$ is not necessarily in $H^1_0(\Omega)$. There are other characterisations of $R_\Omega(\lambda)$. Recall that

$$H^1_{\text{loc}}(\mathbb{R}^N) := \{ u \in L^2_{\text{loc}}(\mathbb{R}^N) : \partial_i u \in L^2_{\text{loc}}(\mathbb{R}^N) \text{ for } i = 1, \ldots, N \}.$$

We then set

$$H^1_{0, \text{loc}}(\Omega) := \{ u \in H^1_{\text{loc}}(\mathbb{R}^N) : \varphi u \in H^1_0(\Omega) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^N) \}.$$

If $f \in L^\infty(\Omega)$ it turns out (see [1, Proposition 1.3]) that $u = R_\Omega(\lambda) f$ if and only if

$$\lambda u - \Delta u = f \quad \text{in } \mathcal{D}'(\Omega),$$

$$u \in H^1_{0, \text{loc}}(\Omega) \cap L^\infty(\Omega).$$

Since we are working with varying domains we wish to define $R_\Omega(\lambda)$ to be an operator on $L^\infty(\mathbb{R}^N)$. To do so let $i_\Omega \in \mathcal{L}(L^p(\Omega), L^p(\mathbb{R}^N))$ denote the natural extension of functions by zero, and $r_\Omega \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\Omega))$ the natural restriction to $\Omega$. Then

$$\tilde{R}_\Omega(\lambda) := i_\Omega \circ R_\Omega(\lambda) \circ r_\Omega \in \mathcal{L}(L^p(\mathbb{R}^N))$$

for all $p \in [1, \infty]$. Since $\| i_\Omega \| = \| r_\Omega \| = 1$ the operator $\tilde{R}_\Omega(\lambda)$ satisfies the same estimate (2.1). The duals of $i_\Omega$ and $r_\Omega$ are $i'_\Omega = r_\Omega$ and $r'_\Omega = i_\Omega$ for all $p \in [1, \infty)$, so $\tilde{R}_\Omega(\lambda)$ has the same duality properties as $R_\Omega(\lambda)$. For this reason we identify $R_\Omega(\lambda)$ with $\tilde{R}_\Omega(\lambda)$. Finally, by convention we set

$$R_\emptyset(\lambda) := 0$$

for all $\lambda > 0$ if $\Omega = \emptyset$.

The operator $R_\Omega(\lambda)$ has some useful monotonicity properties. If $\Omega_1 \subset \Omega_2$ are open sets and $0 \leq f_1 \leq f_2 \in L^\infty(\Omega_2)$, then

$$0 \leq R_{\Omega_1}(\lambda) f_1 \leq R_{\Omega_2}(\lambda) f_2.$$

(2.2)
In the sequel we shall use these properties without further comment. Denote by $B(x,r)$ the open ball in $\mathbb{R}^N$ with radius $r$ and centre $x$. Then clearly $1_{B(0,r)} \to 1$ in $L^\infty(\mathbb{R}^N)$, so by construction of $R_{\mathbb{R}^N} (\lambda)$ we have $R_{\mathbb{R}^N} (\lambda)1_{B(0,r)} \to R_{\mathbb{R}^N} (\lambda)1$. By the monotonicity properties and Dini’s theorem it follows that

$$0 \leq R_{\mathbb{R}^N} (\lambda) 1_{B(0,r)} \nearrow R_{\mathbb{R}^N} (\lambda) 1 = \frac{1}{\lambda}$$

(2.3)

in $C(\mathbb{R}^N)$ as $r \to \infty$, that is, uniformly on compact subsets of $\mathbb{R}^N$. Also, it is well known (see, for instance, [1]) that

$$0 \leq R_{B(0,r)} (\lambda) 1 \nearrow R_{\mathbb{R}^N} (\lambda) 1 = \frac{1}{\lambda}$$

(2.4)

in $C(\mathbb{R}^N)$ as $r \to \infty$. We shall frequently use the two facts in conjunction with the inequalities proved below. We need a characterisation of $H^1_{0, \text{loc}} (\Omega)$ involving capacity. Recall that every $u \in H^1_{0, \text{loc}} (\mathbb{R}^N)$ admits a quasi-continuous version $\tilde{u}$ which is unique up to a polar set (see [17, Theorems 4.4 and 4.12]). Then we have

$$H^1_{0, \text{loc}} (\Omega) = \{ u \in H^1_{0, \text{loc}} (\mathbb{R}^N): \tilde{u} = 0 \text{ quasi-everywhere on } \Omega^c \}.$$

In what follows we do not distinguish between a function $u \in H^1_{0, \text{loc}} (\mathbb{R}^N)$ and its quasi-continuous version. Several times we will make use of the following technical lemma:

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^N$, $\lambda > 0$. For $\delta \geq 0$ let $M := \{ x \in \Omega: u(x) > \delta \}$ and set $u := R_{\Omega} (\lambda) 1$. Then $M \subset \Omega$ is open and $(u - \delta)^+ = R_M (\lambda) (1 - \lambda \delta)$.

**Proof.** Since $u \in C (\Omega)$ the set $M$ is open, and $u \leq \delta$ on $M^c \cap \Omega$. As $(u - \delta)^+ \leq u$ and $u \in H^1_{0, \text{loc}} (\Omega)$ we have $u \leq \delta$ quasi-everywhere on $\Omega^c$. Hence, $u \leq \delta$ quasi-everywhere on $M^c$, so $w := (u - \delta)^+ \in H^1_{0, \text{loc}} (\mathbb{R}^N) \cap L^\infty (\mathbb{R}^N)$ as well. By assumption $w = 0$ on $M^c$, so $w \in H^1_{0, \text{loc}} (M) \cap L^\infty (\mathbb{R}^N)$. As $w = u - \delta$ on $M$, it follows that $\lambda w - \Delta w = 1 - \lambda \delta$ in $\mathcal{D}'(M)$. Now [1, Proposition 1.3] implies that $w = R_M (\lambda) (1 - \lambda \delta)$ as claimed.

Note that the proof of the above lemma can be considerably simplified if $M$ has a smooth boundary and therefore $H^1$ functions have a proper trace on $\partial M$. Next we prove the first key inequality.

**Theorem 2.2** (Intersection inequality). Let $U, V \subset \mathbb{R}^N$ be open and $\lambda > 0$. Then

$$\| R_{U \cap B} (\lambda) 1 - R_{V \cap B} (\lambda) 1 \|_{L^\infty (\mathbb{R}^N)} \leq \| R_{U} (\lambda) 1 - R_{V} (\lambda) 1 \|_{L^\infty (B)}$$

(2.5)

for all open sets $B \subset \mathbb{R}^N$. 


Theorem 2.3

Proof. Fix open sets $U, V, B \subset \mathbb{R}^N$. Since $R_{U \cap B}(\lambda) 1 \leq R_U(\lambda) 1$ on $\mathbb{R}^N$ and $R_V(\lambda) 1 = 0$ quasi-everywhere on $V^c$ we have

$$v := R_{U \cap B}(\lambda) 1 - \|R_U(\lambda) 1 - R_V(\lambda) 1\|_{L^\infty(B)} \leq 0$$

quasi-everywhere on $(U \cap V \cap B)^c$. As $v \in C(U \cap V \cap B)$, the set

$$M := \{x \in U \cap V \cap B : v(x) > 0\}$$

is open. Moreover, by the above $v \leq 0$ quasi-everywhere on $(U \cap V \cap B)^c$. Hence, $\lambda v - \Delta v = f$ in $\mathcal{D}'(U \cap V \cap B)$ if we set $f := 1 - \lambda \|R_U(\lambda) 1 - R_V(\lambda) 1\|_{L^\infty(B)}$. Now [1, Proposition 1.3] implies that $v^+ = R_M(\lambda)f$. By domination $v \leq v^+ \leq R_{U \cap V \cap B}(\lambda) 1 \leq R_{V \cap B}(\lambda) 1$ on $U \cap V \cap B$. Combining this with (2.6) we get

$$R_{U \cap B}(\lambda) 1 - R_{V \cap B}(\lambda) 1 \leq \|R_U(\lambda) 1 - R_V(\lambda) 1\|_{L^\infty(B)}$$

quasi-everywhere on $\mathbb{R}^N$. By interchanging the roles of $U$ and $V$ inequality (2.5) follows. □

Let $\mathcal{T}$ denote the topology of $\mathbb{R}^N$, that is, $\mathcal{T}$ consists of all open subsets of $\mathbb{R}^N$. Then $(\mathcal{T}, \subset)$ is a partially ordered set. We have the following monotonicity properties:

Theorem 2.3 (Monotonicity theorem). Let $\lambda > 0$ and $p \in [1, \infty]$. For $f \in L^p(\mathbb{R}^N)$ non-negative consider the mapping $D_f : \mathcal{T} \times \mathcal{T} \to H^1_{\text{loc}}(\mathbb{R}^N)$ given by

$$D_f(\Omega, B) := R_\Omega(\lambda)f - R_{\Omega \cap B}(\lambda)f.$$

Then, for fixed $B$, the mapping $D_f(\cdot, B) : \mathcal{T} \to L^p(\mathbb{R}^N)$ is increasing. Moreover, for fixed $\Omega$, the mapping $D_f(\Omega, \cdot) : \mathcal{T} \to L^p(\mathbb{R}^N)$ is decreasing.

Proof. We first give a proof in case $p = \infty$. Note that the monotonicity with respect to the second argument immediately follows from (2.2). Hence, it remains to prove the monotonicity in the first argument. To do so fix open sets $B$ and $\Omega_1 \subset \Omega_2$ in $\mathbb{R}^N$. For $f \in L^\infty(\mathbb{R}^N)$ non-negative set

$$h := R_{\Omega_1 \cap B}(\lambda)f - R_{\Omega_2}(\lambda)f + R_{\Omega_1}(\lambda)f - R_{\Omega_1 \cap B}(\lambda)f \in H^1_{\text{loc}}(\mathbb{R}^N).$$

Then $w := h^+ \in H^1_{\text{loc}}(\mathbb{R}^N)$, $w \in C(\Omega_1 \cap B)$ and $M := \{x \in \Omega_1 \cap B : h(x) > 0\}$ is open. Since $h = R_{\Omega_2 \cap B}(\lambda)f - R_{\Omega_2}(\lambda)f \leq 0$ quasi-everywhere on $\Omega_2^c$ and since $h = R_{\Omega_1}(\lambda)f - R_{\Omega_2}(\lambda)f \leq 0$ quasi-everywhere on $B^c$ we get that $w = 0$ quasi-everywhere on $M^c$ and then that $w \in H^1_{0,\text{loc}}(M) \cap L^\infty(\mathbb{R}^N)$. Moreover, since $\lambda w - \Delta w = 0$ in $\mathcal{D}'(M)$ it follows from [1, Proposition 1.3] that $w = R_M(\lambda)0 = 0$ quasi-everywhere on
\[ \mathbb{R}^N = M \cup M^c \] (note that \( M = \emptyset \)). Hence \( h \leq 0 \) on \( \mathbb{R}^N \), completing the proof of the theorem in case \( p = \infty \). If \( f \in L^p(\mathbb{R}^N) \) is non-negative, there exists a sequence of non-negative test functions \( f_n \in D(\mathbb{R}^N) \), such that \( f_n \to f \) in \( L^p(\mathbb{R}^N) \). If \( \Omega_1 \subset \Omega_2 \subset \mathbb{R}^N \) are open sets, then the above implies that \( Df_n(\Omega_1, B) \leq Df_n(\Omega_2, B) \). Taking the limit as \( n \to \infty \), we get \( Df(\Omega_1, B) \leq Df(\Omega_2, B) \) as claimed. \( \square \)

**Remark 2.4.** Let \( \Omega_1 \subset \Omega_2 \subset \mathbb{R}^N \) be open and fixed. It follows from Theorem 2.3 that \( R_{\Omega_1}(\lambda)1 - R_{\Omega_1 \cap B}(\lambda)1 \leq R_{\Omega_2}(\lambda)1 - R_{\Omega_2 \cap B}(\lambda)1 \). Rewriting this inequality we get that \( 0 \leq R_{\Omega_2 \cap B}(\lambda)1 - R_{\Omega_1 \cap B}(\lambda)1 \leq R_{\Omega_2}(\lambda)1 - R_{\Omega_1}(\lambda)1 \). Taking on both sides the norm \( \| \cdot \|_{L^\infty(B)} \) we get Theorem 2.2 in the case when \( U \subset V \) or \( V \subset U \).

### 3. Local versus global uniform convergence

The purpose of this section is to give basic characterisations for local and global uniform convergence. We will also show that local uniform convergence can be obtained by localisation from global uniform convergence. We first prove that uniform convergence of \( R_{\Omega}(\lambda)1 \) for some \( \lambda > 0 \) implies convergence of \( R_{\Omega_n}(\lambda) \) in the operator norm in \( \mathcal{L}(L^\infty(\mathbb{R}^N)) \). The theorem also assures that the notion of global uniform convergence (\( \equiv \)) given in Definition 1.1 is independent of \( \lambda > 0 \).

**Theorem 3.1.** Let \( \Omega, \Omega_n \subset \mathbb{R}^N \) be open and \( \lambda > 0 \). Then the following assertions are equivalent:

1. \( R_{\Omega_n}(\lambda) \to R_{\Omega}(\lambda) \) in \( \mathcal{L}(L^\infty(\mathbb{R}^N)) \);
2. \( R_{\Omega_n}(\lambda)1 \to R_{\Omega}(\lambda)1 \) in \( L^\infty(\mathbb{R}^N) \).

If one of the two assertions holds for some \( \lambda > 0 \), then they both hold for all \( \lambda > 0 \). More generally, if (1) or (2) holds for some \( \lambda > 0 \) and \( \mu \in \mathbb{C} \) is such that \( \sup_{n \in \mathbb{N}} \| R_{\Omega_n}(\mu)\|_{\mathcal{L}(L^\infty)} < \infty \), then \( R_{\Omega_n}(\mu) \to R_{\Omega}(\mu) \) in \( \mathcal{L}(L^\infty(\mathbb{R}^N)) \).

**Proof.** Obviously (1) implies (2). Suppose now that (2) holds. Since the operator norm of a positive operator on \( T \in \mathcal{L}(L^\infty(\mathbb{R}^N)) \) is given by \( \| T1 \|_{\infty} \) we get

\[
\| R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda) \|_{\mathcal{L}(L^\infty)} \\
\leq \| R_{\Omega_n}(\lambda) - R_{\Omega \cap \Omega_n}(\lambda) \|_{\mathcal{L}(L^\infty)} + \| R_{\Omega \cap \Omega_n}(\lambda)1 - R_{\Omega}(\lambda)1 \|_{\mathcal{L}(L^\infty)} \\
= \| R_{\Omega_n}(\lambda)1 - R_{\Omega \cap \Omega_n}(\lambda)1 \|_{\infty} + \| R_{\Omega \cap \Omega_n}(\lambda)1 - R_{\Omega}(\lambda)1 \|_{\infty}.
\]

Applying Theorem 2.2 with \( B = \Omega_n \) and \( \Omega \), respectively, we get

\[
\| R_{\Omega_n}(\lambda)1 - R_{\Omega \cap \Omega_n}(\lambda)1 \|_{\infty} + \| R_{\Omega \cap \Omega_n}(\lambda)1 - R_{\Omega}(\lambda)1 \|_{\infty} \leq 2 \| R_{\Omega_n}(\lambda)1 - R_{\Omega}(\lambda)1 \|_{\infty}.
\]

By assumption the last term converges to zero, and so (1) follows. Next, we prove that if \( R_{\Omega_n}(\lambda)1 \to R_{\Omega}(\lambda)1 \) for some \( \lambda \geq 0 \) then convergence takes place for all \( \lambda > 0 \). In
the light of what we just proved this completes the proof of the theorem. Let \( \lambda, \mu > 0, \mu \neq \lambda \) be fixed. We set \( g := 1 + (\lambda - \mu)R_{\Omega}(\mu)1 \) and \( g_n := g - (\lambda - \mu)R_{\Omega_n}(\lambda)g \). Then by the resolvent equation

\[
R_{\Omega}(\mu)1 = R_{\Omega}(\lambda)g \quad \text{and} \quad R_{\Omega_n}(\mu)g_n = R_{\Omega_n}(\lambda)g.
\] (3.1)

Using assumption (1) we have \( R_{\Omega_n}(\lambda)g \to R_{\Omega}(\lambda)g \) in \( L^\infty(\mathbb{R}^N) \) as \( n \to \infty \). Hence, by (3.1) and the definition of \( g \) we get

\[
\lim_{n \to \infty} g_n = g - (\lambda - \mu)R_{\Omega}(\lambda)g = 1 + (\lambda - \mu)R_{\Omega}(\mu)1 - (\lambda - \mu)R_{\Omega}(\mu)1 = 1
\]

in \( L^\infty(\mathbb{R}^N) \), showing that \( \|R_{\Omega_n}(\mu)(1 - g_n)\|_{L^\infty} \leq \mu^{-1}\|1 - g_n\|_{L^\infty} \to 0 \) as \( n \to \infty \). Taking into account (3.1) we therefore conclude that

\[
\|R_{\Omega_n}(\mu)1 - R_{\Omega}(\mu)1\|_{L^\infty} \leq \|R_{\Omega_n}(\mu)(1 - g_n)\|_{L^\infty} + \|R_{\Omega_n}(\mu)g_n - R_{\Omega}(\mu)1\|_{L^\infty} \\
\leq \frac{1}{\mu}\|1 - g_n\|_{L^\infty} + \|R_{\Omega_n}(\lambda)g - R_{\Omega}(\lambda)g\|_{L^\infty} \xrightarrow{n \to \infty} 0.
\]

Finally, if \( M := \sup_{n \in \mathbb{N}} \|R_{\Omega_n}(\mu)\|_{L(\mathbb{R}^\infty)} < \infty \), then simply replace \( 1/\mu \) by \( M \) in the above estimate. \( \Box \)

We note that convergence in the operator norm implies convergence of finite parts of the spectrum and the corresponding projections (see [1, Section 7]). Before we state the next result let us recall some common notation.

**Definition 3.2** *(Compact inclusion)*. Given \( U, V \subset \mathbb{R}^N \) we write \( U \subset \subset V \) if \( U, V \) are sets, such that \( \overline{U} \) is compact and \( \overline{U} \subset \text{int}(V) \).

In particular note that, \( U \) is bounded if \( U \subset \subset V \).

**Theorem 3.3.** Let \( \Omega, \Omega_n \subset \mathbb{R}^N \) be open sets, \( \lambda > 0 \), \((f_n)\) a bounded sequence in \( L^\infty(\mathbb{R}^N) \) and \( f \in L^\infty(\mathbb{R}^N) \). If \( R_{\Omega_n \cap B}(\lambda)f_n \to R_{\Omega \cap B}(\lambda)f \) in \( L^\infty(\mathbb{R}^N) \) for all open sets \( B \subset \subset \mathbb{R}^N \), then \( R_{\Omega_n}(\lambda)f_n \to R_{\Omega}(\lambda)f \) in \( L^\infty(\mathbb{R}^N) \).

**Proof.** For every compact set \( K \subset \mathbb{R}^N \) we clearly have

\[
\|R_{\Omega_n}(\lambda)f_n - R_{\Omega}(\lambda)f\|_{L^\infty(K)} \leq \|R_{\Omega_n}(\lambda)f_n - R_{\Omega_n \cap B}(\lambda)f_n\|_{L^\infty(K)} \\
+\|R_{\Omega_n \cap B}(\lambda)f_n - R_{\Omega \cap B}(\lambda)f\|_{L^\infty(K)} + \|R_{\Omega \cap B}(\lambda)f - R_{\Omega}(\lambda)f\|_{L^\infty(K)}.
\] (3.2)
Setting $M := \sup\{\|f_n\|_{L^{\infty}(\mathbb{R}^N)} : n \in \mathbb{N}\}$, we have

$$\|R_{\Omega_n}(\lambda) f_n - R_{\Omega_n \cap B}(\lambda) f_n\|_{L^{\infty}(K)} + \|R_{\Omega} \cap B(\lambda) f - R_{\Omega}(\lambda) f\|_{L^{\infty}(K)} \leq M \left( \|R_{\Omega_n}(\lambda) 1 - R_{\Omega_n \cap B}(\lambda) 1\|_{L^{\infty}(K)} + \|R_{\Omega} \cap B(\lambda) 1 - R_{\Omega}(\lambda) 1\|_{L^{\infty}(K)} \right).$$

(3.3)

Fix a compact set $K \subset \mathbb{R}^N$ and $\varepsilon > 0$. By (2.4) there exists $B \subset \subset \mathbb{R}^N$, such that

$$\|R_{\mathbb{R}^N}(\lambda) 1 - R_B(\lambda) 1\|_{L^{\infty}(K)} \leq \frac{\varepsilon}{4M}.$$

Using Theorem 2.3 with $D_1(\Omega, B) \leq D_1(\mathbb{R}^N, B)$ and $D_1(\Omega_n, B) \leq D_1(\mathbb{R}^N, B)$ we see that

$$\|R_{\Omega_n}(\lambda) 1 - R_{\Omega_n \cap B}(\lambda) 1\|_{L^{\infty}(K)} + \|R_{\Omega} \cap B(\lambda) 1 - R_{\Omega}(\lambda) 1\|_{L^{\infty}(K)} \leq 2\|R_{\mathbb{R}^N}(\lambda) 1 - R_B(\lambda) 1\|_{L^{\infty}(K)} \leq \frac{\varepsilon}{2M}$$

(3.4)

for all $n \in \mathbb{N}$. Combining the above inequality with (3.2) and (3.3) we get

$$\|R_{\Omega_n}(\lambda) f_n - R_{\Omega}(\lambda) f\|_{L^{\infty}(K)} \leq \frac{\varepsilon}{2} + \|R_{\Omega_n \cap B}(\lambda) f_n - R_{\Omega \cap B}(\lambda) f\|_{L^{\infty}(K)}$$

for all $n \in \mathbb{N}$. By assumption there exists $n_0 \in \mathbb{N}$, such that

$$\|R_{\Omega_n \cap B}(\lambda) f_n - R_{\Omega \cap B}(\lambda) f\|_{L^{\infty}(K)} \leq \frac{\varepsilon}{2}$$

for all $n \geq n_0$, leading to

$$\|R_{\Omega_n}(\lambda) f_n - R_{\Omega}(\lambda) f\|_{L^{\infty}(K)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq n_0$. Since $\varepsilon > 0$ and $K$ were arbitrary the theorem is proved. □

We next show that the notion of local uniform convergence ($\xrightarrow{lu}$) given in Definition 1.1 is independent of $\lambda > 0$. The theorem can be considered as a counterpart of Theorem 3.1 in case of local uniform convergence. We will need the space

$$L_0^{\infty}(\mathbb{R}^N) := \{u \in L^{\infty}(\mathbb{R}^N) : \text{there exists } v \in C_0(\mathbb{R}^N) \text{ with } |u| \leq v\},$$

which is a closed subspace of $L^{\infty}(\mathbb{R}^N)$. 
Theorem 3.4. Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and $\lambda > 0$. Then the following assertions are equivalent:

1. $R_{\Omega_n}(\lambda)1 \to R_{\Omega}(\lambda)1$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$;
2. $R_{\Omega_n \cap B}(\lambda)1 \to R_{\Omega \cap B}(\lambda)1$ in $L^\infty(\mathbb{R}^N)$ for all open sets $B \subset \subset \mathbb{R}^N$;
3. $R_{\Omega_n}(\lambda) f_n \to R_{\Omega}(\lambda) f$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ in $L^\infty(\mathbb{R}^N)$;
4. $R_{\Omega_n}(\lambda) f \to R_{\Omega}(\lambda) f$ in $L^\infty(\mathbb{R}^N)$ for all $f \in L^\infty(\mathbb{R}^N)$;
5. $R_{\Omega_n}(\lambda) f \to R_{\Omega}(\lambda) f$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ for all $f \in L^\infty(\mathbb{R}^N)$ with compact support.

If one of the above assertions holds for some $\lambda > 0$, then they all hold for every $\lambda > 0$.

Proof. Suppose (1) holds. Given $B \subset \subset \mathbb{R}^N$ we conclude from Theorem 2.2 that

$$
\|R_{\Omega_n \cap B}(\lambda)1 - R_{\Omega \cap B}(\lambda)1\|_{\infty} \leq \|R_{\Omega_n}(\lambda)1 - R_{\Omega}(\lambda)1\|_{L^\infty(B)} \xrightarrow{B \to \infty} 0,$$

proving (2). Next assume (2) holds. Suppose $f_n \rightharpoonup f$ in $L^\infty(\mathbb{R}^N)$ and fix an arbitrary open set $B \subset \subset \mathbb{R}^N$. Then

$$
\|R_{\Omega_n \cap B}(\lambda) f_n - R_{\Omega \cap B}(\lambda) f\|_{L^\infty(\mathbb{R}^N)} \\
\leq \|R_{\Omega_n \cap B}(\lambda) - R_{\Omega \cap B}(\lambda)\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} \|f_n\|_{L^\infty(\mathbb{R}^N)} \\
+ \|R_{\Omega \cap B}(\lambda) f_n - R_{\Omega \cap B}(\lambda) f\|_{L^\infty(\mathbb{R}^N)}
$$

for all $n \in \mathbb{N}$. By assumption and Theorem 2.2 we conclude that

$$
\|R_{\Omega_n \cap B}(\lambda)1 - R_{\Omega \cap B}(\lambda)1\|_{\infty} \leq \|R_{\Omega_n}(\lambda)1 - R_{\Omega}(\lambda)1\|_{L^\infty(B)} \xrightarrow{B \to \infty} 0,$$

and so by Theorem 3.1 we have that $R_{\Omega_n \cap B}(\lambda) \to R_{\Omega \cap B}(\lambda)$ in $\mathcal{L}(L^\infty(\mathbb{R}^N))$ for all open sets $B \subset \subset \mathbb{R}^N$. Since $f_n$ is bounded in $L^\infty(\mathbb{R}^N)$ the first term on the right-hand side of (3.5) converges to zero. Since $\Omega \cap B$ is bounded, $R_{\Omega \cap B}(\lambda)$ is compact (see [1, Theorem 7.2]), so the second term also converges to zero. Hence (3) follows from Theorem 3.3. Suppose now that (3) holds. If $f \in L^\infty_0(\mathbb{R}^N)$, then $u_n := R_{\Omega_n}(\lambda) f \rightarrow u := R_{\Omega}(\lambda) f$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ as $n \to \infty$. By [1, Proposition 2.6] we have that $w := R_{\Omega^N}(\lambda) f \in C_0(\mathbb{R}^N)$. Now fix $\varepsilon > 0$ arbitrary. Since $w \in C_0(\mathbb{R}^N)$ there exists $r > 0$, such that $0 \leq w \leq \varepsilon/2$ on $B(0, r)^c$. Using domination $|u| \leq w$ and $|u_n| \leq w$ for all $n \in \mathbb{N}$, so $|u_n - u| \leq 2w \leq \varepsilon$ on $B(0, r)^c$. Since $u_n \to u$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ there exists $n_0 \in \mathbb{N}$, such that $|u_n - u| \leq \varepsilon$ almost everywhere on $B(0, r)$. Combining the two estimates we get $\|u_n - u\|_{L^\infty} \leq \varepsilon$ for all $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, (4) follows. It is obvious that (4) implies (5), so it remains to show that (5) implies (1). Let $K \subset \mathbb{R}^N$ be a compact set and $\varepsilon > 0$ be arbitrary. By domination we have

$$
0 \leq R_{\Omega_n}(\lambda)1 - R_{\Omega_n}(\lambda)1_{B(0, r)} = R_{\Omega_n}(\lambda)1_{B(0, r)^c} \leq R_{\mathbb{R}^N}(\lambda)1_{B(0, r)^c}
$$
for all \( n \in \mathbb{N} \). Hence (2.4) implies the existence of \( r > 0 \), such that
\[
\| R_{\Omega_n}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} \leq \frac{\varepsilon}{3}
\]
for all \( n \in \mathbb{N} \). Similarly, for the same \( r > 0 \), \( \| R_{\Omega}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} \leq \varepsilon / 3 \). Hence,
\[
\| R_{\Omega_n}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} \leq \| R_{\Omega_n}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} + \| R_{\Omega}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} + 2 \varepsilon
\]
for all \( n \in \mathbb{N} \). By assumption (5) there exists \( n_0 \in \mathbb{N} \) such that
\[
\| R_{\Omega_n}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} \leq \frac{\varepsilon}{3}
\]
for all \( n \geq n_0 \). Therefore, \( \| R_{\Omega_n}(\lambda) 1 - R_{\Omega}(\lambda) 1 \|_{L^\infty(K)} \leq \varepsilon \) for all \( n \geq n_0 \). As \( \varepsilon > 0 \) and \( K \) were arbitrary, (1) follows. To prove the last claim, suppose one of the assertions holds for some \( \lambda > 0 \). Then, by what we proved, all assertions hold for that \( \lambda > 0 \), so in particular (2) holds. By Theorem 3.1 property (2) holds for every \( \lambda > 0 \), so by what we proved, all assertions hold for every \( \lambda > 0 \), completing the proof of the theorem.

In the above theorem, we have only considered \( f_n, f \in L^\infty(\mathbb{R}^N) \). This is not necessary as we show below.

**Corollary 3.5.** If \( \Omega_n \xrightarrow{lu} \Omega \), then \( R_{\Omega_n}(\lambda) f \rightarrow R_{\Omega}(\lambda) f \) in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \) for all \( f \in L^p(\mathbb{R}^N) \) with \( p > N/2 \). If \( \Omega_n \xrightarrow{gu} \Omega \), then convergence is in \( L^\infty(\mathbb{R}^N) \).

**Proof.** We know that there exists a constant \( C > 0 \) independent of the domain \( \Omega \), such that
\[
\| R_{\Omega}(\lambda) \|_{L^p, L^\infty} \leq C
\]
(see [11]). Fix \( f \in L^p(\mathbb{R}^N) \) and \( \varepsilon > 0 \) arbitrary. Since \( L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) is dense in \( L^p(\mathbb{R}^N) \) there exists \( g \in L^\infty(\mathbb{R}^N) \), such that \( \| f - g \|_p < \varepsilon / 4C \). Let now \( B \subset \mathbb{R}^N \) be a bounded set. Then, by assumption, there exists \( n_0 \in \mathbb{N} \), such that
∥Rn(λ)g − RΩ(λ)g∥L∞(B) < ε/2 for all n > n0. Hence,

\[
\|R_{\Omega_n}(\lambda)f - R_{\Omega}(\lambda)f\|_{L^{\infty}(B)} \\
\leq \|R_{\Omega_n}(\lambda)(f - g)\|_{L^{\infty}(B)} + \|R_{\Omega_n}(\lambda)g - R_{\Omega}(\lambda)\|_{L^{\infty}(B)} \\
+ \|R_{\Omega}(\lambda)(g - f)\|_{L^{\infty}(B)} < 2\|f - g\|_p + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

for all n > n0. Since ε > 0 was arbitrary the first assertion of the corollary follows. If Ωn \xrightarrow{gu} Ω we simply replace B by \(\mathbb{R}^N\) in the above argument. □

We next collect some facts about the convergence of various intersections.

**Theorem 3.6 (Intersection theorem).** Let \(U, U_n, V, V_n \subset \mathbb{R}^N\) be open sets. Then the following assertions hold:

1. If \(U_n \xrightarrow{lu} U\) and \(V_n \xrightarrow{lu} V\), then \(U_n \cap V_n \xrightarrow{lu} U \cap V\).
2. If \(U_n \xrightarrow{gu} U\) and \(V_n \xrightarrow{gu} V\), then \(U_n \cap V_n \xrightarrow{gu} U \cap V\).
3. \(U_n \xrightarrow{lu} U\) if and only if \(U_n \cap B \xrightarrow{gu} U \cap B\) for all open sets \(B \subset \mathbb{R}^N\).

**Proof.** To prove (1) fix \(\lambda > 0\). By definition of convergence and Theorem 3.1 we know that \(R_{U_n}(\lambda)1 \rightarrow R_U(\lambda)1\) and \(R_{V_n}(\lambda)1 \rightarrow R_V(\lambda)1\) in \(L^{\infty}(\mathbb{R}^N)\). Hence by Theorem 2.2

\[
\|R_{U \cap V_n}(\lambda)1 - R_{U \cap V}(\lambda)1\|_{\infty} \\
\leq \|R_{U \cap V_n}(\lambda)1 - R_{U \cap V_n}(\lambda)1\|_{\infty} + \|R_{U \cap V_n}(\lambda)1 - R_{U \cap V}(\lambda)1\|_{\infty} \\
\leq \|R_{U_n}(\lambda)1 - R_{U}(\lambda)1\|_{\infty} + \|R_{V_n}(\lambda)1 - R_{V}(\lambda)1\|_{\infty} \xrightarrow{n \to \infty} 0,
\]

so (1) follows. Next we prove (2). It follows form Theorem 3.4 that \(U_n \cap B \xrightarrow{gu} U \cap B\) and \(V_n \cap B \xrightarrow{gu} V \cap B\) for all open sets \(B \subset \mathbb{R}^N\). By (1) we have \((U_n \cap V_n) \cap B \xrightarrow{gu} (U \cap V) \cap B\). Applying Theorem 3.3 we conclude that \(U_n \cap V_n \xrightarrow{lu} U \cap V\), completing the proof of (2). Assertion (3) is a consequence of Theorem 3.4. □

4. Connections to \(L^p\)-convergence

We naturally expect that convergence of \(R_{\Omega_n}(\lambda)\) in \(L^{\infty}(\mathbb{R}^N)\) implies convergence in \(L^p(\mathbb{R}^N)\) for all \(p \in [1, \infty]\). We will show that this is indeed the case. We first look at local uniform convergence.

**Proposition 4.1.** Let \(\Omega, \Omega_n \subset \mathbb{R}^N\) be open sets and \(\lambda > 0\). If \(\Omega_n \xrightarrow{lu} \Omega\) and \(p \in [1, \infty]\), then \(R_{\Omega_n}(\lambda)f \rightarrow R_{\Omega}(\lambda)f\) in \(L^p(\mathbb{R}^N)\) for all \(f \in L^p(\mathbb{R}^N)\).
We show that both terms on the right-hand side converge to zero. Since 

\[ \text{Theorem 3.6.} \]

The second term on the right-hand side of (4.1) converges to zero by 

\[ \text{The last expression converges to zero since} \]

\( \Omega \) and 

\( \Omega \cap \Omega_n \subset \Omega_n \), we have \( R_{\Omega_n}(\lambda) f - R_{\Omega_n \setminus \Omega_n}(\lambda) f \geq 0 \). By definition of the operator on 

\( L^1(\mathbb{R}^N) \) as the dual of the one on \( L^\infty(\mathbb{R}^N) \) we therefore have 

\[
\| R_{\Omega_n}(\lambda) f - R_{\Omega_n \setminus \Omega_n}(\lambda) f \|_{L^1} = \langle R_{\Omega_n}(\lambda) f - R_{\Omega_n \setminus \Omega_n}(\lambda) f, 1 \rangle
\]

\( = \langle f, R_{\Omega_n}(\lambda) 1 - R_{\Omega_n \setminus \Omega_n}(\lambda) 1 \rangle \leq n \| f \|_1 \| R_{\Omega_n}(\lambda) 1 - R_{\Omega_n \setminus \Omega_n}(\lambda) 1 \|_{L^\infty(\Omega)} ).
\]

The last expression converges to zero since \( \Omega_n \xrightarrow{\text{lu}} \Omega \) and thus \( \Omega \cap \Omega_n \xrightarrow{\text{lu}} \Omega \) by 

\[ \text{Theorem 3.6.} \]

The second term on the right-hand side of (4.1) converges to zero by 

a similar argument. Hence, \( R_{\Omega_n}(\lambda) f \to R_{\Omega}(\lambda) f \) in \( L^1(\mathbb{R}^N) \) for all \( f \in L^1(\mathbb{R}^N) \). Let 

now \( p \in (1, \infty) \) and set \( u_n := R_{\Omega_n}(\lambda) f - R_{\Omega}(\lambda) f \). Then clearly 

\[
\| u_n \|_p \leq \| u_n \|_{1/p}^{1/p} \| u_n \|_\infty^{1-1/p}
\]

for all \( n \in \mathbb{N} \). Since \( u_n \to 0 \) in \( L^1(\mathbb{R}^N) \) and \( u_n \) is bounded in \( L^\infty(\mathbb{R}^N) \), it follows 

that \( u_n \to 0 \) in \( L^p(\mathbb{R}^N) \) as claimed. \( \square \)

We next consider global uniform convergence.

**Proposition 4.2.** Let \( \Omega, \Omega_n \subset \mathbb{R}^N \) be open sets and \( \lambda > 0 \). If \( \Omega_n \xrightarrow{\text{gu}} \Omega \) and \( p \in [1, \infty] \), 

then \( R_{\Omega_n}(\lambda) \to R_{\Omega}(\lambda) \) in \( \mathcal{L}(L^p(\mathbb{R}^N)) \).

**Proof.** Recall from Theorem 3.1 that \( R_{\Omega_n}(\lambda) \to R_{\Omega}(\lambda) \) in \( \mathcal{L}(L^\infty(\mathbb{R}^N)) \) if \( \Omega_n \xrightarrow{\text{gu}} \Omega \). Since the operator on \( L^\infty(\mathbb{R}^N) \) is the dual of the one on \( L^1(\mathbb{R}^N) \) it follows that their \n
operator norms are the same, so 

\[
\| R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda) \|_{\mathcal{L}(L^1)} = \| R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda) \|_{\mathcal{L}(L^\infty)} \to 0
\]
as \( n \to \infty \). Hence, \( R_{\Omega_n}(\lambda) \to R_{\Omega}(\lambda) \) in \( \mathcal{L}(L^1(\mathbb{R}^N)) \) if \( \Omega_n \xrightarrow{gu} \Omega \). If \( p \in (1, \infty) \), the Riesz–Thorin interpolation theorem (see [6]) and the above imply that

\[
\| R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda) \|_{\mathcal{L}(L^p)} \leq \| R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda) \|_{\mathcal{L}(L^1)}^{1/p} \| R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda) \|_{\mathcal{L}(L^\infty)}^{1-1/p}
\]

as \( n \to \infty \), completing the proof of the proposition. □

**Remark 4.3.** (a) If \( p = \infty \) the above shows that convergence is in \( \mathcal{L}(L^p(\mathbb{R}^N)) \) if the resolvents converge strongly, that is, \( R_{\Omega_n}(\lambda) f \to R_{\Omega}(\lambda) f \) in \( L^\infty(\mathbb{R}^N) \) for all \( f \in L^\infty(\mathbb{R}^N) \). Note however, that strong convergence for \( p \in [1, \infty) \) does not imply convergence in the operator norm in general (see [12, Example 8.1])! The reason is that functions in \( L^p(\mathbb{R}^N) \) decay at infinity in some sense if \( p \in [1, \infty) \), but not if \( p = \infty \).

(b) Also note that (strong) convergence of \( R_{\Omega_n}(\lambda) \) in \( \mathcal{L}(L^p(\mathbb{R}^N)) \) for all \( p \in (0, \infty) \) does not imply (strong) convergence in \( \mathcal{L}(L^\infty(\mathbb{R}^N)) \) (it does in the vanishing case as shown in Section 5). The reason is that \( L_2 \)-convergence is not equivalent to \( L^\infty \)-convergence in general, and \( L^2 \)-convergence implies \( L^p \)-convergence for all \( p \in (1, \infty) \) (see [12, Section 5]).

### 5. The vanishing case

In this section, we discuss extensively the case where the limit problem is trivial, that is, \( R_{\Omega_n}(\lambda) 1 \to 0 \). For that we simply write \( \Omega_n \to \emptyset \). To derive our result we will make use of the semigroup \( T_{\Omega}(t) \) generated by the Dirichlet Laplacian on \( \Omega \) and represent the resolvent by means of its Laplace transform

\[
R_{\Omega}(\lambda) = \int_0^\infty e^{-\lambda t} T_{\Omega}(t) \, dt \tag{5.1}
\]

for all \( \lambda > 0 \). We recall that \( T_{\Omega}(t) \) is a strongly continuous analytic semigroup of contractions on \( L^p(\Omega) \) for \( 1 \leq p < \infty \) (see [14, Chapter I]), that is

\[
\| T_{\Omega}(t) \|_{\mathcal{L}(L^p(\Omega))} \leq 1 \quad (1 \leq p \leq \infty). \tag{5.2}
\]

It is well known that

\[
0 \leq T_{\Omega_1}(t) \leq T_{\Omega_2}(t) \leq G(t) \tag{5.3}
\]

for all open sets \( \Omega_1 \subset \Omega_2 \subset \mathbb{R}^N \) and \( t > 0 \), where \( G(t) := T_{\mathbb{R}^N}(t) \) is the Gaussian semigroup on \( \mathbb{R}^N \). Also, \( T_{\Omega}(t) \) has a kernel \( k_{\Omega}(t, x, y) \) dominated by the Gauss kernel.
(see [14]). More precisely,

\[ 0 \leq k(t, x, y) \leq g_t(x - y) := (4\pi t)^{-N/2} \exp(-|x - y|^2/4t) \]  

(5.4)

for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \). By convention we set \( k(t, x, y) = 0 \) for \( (x, y) \) outside \( \Omega \times \Omega \). Hence, for every \( 1 \leq p \leq q \leq \infty \) there exists a constant \( C \) only depending on \( N, p \) and \( q \), such that

\[ \| T_\Omega(t) \|_{L(L^p(\Omega), L^q(\Omega))} \leq Ct^{-N/2(1/p - 1/q)} \]  

(5.5)

for all \( t > 0 \). As a first step we characterise the vanishing case for \( L^2 \)-convergence. The result is related to [12, Theorem 4.4]. To do so we use the spectral bound of the Dirichlet–Laplacian on \( \Omega \) given by

\[ \lambda(\Omega) = \inf_{\substack{u \in H_0^1(\Omega) \\setminus \{0\}}} \frac{\| \nabla u \|_2^2}{\| u \|_2^2} \]

**Lemma 5.1.** Let \( \Omega_n \subset \mathbb{R}^N \) be open sets. Then the following assertions are equivalent:

1. \( \lambda(\Omega_n) \to \infty \);
2. \( T_{\Omega_n}(t) \to 0 \) in \( L(L^2(\mathbb{R}^N)) \) for some (all) \( t > 0 \);
3. \( R_{\Omega_n}(\lambda) \to 0 \) in \( L(L^2(\mathbb{R}^N)) \) for some (all) \( \lambda > 0 \).

**Proof.** Since \( -\Delta \) and thus \( T_{\Omega_n}(t) \) and \( R_{\Omega_n}(\lambda) \) are self-adjoint on \( L^2(\Omega_n) \) it follows from standard spectral mapping theorems (see [20, Section V.3.5] and [2, Corollary A-III.6.5]) that

\[ \| T_{\Omega_n}(t) \|_{L(L^2(\Omega_n))} = e^{-t\lambda(\Omega_n)} \]  

(5.6)

and

\[ \| R_{\Omega_n}(\lambda) \|_{L(L^2(\Omega_n))} = \frac{1}{\lambda + \lambda(\Omega_n)}. \]

Hence (1)–(3) are equivalent. \( \square \)

We next show that \( L^2 \)-convergence implies \( L^\infty \)-convergence for the semigroups.

**Theorem 5.2.** Let \( \Omega_n \subset \mathbb{R}^N \) be open sets and suppose that \( T_{\Omega_n}(t) \to 0 \) in \( L(L^2(\mathbb{R}^N)) \) for some \( t > 0 \). If \( 1 \leq p \leq q \leq \infty \), then \( T_{\Omega_n}(t) \to 0 \) in \( L(L^p(\mathbb{R}^N), L^q(\mathbb{R}^N)) \) uniformly with respect to \( t \) in closed subsets of \( (0, \infty) \).
Proof. By Lemma 5.1 it follows that \( T_{\Omega_n}(t) \to 0 \) in \( \mathcal{L}(L^2(\mathbb{R}^N)) \) for all \( t > 0 \). By the semigroup property and (5.5), we have

\[
\| T_{\Omega_n}(t) \|_{\mathcal{L}(L^p,L^q)} \leq \| T_{\Omega_n}(\delta/2) \|_{\mathcal{L}(L^p)} \| T_{\Omega_n}(t - \delta/2) \|_{\mathcal{L}(L^p,L^q)}
\]

\[
\leq C \left( t - \frac{\delta}{2} \right)^{-\frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| T_{\Omega_n}(\delta/2) \|_{\mathcal{L}(L^p)} \leq C \left( \frac{\delta}{2} \right)^{-\frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| T_{\Omega_n}(\delta/2) \|_{\mathcal{L}(L^q)}
\]

for all \( t \geq \delta > 0 \) and \( n \in \mathbb{N} \). To get uniform convergence with respect to \( t \) in closed subsets of \((0, \infty)\), it is therefore sufficient to prove that \( T_{\Omega_n}(t) \to 0 \) in \( \mathcal{L}(L^q(\mathbb{R}^N)) \) for all \( t > 0 \). We first assume that \( q \in (1, \infty) \). By the Riesz–Thorin interpolation theorem (see [6]) it follows from (5.2) and (5.6) that

\[
\| T_{\Omega_n}(t) \|_{\mathcal{L}(L^q(\mathbb{R}^N))} \leq e^{-\theta_q \lambda(\Omega_n)t}, \tag{5.7}
\]

where \( \theta_q \in (0, 1] \) is given by \( \theta_q = 2/q \) if \( 2 \leq q < \infty \) and \( \theta_q = 2 - 2/q \) if \( 1 < q \leq 2 \). Hence, by Lemma 5.1 we have \( T_{\Omega_n}(t) \to 0 \) in \( \mathcal{L}(L^q(\mathbb{R}^N)) \) for all \( t > 0 \). We next look at the case \( q = \infty \). Since \( T_{\Omega_n}(t) \) is a positive operator it is sufficient to show that \( T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty(\mathbb{R}^N) \). Suppose that this is not the case. Then, after possibly passing to a subsequence, there exist \( \varepsilon > 0 \) and \( x_n \in \Omega_n \), such that \( \left( T_{\Omega_n}(t)1 \right)(x_n) > \varepsilon \)

for all \( n \in \mathbb{N} \). Now observe that

\[
\left( T_{\Omega_n}(t)1 \right)(x_n) = \left( T_{\Omega_n}(t)1_{B(x_n,r)} \right)(x_n) + \left( T_{\Omega_n}(t)1_{B(x_n,r)^c} \right)(x_n)
\]

for all \( n \in \mathbb{N} \) and \( r > 0 \). By (5.4)

\[
\left( T_{\Omega_n}(t)1_{B(x_n,r)^c} \right)(x_n) \leq \int_{\mathbb{R}^N} g_t(x_n - y)1_{B(x_n,r)^c}(y) \, dy = \int_{|y| \geq r} g_t(y) \, dy
\]

for all \( n \in \mathbb{N} \) and \( r > 0 \). Hence, we can choose \( r > 0 \) such that

\[
\left( T_{\Omega_n}(t)1_{B(x_n,r)^c} \right)(x_n) < \frac{\varepsilon}{2}
\]

for all \( n \in \mathbb{N} \). Using (5.5) and what we already proved

\[
\| T_{\Omega_n}(t)1_{B(x_n,r)} \|_\infty = \| T_{\Omega_n}(t/2)T_{\Omega_n}(t/2)1_{B(x_n,r)} \|_\infty
\]

\[
\leq \| T_{\Omega_n}(t/2) \|_{\mathcal{L}(L^2,L^\infty)} \| T_{\Omega_n}(t/2)1_{B(x_n,r)} \|_2
\]

\[
\leq C \left( \frac{t}{2} \right)^{-N/4} \| T_{\Omega_n}(t/2) \|_{\mathcal{L}(L^2)} \| 1_{B(0,r)} \|_2 \xrightarrow{n \to \infty} 0.
\]
By choice of \(x_n\) and \(r > 0\) we have

\[
\varepsilon \leq \left( T_{\Omega_n}(t)1\right)(x_n) \leq \frac{\varepsilon}{2} + \|T_{\Omega_n}(t)1_{B(x_n, r)}\|_{\infty}
\]

for all \(n \in \mathbb{N}\), so letting \(n \to \infty\) we get \(0 < \varepsilon \leq \varepsilon/2\). As this is not possible it follows that \(T_{\Omega_n}(t)1 \to 0\) in \(L^\infty(\mathbb{R}^N)\). The case \(p = q = 1\) follows since by duality

\[
\|T_{\Omega_n}(t)\|_{L(L^1)} = \|T_{\Omega_n}(t)\|_{L(L^\infty)}.
\]

□

We next provide a version of the above theorem for the elliptic problem.

**Theorem 5.3.** Let \(\Omega_n \subset \mathbb{R}^N\) be open sets and suppose that \(R_{\Omega_n}(\lambda) \to 0\) in \(L(L^2(\mathbb{R}^N))\) for some \(\lambda > 0\). If \(1 \leq p \leq q \leq \infty\) with

\[
\frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right) < 1,
\]

(5.8)

then \(R_{\Omega_n}(\lambda) \to 0\) in \(L(L^p(\mathbb{R}^N), L^q(\mathbb{R}^N))\).

**Proof.** Fix \(\varepsilon > 0\) arbitrary. By (5.5) and (5.8) there exists \(s > 0\) such that

\[
\int_0^s \|T_{\Omega_n}(t)\|_{L(L^p, L^q)} e^{-\lambda t} dt \leq C \int_0^s t^{-\frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} e^{-\lambda t} dt < \frac{\varepsilon}{2}
\]

for all \(n \in \mathbb{N}\). Using the Laplace transform representation (5.1) we get

\[
\|R_{\Omega_n}(\lambda)\|_{L(L^p, L^q)} \leq \left\| \int_0^s e^{-\lambda t} T_{\Omega_n}(t) dt \right\|_{L(L^p, L^q)} + \left\| \int_s^\infty e^{-\lambda t} T_{\Omega_n}(t) dt \right\|_{L(L^p, L^q)}
\]

\[
\leq \frac{\varepsilon}{2} + \int_s^\infty e^{-\lambda t} \|T_{\Omega_n}(t)\|_{L(L^p, L^q)} dt
\]

for all \(n \in \mathbb{N}\). By Lemma 5.1 and Theorem 5.2 \(\|T_{\Omega_n}(t)\|_{L(L^p, L^q)} \to 0\) uniformly with respect to \(t \geq s\). Hence, there exists \(n_0 \in \mathbb{N}\), such that \(\|T_{\Omega_n}(t)\|_{L(L^p, L^q)} < \lambda \varepsilon /2\) for all \(n > n_0\) and \(t \geq s\), so

\[
\|R_{\Omega_n}(\lambda)\|_{L(L^p, L^q)} \leq \frac{\varepsilon}{2} + \frac{\lambda \varepsilon}{2} \int_0^\infty e^{-\lambda t} dt = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

for all \(n > n_0\). As \(\varepsilon > 0\) was arbitrary, the assertion of the proposition follows. □

The only new case covered in the above proposition is that \(R_{\Omega_n}(\lambda) \to 0\) in \(L(L^p(\mathbb{R}^N))\) for some \(p \in (1, \infty)\) implies convergence in \(L(L^q(\mathbb{R}^N), L^\infty(\mathbb{R}^N))\) for \(N/2 < q \leq \infty\). The other cases are covered in [12, Theorem 5.4]. The following corollary is a simple consequence of Lemma 5.1, Theorems 5.2 and 5.3.
Corollary 5.4. Let \( \lambda > 0 \) and \( (\Omega_n) \) a sequence of open sets. Then the following assertions are equivalent:

1. \( R_{\Omega_n}(\lambda) \to 0 \) in \( L(L^2(\mathbb{R}^N)) \);
2. \( R_{\Omega_n}(\lambda) \to 0 \) in \( L(L^\infty(\mathbb{R}^N)) \);
3. \( T_{\Omega_n}(t) \to 0 \) in \( L(L^2(\mathbb{R}^N)) \) for all \( t > 0 \);
4. \( T_{\Omega_n}(t) \to 0 \) in \( L(L^\infty(\mathbb{R}^N)) \) for all \( t > 0 \);
5. \( \lambda(\Omega) \to \infty \).

From the above we deduce a version on local uniform convergence.

Corollary 5.5. Let \( \lambda > 0 \) and \( (\Omega_n) \) a sequence of open sets. Then the following assertions are equivalent:

1. \( R_{\Omega_n}(\lambda)1 \to 0 \) in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \), that is, \( \Omega_n \rightharpoonup \emptyset \);
2. there exists \( p \in (1, \infty) \), such that \( R_{\Omega_n}(\lambda)f \to 0 \) in \( L^p(\mathbb{R}^N) \) for all \( f \in L^p(\mathbb{R}^N) \);
3. \( \lambda(\Omega_n \cap B) \to \infty \) for every bounded open set \( B \subset \mathbb{R}^N \).

Proof. The implication \( (1) \Rightarrow (2) \) follows from Proposition 4.1. By the uniform bound (2.1) and interpolation, convergence in \( L^p(\mathbb{R}^N) \) for some \( p \in (1, \infty) \) implies convergence in \( L^2(\mathbb{R}^N) \). Now \( (2) \Rightarrow (3) \) follows from \cite[Theorem 6.1]{12}. Suppose now that (3) holds. Then by Corollary 5.4 \( R_{\Omega_n \cap B}(\lambda)1 \to 0 \) in \( L^\infty(\mathbb{R}^N) \) for every bounded open set \( B \subset \mathbb{R}^N \). Now Theorem 3.4 implies (1). \( \Box \)

Note that in (2) of the above theorem, we cannot admit \( p = \infty \) since this would imply global uniform convergence by Theorem 3.1. Hence (2) would not be equivalent to (1). Also compare to Remark 4.3.

6. Tools for localisation

In this section, we collect some more properties of heat semigroup \( T_\Omega(t) \) introduced in Section 5. These properties will be useful to prove localisation results. For every \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) we define

\[
C_\varepsilon := \min \left\{ C \geq 0 : \int_{C/2}^{\infty} e^{-s^2} s^{N-1} ds \leq \frac{\varepsilon \pi^{N/2}}{\sigma_N} \right\},
\]

(6.1)

where \( \sigma_N \) is the surface area of the unit sphere in \( \mathbb{R}^N \). Clearly the function

\[
C \to \int_{C/2}^{\infty} e^{-s^2} s^{N-1} ds
\]
is continuous on \([0, \infty)\) and decreasing to zero as \(C \to \infty\), so the above minimum exists. Moreover, since
\[
\sigma_N := \frac{2\pi^{N/2}}{\Gamma(N/2)} \quad \text{and} \quad \int_0^\infty e^{-s^2} s^{N-1} ds = \frac{1}{2} \Gamma(N/2),
\]
we have \(C_\varepsilon = 0\) whenever \(\varepsilon \geq 1\). For an arbitrary set \(A \subset \mathbb{R}^N\) or point \(A \in \mathbb{R}^N\) and \(\delta > 0\) we denote the open \(\delta\)-neighbourhood of \(A\) by
\[
\mathcal{B}(A, \delta) := \{x \in \mathbb{R}^N : \text{dist}(x, A) < \delta\}.
\]
(6.2)
The distance between two sets \(A, B \subset \mathbb{R}^N\) is defined by
\[
\text{dist}(A, B) := \inf_{(x,y) \in A \times B} \|x - y\|.
\]
(6.3)

**Lemma 6.1.** Suppose that \(\Omega \subset \mathbb{R}^N\) is an open set and that \(A, B \subset \mathbb{R}^N\) are two measurable sets, such that \(\text{dist}(A, B) > 0\). Then for every \(\varepsilon > 0\)
\[
\|T_\Omega(t) 1_A\|_{L^\infty(B)} \leq \varepsilon
\]
for all \(t > 0\) with \(t^{-1/2} \cdot \text{dist}(A, B) \geq C_\varepsilon\).

**Proof.** We first prove an auxiliary inequality involving the Gaussian semigroup \(G(t)\). If we fix \(\delta > 0\) and represent \(G(t)\) by means of the Gauss kernel we get
\[
G(t)1_{\mathcal{B}(0, \delta)^c}(0) = (4\pi t)^{-N/2} \int_{|y| \geq \delta} e^{-|y|^2/(4t)} dy.
\]
Evaluating the integral using spherical coordinates we see that
\[
G(t)1_{\mathcal{B}(0, \delta)^c}(0) \leq \frac{\sigma_N}{(4\pi t)^{N/2}} \int_\delta^\infty e^{-r^2/(4t)} r^{N-1} dr = \frac{\sigma_N}{\pi^{N/2}} \int_{\delta/\sqrt{4t}}^\infty e^{-s^2} s^{N-1} ds \leq \varepsilon
\]
for all \(t > 0\), such that \(\delta t^{-1/2} \geq C_\varepsilon\). Now set \(\delta := \text{dist}(A, B)\) and fix \(x \in B \cap \Omega\) arbitrary. Given \(\varepsilon > 0\) the above inequality implies that
\[
0 \leq T_\Omega(t) 1_A(x) \leq G(t)1_A(x) \leq G(t)1_{\mathcal{B}(x, \delta)^c}(x) = G(t)1_{\mathcal{B}(0, \delta)^c}(0) \leq \varepsilon
\]
for all \(t > 0\) with \(t^{-1/2} \delta \geq C_\varepsilon\). Since \(x \in B \cap \Omega\) was arbitrary and \(T_\Omega(t) 1_A = 0\) on \(\Omega^c \cap B\), the assertion of the lemma follows. \(\square\)
We next prove a weak parabolic maximum principle. Note that the assertion follows from the classical maximum principle if all sets involved have a $C^2$ boundary.

**Theorem 6.2 (Parabolic maximum principle).** Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ be open sets, $\varepsilon, \tau > 0$, and $f \in L^\infty(\mathbb{R}^N)$ non-negative. If $T_{\Omega_2}(t) f(x) \leq \varepsilon$ for all $x \in \partial \Omega_1 \cap \Omega_2$ and all $t \in (0, \tau]$, then $T_{\Omega_2}(t) f(x) = T_{\Omega_1}(t) f(x) + \varepsilon$ for all $x \in \Omega_1$ and all $t \in (0, \tau]$.

**Proof.** Since $T_{\Omega_2}(t)$ is weak*-continuous and $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is weak*-dense in $L^\infty(\mathbb{R}^N)$ we may assume without loss of generality that $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is non-negative. Set $u_2(t) := T_{\Omega_2}(t) f$, $u_1(t) := T_{\Omega_1}(t) f$ and $u := u_2 - \varepsilon - u_1$. Then $u \in C([0, \infty), L^2(\Omega_1)) \cap C^\infty((0, \infty), L^2(\Omega_1))$, $u' - \Delta u = 0$ and $u^+(t) \in H^1_0(\Omega_1)$ for all $t \in (0, \tau)$. We claim that

$$
\frac{1}{2} \left( \|u^+(t)\|^2_{L^2(\Omega_1)} - \|u^+(\delta)\|^2_{L^2(\Omega_1)} \right) = - \int_\delta^t \|\nabla u^+(s)\|^2_{L^2(\Omega_1)} ds
$$

(6.4)

for every $\delta, t \in (0, \tau]$. Letting $\delta \to 0^+$ and using that $u^+ \in C([0, \tau], L^2(\Omega_1))$ we get

$$
\frac{1}{2} \|u^+(t)\|^2_{L^2(\Omega_1)} - \frac{1}{2} \|u^+(0)\|^2_{L^2(\Omega_1)} = 0,
$$

that is, $u(t, \cdot) \leq 0$ on $\Omega_1$ for $t \in (0, \tau]$. Since $t \in (0, \tau)$ was arbitrary, $u \leq 0$ and the assertion of the theorem follows. Hence, it remains to prove (6.4). For $n \in \mathbb{N}$ let $j_n(\xi) := \sqrt{\xi^2 + 1/n^2} - 1/n$ if $\xi \geq 0$ and $j_n(\xi) := 0$ if $\xi < 0$. Then obviously $j_n \in C^1(\mathbb{R})$, and $0 \leq j'_n \leq 1$ for all $n \in \mathbb{N}$. Hence, $j_n \circ u \in C^1((0, \infty), L^2(\Omega_2))$ and

$$
\frac{1}{2} \frac{d}{dt} \|j_n \circ u(t)\|^2_{L^2(\Omega_2)} = \langle j'_n(u(t))u'(t), j_n(u(t)) \rangle
$$

$$
= \langle u'(t), j'_n(u(t)) j_n(u(t)) \rangle = \langle \Delta u, j'_n(u(t)) j_n(u(t)) \rangle
$$

for all $t > 0$. If $t, \delta > 0$ we therefore get

$$
\frac{1}{2} \left( \|j_n \circ u(t)\|^2_{L^2(\Omega_1)} - \|j_n \circ u(\delta)\|^2_{L^2(\Omega_1)} \right) = \int_\delta^t \langle \Delta u, j'_n(u(s)) j_n(u(s)) \rangle ds.
$$

Next observe that $j'_n(u) j_n(u) \not\to u^+$ and $j_n(u) \not\to u^+$ as $n \to \infty$. Hence, by the dominated convergence theorem

$$
\frac{1}{2} \left( \|u^+(t)\|^2_{L^2(\Omega_1)} - \|u^+(\delta)\|^2_{L^2(\Omega_1)} \right) = \int_\delta^t \langle \Delta u(s), u^+(s) \rangle ds.
$$
As \( u^+(s) \in H^1_0(\Omega_1) \) for \( s \in (0, \tau) \) we have \( \langle \Delta u(s), u^+(s) \rangle = -\langle \nabla u, \nabla u^+ \rangle = -\|\nabla u^+\|_{L^2(\Omega_1)}^2 \) and hence (6.4) follows. \( \square \)

For two sets we denote the symmetric difference by \( U \triangle V := (U \cap V^c) \cup (V \cap U^c) \).

**Lemma 6.3.** Suppose that \( \Omega_1, \Omega_2 \subset \mathbb{R}^N \) are open sets and that \( A \subset \mathbb{R}^N \) is a measurable set, such that \( \text{dist}(A, \Omega_2 \triangle \Omega_1) > 0 \). Then

\[
\| T_{\Omega_2}(t)1_A - T_{\Omega_1}(t)1_A \|_{L^\infty(\mathbb{R}^N)} \leq \epsilon
\]

for all \( t > 0 \), such that \( t^{-1/2} \text{dist}(A, \Omega_2 \triangle \Omega_1) \geq C_{\epsilon/2} \), where \( C_{\epsilon/2} \) is defined by (6.1).

**Proof.** First we look at the case where \( \Omega_1 \subset \Omega_2 \). Since \( \delta := \text{dist}(A, \Omega_2 \triangle \Omega_1) = \text{dist}(A, \Omega_2 \setminus \Omega_1) > 0 \), Lemma 6.1 implies that \( \| T_{\Omega_2}(t)1_A \|_{L^\infty(\Omega_2 \setminus \Omega_1)} \leq \epsilon \) for all \( t > 0 \) with \( t^{-1/2} \delta \geq C_{\epsilon} \). By the continuity of \( T_{\Omega_2}(t)1_A \) in \( \Omega_2 \) it follows in particular that \( T_{\Omega_2}(t)1_A(x) \leq \epsilon \) for all \( x \in \partial \Omega_1 \cap \Omega_2 \) and all \( t > 0 \) with \( t^{-1/2} \delta \geq C_{\epsilon} \). Applying Theorem 6.2 we get that \( T_{\Omega_1}(t)1_A \leq T_{\Omega_2}(t)1_A \leq T_{\Omega_1}(t)1_A + \epsilon \) on \( \mathbb{R}^N \), and thus \( \| T_{\Omega_2}(t)1_A - T_{\Omega_1}(t)1_A \|_{L^\infty(\mathbb{R}^N)} \leq \epsilon \) for all \( t > 0 \) with \( t^{-1/2} \delta \geq C_{\epsilon} \). Hence, the assertion of the lemma follows if \( \Omega_1 \subset \Omega_2 \). Let now \( \Omega_1, \Omega_2 \) be arbitrary open sets in \( \mathbb{R}^N \). Then by what we just proved we have

\[
\| T_{\Omega_1}(t)1_A - T_{\Omega_2}(t)1_A \|_{L^\infty(\mathbb{R}^N)} \leq \| T_{\Omega_1}(t)1_A - T_{\Omega_1 \cap \Omega_2}(t)1_A \|_{L^\infty(\mathbb{R}^N)}
\]

\[
+ \| T_{\Omega_1 \cap \Omega_2}(t)1_A - T_{\Omega_2}(t)1_A \|_{L^\infty(\mathbb{R}^N)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

all \( t > 0 \) with \( t^{-1/2} \delta \geq C_{\epsilon/2} \) as claimed. \( \square \)

**7. Localisation theorems**

Localisation of Convergence is an important tool to compare the behaviour of \( T_{\Omega_n}(t) \) with the behaviour of \( T_{\Omega \cap U}(t) \) on a fixed open set \( U \). Moreover, it allows us to generalise earlier results. For example, Corollary 5.4 is a particular case of Theorem 8.8 if we replace \( \Omega \) by the empty set. It seems to be more difficult to prove such a localisation theorem directly for the elliptic case. So we prove it for the parabolic case (Theorem 7.3) first.

To simplify the statements of our results we need the following basic definitions.

**Definition 7.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set. Then for \( n \in \mathbb{N} \) we set \( A_n := \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega^c) \geq 1/n \} \) and for measurable functions \( f \) we set \( |f|_n := \| f \|_{L^\infty(A_n)} \). We consider the space

\[
L^\infty_d(\Omega) := \{ f : \Omega \to \mathbb{R} \ \text{measurable}: |f|_n < \infty \text{ for all } n \in \mathbb{N} \}
\]
equipped with the metric

\[ d(f, g) := \sum_{n=1}^{\infty} 2^{-n} \min(1, |f - g|_n). \]

It is obvious that \( L^\infty_d(\mathbb{R}^N) = L^\infty(\mathbb{R}^N). \) Moreover, if \( \Omega \) is a bounded open set, then \( L^\infty_d(\Omega) = L^\infty_{\text{loc}}(\Omega). \) Let \( \Omega \) be an arbitrary open set. Saying that \( f_n \to f \) in \( L^\infty_d(\Omega) \) is equivalent to \( \|f_n - f\|_{L^\infty(W)} \to 0 \) for all open sets \( W \subset \Omega \) with \( \text{dist}(W, \Omega^c) > 0. \) We introduce the following notions of local convergence.

**Definition 7.2.** Let \( W \) be an open set. We write \( \Omega_n \xrightarrow{\text{du}(W)} \Omega \) if \( R_{\Omega_n}(\lambda)1 \to R_\Omega(\lambda)1 \) in \( L^\infty_d(W) \) for some \( \lambda > 0. \) We say that \( \Omega_n \) converges to \( \Omega \) **distantly uniformly** on \( W. \) We furthermore write \( \Omega_n \xrightarrow{\text{lu}(W)} \Omega \) and \( \Omega_n \xrightarrow{\text{gu}(W)} \Omega \) if, for some \( \lambda > 0, R_{\Omega_n}(\lambda)1 \to R_\Omega(\lambda)1 \) in \( L^\infty_{\text{loc}}(W) \) and \( L^\infty(W) \), respectively.

Theorems 3.1 and 3.4 show that convergence in case \( W = \mathbb{R}^N \) is independent of \( \lambda > 0, \) that is, \( R_{\Omega_n}(\lambda)1 \to R_\Omega(\lambda)1 \) in the respective metrics for all \( \lambda > 0 \) if this is true for one particular \( \lambda > 0. \) Also note that

\[ \Omega_n \xrightarrow{\text{gu}(W)} \Omega \implies \Omega_n \xrightarrow{\text{du}(W)} \Omega \implies \Omega_n \xrightarrow{\text{lu}(W)} \Omega. \]

We continue with a first localisation theorem for the semigroup.

**Theorem 7.3 (Parabolic localisation theorem).** Suppose that \( U, \Omega_n \subset \mathbb{R}^N \) are open sets.

1. If \( T_{\Omega_n \cap U}(t)1 \to 0 \) in \( L^\infty(\mathbb{R}^N) \) for all \( t > 0, \) then \( T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty_d(U) \) uniformly with respect to \( t \) in closed subsets of \( (0, \infty). \)
2. If \( T_{\Omega_n \cap U}(t)1 \to 0 \) in \( L^\infty_{\text{loc}}(U) \) for all \( t > 0 \) if and only if \( T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty_{\text{loc}}(U) \) for all \( t > 0. \)

**Proof.** We start by proving (1). Fix \( W \subset U \) such that \( \delta := \text{dist}(W, \partial U) > 0. \) We need to prove that \( T_n(t)1 := T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty(W) \) uniformly with respect to \( t \) in \( [s, \infty) \) for all \( s > 0. \) Note that \( 0 \leq T_n(t)1 \leq 1 \) for all \( n \in \mathbb{N} \) and \( t \geq 0 \) and thus \( 0 \leq T_n(t)1 = T_n(s)T_n(t-s)1 \leq T_n(s)1 \) for all \( t \geq s. \) Hence to prove (1) it is sufficient to show that \( T_n(t)1 \to 0 \) in \( L^\infty(W) \) for all \( t > 0. \) Set \( U_n := \Omega_n \cap U \) and \( S_n(t) := T_{U_n}(t). \) Let \( V := \{ x \in \mathbb{R}^N : \text{dist}(x, W) < \delta/2 \} \) and fix \( \varepsilon > 0 \) arbitrary. Since \( \text{dist}(V^c, W) \geq \delta/2, \) it follows from Lemma 6.1 that there exists \( t_\varepsilon > 0, \) such that \( \|T_n(t)V\|_{L^\infty(W)} < \varepsilon/4 \) for all \( t \in (0, t_\varepsilon) \) and \( n \in \mathbb{N}. \) Since \( \text{dist}(\overline{V}, \Omega_n \setminus U_n) \geq \text{dist}(\overline{V}, U^c) \geq \delta/2 \) it follows from Lemma 6.3 that

\[ \|T_n(t)V - S_n(t)V\|_{L^\infty} < \frac{\varepsilon}{4}. \]
for all \( t \in (0, t_\varepsilon) \) and \( n \in \mathbb{N} \). Combining the above,

\[
\|T_n(t)1\|_{L^\infty(W)} \leq \|T_n(t)1\|_{L^\infty(W)} + \|T_n(t)1\|_{L^\infty(W)} - \|S_n(t)1\|_{L^\infty(W)} + \|S_n(t)1\|_{L^\infty(W)} \\
\leq \frac{\varepsilon}{2} + \|S_n(t)1\|_{L^\infty(W)}
\]

for all \( t \in (0, t_\varepsilon) \) and \( n \in \mathbb{N} \). Fix now \( t \in (0, t_\varepsilon) \). By assumption \( S_n(t)1 \to 0 \) in \( L^\infty(W) \), so there exists \( n_0 \in \mathbb{N} \), such that \( \|S_n(t)1\|_{L^\infty(W)} < \varepsilon/2 \) for all \( n \geq n_0 \). Hence by the above \( \|T_n(t)1\|_{L^\infty(W)} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \) for all \( n > n_0 \). If \( t > t_\varepsilon \), then

\[
\|T_n(t)1\|_{L^\infty(W)} = \|T_n(t_\varepsilon)T_n(t - t_\varepsilon)1\|_{L^\infty(W)} \leq \|T_n(t_\varepsilon)1\|_{L^\infty(W)}.
\]

By assumption \( S_n(t_\varepsilon)1 \to 0 \) in \( L^\infty(W) \), so there exists \( n_0 \in \mathbb{N} \), such that

\[
\|S_n(t_\varepsilon)1\|_{L^\infty(W)} < \varepsilon/2
\]

for all \( n \geq n_0 \), and as before \( \|T_n(t)1\|_{L^\infty(W)} \leq \varepsilon \) for all \( n > n_0 \). As \( \varepsilon, t > 0 \) were arbitrary (1) follows. We now prove (2). As \( 0 \leq T_{\Omega_n \cap U}(t)1 \leq T_{\Omega_n}(t)1 \) one of the implications is obvious. To prove the other let \( K \subset U \) be a compact set. Then there exists an open set \( V \subset U \) containing \( K \). One has that \( T_{\Omega_n \cap V}(t)1 \to 0 \) in \( L^\infty(\mathbb{R}^N) \) for all \( t > 0 \). By (1) it follows that \( T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty(K) \) for all \( t > 0 \). Since \( K \subset U \) was arbitrary, (2) follows.

The following result is a version of Corollary 5.5 for the parabolic case.

**Corollary 7.4.** Let \( \Omega_n \subset \mathbb{R}^N \) be open sets. Then the following assertions are equivalent:

1. \( \Omega_n \xrightarrow{\text{lu}} \emptyset \);
2. \( T_{\Omega_n \cap B}(t)1 \to 0 \) in \( L^\infty(\mathbb{R}^N) \) for all \( t > 0 \) and all open sets \( B \subset \subset \mathbb{R}^N \);
3. \( T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \).

**Proof.** Statements (1) and (2) are equivalent by Corollaries 5.4 and 5.5. Moreover, (3) implies (2) by domination. Suppose now (2) holds. Given a compact set \( K \subset \mathbb{R}^N \) we choose an open set \( B \subset \subset \mathbb{R}^N \), such that \( K \subset B \). By (2) we know \( T_{\Omega_n \cap B}(t)1 \to 0 \) in \( L^\infty(\mathbb{R}^N) \). Hence, Theorem 7.3 implies that \( T_{\Omega_n}(t)1 \to 0 \) in \( L^\infty_{\text{loc}}(B) \) for all \( t > 0 \). In particular convergence is in \( L^\infty(K) \), so (3) follows.

Now we are ready to transfer the above to the elliptic case.

**Theorem 7.5 (Elliptic localisation theorem).** Let \( U, \Omega_n \subset \mathbb{R}^N \) be open sets and \( \lambda > 0 \). Then, the following assertions are equivalent:

1. \( R_{\Omega_n \cap U}(\lambda)1 \to 0 \) in \( L^\infty_d(U) \), that is, \( \Omega_n \cap U \xrightarrow{\text{dau}(U)} \emptyset \).
2. \( R_{\Omega_n}(\lambda)1 \to 0 \) in \( L^\infty_d(U) \), that is, \( \Omega_n \xrightarrow{\text{dau}(U)} \emptyset \).
Proof. The implication (2) \(\implies\) (1) is obvious by domination. Assume now that (1) holds. Then for every open set \(W \subset U\) with \(\text{dist}(W, U^c) > 0\) we have \(\Omega_n \cap W \xrightarrow{\text{gu}} \emptyset\), so Corollary 5.4 implies that \(T_{\Omega_n \cap W}(t)1 \to 0\) in \(L^\infty(\mathbb{R}^N)\). Applying Theorem 7.3 we get \(T_{\Omega_n}(t)1 \to 0\) in \(L^\infty_d(U)\) for all \(t > 0\). Hence, \(T_{\Omega_n}(t)1 \to 0\) in \(L^\infty(U)\) for all \(t > 0\). Using the Laplace transform (5.1) and the dominated convergence theorem we get that

\[
\|R_{\Omega_n}(\lambda)1\|_{L^\infty(W)} \leq \int_0^\infty e^{-\lambda t}\|T_{\Omega_n}(t)1\|_{L^\infty(W)} dt \to 0
\]

for all \(\lambda > 0\) and all open sets \(W \subset U\) with \(\text{dist}(W, U^c) > 0\). Hence (2) follows. \(\square\)

**Corollary 7.6.** Let \(U, \Omega_n \subset \mathbb{R}^N\) be open sets. Then the following assertions are equivalent:

1. \(R_{\Omega_n}(\lambda)1 \to 0\) in \(L^\infty_{\text{loc}}(U)\), that is, \(\Omega_n \xrightarrow{\text{lu}(U)} \emptyset\).
2. \(R_{\Omega_n \cap U}(\lambda)1 \to 0\) in \(L^\infty_{\text{loc}}(U)\), that is, \(\Omega_n \cap U \xrightarrow{\text{lu}(U)} \emptyset\).
3. \(R_{\Omega_n \cap (U \cup \{0\})}(\lambda)1 \to 0\) in \(L^\infty_{\text{loc}}(\mathbb{R}^N)\), that is, \(\Omega_n \cap U \xrightarrow{\text{lu}} \emptyset\).

**Proof.** The implication (1) \(\implies\) (2) is obvious since \(0 \leq R_{\Omega_n \cap U}(\lambda)1 \leq R_{\Omega_n}(\lambda)1\) by domination. Suppose now (2) holds. By Theorem 3.6 we have \(\Omega_n \xrightarrow{\text{lu}} \emptyset\) if and only if \(\Omega_n \cap B \to \emptyset\) for all bounded open sets \(B \subset \mathbb{R}^N\). Hence we assume without loss of generality that \(U\) is bounded. If \(f \in L^\infty(U)\), then by domination \(|R_{\Omega_n}(\lambda)f| \leq \|f\|_{\infty}R_{\Omega_n}(\lambda)1 \leq \|f\|_{\infty}/\lambda\) for all \(n \in \mathbb{N}\). By assumption \(R_{\Omega_n}(\lambda)1 \to 0\) in \(L^\infty_{\text{loc}}(U)\), so by the dominated convergence theorem \(\|R_{\Omega_n}(\lambda)f\|_2 \to 0\) for all \(f \in L^\infty(U)\). By the uniform bound (2.1) and the density of \(L^\infty(U)\) in \(L^2(U)\) we have \(\|R_{\Omega_n}(\lambda)f\|_2 \to 0\) for all \(f \in L^2(U)\). Hence Corollary 5.5 implies (3). If (3) holds, then by Theorem 3.6 we have \(\Omega_n \cap B \to \emptyset\) whenever \(B\) is an open set with \(B \subset \subset U\). Suppose now that \(K \subset U\) is compact and that \(B\) is an open subset with \(K \subset B \subset \subset U\). Since \(B\) is bounded (3) implies that \(\Omega_n \cap B \xrightarrow{\text{du}(B)} \emptyset\). As \(\text{dist}(K, B^c) > 0\) we conclude from Theorem 7.5 that \(R_{\Omega_n}(\lambda)1 \to 0\) in \(L^\infty(K)\), so (1) follows. \(\square\)

8. **Conditions for global uniform convergence**

In this section, we want to give conditions for global uniform convergence. Note that they also provide conditions for local uniform convergence since \(\Omega_n \xrightarrow{\text{lu}} \Omega\) if and only if \(\Omega_n \cap B \xrightarrow{\text{gu}} \Omega \cap B\) for all open sets \(B \subset \subset \mathbb{R}^N\) by Theorem 3.6. Unlike in the vanishing case, \(L^2\)-convergence does not in general imply \(L^\infty\)-convergence. Indeed, if we let \(\Omega_n := \mathbb{B}(0, 1) \setminus \mathbb{B}(0, 1/n)\) and \(\Omega := \mathbb{B}(0, 1)\), then \(R_{\Omega_n}(\lambda)1 \to R_{\Omega}(\lambda)1\) in \(L^2(\mathbb{R}^N)\), but not in \(L^\infty(\mathbb{R}^N)\). We will show next that in order to get \(L^\infty\)-convergence, we cannot cut holes of positive capacity in the interior of \(\Omega\). (By capacity we mean the usual 2-capacity as defined in [17].) More precisely, we have the following necessary
condition for $L^\infty$-convergence. We recall that $\text{dist}(A, B)$ denotes the distance between the sets $A$ and $B$ as defined in (6.3).

**Theorem 8.1.** Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open and $\lambda > 0$. If $\Omega_n \xrightarrow{\text{gu}} \Omega$, then

For every set $C \subset \Omega$ with $\inf_C R_\Omega(\lambda)1 > 0$ there exists $n_0 \in \mathbb{N}$, such that $\text{cap}(\Omega_n^C \cap C) = 0$ for all $n > n_0$. (8.1)

If $\text{dist}(C, \partial \Omega) > 0$, then $\inf_C R_\Omega(\lambda)1 > 0$. Finally, if $\Omega_n \xrightarrow{\text{lu}} \Omega$, then for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$, such that $\text{cap}(K \cap \Omega_n^C) = 0$ for all $n > n_0$.

**Proof.** To prove (8.1) set $u := R_\Omega(\lambda)1$, $u_n := R_{\Omega_n}(\lambda)1$ and suppose that $C \subset \Omega$ is such that $\varepsilon := \inf_{x \in C} u(x) > 0$. By assumption there exists $n_0 \in \mathbb{N}$, such that $\|u - u_n\|_{L^\infty(C, B)} \leq \varepsilon/2$ for all $n > n_0$. Hence $|u - u_n| \leq \varepsilon/2$ quasi-everywhere on $\mathbb{R}^N$. Since $u(x) \geq \varepsilon$ for all $x \in C$ one has that $u_n \geq \varepsilon/2$ quasi-everywhere on $C$ for all $n > n_0$. Since $u_n = 0$ quasi-everywhere on $\Omega_n^C$ it follows that $\text{cap}(C \cap \Omega_n^C) = 0$ for all $n > n_0$, proving (8.1). Suppose now that $\delta := \text{dist}(C, \partial \Omega) > 0$. Then by domination

$$
(R_\Omega(\lambda)1)(x) \geq (R_{B(x, \delta)}(\lambda)1)(x) = (R_{B(0, \delta)}(\lambda)1)(0) > 0
$$

(8.2)

for all $x \in C$. To prove the remaining assertion let $K \subset \Omega$ be a compact set. Then there exists an open set $B \subset \subset \Omega$ containing $K$. By Theorem 3.6 it follows that $\Omega_n \cap B \xrightarrow{\text{gu}} \Omega \cap B$. Since $K$ is compact $\delta := \text{dist}(K, \partial(K \cap \Omega)) > 0$. Hence, condition (8.1) is satisfied, and so there exists $n_0 \in \mathbb{N}$, such that $K \cap \Omega_n \cap B \subset K \cap \Omega_n^C$ has capacity zero for all $n > n_0$. \(\square\)

**Remark 8.2.** If every $\Omega_n$ is Dirichlet (Wiener) regular (that is, bounded solutions of the Dirichlet problem on $\Omega$ are continuous up to the boundary) and $\Omega_n \xrightarrow{\text{lu}} \Omega$, then for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$, such that $K \setminus \Omega_n = \emptyset$ for all $n > n_0$. If $\Omega$ is not Dirichlet regular, then it may happen that $K \setminus \Omega_n$ is never empty. For example, take $\Omega_n := \mathbb{B}(0, 1) \setminus \{0\}$, $\Omega := \mathbb{B}(0, 1)$ and $K := \overline{\mathbb{B}(0, 1/2)}$. Then $\Omega_n \xrightarrow{\text{lu}} \Omega, \ K \subset \Omega$ is compact and $K \setminus \Omega_n = \{0\} \neq \emptyset$.

We next show that the necessary condition in Theorem 8.1 is sufficient if $\Omega_n$ approaches $\Omega$ from the inside.

**Theorem 8.3.** Let $\Omega, \Omega_n \subset \mathbb{R}^N$ be open sets with $\Omega_n \subset \Omega$ for all $n \in \mathbb{N}$ and $\lambda > 0$. If (8.1) holds, then $\Omega_n \xrightarrow{\text{gu}} \Omega$.

**Proof.** Let $\lambda, \varepsilon > 0$ be fixed and let $u := R_\Omega(\lambda)1$. Setting $\Omega_\varepsilon := \{x \in \Omega : u(x) > \varepsilon\}$ Lemma 2.1 shows that $(u - \varepsilon)^+ = R_{\Omega_\varepsilon}(\lambda)(1 - \lambda \varepsilon)$. By assumption (8.1) there exists
There exists Theorem 8.4. Let $\Omega, \Omega_n \subset \mathbb{R}^N$ be open sets. Then the following assertions are equivalent:

(1) $\Omega_n \xrightarrow{\text{gu}} \Omega$.

(2) $\Omega_n \cap B(\Omega, 1/k) \xrightarrow{\text{gu}} \Omega$ as $n, k \to \infty$ and $\Omega_n \xrightarrow{\text{dul}(\Omega)} \emptyset$.

(3) There exists $\lambda > 0$, such that $R_{\Omega_n}(\lambda) \to 0$ in $L_{d}^{\infty}(\Omega^c)$ and for every $\varepsilon > 0$ there exist an open set $B_{0} \subset \mathbb{R}^N$ with $\text{dist}(\Omega, B_{0}) > 0$ and $N_0 \in \mathbb{N}$ such that

$$\|R_{\Omega_n \cap B_{0}}(\lambda) - R_{\Omega}(\lambda)\|_{\infty} \leq \varepsilon$$

for all $n > N_0$.

**Proof.** Throughout the proof we set $B_k := B(\Omega, 1/k)$. If (1) holds, then by Theorem 2.2

$$\|R_{\Omega_n \cap B_k}(\lambda) - R_{\Omega}(\lambda)\|_{\infty} = \|R_{\Omega_n \cap B_k}(\lambda) - R_{\Omega \cap B_k}(\lambda)\|_{\infty} \leq \|R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda)\|_{\infty} \to 0$$

as $n, k \to \infty$. Similarly, for every open set $U$ with $\text{dist}(U, \Omega) > 0$ we have

$$\|R_{\Omega_n \cap U}(\lambda)\|_{\infty} = \|R_{\Omega_n \cap U}(\lambda) - R_{\Omega \cap U}(\lambda)\|_{\infty} \leq \|R_{\Omega_n}(\lambda) - R_{\Omega}(\lambda)\|_{\infty} \to 0$$

as $n \to \infty$. Hence (2) follows. If (2) holds, then for every $\lambda, \varepsilon > 0$ there exists $N_0$, such that $\|R_{\Omega}(\lambda) - R_{\Omega_n \cap B_k}(\lambda)\|_{L_{d}^{\infty}(\Omega^c)} \leq \varepsilon$ for all $n, k \geq N_0$, so (3) follows if we set
The following theorem is an extension of Corollary 5.4. Note that (8.1) is always the same as the stability of Remark 8.7.

(a) In Appendix A, we show that our notion of uniform stability of $B \in \mathbb{R}^N$ satisfied in the vanishing case. Recall that $\lambda(U)$ is the spectral bound of the Dirichlet Laplacian on the open set $U \subset \mathbb{R}^N$.

(b) It also turns out that every uniformly stable domain is regular since we can approach every open set by a sequence of smooth sets from the outside. Given that $R_{\Omega_n}(\lambda)f$ converges uniformly to $R_{\Omega}(\lambda)f$ for all $f \in L^\infty(\mathbb{R}^N)$ implies that $R_{\Omega}(\lambda)f \in C(\mathbb{R}^N)$ for all such $f$. Hence $\Omega$ is regular by [1, Proposition 4.2].

The following theorem is an extension of Corollary 5.4. Note that (8.1) is always satisfied in the vanishing case. Recall that $\lambda(U)$ is the spectral bound of the Dirichlet Laplacian on the open set $U \subset \mathbb{R}^N$. 

$B_0 := B(\Omega, 1/N_0)$. Suppose now that (3) holds. Then, given $\varepsilon > 0$, there exists an open set $B_0$ and $N_1 \in \mathbb{N}$, such that $\|R_{\Omega}(\lambda)1 - R_{\Omega_n \cap B_0}(\lambda)1\|_\infty < \varepsilon$ for all $n > N_1$. Set $w_n := R_{\Omega_n}(\lambda)1$ and $M_{\varepsilon,n} := \{x \in \Omega_n \cap B_0 : w_n(x) > \varepsilon\}$. Since $w_n \to 0$ in $L^\infty(\Omega^c)$ we can choose $N_2 \in \mathbb{N}$ such that $w_n \leq \varepsilon$ quasi-everywhere on $B_0^c$ for all $n > N_2$. Since $w_n \in C(\Omega_n \cap B_0)$ we also have $w_n \leq 0$ on $(\Omega_n \cap B_0) \setminus M_{\varepsilon,n}$. Finally, $w_n = 0$ quasi-everywhere on $\Omega^c$, so $w_n - \varepsilon \leq 0$ quasi-everywhere on $M_{\varepsilon,n}^c$ for all $n > N_0 := \max\{N_1, N_2\}$. Hence Lemma 2.1 implies that $(w_n - \varepsilon)^+ = R_{M_{\varepsilon,n}}(\lambda)(1 - \lambda)\varepsilon$, and thus by domination

$$R_{\Omega_n \cap B_0}(\lambda)1 \leq w_n \leq R_{M_{\varepsilon,n}}(\lambda)(1 - \lambda)\varepsilon + \varepsilon \leq R_{\Omega_n}(\lambda)1 + \varepsilon$$

for all $n > N_0$. Hence

$$\|R_{\Omega}(\lambda)1 - R_{\Omega_n}(\lambda)1\|_{L^\infty(\mathbb{R}^N)} \leq \|R_{\Omega}(\lambda)1 - R_{\Omega_n \cap B_0}(\lambda)1\|_{L^\infty(\mathbb{R}^N)} + \|R_{\Omega_n \cap B_0}(\lambda)1 - R_{\Omega_n}(\lambda)1\|_{L^\infty(\mathbb{R}^N)} < \varepsilon + \varepsilon = 2\varepsilon$$

for all $n > N_0$. As $\varepsilon > 0$ was arbitrary (1) follows. □

To get approximation from the outside we need some regularity properties on $\Omega$.

Definition 8.5 (Capacity regular). Let $\Omega \subset \mathbb{R}^N$ be an open set. Then $z \in \partial \Omega$ is called regular in capacity for $\Omega$, if $\text{cap}(B(z, r) \cap \Omega^c) > 0$ for all $r > 0$. If every point of $\partial \Omega$ is regular in capacity for $\Omega$, then $\Omega$ is called regular in capacity.

Definition 8.6 (Uniform stability). Let $\Omega \subset \mathbb{R}^N$ be open and $\Gamma \subset \partial \Omega$ closed. Then $\Omega$ is called uniformly (Dirichlet) stable on $\Gamma$, if $\Omega$ is regular in capacity and $\Omega \cup \overline{B}(\Gamma, 1/n) \xrightarrow{\text{ru}} \Omega$ as $n \to \infty$. We call $\Omega$ uniformly stable if the above holds with $\Gamma = \partial \Omega$.

Remark 8.7. (a) In Appendix A, we show that our notion of uniform stability of $\Omega$ is the same as the stability of $\Omega$ introduced in Keldyš [21]. We emphasise that this is not the same as the stability of $\Omega$ as used by Hedberg [16] and most papers on $L^2$-convergence on varying domains! We explain the difference in Remark A.2 in the appendix.

(b) It also turns out that every uniformly stable domain is regular since we can approach every open set by a sequence of smooth sets from the outside. Given that $R_{\Omega_n}(\lambda)f$ converges uniformly to $R_{\Omega}(\lambda)f$ for all $f \in L^\infty(\mathbb{R}^N)$ implies that $R_{\Omega}(\lambda)f \in C(\mathbb{R}^N)$ for all such $f$. Hence $\Omega$ is regular by [1, Proposition 4.2].
Theorem 8.8. Let $\Omega \subset \mathbb{R}^N$ be a uniformly stable open set and $\lambda > 0$. Then, the following assertions are equivalent:

1. $R_{\Omega_n}(\lambda) \rightarrow R_{\Omega}(\lambda)$ in $L(L^\infty(\mathbb{R}^N))$, that is, $\Omega_n \overset{\text{gu}}{\rightarrow} \Omega$.
2. $R_{\Omega_n}(\lambda) \rightarrow R_{\Omega}(\lambda)$ in $L(L^2(\mathbb{R}^N))$ and (8.1) holds.
3. $\lambda(\Omega_n \cap \overline{\Omega}) \rightarrow \infty$ and (8.1) holds.

If $\Omega$ is bounded, then the above assertions are equivalent to the following statement.

4. $\lambda(\Omega_n \setminus \overline{\Omega}) \rightarrow \infty$ and for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$, such that $\text{cap}(K \cap \Omega_n^c) = 0$ for all $n > n_0$.

Proof. By Theorem 8.1 and Proposition 4.2 statement (1) implies (2). If (2) holds, then $R_{\Omega_n}(\lambda) \rightarrow 0$ in $L(L^2(\mathbb{R}^N))$ and hence by Lemma 5.1 we get (3). If (3) holds, then by Corollary 5.4 and Theorem 7.5 we have that $\Omega_n \overset{du(\overline{\Omega})}{\rightarrow} \emptyset$. Next note that

$$
\Omega_n \cap \Omega \subset \Omega_n \cap \mathbb{B}(\Omega, 1/k) \subset \mathbb{B}(\Omega, 1/k)
$$

(8.3)

for all $n, k \in \mathbb{N}$. Clearly $\Omega_n$ satisfies (8.1) if and only if that is the case for $\Omega \cap \Omega_n$. Hence $\Omega_n \cap \Omega \overset{\text{gu}}{\rightarrow} \Omega$ by Theorem 8.3. Fix $\lambda, \varepsilon > 0$. There exists $n_1 \in \mathbb{N}$, such that $R_{\Omega_n}(\lambda)1 - \varepsilon < R_{\Omega_n \cap \Omega}(\lambda)1$ for all $n > n_0$. By assumption $\mathbb{B}(\Omega, 1/k) \overset{\text{gu}}{\rightarrow} \Omega$, so there exists $n_2 \in \mathbb{N}$ such that $R_{\mathbb{B}(\Omega, 1/k)}(\lambda)1 < R_{\Omega}(\lambda)1 + \varepsilon$ for all $k > n_2$. Hence by (8.3) and domination

$$
R_{\Omega}(\lambda)1 - \varepsilon \leq R_{\Omega_n \cap \Omega}(\lambda)1 \leq R_{\Omega_n \cap \mathbb{B}(\Omega, 1/k)}(\lambda)1 \leq R_{\mathbb{B}(\Omega, 1/k)}(\lambda)1 < R_{\Omega}(\lambda)1 + \varepsilon
$$

for all $n, k > n_0 := \max\{n_1, n_2\}$. Since $\varepsilon > 0$ was arbitrary $\Omega_n \cap \mathbb{B}(\Omega, 1/k) \overset{\text{gu}}{\rightarrow} \Omega$ as $n, k \rightarrow \infty$. Now Theorem 8.4 implies (1). Finally, note that if $\Omega$ is bounded and regular, then $u := R_{\Omega_n}(\lambda)1 \in C_0(\overline{\Omega})$. Since $\partial \Omega$ is compact, the set $\{x \in \Omega: u(x) \geq \varepsilon\}$ is compact in $\Omega$. Hence, (8.1) is equivalent to the assumption that for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$, such that $\text{cap}(K \cap \Omega_n^c) = 0$ for all $n > n_0$. This completes the proof of the theorem. \qed

As a consequence of the above theorem we easily deduce the following facts.

Remark 8.9. (a) Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open sets and assume that $\Omega$ is uniformly stable. If $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$ and $\Omega_n \setminus \overline{\Omega} \overset{\text{gu}}{\rightarrow} \emptyset$, then $\Omega_n \overset{\text{gu}}{\rightarrow} \Omega$.

(b) Suppose that $\Omega_n$ satisfies (8.1). If $\Omega$ is uniformly stable on $\Gamma \subset \partial \Omega$ and for every $k \in \mathbb{N}$ we have $\text{dist}(\Omega_n \cap (\mathbb{B}(\Gamma, 1/k) \cup \Omega)^c, \overline{\Omega}) > 0$ for $n$ large, then $\Omega_n \overset{\text{gu}}{\rightarrow} \Omega$. To see this is a simple modification of the proof of Theorem 8.8. This covers examples like the cracking domain shown in Fig. 3, where $\Omega_n = \Omega \setminus C_n$ and $C_n$ are closed sets with $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$ and $\Omega$ open. The set $\Gamma$ is given by $\bigcap_{n \in \mathbb{N}} C_n$ (the end point of the crack in Fig. 3).
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Appendix A. Stability of bounded sets

The purpose of this section is to show that our definition of uniform stability of an open set \( \Omega \) coincides with the stability of \( \overline{\Omega} \) introduced in Keldyš [21, Section V] if \( \Omega \) is bounded and \( \text{int}(\Omega) = \Omega \).

We first recall the definition of stability given by Keldyš. Suppose that \( \Omega \) is an open bounded set with \( \text{int}(\Omega) = \Omega \). Let \( \Omega_n \) be Dirichlet regular open sets (called normal in [21]), such that \( \Omega \Subset \Omega_{n+1} \Subset \Omega_n \) for all \( n \in \mathbb{N} \) and such that \( \bigcap_{n \in \mathbb{N}} \Omega_n = \Omega \). Given \( \phi \in C(\mathbb{R}^N) \), denote the unique solution of

\[
-\Delta v = 0 \quad \text{in } \Omega_n, \\
v = \phi \quad \text{in } \Omega_n^c
\]

by \( v_{\phi,n} \). By assumption \( \Omega_n \) is Dirichlet regular, which by definition means that \( v_{\phi,n} \in C(\mathbb{R}^N) \). Furthermore, denote by \( v_{\phi} \) the Perron solution (see [21]) of the problem

\[
-\Delta v = 0 \quad \text{in } \Omega, \\
v = \phi \quad \text{in } \Omega^c
\]

on the possibly irregular domain \( \Omega \). Then Keldyš calls \( \overline{\Omega} \) stable if \( v_{\phi,n} \to v_{\phi} \) uniformly on \( \overline{\Omega} \) for all \( \phi \in C(\mathbb{R}^N) \). It turns out that the definition of stability is independent of the particular sequence \( \Omega_n \) chosen. Since the uniform limit of continuous functions is continuous it follows that \( v_{\phi} \) is continuous. Hence, if \( \overline{\Omega} \) is stable, then \( \Omega \) is regular.

Further note that if \( \phi, \psi \in C(\mathbb{R}^N), \varepsilon > 0 \) and \( |\phi - \psi| < \varepsilon \) on \( \Omega_1 \), then by the maximum principle for harmonic functions \( |v_{\phi,n} - v_{\psi,n}| < \varepsilon \) on \( \Omega_1 \) for all \( n \in \mathbb{N} \). Hence, \( \Omega \) is stable if and only if \( v_{\phi,n} \to v_{\phi} \) uniformly on \( \overline{\Omega} \) for all \( \phi \) in a dense subset of \( C(\mathbb{R}^N) \).

In particular we can use \( \phi \in C^2(\mathbb{R}^N) \). Since all sets \( \Omega_n \) are contained in a suitable bounded subset of \( \mathbb{R}^N \) (recall Definition 3.2) we have that \( \sup_{n \in \mathbb{N}} \|R_{\Omega_n}(0)\|_{L(\infty)} < \infty \), so by Theorem 3.1 we can use \( \lambda = 0 \) in our proofs. We now have the following result.

**Theorem A.1.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded set with \( \text{int}(\overline{\Omega}) = \Omega \). Then \( \overline{\Omega} \) is stable in the sense of Keldyš if and only if \( \Omega \) is uniformly stable, that is, \( B(\Omega, 1/n) \overset{\text{gr}}{\to} \Omega \) as \( n \to \infty \).

**Proof.** First note that there always exist regular domains \( \Omega_n \), such that

\[
B(\Omega, 1/(n + 1)) \subset \Omega_n \subset B(\Omega, 1/n)
\]
(see [15, V.4.8]). Then by domination $\Omega_n \stackrel{\text{gu}}{\rightarrow} \Omega$ if and only if $B(\Omega, 1/n) \stackrel{\text{gu}}{\rightarrow} \Omega$. Hence we show that $\overline{\Omega}$ is stable in the sense of Keldyš if and only if $\Omega_n \stackrel{\text{gu}}{\rightarrow} \Omega$. First we assume that $\overline{\Omega}$ is stable. Set $u_n := R_{\Omega_n}(0)1$ and $\phi(x) := |x|^2/2N$. Then $v_n := u_n + \phi$ satisfies (A.1). Since $\overline{\Omega}$ is stable it follows that $v_n \rightarrow v$ uniformly on $\overline{\Omega}$, where $v$ is the solution of (A.2). Since $v$ is continuous and by monotonicity $v_n \nearrow v$, Dini’s theorem implies that $v_n \rightarrow v$ uniformly on $\mathbb{R}^N$. Clearly $u := v - \phi \in C_0(\Omega)$ satisfies $-\Delta u = 1$. Hence, $u = R_{\overline{\Omega}}(0)1$ by [1, Theorem 2.5], showing that $\Omega_n \stackrel{\text{gu}}{\rightarrow} \Omega$. Fix $\varphi \in C^2(\mathbb{R}^N)$ and set $f := \Delta \varphi$. By assumption $u_n := R_{\Omega_n}(0)f \rightarrow u := R_{\overline{\Omega}}(0)f$. Since $u_n, u \in C(\mathbb{R}^N)$ we have $v_{\varphi,n} = u + \varphi$ and $v_{\varphi} = u_n + \varphi$, so $v_{\varphi,n} \rightarrow v_{\varphi}$ uniformly on $\overline{\Omega}$. Hence $\overline{\Omega}$ is stable as claimed. 

**Remark A.2.** (a) As mentioned before, stability of $\overline{\Omega}$ in Keldyš [21] is not the same as stability in Hedberg [16]. The difference is that Keldyš (see [21, Section V] calls $\overline{\Omega}$ stable if $v_{\varphi,n} \rightarrow v_{\varphi}$ uniformly on $\overline{\Omega}$ for all $\varphi \in C(\mathbb{R}^N)$, whereas Hedberg’s notion of stability is equivalent to the requirement that $v_{\varphi,n} \rightarrow v_{\varphi}$ uniformly on $\overline{\Omega}$ for all $\varphi \in C(\mathbb{R}^N)$ for which $v_{\varphi} \in C(\overline{\Omega})$ (see [16, Theorem 11.8]).

(b) In both cases discussed in (a), stability is a local property of $\partial \Omega$. We say that $x \in \partial \Omega$ is a stable point of $\partial \Omega$ if $v_{\varphi,n}(x) \rightarrow v_{\varphi}(x)$ for all $\varphi \in C(\mathbb{R}^N)$ (see [21, Section V]). As usual, we call $x$ regular if $v_{\varphi}$ is continuous at $x$ for all $\varphi \in C(\mathbb{R}^N)$. It turns out that the notion of stability used by Hedberg (and most papers on $L^2$-convergence for varying domains) is equivalent to the fact that the set of regular and stable points of $\partial \Omega$ coincide (see [21, Theorem XIX] and [16, Theorem 11.8]). Stability of $\overline{\Omega}$ in the sense of Keldyš is equivalent to saying that all points of $\partial \Omega$ are stable points (this follows from the definition and [21, Theorem XVII]). A domain in $\mathbb{R}^3$ with a Lebesgue cusp for instance is stable in the sense of Hedberg, but not in the sense of Keldyš, so the two definitions do not coincide. As a consequence we could assume in Theorem 8.8 that $\Omega$ be stable for $L^2$-convergence (that is, in the sense of Hedberg) and regular as this is the same as the uniform stability.

**References**


